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THE PRICE OF NATURAL FLAVOUR CONSERVATION IN NEUTRAL WEAK INTERACTIONS

Michael S. Chanowitz*), John Ellis and Mary K. Gaillard**)

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ABSTRACT

The natural conservation of flavours to $O(G_F^2)$ in neutral weak interactions severely constrains choices of gauge groups as well as their fermion representations. In the absence of exactly conserved quantum numbers other than charge, and of $|\Delta Q| \geq 2$ charged currents, essentially the only weak and electromagnetic gauge groups whose neutral interactions naturally conserve all flavours are $SU(2)_L \times U(1)$ and $SU(2)_L \times [U(1)]^2$. The plausible extensions of these gauge groups to grand unified models including the strong interactions are based on SU(5) and SO(10) respectively. Making the SU(5) model completely natural, including in the Higgs sector, gives the prediction

$$m_d/m_e \simeq m_s/m_\mu \simeq m_b/m_\tau ,$$

where τ is the probable new heavy lepton and b is the conjectured third flavour of charge -1/3 quark. The SO(10) model contains a potential $SU(2)_L \times SU(2)_R \times U(1)$ weak and electromagnetic gauge group, and has a complicated Higgs structure which does not naturally conserve quark flavours. However, SO(10) is better than SU(5) at reconciling data on the weak neutral current mixing angle and proton stability.

*) Present address: Lawrence Berkeley Laboratory, Berkeley, CA 94720.

**) Laboratoire de Physique Théorique et Particules Elémentaires, Orsay (Laboratoire associé au CNRS)

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1. INTRODUCTION

It seems that flavour-changing neutral currents are greatly suppressed in both the leptonic and quark sectors. Muon and electron numbers are separately conserved to a high degree of accuracy, and strangeness-changing neutral interactions are apparently $O(G_F^2)$. This strong suppression of $\Delta S = 1$ neutral currents was the prime motivation for the existence of charm, which seems experimentally to play the rôle proposed for it by Glashow, Iliopoulos and Maiani[1]. Now it also seems likely that $\Delta C = 1$ neutral interactions are very strongly suppressed. In the prototype $SU(2)_L \otimes U(1)$ gauge model[2),3] all flavour-changing neutral currents are indeed suppressed to order G_F^2. Direct second order (Fig. 1) and indirect fourth order (Fig. 2) neutral weak currents are subject to the GIM cancellation, and even the Higgs system can be chosen so that neutral Higgs boson exchanges (Fig. 3) conserve flavour.

Several authors[4,5,6] have recently proposed that the $O(G_F^2)$ suppression of flavour-changing neutral currents be promoted to a general principle. In particular, Glashow and Weinberg[4] have enunciated the conditions which ensure that the direct and induced neutral currents in an $SU(2) \otimes U(1)$ gauge theory conserve all flavours "naturally", i.e., for all values of the parameters of the theory*). They found that all fermions of the same charge and helicity should have the same $SU(2)$ transformation properties and acquire their masses from the same unique source, either a single neutral Higgs boson or a gauge invariant bare mass term. Thus the "naturalness" requirement severely restricts the possible representation content of an $SU(2) \otimes U(1)$ gauge theory.

We do not know whether all quark and lepton flavours are indeed naturally conserved by all neutral weak interactions. Indeed, there are several suggestions that the standard "natural" $SU(2)_L \otimes U(1)$ gauge theory is inadequate, such as the high y anomaly in antineutrino scattering, the negative results of atomic physics experiments searching for parity violation, possible radiative decays of charged leptons, and trimuon events in neutrino scattering. However, the phenomenological relevance, aesthetic**) appeal and restrictive power of the principle of naturalness make us seek to push it to its limit. In this paper we systematically study the restrictions it imposes on the choice of gauge group as well as

*) They and we assume that quarks are fractionally charged and interact via massless, unbroken $SU(3)$ colour gauge gluons.

**) We are humbly aware that aesthetic judgements are subjective and time-dependent. The cockroaches of Troy probably did not understand why the Greeks were making so much fuss.

the representation content. Almost any group can probably be made natural by some trick: our purpose is not so much to rigorously exclude any possibility, but rather to assess the aesthetic price you must pay for inflicting natural flavour conservation on any given group. Most of the time we will talk about quarks, but most of the arguments apply equally to leptons. However, the lepton mass spectrum looks qualitatively different from that of quarks ($m_\nu \simeq 0$), and it is perhaps not clear that the same naturalness conditions should be applied.

We will first be concerned with the form of the spontaneously broken unified gauge theory of weak and electromagnetic interactions[3)], which we call Quantum Asthenodynamics[*)] or QAD. We will then study the incorporation of natural QAD theories together with the strong interactions in a completely unified gauge theory[7)], which we call Quantum Holodynamics[*)] or QHD. It will emerge that in the absence of any absolutely conserved quantum number besides electric charge Q, or of exotic $|\Delta Q| \geq 2$ charged currents, the only plausible, completely natural, QAD models are based on the groups $SU(2) \otimes U(1)$ and $SU(2) \otimes [U(1)]^2$. The most plausible QHD theories containing these possibilities are based on the groups SU(5) [7)] and SO(10) [8),9),10)], respectively. The SU(5) model can be made completely natural by using[7)] a single Higgs multiplet, in which case it predicts that at present energies $\mu \simeq 10$ GeV

$$\frac{m_d}{m_e} \approx \frac{m_s}{m_\mu} \approx \frac{m_b}{m_\tau} \approx (2 \text{ to } 5) \tag{1.1}$$

where m_b is the effective mass of a conjectured third flavour of charge $-1/3$ quark, and m_τ is the mass of the heavy lepton probably discovered recently. However, the SU(5) QHD model is only marginally consistent with present data on the weak neutral current mixing angle θ, and the lower limit of $\sim 10^{30}$ years on the proton lifetime τ_{proton}. In contrast, the SO(10) model is natural except for the (possibly negligible) Higgs exchanges. It predicts that at present energies

$$\sin^2\theta \approx 0.3 \;, \; \alpha_s(\mu) = O(0.1), \; \tau_{proton} \gg 10^{30} \text{ years} \tag{1.2}$$

for a reasonable range of grand unification masses.

*) The Greek word ἀσθενής means weak, without strength. The greek word ὅλος means whole or complete.

The layout of our paper is as follows. In Section 2 we restate for general groups the conditions Glashow and Weinberg found necessary for naturalness in $SU(2) \otimes U(1)$ models. When proceeding to higher groups, we find it convenient to state a "zeroth" condition that the unbroken symmetric gauge theory has no flavour-changing neutral currents. We also consider that "natural" theories should have arbitrary Higgs potentials and vacuum expectation values. We will assume the absence of exactly conserved quantum numbers other than electric charge Q. In Section 3 we first consider an arbitrary QAD gauge group $G_{QAD} = \hat{\otimes}_i G_i$ where the G_i are simple groups, and first show that if any of the G_i has rank ≥ 2, then a natural theory will have $|\Delta Q| \geq 2$ charged currents. If G_{QAD} is itself simple, it and its representations must be somewhat weird. We are then reduced to $[SU(2)]^N \otimes [U(1)]^M$, and show that completely natural theories with $N > 1$ generally have exotic quark charges and often $|\Delta Q| \geq 2$ charged currents. Reduced to $SU(2) \otimes [U(1)]^M$, we first reiterate that a natural $SU(2)$ acts on a unique helicity[3),4),5)], and then show that requiring the cancellation of triangle anomalies within the observed spectrum of fundamental fermions strongly suggests $M \leq 2$.

In Section 4 we turn to QHD theories, motivating and studying the possibility that rank (G_{QHD}) = rank $(G_{QAD}) + 2$ $[\text{the } 2 \text{ for } SU(3)_{QCD}]$. For G_{QAD}, the unique plausible G_{QHD} is $SU(5)$[7)]. But $\sin^2\theta$ is strongly renormalized[11)] from its Clebsch value of $3/8$ at the grand unification mass scale, becoming $\sin^2\theta \leq 0.2$ at present energies, and the proton lifetime is also a bit dodgy. If $SU(5)$ is to be completely natural, a unique Higgs representation must be responsible for quark and lepton masses. Taking this to be a $\underline{5}$ gives[7)] at the grand unification mass scale (GUM):

$$\left.\frac{m_d}{m_e}\right|_{GUM} = \left.\frac{m_s}{m_\mu}\right|_{GUM} = \left.\frac{m_b}{m_\tau}\right|_{GUM} = 1 \tag{1.3}$$

which is renormalized to (1.1) at present energies. For $G_{QAD} = SU(2) \otimes [U(1)]^2$, the unique plausible G_{QHD} of rank 5 is $SO(10$[8),9),10)], which we analyze in some detail. The group $SO(10)$ actually contains a possible G_{QAD} subgroup of $SU(2)_L \otimes SU(2)_R \otimes U(1)$ [12),13),14)]. It needs more than one Higgs multiplet to give realistic fermion masses and mixings, and is hence not completely natural, even though the $SU(2)_R$ bosons can in principle be zapped to arbitrarily high masses. We derive the predictions (1.2) for $\sin^2\theta$, $\alpha_S(\mu)$ and τ_{proton}, but find no useful mass relations. In Section 5 we comment on some QAD theories, and draw some conclusions from our analysis of naturalness. We feel that naturalness almost forces you into a QAD gauge group of the form $SU(2) \otimes [U(1)]^M$, and that SU(5) and SO(10) are then the least unlikely QHD gauge groups. Of these two, SU(5) is more natural but has more phenomenological problems. Maybe naturalness will not long remain a viable assumption.

2. CONDITIONS FOR NATURAL FLAVOUR CONSERVATION

It seems that neutral current cross-sections are generally of the same order as charged current cross-sections, so that neutral current amplitudes are generally $O(G_F)$. This is apparently untrue for $\Delta S = 1$ neutral currents, which are experimentally of $O(G_F^2\, m_h^2)$, with m_h a typical hadron mass. Thus $\Delta S = 1$ neutral currents are apparently not only absent in second order $O(G_F)$ weak interactions, but also suppressed beyond the $O(G_F\, \alpha)$ naïvely expected from the fourth order weak interactions of Fig. 2. These properties are guaranteed by the Glashow, Iliopoulos and Maiani (GIM)[1] form of charm-changing charged current, whatever the masses of the u and c quarks and the value of the Cabibbo mixing angle might be. This is what Glashow and Weinberg[4] term the "natural" suppression of $\Delta S = 1$ neutral currents. From upper limits on the fraction of $\Delta S = 2$ final states at SPEAR, it seems that mass mixing of D^o and $\bar{D}^o$ is not larger than their decay rates. This suggests that $\Delta C = 1$ neutral currents are also $O(G_F^2\, m_h^2)$, again as guaranteed "naturally" by the GIM current. It is, therefore, tempting to pursue the suggestion of Glashow and Weinberg that all flavour-changing neutral currents be "naturally" suppressed -- i.e., $O(G_F^2\, m_h^2)$ independently of the particular values of elements in the quark mass-matrix. Glashow and Weinberg[4], and Paschos[5] and co-authors[6], have studied the consequences of this Ansatz for $SU(2) \otimes U(1)$ QAD theories, finding severe constraints on the fermion representation content. We first set out the conditions necessary for natural flavour conservation by neutral interactions in a general gauge group.

Consider a semisimple gauge group G_{QAD} with several neutral generators Y^i coupled to electrically neutral gauge boson fields Z^i in the symmetric, unbroken theory. Suppose that the Z^i are coupled to left-handed (right-handed) quarks via the coupling matrices $Y_L^i (Y_R^i)$. The QAD Lagrangian contains terms

$$\sum_i g_i\, J_i^\mu Z_\mu^i$$

where

$$J_i^\mu = \bar{q}\gamma^\mu(1-\gamma_5)Y_L^i q + \bar{q}\gamma^\mu(1+\gamma_5)Y_R^i q \qquad (2.1)$$

In order that there be no flavour-changing neutral currents in $O(G_F)$, it is first necessary that there be no flavour-changing neutral current in the QAD theory before spontaneous symmetry breakdown. This means that in (2.1) all the coupling matrices $Y^i_{L,R}$ must be diagonal in quark fields. Any such flavour-changing neutral current in the unbroken theory would also show up in the spontaneously broken theory. The absence of such a current is trivial in $SU(2) \otimes U(1)$ gauge theories, but not for larger gauge groups G_{QAD}. For example, in the $SU(3)$ scheme of Fritzsch and Minkowski[15] there is a $\Delta F = 1$ neutral current in the unbroken theory corresponding to the non-vanishing root indicated by an arrow in the quark representation of Fig. 4. Even if the quarks d and e did not mix in the broken theory, there would be a flavour-changing neutral current. All generators $E_{\pm\alpha}$ of G_{QAD} with non-vanishing roots $\pm\alpha$ must have $\pm\Delta Q \geq 1$. Mathematically, the condition can be stated as

(A) All neutral generators Y^i must lie in H_{QAD}, the set of mutually commuting, simultaneously diagonalizable linear operators in G_{QAD}.

Now consider the spontaneously broken gauge theory with the quark mass term

$$-\bar{q}(1-\gamma_5)Mq \quad - \bar{q}(1+\gamma_5)M^+q \tag{2.2}$$

The mass matrix M is arbitrary, but must commute with the electric charge Q:

$$[M, Q] = 0 \tag{2.3}$$

The quark mass matrix M is then transformed by introducing unitary matrices U_L, U_R and the new quark fields

$$q' = \tfrac{1}{2}(1-\gamma_5)U_L q + \tfrac{1}{2}(1+\gamma_5)U_R q \tag{2.4}$$

so as to take the diagonal form

$$M^D = U_L M U_R^+ \tag{2.5}$$

The neutral current couplings Y_L^i and Y_R^i of Eq. (2.1) then become

$$Y_L^{iD} = U_L Y_L^i U_L^{-1} \quad , \quad Y_R^{iD} = U_R Y_R^i U_R^{-1} \tag{2.6}$$

The naturalness postulate of Glashow and Weinberg[4)] then demands that the Y_L^{iD} and Y_R^{iD} be diagonal for any choice of M obeying the condition (2.3), and hence for any unitary U_L and U_R commuting with Q. There might be other exactly conserved quantum numbers P^a in the theory, in which case M, U_L and U_R would also have to commute with the P^a:

$$[M, P^a] = 0 \quad , \quad [U_{L,R}, P^a] = 0 \tag{2.7}$$

In order for the $Y_{L,R}^{iD}$ to be diagonal for all such U_L and U_R, the $Y_{L,R}^i$ must have the same eigenvalues $Y_{L,R}^i$ for all quarks with the same charges and P^a quantum numbers. Therefore the $Y_{L,R}^i$ must be functions of the matrices Q and P^a:

$$Y_{L,R}^i = Y_{L,R}^i(Q, P^a) \tag{2.8}$$

Glashow and Weinberg[4)] did not consider in full detail the implications of such exactly conserved quantum numbers P^a. As emphasized by Paige, Paschos and Trueman[6)], the exclusion of such P^a is necessary to eliminate large classes of flavour-conserving gauge models. In the rest of our analysis, we will <u>assume no exactly conserved quantum number exists except charge</u>[*)]. In this case, Eq. (2.8) reduces to

$$Y_{L,R}^i = Y_{L,R}^i(Q) \tag{2.9}$$

so that there is an extended version of a deduction of Glashow and Weinberg.

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*) In Section 3.3 we discuss the $SU(2)_L \otimes SU(2)_R \otimes U(1)$ model of Ma[12)], which conserves flavour to $O(G_F^2)$ in neutral interactions by virtue of a new exactly conserved quantum number.

(B) All quarks of the same charge and helicity must have the same eigenvalues for each of the Y^i.

Since it follows from condition (A) that Q and the Y^i together form H_{QAD}, the set of mutually commuting linear operators in G_{QAD}, condition (B) means that all quarks of the same charge and helicity must have the same weights. Conditions (A) and (B) together ensure that there are no neutral weak interactions in $O(G_F)$: we now turn to the fourth order diagrams illustrated in Fig. 2.

If we consider the diagrams of Fig. 2a, then Glashow and Weinberg[4)] pointed out that together they produce a change ΔY^i in the coupling of Z^i of the form[*)]

$$\Delta Y^i \propto T_{+\alpha} Y^i T_{-\alpha} + T_{-\alpha} Y^i T_{+\alpha} - \tfrac{1}{2}\{T_{+\alpha}, T_{-\alpha}\} Y^i - \tfrac{1}{2} Y^i \{T_{+\alpha}, T_{-\alpha}\} \tag{2.10}$$

where we denote by $T_{+\alpha}(T_{-\alpha})$ the quark coupling matrix of the gauge boson $W_{+\alpha}$ $(W_{-\alpha})$ corresponding to the non-diagonal generator $E_{+\alpha}(E_{-\alpha})$ of G_{QAD}. If we denote[*)] the root vectors r^i_α[**)]

$$[Y^i, T_{\pm\alpha}] = \pm r^i_\alpha T_{\pm\alpha} \tag{2.11}$$

then (2.10) becomes[4)]

$$\begin{aligned}\Delta Y^i &\propto T_{+\alpha} T_{-\alpha}(Y^i - r^i_\alpha) + T_{-\alpha} T_{+\alpha}(Y^i + r^i_\alpha) - \{T_{+\alpha}, T_{-\alpha}\} Y^i \\ &= -r^i_\alpha [T_{+\alpha}, T_{-\alpha}]\end{aligned} \tag{2.12}$$

But

$$[T_{+\alpha}, T_{-\alpha}] = r^j_\alpha Y^j \tag{2.13}$$

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*) In this and subsequent equations, the summation convention is not used for the index α.

**) If necessary, we will think of Q as Y^0, so that the charge of the boson $W_{+\alpha}$ is r^0_α.

so that the changed coupling $Y^i + \Delta Y^i$ of the boson Z^i is still contained in H_{QAD}, and conserves flavours because of condition (B). [Equations (2.10) to (2.13) apply separately to each helicity L and R.] As pointed out by Glashow and Weinberg[4)], two of the diagrams in Fig. 2a also require fermion mass renormalizations: these cause no problems because condition (B) ensures flavour conservation for any mass matrix. The diagrams of Fig. 2b are proportional to (2.13) and therefore conserve flavour through condition (B). Following Glashow and Weinberg[4)], we see that Fig. 2c gives an effective current-current Fermi interaction of strength $G_F\alpha$ with left- and right-handed coupling matrices $X^{+\alpha-\beta}_{L,R}$ which contain both $[T^{+\alpha}_{L,R}, T^{-\beta}_{L,R}]$ and $\{T^{+\alpha}_{L,R}, T^{-\beta}_{L,R}\}$ pieces. Taking the $\alpha = \beta$ case, we find

$$X^{+\alpha-\alpha}_{L,R} \propto 3\left(\underline{\hat{T}}^{\alpha^2}_{L,R} - \hat{T}^{\alpha^2}_{3_{L,R}}\right) \pm 5\,\hat{T}^{\alpha}_{3_{L,R}} \tag{2.14}$$

where we have introduced

$$\hat{E}_{\pm\alpha} \equiv \frac{\sqrt{2}}{|r_\alpha|} E_{\pm\alpha}\,, \quad \hat{E}_{3\alpha} \equiv \frac{r^j_\alpha Y^j}{|r_\alpha|^2} \tag{2.15}$$

which together form an SU(2) group (α spin), and have denoted the coupling matrices of $\underline{\hat{E}}_\alpha$ by $\underline{\hat{T}}_\alpha$. Just as for Glashow and Weinberg, the couplings $X^{+\alpha-\alpha}_{L,R}$ are diagonal whenever the quark masses are diagonal if and only if

$$\underline{\hat{T}}^{\alpha^2}_{L,R} = g^\alpha_{L,R}(Q) \tag{2.16}$$

Hence all quarks of the same charge and helicity must have the same value of each α spin, as well as the same weight. It is then clear that the following condition must hold:

(C) All irreducible quark representations of G_{QAD} must either be identical or else be completely non-overlapping as far as quark charges are concerned.

The cases $\alpha \neq \beta$ in Fig. 2c give no further restrictions. If α and β have different charges, they cannot give a neutral current. If α and β have the same charge $[E_{+\alpha}, E_{-\beta}] = 0$ because of naturalness condition (A). Then $\{E_{+\alpha}, E_{-\beta}\} \neq 0$, but seeks to connect quarks of the same charge but different Y^i, which are forbidden by condition (B).

This completes the discussion of second- and fourth-order intermediate vector boson exchange diagrams. The one remaining interesting possible source of flavour-changing neutral weak interactions is the second-order exchange of some neutral Higgs boson shown in Fig. 3. The strength of such a graph is

$$g^2_{Hq\bar{q}}/m^2_H \tag{2.17}$$

where one might expect

$$g^2_{Hq\bar{q}} = O(m_q^2 G_F) \tag{2.18}$$

Higgs exchange will therefore be of order $G_F^2 m_q^2$ if

$$m_H^2 \gtrsim O\left(\frac{1}{G_F}\right) \approx (300\,\mathrm{GeV})^2 \tag{2.19}$$

The Higgs system becomes strongly interacting if $m_H \geq 1$ TeV, so this may give some order of magnitude upper limit on reasonable Higgs particle masses. If the H^4 coupling constant λ is $O(e^2)$, which might seem a plausible possibility, then

$$m_H = O(100)\,\mathrm{GeV} \tag{2.20}$$

It is therefore not[13,14] completely clear that flavour-changing Higgs exchanges are strong enough to worry about. If one does, then as shown by Glashow and Weinberg[4], there is the condition:

(D) All quarks of a given charge must get their masses from the same, unique Higgs boson*).

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*) We dismiss the alternative of an invariant mass term as impossible to realize in realistic natural models.

We would interpret the naturalness philosophy as requiring that the condition (D) apply for the most general Higgs-fermion and inter-Higgs couplings, and Higgs vacuum expectation values, consistent with the G_{QAD} symmetry properties of the various fields. Among other things, this means that in general not more than one Higgs multiplet in the same G_{QAD} representation can be allowed.

Having made the routine generalization of the Glashow-Weinberg naturalness conditions to QAD groups different from $SU(2) \otimes U(1)$, we are now in a position to study the restrictions they imply for general QAD theories. We will apply mercilessly conditions (A) to (C), but condition (D) on the Higgs system we may sometimes relax.

3. NATURALNESS RESTRICTIONS ON THE CHOICE OF G_{QAD}

We will now consider a weak and electromagnetic gauge group

$$G_{QAD} = \bigotimes_i G_i \tag{3.1}$$

where the G_i are simple groups or $U(1)$ factors. Almost any group can probably yield a natural gauge theory if sufficiently exotic charge assignments are made for the bosons and fermions, or sufficiently bizarre fermion representations are chosen. Therefore we will not exclude rigorously any choice (3.1) of G_{QAD}, but rather indicate the aesthetic price*) to be paid in each case.

3.1 G_{QAD} with a factor G_i of rank ≥ 2

Every simple group of rank > 2 contains some simple subgroup of rank = 2, and it will suffice to consider this latter case. We therefore examine $A_2 = SU(3)$, $B_2 = C_2 = SO(5)$ and G_2. Condition (A) of Section 2 implies that all the root vectors must carry $\Delta Q \neq 0$. Because rank 2 groups have three or more roots, we are therefore forced into roots (and hence charged currents and vector bosons) with $|\Delta Q| \geq 2$. The least exotic choices for $SU(3)$, $SO(5)$ and G_2 are shown in Fig. 5. The lowest "natural" quark representation in $SU(3)$ is shown in Fig. 6. The weirdness of quark and boson charges rapidly grows with rank > 2. For example, the simplest natural QAD theory with an $SU(4)$ factor would have $|\Delta Q| = 3$ charged currents and quarks with charges $5/3$ and $-4/3$.

*) See, however, the footnote on page 1.

3.2 G_{QAD} simple

Since neutral currents exist, G_{QAD} must have rank ≥ 2, so that we run into all the problems of Section 3.1. However, more problems arise when G_{QAD} is simple, for any representation R must have

$$\sum_{q \in R} Q_q = 0 \tag{3.2}$$

Since we want quarks with charges $-1/3$ and $+2/3$, the condition (3.2) means that we must use irreducible representations with dimensionality divisible by 3. In the case of SU(3), the simplest three-dimensional representation of Fig. 6. does not have the property (3.2). No natural SU(3) model based on a six- or 15-dimensional representation exists either, the crucial stumbling block being condition (B) of Section 2. It is barely conceivable that a higher dimensional representation of SU(3) might work: if so it would contain quarks with charges $Q = O(10)$. The lowest candidate representations of SO(5) and G_2 have dimensions 30 and 27 respectively: in view of Figs. 5b and 5c we have not studied them. We see no way things can improve for simple groups of rank > 2, and conclude that natural simple G_{QAD} theories are outlandish.

3.3 G_{QAD} containing $SU(2) \otimes SU(2)$

If we exclude factors G_i of rank ≥ 2, then

$$G_{QAD} = [SU(2)]^N \otimes [U(1)]^M : N, M \geq 1 \tag{3.3}$$

We start with $N = 2$: the constraints of naturalness are more difficult to satisfy if $N > 2$. If either of the SU(2) groups is ambidextrous, that is it acts on quarks of both left and right helicities, then we run into a more complicated version of the problems found by Glashow and Weinberg[4)]. They showed that ambidextrous $SU(2) \otimes U(1)$ models were all unnatural: the phenomenologically excluded vector-like models were unnatural only in the Higgs sector condition (D) of Section 2 while other ambidextrous models had more unnatural sins. Accordingly we only consider single-handed SU(2) groups, and so analyze the two distinct cases of $SU(2)_L \otimes SU(2)_L$ and $SU(2)_L \otimes SU(2)_R$.

$SU(2)_L \otimes SU(2)_L$

Suppose we placed some quarks in an (n,m) $(n > 1,\ m > 1)$ matrix representation of $SU(2)_L \otimes SU(2)_L$. Then by condition (B) of Section 2 all the quarks in the matrix must have different charges. Hence there must be at least $n \times m$ distinct quark charges. Furthermore, if we take the W's acting up and down columns to have $|\Delta Q| = 1$, the non-overlapping charges mean the row W's must have $|\Delta Q| \geq n$. Even the simplest case with a $(2,2)$ representation

$$\begin{pmatrix} Q = 2/3 & Q = 8/3 \\ Q = -1/3 & Q = 5/3 \end{pmatrix} \tag{3.4}$$

of $SU(2)_L \otimes SU(2)_L$ is rather unappealing.

If we try to put quarks into $(n,1)$ and $(1,m)$ representations of $SU(2)_L \otimes SU(2)_L$, then condition (C) will tolerate no overlap of quark charges between the representations. The total range of quark charges then ranges over at least $(n+m-1)$ units. Also there is no communication at all between the two left-handed worlds: possible, but uninteresting.

$SU(2)_L \otimes SU(2)_R$

This gauge subgroup has been considered by Fritzsch and Minkowski[8], Ma[12], Mohapatra and Sidhu[13], de Rujula, Georgi and Glashow[14], and others. Their models are very interesting for phenomenology, as they introduce right-handed currents without being trapped in the straitjacket fatal to vector-like models[16]. These models include at least some non-trivial representations (i.e., dimension ≥ 2) of each of the $SU(2)$ groups. As shown by de Rujula, Georgi and Glashow[14], the naturalness conditions (B) and (C) then require all quarks to be in identical representations of each $SU(2)$. Suppose we put them in n (≥ 2) dimensional representations of $SU(2)_L$ and m (≥ 2) dimensional representations of $SU(2)_R$. The only unnaturalness arises in the Higgs sector. We need Higgses H in (n,m) dimensional representations Σ of $SU(2)_L \otimes SU(2)_R$ to get $m_q \neq 0$. If we try a single real representation, the Higgs quark-antiquark interaction is

$$h_{rs}\, \bar{q}_L^r H q_R^s + h_{rs}^* \,\bar{q}_R^s H^\dagger q_L^r \tag{3.5}$$

Let us suppose that p quarks are common to both left- and right-handed representations and that H and $H^\dagger$ acquire vacuum expectation values which we may as well write as

$$H_0 = \left(\begin{array}{cc|c} a_1 & 0 & \\ & \ddots & 0 \\ 0 & a_p & \\ \hline & 0 & 0 \end{array}\right), \quad H_0^\dagger = \left(\begin{array}{cc|c} a_1^* & 0 & \\ & \ddots & 0 \\ 0 & a_p^* & \\ \hline & 0 & 0 \end{array}\right) \tag{3.6}$$

The quark mass matrix (2.2) is then

$$\bar{q}_L^r \left(\begin{array}{cc|c} h_{rs}a_1 & 0 & \\ & \ddots & 0 \\ 0 & h_{rs}a_p & \\ \hline & 0 & 0 \end{array}\right) q_R^s + \bar{q}_R^s \left(\begin{array}{cc|c} h_{rs}^*a_1^* & 0 & \\ & \ddots & 0 \\ 0 & h_{rs}^*a_p^* & \\ \hline & 0 & 0 \end{array}\right) q_L^r \tag{3.7}$$

We can diagonalize h_{rs} and hence the matrix (3.7) by appropriate unitary transformations on the left and right fields, but then all the generalized Cabibbo angles amongst the first p quarks vanish if $p > 1$ as usually wanted. We are then forced[13),14)] into more than one real Higgs multiplet, in <u>conflict with naturalness</u> condition (D) of Section 2[*)]. Indeed, with two Higgs multiplets $H^{(i)}$, with quark couplings $h_{rs}^{(i)}$, the generalized Cabibbo angles are non-zero if and only if the $h_{rs}^{(i)}$ are not proportional, in which case the <u>neutral Higgs exchanges necessarily violate flavour conservation</u>. Of course, as discussed in Section 2, you may not care about the Higgs couplings. If you want to stay natural in $SU(2)_L \otimes SU(2)_R$, then you must get the left- and right-handed multiplets to have charge overlaps ≤ 1[**)]. Such a model would have quark charges $\geq 5/3$ or $\leq -4/3$, and is not wanted by anyone at the moment[***)].

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*) This conflict with naturalness is ignored in Refs. 13) and 14), but recognized by de Rujula in Ref. 17).

**) In a sense, this happens in the standard $SU(2)_L \otimes U(1)$ model[2),3)], where all q_R are singlets, so that $SU(2)_R$ is inactive.

***) Of course, a $\begin{pmatrix} d \\ -\frac{4}{3} \end{pmatrix}_R$ doublet would fix the $\bar{\nu}$ high y anomaly.

In this analysis of $SU(2)_L \otimes SU(2)_R$, as elsewhere and presaged in Section 2, we have assumed there are no exactly conserved quantum numbers besides charge. An amusing model which is natural, violates the conditions (B) and (C), and has such a quantum number is a $SU(2)_L \otimes SU(2)_R \otimes U(1)$ model of Ma. It has quark multiplets

$$\begin{pmatrix} u \\ d \end{pmatrix}_L , \begin{pmatrix} c \\ s \end{pmatrix}_L ; \begin{pmatrix} u \\ b \end{pmatrix}_R , \begin{pmatrix} t \\ d \end{pmatrix}_R ; c_R, s_R ; b_L , t_L \tag{3.8}$$

and (1,2), (2,1) Higgs multiplets so that $m_u = m_d = 0$. We can define a quantum number $\mathcal{M} = +1$ for $u_L, d_L, c_L, s_L, c_R, s_R$, $= -1$ for $u_R, d_R, t_R, b_R, t_L, b_L$, $= 0$ for the W's and the Higgs. The quantum number $\mathcal{M}$ is exactly conserved, and the u and d quarks remain massless in the renormalized theory. Because of the vanishing masses, as well as other phenomenological reasons, Ma's original model[12] (3.8) is not of direct interest, but it does point up how naturalness conditions may be evaded.

3.4 G_{QAD} of the form $SU(2) \otimes [U(1)]^M$

As shown by Glashow and Weinberg[4], there is no completely natural theory with an ambidextrous SU(2) subgroup and quark charges of -1/3 and 2/3. The conditions (A), (B) and (C) of Section 2 force us into the standard $SU(2)_L$ or vector-like $SU(2)_A$, and the latter is not natural in the Higgs sector (D). Ambidextrous natural $SU(2)_A \otimes [U(1)]^M$ theories with very exotic quark charges could in principle be constructed, but we assume that only $SU(2)_L \otimes [U(1)]^M$ theories are now of practical interest. Is there any restriction on the number M? The naturalness conditions (B) and (C), together with the observed doublets

$$\begin{pmatrix} u \\ d' \end{pmatrix}_L , \begin{pmatrix} c \\ s' \end{pmatrix}_L , \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L , \begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix}_L , \begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix}_L ?$$

force us into sequences of quark and lepton doublets with the same charges and hypercharges[*]. Let us call the $T = +\frac{1}{2}$ $(-\frac{1}{2})$ fermions anofermions (cathofermions). The $SU(2)_L \otimes [U(1)]^M$ groups are not safe from anomalies: their absence

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*) Unless we want fermions differing in charge by ≥ 2 from those seen so far.

restricts the possible representation content. It is well-known that in the standard $SU(2)_L \otimes U(1)$ each (coloured) quark isodoublet cancels against an (uncoloured) lepton isodoublet. The anomaly condition is

$$\sum T_{3_L}^2 Q = 0 \tag{3.9}$$

which is obeyed[18] by the observed doublets if and only if there are equal numbers of (coloured) quark doublets and (uncoloured) lepton doublets. Because of their non-zero mass, we are then forced into equal numbers of right-handed anoquarks ($Q = 2/3$), cathoquarks ($Q = -1/3$) and catholeptons ($Q = -1$). Our only residual freedom is in the number of right-handed massless anoleptons (neutrinos). We are therefore led to consider the following possible fermion structures in $SU(2)_L \otimes [U(1)]^M$

$$\begin{pmatrix} q_A \\ q_C \end{pmatrix}_L , \; q_{A_R} , \; q_{C_R} ; \; \begin{pmatrix} l_A \\ l_C \end{pmatrix}_L , \; l_{C_R} \tag{3.10}$$

$$\begin{pmatrix} q_A \\ q_C \end{pmatrix}_L , \; q_{A_R} , \; q_{C_R} ; \; \begin{pmatrix} l_A \\ l_C \end{pmatrix}_L , \; l_{A_R} , \; l_{C_R} \tag{3.11}$$

though the number of right-handed neutrinos is in principle arbitrary. We now ask what values of M can have their anomalies cancelled within the representation structures (3.10) and (3.11).

Our gauge group has the diagonal generators T_{3_L}, $Q = T_{3_L} + Y/2$, and possibly other hypercharges which we denote by Y^i. The general set of anomaly conditions is

$$\begin{array}{llllll}
\sum T_{3_L}^3 = 0 & (a) & \sum T_{3_L}^2 Y_L^i = 0 & (d) & \sum T_{3_L} Y_L^i Y_L^j = 0 & (g) \\
\sum T_{3_L}^2 Q = 0 & (b) & \sum T_{3_L} Q Y_L^i = 0 & (e) & \sum Q(Y_L^i Y_L^j - Y_R^i Y_R^j) = 0 & (h) \\
\sum T_{3_L} Q^2 = 0 & (c) & \sum Q^2 (Y_L^i - Y_R^i) = 0 & (f) & \sum (Y_L^i Y_L^j Y_L^k - Y_R^i Y_R^j Y_R^k) = 0 & (i)
\end{array} \tag{3.12}$$

Of these conditions, (3.12a and g) are trivially satisfied because of the representation contents (3.10) and (3.11). Condition (3.12b and c) are already obeyed through the usual charge assignments, while condition (3.12e) is equivalent to (3.12d). The only remaining conditions are therefore

$$\begin{aligned} &\sum Y_L^i = 0 \quad (a) && \sum Q\left(Y_L^i Y_L^j - Y_R^i Y_R^j\right) = 0 \quad (c) \\ &\sum Q^2\left(Y_L^i - Y_R^i\right) = 0 \quad (b) && \sum \left(Y_L^i Y_L^j Y_L^k - Y_R^i Y_R^j Y_R^k\right) = 0 \quad (d) \end{aligned} \qquad (3.13)$$

which we try to satisfy using the representations (3.11). A little algebra shows that the only linearly independent assignments of hypercharges to the objects

$$\left(q_{A,C_L}\ ,\ l_{A,C_L}\ ;\ q_{A_R}, q_{C_R}\ ;\ l_{A_R}\ ,\ l_{C_R}\right)$$

are

$$Y = \left(\tfrac{1}{3}\ ,\ -1\ ;\ \tfrac{4}{3}\ ,\ -\tfrac{2}{3}\ ;\ 0\ ,\ -2\right) \qquad (3.14)$$

or

$$Y^1 = \left(0\ ,\ 0\ ;\ 1\ ,\ -1\ ;\ 1\ ,\ -1\right) \qquad (3.15)$$

or

$$Y^2 = \left(0\ ,\ 0\ ;\ -7\ ,\ 5\ ;\ 1\ ,\ -(35)^{\frac{1}{3}}\right) \qquad (3.16)$$

Of these possibilities*), (3.14) is just the conventional hypercharge of $SU(2)_L \otimes U(1)$, and is the only one of the three which does not need a right-handed neutrino. The solution (3.16) looks pathological, and certainly could not be embedded in any grand unified theory involving other interactions. We are then left with (3.15) as the only new solution**). Thus, unless we introduce hypercharges which are zero for fermions but non-zero for Higgs multiplets, we can have at most $M = 2$:

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*) A theory with both Y^1 and Y^2 simultaneously would not obey all the conditions (3.13).

**) Notice that it has the structure $Y^1 \propto T_{3R}$, which could[8] be part of an $SU)2)_R$ subgroup. This analysis also suggests that not more than two SU(2) groups can be used in a gauge theory exploiting known fermions.

$$G_{QAD} = SU(2)_L \otimes [U(1)]^{1 \text{ or } 2} \tag{3.17}$$

Thus our chain of plausibility arguments based on the naturalness requirements - (A) to (D) of Section 2, and on the exclusion of pathological models leads us to the two candidates (3.17) for G_{QAD}. There is however an unaesthetic*) way to incorporate $SU(2)_L \otimes U(1)^M$ with arbitrary M, by choosing a family of M hypercharges $Y_{\alpha,\beta} = \alpha Y + \beta Y'$ with α,β arbitrary real numbers. To each hypercharge we associate a boson field $Z_{\alpha,\beta}$ which gets an arbitrary mass $M_{\alpha,\beta}$ from (for example) a singlet Higgs field ϕ. In the unbroken version such models collapse degenerately into $SU(2)_L \otimes U(1) \otimes U(1)$; they are distinguishable from the latter only by virtue of the symmetry breaking in the Higgs sector.

4. NATURAL QHD THEORIES

Having pursued the naturalness conditions for QAD theories, and found just the two plausible theories (3.17), it is reasonable to ask how they can be embedded in unified QHD theories[7),8),19)] of the strong, weak and electromagnetic interactions.

4.1 The rank of G_{QHD}

We assume that the strong interactions are described by QCD, and so by a gauge group $SU(3)_c$ with rank 2. The G_{QAD} groups (3.17) have ranks 2 and 3. Clearly

$$\text{rank}(G_{QHD}) \geq \text{rank}(G_{QAD}) + \text{rank}(G_{QCD}) = 4 \text{ or } 5 \tag{4.1}$$

Should the equality in (4.1) be realized? Candidate additions to the set H_{QHD} of mutually commuting linear operators in G_{QHD} are objects with non-trivial transformation properties under both colour and flavour groups, i.e., operators of the form

$$O \equiv \bar{q}\, \gamma_\mu \lambda^c_{3,8} F\, q \tag{4.2}$$

*) See, however, footnote on page 1.

where F is a matrix diagonal in helicity and flavour space but not necessarily proportional to unity. We first ask whether F can discriminate between different QAD isodoublets, such as

$$O \propto \bar{u}\gamma_\mu \lambda^c_{3,8} u - \bar{c}\gamma_\mu \lambda^c_{3,8} c$$

When we allow an arbitrary quark mass mixing matrix, such operators O will in general become non-diagonal in flavour space, for example

$$O \ni \bar{u}\gamma_\mu \lambda^c_{3,8} c$$

Fourth-order diagrams involving the combined exchange of a boson V coupled to O and of a gluon as shown in Fig. 7, will then generate flavour-changing neutral interactions. The bosons V are presumably superheavy, so that such flavour-changing neutral interactions are very suppressed. If V exchange is of order $g^2/m_V^2 = G_V$, they would tend to be of order $G_V\,\alpha_{strong}$. To exclude such flavour-changing neutral interactions needs a new "supernaturalness" principle for which there is absolutely no phenomenological justification, though it may seem to be an aesthetic*) extrapolation of the previous ideas of naturalness. If we make this important assumption, then the only linearly independent non-trivial forms of F that we have to consider are T_{3L}, T_{3R}, and $Y_L - Y_R$. It is then easy to see that the triangle diagrams in Figs. 8a and 8b will be anomalous, given the representation contents (3.11) and (3.12). So F should be a unit matrix in flavour space. The "supernaturalness" assumption therefore implies that the equality is realized in Eq. (4.1). We now consider only rank 4 {containing (3.10) and $SU(2)_L \otimes U(1)$} and rank 5 {containing (3.11) and $SU(2)_L \otimes [U(1)]^2$} as possibilities for G_{QAD}.

4.2 Rank 4: SU(5)

As was shown by Georgi and Glashow[7)], the only possible rank 4 G_{QHD} is SU(5). It breaks down into $SU(3)_c \otimes SU(2)_L \otimes U(1)$. The fifteen left-handed fermions

$$q^{R,Y,B}_{A_L}\,,\ q^{R,Y,B}_{c_L}\ ;\ \bar{q}^{R,Y,B}_{A_L}\,,\ \bar{q}^{R,Y,B}_{c_L}\ ;\ \ell_{A_L}\,,\ \ell_{c_L}\ ;\ \bar{\ell}_{c_L} \qquad (4.3a)$$

*) See, however, footnote on page 1.

are put into a $\underline{\bar{5}} + 10$ reducible representation of SU(5)

$$(\bar{q}_c^R, \bar{q}_c^Y, \bar{q}_c^B ; l_c, l_A)_L ; \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \bar{q}_A^B & -\bar{q}_A^Y & -q_A^R & -q_c^R \\ -\bar{q}_A^B & 0 & \bar{q}_A^R & -q_A^Y & -q_c^Y \\ \bar{q}_A^Y & -\bar{q}_A^R & 0 & -q_A^B & -q_c^B \\ q_A^R & q_A^Y & q_A^B & 0 & -\bar{l}_c \\ q_c^R & q_c^Y & q_c^B & \bar{l}_c & 0 \end{pmatrix}_L \tag{4.3b}$$

If we introduce, following Georgi, Quinn and Weinberg[11),

$$Q = T_{3_L} + D T_0 \tag{4.4}$$

where T_o is a normalized generator of SU(5), then the representation content (4.3b) implies that $D^2 = 5/3$. The neutral weak interaction mixing angle θ defined so that the neutral Z^o boson couples to $T_3 - \sin^2 \theta Q$ then takes the value[7),11)

$$\sin^2\theta = \frac{1}{1+D^2} = \frac{3}{8} \tag{4.5}$$

at M the grand unification mass. An arbitrary quark mass matrix can be obtained with just one Higgs multiplet. As pointed out by Georgi and Glashow[7), if the Higgs is taken in a $\underline{5}$ multiplet, then

$$m_{q_c}(M) = m_{l_c}(M) \tag{4.6}$$

where we have indicated explicitly that the relation (4.6) holds at the grand unification mass where SU(5) is a good symmetry. Equation (4.6) implies that

$$\frac{m_e(M)}{m_d(M)} = \frac{m_\mu(M)}{m_s(M)} = \frac{m_\tau(M)}{m_b(M)} = 1 \tag{4.7}$$

where τ is the charged heavy lepton apparently discovered recently, and b is the conjectured related third charge $-1/3$ quark.

The relations (4.5), (4.6) and (4.7) are not exact at a present energy scale μ, because below the grand unification mass the $SU(3)_c$, $SU(2)_L$ and $U(1)$ coupling constants g_3, g_2 and g_1 are renormalized differently. Georgi, Quinn and Weinberg[11)] showed that[*)] (where $\alpha \equiv e^2/4\pi$, $\alpha_s \equiv g_3^2/4\pi$):

$$\sin^2\Theta(\mu) = \frac{1 + 2D^2\,\alpha/\alpha_s(\mu)}{1+3D^2} \tag{4.8}$$

while

$$\ln\left(\frac{M}{\mu}\right) = \frac{12\pi}{22(1+3D^2)\alpha}\left(1 - \frac{\alpha}{\alpha_s(\mu)}(1+D^2)\right) \tag{4.9}$$

Their analysis can be extended to the renormalization of the mass relations (4.6) and (4.7) by using the anomalous dimension

$$\gamma_m = -\frac{g_3^2}{2\pi^2} \tag{4.10}$$

of the mass operator in QCD. If there are N_D sets of fundamental fermions (3.10) then

$$\ln\left(\frac{m_{q_c}(\mu)}{m_{L_c}(\mu)}\right) \approx \frac{4}{11-\frac{4}{3}N_D}\ln\left(\frac{\alpha_s(\mu)}{\alpha_{GUM}}\right) \tag{4.11}$$

where α_{GUM} is the $SU(5)$ coupling at the grand unification mass M: $\alpha_{GUM} \equiv g^2_{GUM}/4\pi$:

---

*) Higgs multiplets could in principle contribute asymmetrically to the renormalization of the $SU(3)$, $SU(2)_L$ and $U(1)$ coupling constants, but their effects are negligible for $SU(5)$ with a single $\underline{5}$ Higgs multiplet.

$$\frac{1}{g^2_{GUM}} = \frac{1}{g_i^2(\mu)} + 2b_i \ln\left(\frac{\mu}{M}\right) \quad (i=1,2,3) \tag{4.12}$$

The b_i are defined by

$$\mu \frac{\partial}{\partial\mu} g_i(\mu) \equiv \beta_i \approx b_i\, g_i^3(\mu) \quad \text{for } |g_i| << 1 \tag{4.13}$$

where

$$b_1 = \frac{N_D}{12\pi^2} \quad , b_2 = -\frac{1}{16\pi^2}\left[\frac{22}{3} - \frac{4}{3}N_D\right] , b_3 = -\frac{1}{16\pi^2}\left[11 - \frac{4}{3}N_D\right] \tag{4.14}$$

and the $SU(2)_L$ Higgs doublets in the simplest[3,7] $\underline{5}$ representation of SU(5) make a negligible contribution. A lower limit on the choice of M is provided[11] by the limit of $\sim 10^{30}$ years on the proton lifetime τ_{proton}, expected to be

$$\tau_{proton} = O\left(\frac{M^4}{m^5_{proton}}\right) \tag{4.15}$$

in this model*). An upper limit on the choice of M follows if we demand that α_s (μ = 10 GeV) $\lesssim$ 0.3 as suggested by the success of asymptotically free perturbation theory. In Fig. 9 we have plotted

$\sin^2\theta(\mu)$, $\alpha_s(\mu)$, $m_{q_c}(\mu)/m_{l_c}(\mu) = m_b(\mu)/m_\tau(\mu)$ and τ_{proton}

as functions of M, taking μ = 10 GeV. The constraints mentioned above allow

$$O(3\times 10^{15})\,\text{GeV} \lesssim M \lesssim O(10^{17})\,\text{GeV} \tag{4.16}$$

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*) The proton decay rate can always be suppressed by a Cabibbo rotation, but this seems an unreasonable ploy.

which excludes grand unification at the Planck Mass $\sim 10^{19}$ GeV. For the range (4.16) we find

$$\sin^2\theta(\mu) \sim (0.19 \text{ to } 0.20),\ \alpha_s(\mu) \gtrsim 0.15, \qquad (4.17)$$
$$\text{and} \quad m_{q_c}(\mu)/m_{l_c}(\mu) \sim (2.5 \text{ to } 4.5)$$

Experimentally, $\sin^2\theta \approx 0.3$, so that SU(5) may have problems with this quantity, but the values of $\alpha_s(\mu)$ and the quark-lepton mass ratio in (4.17) seem quite respectable. We can use (4.17) to predict the mass of the conjectured third charge -1/3 quark b, if a heavy lepton τ with mass ~ 1.9 GeV indeed exists:

$$m_b \simeq (4 \text{ to } 10) \text{ GeV} \qquad (4.18)$$

This prediction rests on the naturalness condition that only a 5 Higgs representation be responsible for giving masses to fermions.

4.3 Rank 5: SO(10)

The only rank 5 groups which are simple, or admit a discrete symmetry and so can have a unique coupling constant, are $[SU(2)]^5$, SO(10), SO(11), SU(6) and SP(10). Of these possibilities, $[SU(2)]^5$ has no SU(3) subgroup, while SO(11), SU(6) and SP(10) have no 16-dimensional representations suitable for the fundamental fermions (3.11). SO(11) has an 11-dimensional "vector" representation, 55-dimensional adjoint representation, and 32-dimensional "spinorial" representation. The corresponding numbers for SP(10) are 10, 55 and 32 respectively. The irreducible 15-dimensional representation of SU(6) has the unacceptable $SU(3)_c \otimes SU(2)$ content $(3,2) \oplus (3,1) \oplus (\bar{3},1) \oplus (1,2) \oplus (1,1)$. This leaves us with the SO(10) group of Fritzsch and Minkowski[8)], and of Georgi and Glashow[9),10)]. The tensor analysis of SO(10) in terms of fundamental spinors is set out in the Appendix. The irreducible 16-dimensional representation of SO(10) can be used for (3.11) as

$$\left(q_A^{R,Y,B},\ q_C^{R,Y,B};\ l_A, l_C;\ \bar{q}_A^{R,Y,B},\ \bar{q}_C^{R,Y,B};\ \bar{l}_A, \bar{l}_C\right)_L \qquad (4.19)$$

which has a decomposition

$$\underline{16} = (4,2,1) \oplus (\bar{4},1,2) \qquad (4.20)$$

into $SU(4) \otimes SU(2)_L \otimes SU(2)_R$, or

$$\underline{16} = (3,2,1) \oplus (1,2,1) \oplus (\bar{3},1,2) \oplus (1,1,2) \qquad (4.21)$$

into

$$SU(3)_c \otimes SU(2)_L \otimes SU(2)_R \otimes U(1)$$

Therefore the QAD subgroup of SO(10) is in general $SU(2)_L \otimes SU(2)_R \otimes U(1)$[12)-14)] rather than $SU(2) \otimes [U(1)]^2$ (see also the Appendix). In the SU(5) model[7)] the neutrinos were necessarily massless because there were no right-handed neutrino fields. In SO(10) the neutrinos can in general acquire masses which can be made finite and calculable in the Higgs systems discussed below. The multiplet structure of Refs. 12), 13) and 14) with some massless, some massive anoleptons can be achieved by introducing extra SO(10) singlets, but at the expense of naturalness conditions (B) and (C) being violated in the lepton sector.

Chains of symmetry breaking for SO(10) can be found leading to the full $SU(2)_L \otimes SU(2)_R \otimes U(1)$ or the restricted $SU(2)_L \otimes [U(1)]^2$ or $SU(2)_L \otimes U(1)$ forms for G_{QAD} being effective at low energies*). As an example, we consider the breaking scheme of Ref. 14), where the neutral weak mixing angle is defined by the photon field being

$$A^\mu \equiv \sin\Theta \left(W^\mu_{3_L} + W^\mu_{3_R} \right) + \sqrt{\cos 2\Theta}\; X^\mu \qquad (4.22)$$

with the gauge-covariant derivative

$$D^\mu \equiv \partial^\mu + i \left\{ \frac{e}{\sin\Theta} \left(\underline{W}^\mu_L \cdot \underline{I}_L + \underline{W}^\mu_R \cdot \underline{I}_R \right) + \frac{eD}{\sqrt{\cos 2\Theta}} Y_0 X^\mu \right\} \qquad (4.23)$$

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*) The Higgs systems discussed below do not admit a breaking to $SU(2)_L \otimes [U(1)]^2$ for G_{QAD} unlike the scheme of Ref. 8).

The coupling matrix T_o is normalized so that traced over the representation (4.19)

$$\mathrm{Tr}(T_0^2) = \mathrm{Tr}(T_{3_L}^2) = \mathrm{Tr}(T_{3_R}^2) = 2 \tag{4.24}$$

in which case $D^2 = 2/3$. At the grand unification mass M

$$\frac{g_2(M)}{g_1(M)} = 1 = \frac{\sqrt{\cos 2\theta}}{D \sin\theta}$$

so that[14])

$$\sin^2\theta = \frac{1}{2+D^2} = \frac{3}{8} \tag{4.25}$$

as in the SU(5) model. To give the intermediate vector bosons realistic masses requires at least one Higgs multiplet in a (1,2) representation of $SU(2)_L \otimes SU(2)_R$. The simplest way of finding such a Higgs in SO(10) is in a $\underline{16}$, in which case there is also a (2,1) Higgs which could acquire a vacuum expectation value. We assume this can be neglected as a first approximation.

Higgs multiplets which can*) give masses to the fundamental fermions in SO(10) are those in $\underline{16} \otimes \underline{16}$:

$$\underline{16} \otimes \underline{16} = \underline{10} \oplus \underline{120} \oplus \underline{126} \tag{4.26}$$

which have the $SU(4) \otimes SU(2)_L \otimes SU(2)_R$ decompositions

$$\left.\begin{aligned}
\underline{10} &= (6,1,1) \oplus (1,2,2) \\
\underline{120} &= (10,1,1) \oplus (\overline{10},1,1) \oplus (\bar{6},3,1) \\
&\qquad \oplus (6,1,3) \oplus (15,2,2) \oplus (1,2,2) \\
\underline{126} &= (\bar{6},1,1) \oplus (10,3,1) \oplus (\overline{10},1,3) \oplus (15,2,2)
\end{aligned}\right\} \tag{4.27}$$

*) See the Appendix for an explicit construction.

Since

$$\left(q_A^{R,Y,B}\,;\,l_A\right) \qquad \text{and} \qquad \left(q_c^{R,Y,B}\,;\,l_c\right)$$

form $\underline{4}$'s of SU(4), it is clear from the SU(4) representation contents of the Higgs multiplets in (4.27) that we get the mass relations

$$m_{q_A}(M) = m_{l_A}(M) \quad , \quad m_{q_c}(M) = m_{l_c}(M) \tag{4.28}$$

if only the $\underline{10}$ contributes to fermion masses, and

$$m_{q_A}(M) = 3m_{l_A}(M) \quad , \quad m_{q_c}(M) = 3m_{l_c}(M) \tag{4.29}$$

if only the $\underline{126}$ contributes (see the Appendix). If just the $\underline{120}$ gives fermion masses,

$$m_u : m_d : m_{\nu_e} : m_e = m_c : m_s : m_{\nu_\mu} : m_\mu = m_t : m_b : m_{\nu_\tau} : m_\tau = \cdots \tag{4.30}$$

and the Cabibbo angles are zero, even though there are two complex (2,2) multiplets because their coupling matrices are proportional -- see Section 3.3. Nor are there Cabibbo angles if only one of the $\underline{10}$ or $\underline{126}$ representations is responsible for the fermion masses. The $\underline{10}$'s and $\underline{126}$'s have antisymmetric coupling matrices (see the Appendix) and so would by themselves yield at least some degenerate masses. Taking more than one $\underline{120}$ but no $\underline{10}$ or $\underline{126}$ cannot give the desired[14)] pattern of left- and right-handed Cabibbo angles. We conclude that unless the right-handed currents are suppressed very strongly by breaking the symmetry differently from Refs. 12), 13), 14), and giving the $W_R^\pm$ very high masses, the observed fermion masses and phenomenological constraints on mixing angles require at least one $\underline{120}$, and one $\underline{10}$ or $\underline{126}$ Higgs representation to contribute. These and the $\underline{16}$ necessary to give intermediate vector boson masses, to say nothing of those connected with superheavy bosons, motivate Glashow's[10)] characterization of the Higgs system of SO(10) as baroque. The

SO(10) model is even more unnatural in the Higgs sector than are the plain $SU(2)_L \otimes SU(2)_R \otimes U(1)$ models discussed in Section 3.3.

The relations (4.25), (4.28) and (4.29) apply at the grand unification mass, and their renormalization at present energies can be calculated using the ideas of Georgi, Quinn and Weinberg[11]. In this model $\sin^2\theta$ at present energies is

$$\sin^2\theta(\mu) \approx \frac{1}{2+3D^2}\left[1+\frac{\alpha}{\alpha_s(\mu)}2D^2\right] \tag{4.31}$$

where $D^2 = 2/3$, and we have again neglected the possible effects of Higgs multiplets*). The renormalization (4.31) of $\sin^2\theta$ is somewhat less than in the SU(5) model: in the limit $\alpha/\alpha_s \to 0$:

$$\sin^2\theta\Big|_{SU(5)} \to \frac{1}{6} \quad , \quad \sin^2\theta\Big|_{SO(10)} \to \frac{1}{4} \tag{4.32}$$

The grand unification mass M is given by

$$\ln\left(\frac{M}{\mu}\right) = \frac{6\pi}{11(2+3D^2)\alpha}\left[1-\frac{\alpha}{\alpha_s}(2+D^2)\right] \tag{4.33}$$

Comparing (4.9) and (4.33) we see that M tends to be larger in the SO(10) model: as $\alpha/\alpha_s \to 0$:

$$\ln\left(\frac{M}{\mu}\right)\Big|_{SU(5)} \to \frac{\pi}{11\alpha} \quad , \quad \ln\left(\frac{M}{\mu}\right)\Big|_{SO(10)} \to \frac{3\pi}{22\alpha} \tag{4.34}$$

We have not calculated any mass renormalization relations, because neither of the conditions (4.28), (4.29) can be realistically imposed. We have plotted in Fig. 10 the quantities $\sin^2\theta(\mu)$, $\alpha_s(\mu)$ and τ_{proton} as functions of M, taking $\mu \simeq 10$ GeV. We notice that in contrast to the SU(5) model there is no squeeze between proton stability and realistic values of $\sin^2\theta(\mu)$ and $\alpha_s(\mu)$. We can easily take M to be the Planck mass of $\sim 10^{19}$ GeV, in which case

*) We have calculated their effect on (4.31) to be ≤ 0.01, despite the plethora of multiplets discussed above.

$$\sin^2\theta(\mu) \sim (0.29 \text{ to } 0.30),\ \alpha_s(\mu) \sim (0.07 \text{ to } 0.1) \text{ and } \tau_{proton} \sim 10^{44} \text{ years} \qquad (4.35)$$

where the uncertainties arise from the Higgs multiplets. Comparing the SU(5) results (4.16) and (4.17) with the SO(10) results (4.35), we find the SO(10) may fare better. The results (4.16), (4.17) and (4.18) show the price to be paid for complete naturalness in QHD theories.

5. DISCUSSION

In this paper we have pushed to its limit the possibility that the neutral weak interactions may naturally conserve all fermion flavours to $O(G_F^2)$. Phenomenology does not force naturalness upon us -- indeed there are various indications that weak interactions may not be completely natural -- but it is a powerful tool for organizing our thoughts about gauge models. Gauge models have so much freedom that we may need an organizing principle[4)-6)] to regulate our imaginations. In Section 2 we extended the naturalness conditions of Glashow and Weinberg[4)] to general weak and electromagnetic gauge groups. In Section 3 we showed how, modulo the existence of a new exactly conserved quantum number or $|\Delta Q| \geq 2$ charged currents, the only completely natural anomaly-free gauge groups were $SU(2) \otimes [U(1)]^M$. Barring inelegant tricks with the Higgs sector, only $M = 1$ and 2 were completely natural. In Section 4 we studied extensions of these natural theories to include the strong interactions, studying the possibility that the rank of the super unification group should be 4 or 5. The only possible groups were SU(5)[7)] and SU(10)[8)-10)]. If it is made natural by using a unique five-dimensional Higgs multiplet, the SU(5) model needs the third charge $-1/3$ quark to have a mass O(5 to 10) GeV. If the SU(5) grand unification scale is chosen to make the proton lifetime obey the present limit of 10^{30} years, then the neutral weak interaction mixing angle has $\sin^2\theta \lesssim 0.2$, which is on the verge of inconsistency with present neutral current data. The SO(10) model is even more Higgs unnatural than the $SU(2)_L \otimes SU(2)_R \otimes U(1)$ group which is its maximal weak and electromagnetic subgroup. Also, it has no quark-lepton mass relations. However, it gives $\sin^2\theta \simeq 0.3$ which is nicely consistent with present data. If we choose the SO(10) grand unification mass to be the Planck mass $\sim 10^{19}$ GeV, then at present energies $\alpha_{strong} \simeq O(0.1)$ and the proton lifetime is $O(10^{44})$ years. If so desired, the $SU(2)_R$ subgroup of SO(10) can be postponed to arbitrarily high energies, so that the low energy weak and electromagnetic gauge group is essentially $SU(2)_L \otimes U(1)$: this may not be satisfactory phenomenologically[14)].

Based on our results, we can make a few remarks about some gauge models in the literature. Theories based on a simple weak and electromagnetic gauge group such as SU(3) cannot be made natural without introducing exotic charged currents and complicated fermion representations. Existing SU(3) models[8] are not natural. Models[20),21] exist which are based on $SU(3) \otimes U(1)$, which generally violate naturalness condition (A) by having flavour-changing neutral currents in the symmetric Lagrangian. The recent model of Lee and Weinberg[21] also disobeys conditions (B) and (C): this does not however yield more violation of naturalness because they have a new absolutely conserved quantum number, analogous to that introduced by Ma[12]. Of models based on $SU(2)_L \otimes SU(2)_R \otimes U(1)$ [12-14], the only one which is completely natural is one due to Ma[12], which has a new exactly conserved quantum number and is forced to have u and d quarks exactly massless even after renormalization. The observed low masses of the u and d quarks make the Ma model rather appealing, and models of this class may repay further study. The phenomenologically attractive $SU(2)_L \otimes SU(2)_R \otimes U(1)$ models of Mohapatra and Sidhu[13] and of de Rujula, Georgi and Glashow[14] are natural as far as single and double intermediate vector boson exchanges are concerned, but their neutral Higgs couplings necessarily violate flavour conservation[17]. In view of the unknown and possibly large masses of the neutral Higgs bosons, it is not clear this Higgs unnaturalness is a serious problem. Turning to $SU(2) \otimes U(1)$ models, it was pointed out in the original papers[4-6] on naturalness that an ambidextrous SU(2) group violated naturalness in the Higgs sector, though the phenomenologically disfavoured vector-like models could be natural for vector boson exchanges. As for $SU(2)_L \otimes U(1)$, Poggio and Schnitzer[22] have emphasized that the naturalness requirement is not very sensible if fermion masses are allowed to approach the W and Z masses.

Turning to super-unified theories including the strong interactions, the class of models based on exceptional groups[23] is generally unnatural because such groups contain $SU(3) \otimes SU(3)$, one SU(3) to be identified with colour, while the other SU(3) is part of the weak and electromagnetic gauge group, so that the problems of Section 3.1 arise. The original SU(5) model of Georgi and Glashow[7] can be made completely natural by a suitable choice of Higgs representations, but the related SU(6) model of Abud et al.[24] violates naturalness conditions (B) and (C) (connected with W and Z exchanges) as well as needing unnatural Higgs representations.

Since pursuing naturalness to the bitter end is such a theoretical straitjacket, it is sensible to seek ways of relaxing the assumption. Possible ways to go might include allowing exotic quark charges and $|\Delta Q| \geq 2$ charged currents. Alternatively, one might abandon[14] the naturalness restrictions on Higgs boson couplings, which are of dubious phenomenological relevance. Another possibility would be to pursue[21] the line of the $SU(2)_L \otimes SU(2)_R \otimes U(1)$ model of Ma[12], and have an extra (approximately) conserved quantum number. It should be remembered also that flavour conservation is on a different footing in the quark and lepton sectors, since in the latter case muon number may be conserved to a good approximation because the neutrinos are essentially degenerate in mass, as seems to be the case for ν_e and ν_μ. Since applying naturalness to the quark sector alone severely constrains the choice of gauge group, perhaps the easiest freedom to exercise is in the choice of lepton representation content. As for constructing grand unified theories, higher rank schemes violating "supernaturalness" certainly warrant study. Even if naturalness is not an absolute principle, and is broken in one of the above-or-other-ways, we hope it may be a useful starting approximation.

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APPENDIX

TENSOR FORMULATION OF SO(10)

The spinorial representation of SO(10) has the generators represented by generalized σ matrices

$$\sigma_{ij} \equiv \frac{1}{2i}[\gamma_i, \gamma_j] = -\sigma_{ji} \qquad (i,j = 1, \ldots, 10) \tag{A.1}$$

where the generalized γ matrices obey

$$\{\gamma_i, \gamma_j\} = 2\delta_{ij} \qquad (i,j = 1, \ldots, 10) \tag{A.2}$$

From (A.1) and (A.2) it follows that σ_{ij} commutes with σ_{kl} if there are no overlapping indices, so we may diagonalize five generators simultaneously, for example:

$$\sigma_{diag} \equiv \sigma_{12}, \sigma_{34}, \sigma_{56}, \sigma_{78}, \sigma_{910} \tag{A.3}$$

which have eigenvalues ± 1 since $\sigma_{ij}^2 = 1$. The eigenstates of σ_{diag} therefore span a space of dimension $2^5 = 32$. However, we may define a "chiral" operator

$$\gamma_\chi \equiv -i \prod_i \gamma_i = \prod \sigma_{diag} \tag{A.4}$$

which commutes with all the σ_{ij}. Therefore we can divide the 32-dimensional multiplet into a $\underline{16}$ and a $\overline{\underline{16}}$:

$$\xi_+ = \left(\frac{1+\gamma_\chi}{2}\right)\xi_+ \quad , \quad \xi_- = \left(\frac{1-\gamma_\chi}{2}\right)\xi_- \tag{A.5}$$

corresponding to two irreducible representations which are inequivalent and conjugate to each other. If ξ_+ transforms as

$$\delta \xi_+ = i\,\lambda^{ij}\sigma_{ij}\,\xi_+ \qquad : \lambda \text{ real} \tag{A.6}$$

then the conjugate representation transforms as

$$\delta \xi_+^* = -i\,\lambda^{ij}\sigma_{ij}^*\,\xi_+^* = -i\,\lambda^{ij}\sigma_{ij}^T\,\xi_+^* \tag{A.7}$$

where we have chosen the σ's Hermitian. If we introduce a unitary matrix $\mathcal{B}$ such that

$$\mathcal{B}^{-1}\,\sigma_{ij}^T\,\mathcal{B} = -\sigma_{ij} \tag{A.8}$$

then

$$\tilde{\xi} \equiv \mathcal{B}^{-1}\xi^*$$

transforms as

$$\delta\tilde{\xi} = -i\,\lambda^{ij}\,\mathcal{B}^{-1}\sigma_{ij}^T\,\xi_+^* = i\,\lambda^{ij}\sigma_{ij}\,\tilde{\xi} \tag{A.9}$$

From (A.4) and the defining equation (A.8) we have

$$\mathcal{B}^{-1}\gamma_\chi\,\mathcal{B} = -\gamma_\chi^T = -\gamma_\chi \tag{A.10}$$

because the diagonal σ's are real and symmetric. Therefore $\mathcal{B}$ flips chirality, and we can identify

$$\tilde{\xi} = \xi_- \tag{A.11}$$

Using curly letters for 32-dimensional quantities, capital letters for 16-dimensional quantities, and small letters for eight dimensions, we choose:

$$\left.\begin{array}{ll} \mathcal{B} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \mathcal{B}^{-1} \quad , & \sigma_{ij} = \begin{pmatrix} S_{ij} & 0 \\ 0 & -S^{*}_{ij} \end{pmatrix} \\ \xi_{+} = \begin{pmatrix} F \\ 0 \end{pmatrix} \quad , & \xi_{-} = \begin{pmatrix} 0 \\ F^{*} \end{pmatrix} \end{array}\right\} \qquad \text{(A.12)}$$

We classify the 16 left-handed fermions in the representation <u>16</u> according to

$$F = \begin{pmatrix} f \\ g \end{pmatrix}_L \; ; \quad \left.\begin{array}{l} f^T = \left(q_A^R, q_A^Y, q_A^B ; q_C^R, q_C^Y, q_C^B ; l_A, l_C\right) \\ g^T = \left(\bar{q}_A^{R'}, \bar{q}_A^{Y'}, \bar{q}_A^{B'} ; \bar{q}_C^{R'}, \bar{q}_C^{Y'}, \bar{q}_C^{B'} ; l'_A, l'_C\right) \end{array}\right\} \qquad \text{(A.13)}$$

where q_A, q_C, l_A, l_C are anoquarks, cathoquarks, anoleptons (neutrinos) and catholeptons (charged leptons) (3.11) respectively, and $\bar{q}'_A$ etc. are the corresponding antiparticles arbitrarily mixed with respect to charge-degenerate flavours. We may then identify:

$$\left.\begin{array}{ll} \sigma_{12} = 2(T_3^L + T_3^R) & \\ \sigma_{34} = 2(T_3^L - T_3^R) & \sigma_{78} = 2T_3^C - Y + Y^C \\ \sigma_{56} = 2T_3^C + Y - Y^C & \sigma_{910} = 2Y^C + Y \end{array}\right\} \qquad \text{(A.14)}$$

where Y is defined by $Q = T_3^L + T_3^R + Y/2$, T_3^C and Y^C are colour SU(3) matrices, and $T_3^{L,R}$ are generators of the left and right flavour isospin groups.

To construct the Higgs couplings we must find the irreducible representations in the product <u>16</u> ⊗ <u>16</u>. Writing the product of two spinors in a matrix

$$\xi_{+} \xi_{+}^{T} = \mathcal{M} \qquad \text{(A.15)}$$

we may expand $\mathfrak{M}$ in terms of a complete set of matrices in the 32-dimensional space. A choice is

$$\{\Gamma^\alpha\} = \{\Gamma^0, \dots, \Gamma^5, \gamma_\chi \Gamma^0, \dots, \gamma_\chi \Gamma^4\}$$

where

$$\left.\begin{array}{l} \Gamma^0 = 1 \ , \quad \Gamma^1_i = \gamma_i \ , \quad \Gamma^2_{ij} \equiv \sigma_{ij} \\ \Gamma^3_{ijk} = -i\,\gamma_i\gamma_j\gamma_k \\ \Gamma^4_{ijkl} = \gamma_i\gamma_j\gamma_k\gamma_l \\ \Gamma^5_{ijklm} = \gamma_i\gamma_j\gamma_k\gamma_l\gamma_m \end{array}\right\} i \neq j \neq k \neq l \neq m \tag{A.16}$$

Then if we write

$$\mathfrak{M} = \sum_\alpha m_\alpha \Gamma^\alpha \mathcal{B}^{-1} \tag{A.17}$$

from the transformation property

$$\delta(\xi_+ \xi_+^T) = i\lambda^{ij}\sigma_{ij}\xi_+\xi_+^T + i\lambda^{ij}\xi_+\xi_+^T\sigma_{ij}^T \tag{A.18}$$

we have, using (A.8)

$$\delta\mathfrak{M} = i\lambda^{ij}\sum_\alpha m_\alpha [\sigma_{ij}, \Gamma^\alpha]\,\mathcal{B}^{-1} \tag{A.19}$$

It follows that the Γ^α of fixed tensor rank form irreducible representations. Furthermore, because of the properties (A.5) and (A.10), the representations included in $\underline{16} \otimes \underline{16}$ (and in $\overline{\underline{16}} \otimes \overline{\underline{16}}$) have the property

$$\{\Gamma^{\alpha}, \gamma_{\chi}\} = 0 \tag{A.20}$$

while those in $\underline{16} \otimes \overline{\underline{16}}$ have the property

$$[\Gamma^{\alpha}, \gamma_{\chi}] = 0 \tag{A.21}$$

Since

$$\gamma_{\chi}\{\Gamma^5\} = \{\Gamma^5\}$$

only half the $\{\Gamma^5\}$ are linearly independent. Thus we have the product decompositions

$$\left.\begin{matrix} \underline{16} \otimes \underline{16} \\ \overline{\underline{16}} \otimes \overline{\underline{16}} \end{matrix}\right\} = \left\{\begin{matrix} \underline{10} \\ \overline{\underline{10}} \end{matrix}\right\} \oplus \left\{\begin{matrix} \underline{120} \\ \overline{\underline{120}} \end{matrix}\right\} \oplus \left\{\begin{matrix} \underline{126} \\ \overline{\underline{126}} \end{matrix}\right\}$$
$$= \{(1 \pm \gamma_{\chi})\gamma_i\} \oplus \{(1 \pm \gamma_{\chi})\Gamma^3\} \oplus \{(1 \pm \gamma_{\chi})\Gamma^5\} \tag{A.22}$$

and

$$\underline{16} \otimes \overline{\underline{16}} = \underline{1} \oplus \underline{45} \oplus \underline{210}$$
$$= 1 \oplus \{\sigma_{ij}\} \oplus \{\Gamma^4\} \tag{A.23}$$

with the appropriate chiral projection operators.

Scalar Higgs mesons can therefore belong to the odd chirality representations $\underline{10}$, $\underline{120}$ or $\underline{126}$. Since $\mathcal{M}^{\dagger}$ transforms as $\xi^{*}\xi^{\dagger}$, the Yukawa coupling

$$\mathcal{L}_Y = \xi^T \mathcal{M}^{\dagger} \xi + \text{h.c.} = \sum_{\alpha} m_{\alpha}^{*} \xi^T \mathcal{B} \Gamma^{\alpha} \xi + \text{h.c.} \tag{A.24}$$

is invariant. If we now define the matrix

$$\mathcal{C} \equiv \begin{pmatrix} 0 & B \\ B & 0 \end{pmatrix} \quad ; \quad B \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

then (A.24) may be written as

$$\mathcal{L}_Y = \sum_\alpha m_\alpha^* \tilde{\xi}^T \mathcal{C} \Gamma^\alpha \xi + h.c. \tag{A.25}$$

with

$$\tilde{\xi} = \mathcal{C} \mathcal{B} \xi = \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix} \xi = \begin{pmatrix} g \\ f \\ 0 \\ 0 \end{pmatrix}_L$$

The non-vanishing expectation values $\langle m_a^* \rangle \neq 0$ must be such that the matrix $\mathcal{D}^\alpha = \mathcal{C}\Gamma^\alpha$ is diagonal. Since we have

$$\{\gamma_\chi , \mathcal{C}\} = \{\gamma_\chi , \Gamma^\alpha\} = 0$$

it follows that

$$[\gamma_\chi , \mathcal{D}] = 0 \tag{A.26}$$

Then $\mathcal{D}$ must be formed from even rank tensors Γ^α which can be constructed from products of σ_{ij} which are of the form:

$$\sigma_{ij} = \begin{pmatrix} S_{ij} & 0 \\ 0 & -S_{ij}^* \end{pmatrix} \quad : \quad S = S^+ \tag{A.27}$$

Those which do not mix fermion and antifermion may further be reduced according to

$$S_{ij}^V = \begin{pmatrix} s_{ij} & 0 \\ 0 & -s_{ij}^* \end{pmatrix} \tag{A.28a}$$

$$: \quad s = s^+$$

$$S_{ij}^A = \begin{pmatrix} s_{ij} & 0 \\ 0 & s_{ij}^* \end{pmatrix} \tag{A.28b}$$

where we denote generators as "axial" (A) or "vector" (V) according to the commutation relations:

$$\begin{aligned} [\sigma^V_{ij}\ ,\ \mathcal{C}] &= 0 \\ \{\sigma^A_{ij}\ ,\ \mathcal{C}\} &= 0 \end{aligned} \tag{A.29}$$

The attribution (A.13) implies that the SU(4) generators (i,j = 5, ..., 10) which include colour SU(3) are of the form (A.28a). Of the diagonal generators $\sigma_{34} = 2(T^L_3 - T^R_3)$ is "axial" and the others are "vector". Then since

$$\begin{aligned} [\sigma^V, \sigma^V] &\to \sigma^V \\ [\sigma^A, \sigma^A] &\to \sigma^V \\ [\sigma^V, \sigma^A] &\to \sigma^A \end{aligned}$$

and

$$[\sigma_{im}, \sigma_{jm}] \to 2i\,\sigma_{ij}$$

we may choose σ_{3i} as "axial" and σ_{ij}, $i,j \neq 3$, as "vector". Then writing

$$\Gamma^\alpha = \mathcal{C}\,\mathcal{D}^\alpha \tag{A.30}$$

we see from (A.29) that the Γ^α (A.30) commute with all the generators except σ_{34}. The only such matrices in the 10 representation are γ_3 and γ_4, which moreover commute with all the σ_{ij} : $i,j \neq 3,4$. The only identification possible [see (A.29)] is therefore

$$\gamma_3 = \mathcal{C} \quad , \quad \gamma_4 = -i\,\mathcal{C}\sigma_{34} \tag{A.31}$$

which correspond to the two neutral components of the (1,2,2) multiplet of $SU(4) \otimes SU(2)_L \otimes SU(2)_R$ displayed in Eq. (4.27) of the text.

For the 120 representation, we have (4.27) two (1,2,2) multiplets with a total of four neutral members. From the representation (A.30) they are seen to be

$$\left.\begin{aligned}
\Gamma^{3}_{312} &= \gamma_3 \sigma_{12} = \mathcal{C} \sigma_{12} \\
\Gamma^{3}_{412} &= \gamma_4 \sigma_{12} = -i\, \mathcal{C} \sigma_{34} \sigma_{12} \\
\Gamma^{3}_{3Y} &= \mathcal{C}\, \mathcal{Y} \\
\Gamma^{3}_{4Y} &= -i\, \mathcal{C} \sigma_{34}\, \mathcal{Y}
\end{aligned}\right\} \qquad \text{(A.32)}$$

where $\mathcal{Y}$ is the $SU(3)_c$ singlet amongst the $SU(4)$ generators. Finally, for the 126 representation [see Eq. (4.27)] there is one (1,2,2) and we get

$$\Gamma^{5}_{312Y} = \mathcal{C} \sigma_{12}\, \mathcal{Y} \quad , \quad \Gamma^{5}_{412Y} = -i\, \mathcal{C} \sigma_{34} \sigma_{12}\, \mathcal{Y} \qquad \text{(A.33)}$$

From (A.24) the mass matrix is now given by

$$\mathcal{L}_m = \sum_{\alpha} \langle m^*_\alpha \rangle\, \tilde{\xi}^T \mathcal{D}^\alpha \xi + h.c.$$

$$\left.\begin{aligned}
&\equiv \sum_{\alpha} \langle m^*_\alpha \rangle\, \tilde{F}^T D^\alpha F + h.c. \\
\text{where} \quad &\tilde{F} \equiv \begin{pmatrix} g \\ f \end{pmatrix}_L \quad , \left(\frac{1+\gamma_\chi}{2}\right) \mathcal{D} \left(\frac{1+\gamma_\chi}{2}\right) \equiv \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}
\end{aligned}\right\} \qquad \text{(A.34)}$$

For the diagonal generators S in Eq. (A.27) is real and diagonal with $S_{12} = S_{34}$. Then, since σ_{34} is "axial", D is of the form

$$D = \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} \text{ for } \underline{10}, \underline{126} \quad ; \quad D = \begin{pmatrix} d & 0 \\ 0 & -d \end{pmatrix} \text{ for } \underline{120} \qquad \text{(A.35)}$$

Thus we obtain mass matrices of the form

$$\begin{aligned}\mathcal{L}_m &= v_\alpha\, g^\alpha_{ab}\, \tilde{F}^T_a\, D^\alpha F_b + h.c.\\ &= v_\alpha\, g^\alpha_{ab}\left(h^a d f^b \pm f^a d h^b\right) + h.c. \\ &= v_\alpha\, h^a d \left(g^\alpha_{ab} \mp g^\alpha_{ba}\right) f^b + h.c.\end{aligned} \tag{A.36}$$

where we have denoted vacuum expectation values of Higgs fields by v_α, used the indices a,b for different 16 fermion multiplets, and anticommuted fermion fields to get (A.36). If only the 10 contributes, "colour SU(4)" is conserved and the mass matrices yield [see also (4.28)]

$$m_{q_A} = m_{\ell_A} \quad , \quad m_{q_C} = m_{\ell_C} \tag{A.37}$$

If only the 126 contributes we get [see also (4.29)]

$$m_{q_A} = -3m_{\ell_A} \quad , \quad m_{q_C} = -3m_{\ell_C} \tag{A.38}$$

since the matrices (A.33) transform as the 15$^{\text{th}}$ member of a 15 of SU(4). The minus sign in (A.38) can be trivially removed by an appropriate chiral phase transformation. Finally, if there is a single (complex) Higgs multiplet contributing to fermion masses, the coupling constant g^α_{ab} in (A.36) is independent of α. As a consequence the mass matrices for ano- and catho-quarks and leptons will all be proportional, giving

$$m_u : m_d : m_{\nu_e} : m_e = m_c : m_s : m_{\nu_\mu} : m_\mu = m_t : m_b : m_{\nu_\tau} : m_\tau = \cdots \tag{A.39}$$

Note moreover that if only the 10 and 126 contribute, then the mass matrix is antisymmetric in the multiplet (a,b) space, whereas if only the 120 contributes the mass matrix is symmetric. For any complex matrix G we can find unitary matrices U and V such that

$$V^{\dagger} G U = M$$

is real and diagonal. U and V are the matrices which diagonalize the Hermitian matrices $G^{\dagger}G$ and $GG^{\dagger}$ respectively, and if there are no degenerate eigenvalues they are unique up to a phase transformation on the eigenvectors. Since $M = M^T$ by hypothesis, if $G = G^T$ then

$$M = U^T G V^*$$

so that V^* and U must be the same up to a phase transformation. In our context this means that if representations $\underline{120}$ alone were responsible for fermion masses, then the left- and right-handed currents' generalized Cabibbo angles would be related; in particular the coupling $(\bar{u}\, d')_R$ cannot be made arbitrarily small. This is inconsistent[16] with $SU(2)_R$ playing a significant phenomenological rôle[14] at present energies. If $\underline{10}$ and $\underline{126}$ representations alone were responsible for fermion masses, their antisymmetric couplings would generate some degenerate fermion masses. Therefore a realistic Higgs structure must contain at least one $\underline{120}$ representation and at least one $\underline{10}$ or $\underline{126}$ representation if it is to give a realistic fermion mass matrix.

Further, we observe that since the Γ^{α} corresponding to non-vanishing Higgs vacuum expectation values commute separately with $T_3^L + T_3^R$ and Y, only the "axial" neutral vector boson coupling to $T_3^L - T_3^R$ can acquire a mass through the Higgs multiplets in $\underline{16} \times \underline{16}$, and the charged bosons coupling to $T_{\pm}^L$ and $T_{\pm}^R$ will be degenerate. For this reason an extra Higgs multiplet is necessary, the simplest choice being a $\underline{16}$ spinorial representation.

Finally, let us remark on the stability of the mass relations, and the possibility of massless neutrinos. The neutrinos will be massless before renormalization if the vacuum expectation values point in specified directions in $SO(10)$ space (for example, $v_3^{10} = i v_4^{10}$). This and other mass relations will not be affected by infinite renormalization. However, as in $SU(5)$, the zeroth order mass relations will receive a finite renormalization due to the fact that the Higgs sector necessary to give appropriate masses to the W bosons require a different symmetry breaking pattern. We have not determined whether or not the condition for massless neutrinos can be phrased in terms of a symmetry which could be respected by the latter Higgs sector, in which case neutrinos would remain massless. An amusing possibility -- limiting the unpleasant abundance of Higgs fields -- would be that only $\underline{10}$ contributes, but that the finite fermion mass renormalization produces an acceptable mass pattern. In addition to relations (A.37), if there are only three fermion 16-plets, the antisymmetry of the mass matrix requires one massless fermion of each type $(u, d, e, \nu_e\ ?)$ and the other two degenerate in zeroth order. Unfortunately Higgs in $\underline{10}$ and $\underline{16}$ leave some vector-like lepto-quark bosons massless, so another high multiplicity representation must appear. One can also guarantee some massless left-handed neutrinos by introducing left-handed neutrino singlets which mix with the fermion 16-plet via the Higgs 16-plet. However, this will in general introduce a Cabibbo angle into, for example, the $(e^- \nu_e)_L$ coupling, where ν_e is a mass eigenstate.

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Figure captions

Fig. 1 : Second-order neutral vector boson exchange diagram.

Fig. 2 : Fourth-order diagrams involving two vector bosons.

Fig. 3 : Second-order neutral Higgs boson exchange diagram.

Fig. 4 : The most straightforward triplet quark representation of SU(3).

Fig. 5 : The root vector diagrams for simple rank 2 groups with the least charged currents consistent with naturalness.

Fig. 6 : One of the first triplet quark representations of SU(3) consistent with naturalness.

Fig. 7 : Fourth-order diagrams involving a gluon and a coloured and flavoured vector boson V.

Fig. 8 : Triangle diagrams giving anomalies in grand unified gauge theories.

Fig. 9 : The neutral weak mixing angle θ, α_{strong}, m_d/m_e and the proton lifetime τ_{proton} plotted as a function of the grand unification mass M in SU(5).

Fig. 10 : The neutral weak mixing angle θ, α_s and the proton lifetime τ_{proton} plotted as a function of the grand unification mass M in SO(10). The values of $\sin^2\theta$ and α_s may be slightly larger if the Higgs multiplets contribute asymmetrically to coupling constant renormalia-tions.

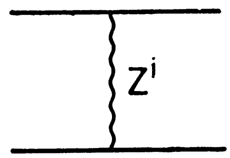

FIG.1

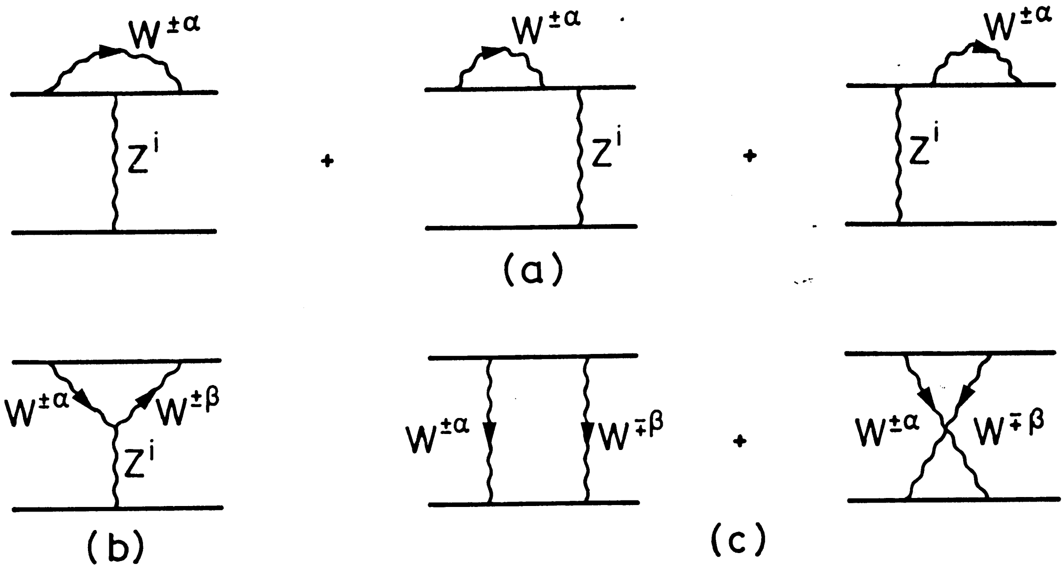

FIG. 2

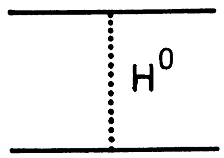

FIG.3

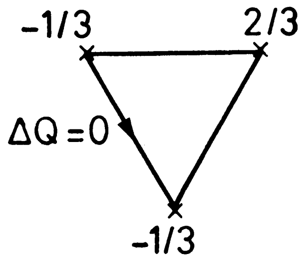

FIG.4

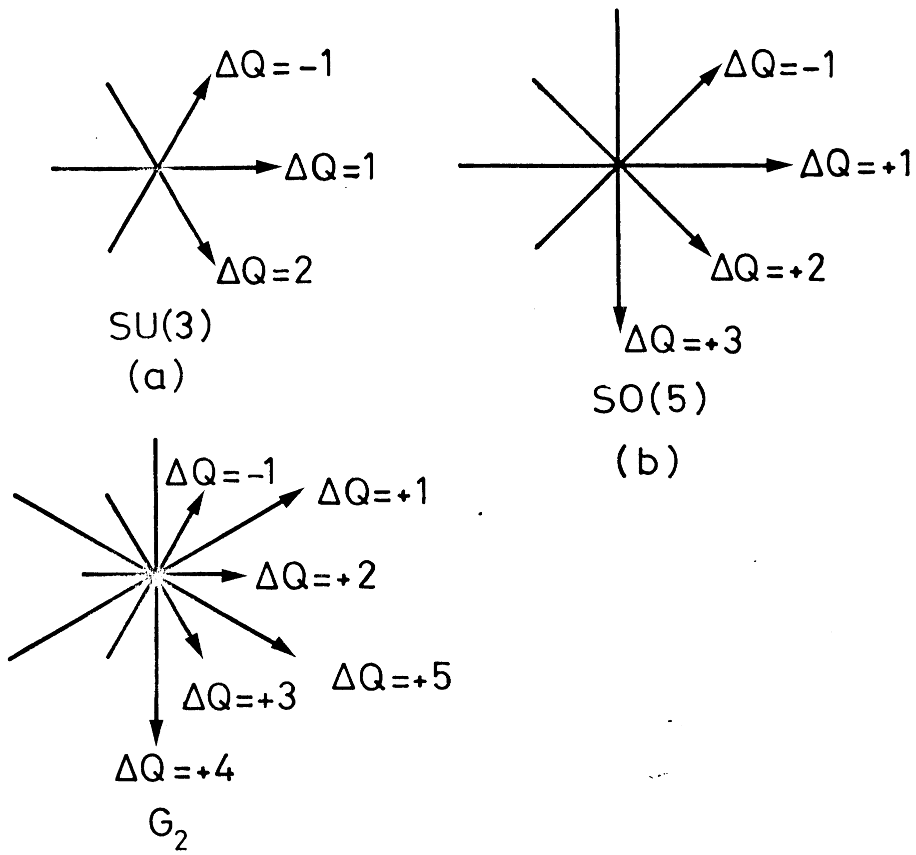

FIG. 5

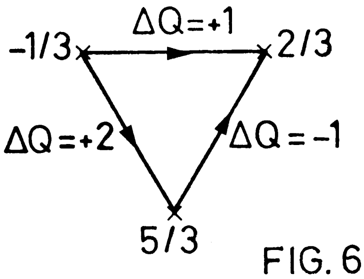

FIG. 6

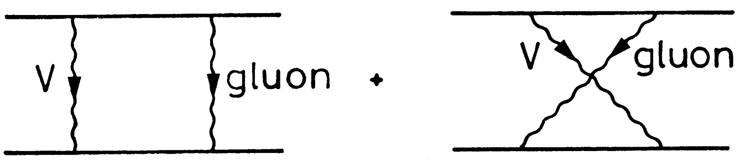

FIG. 7

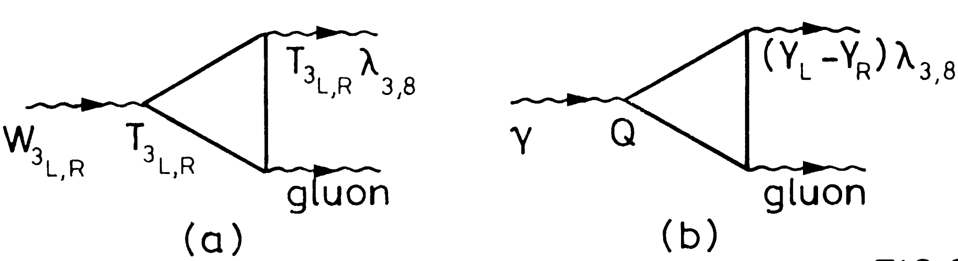

FIG. 8

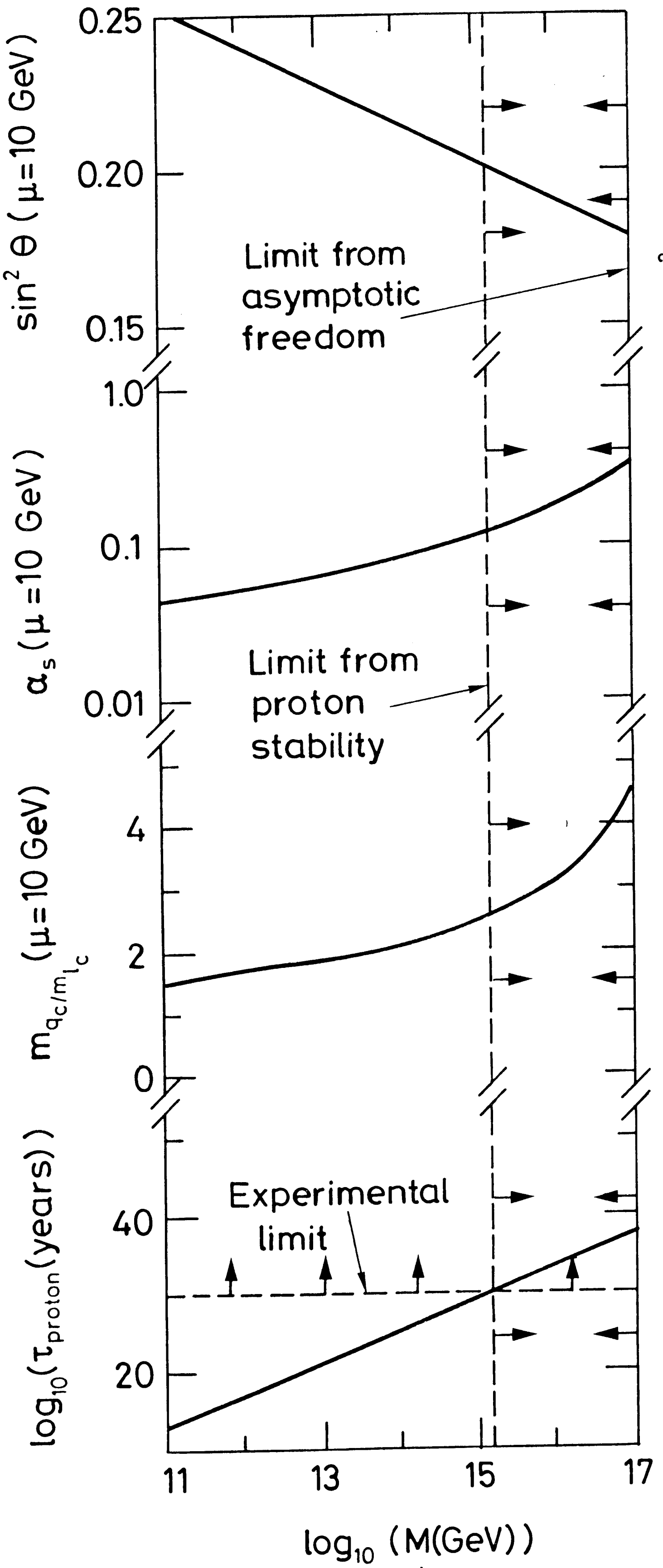
0.25
0.20
0.15
$\sin^2\theta$ (μ=10 GeV)
Limit from asymptotic freedom
1.0
0.1
0.01
α_s (μ=10 GeV)
Limit from proton stability
4
2
0
$m_{q_c/m_{l_c}}$ (μ=10 GeV)
40
20
$\log_{10}(\tau_{proton}$(years))
Experimental limit
11
13
15
17
$\log_{10}$ (M(GeV))

FIG.9

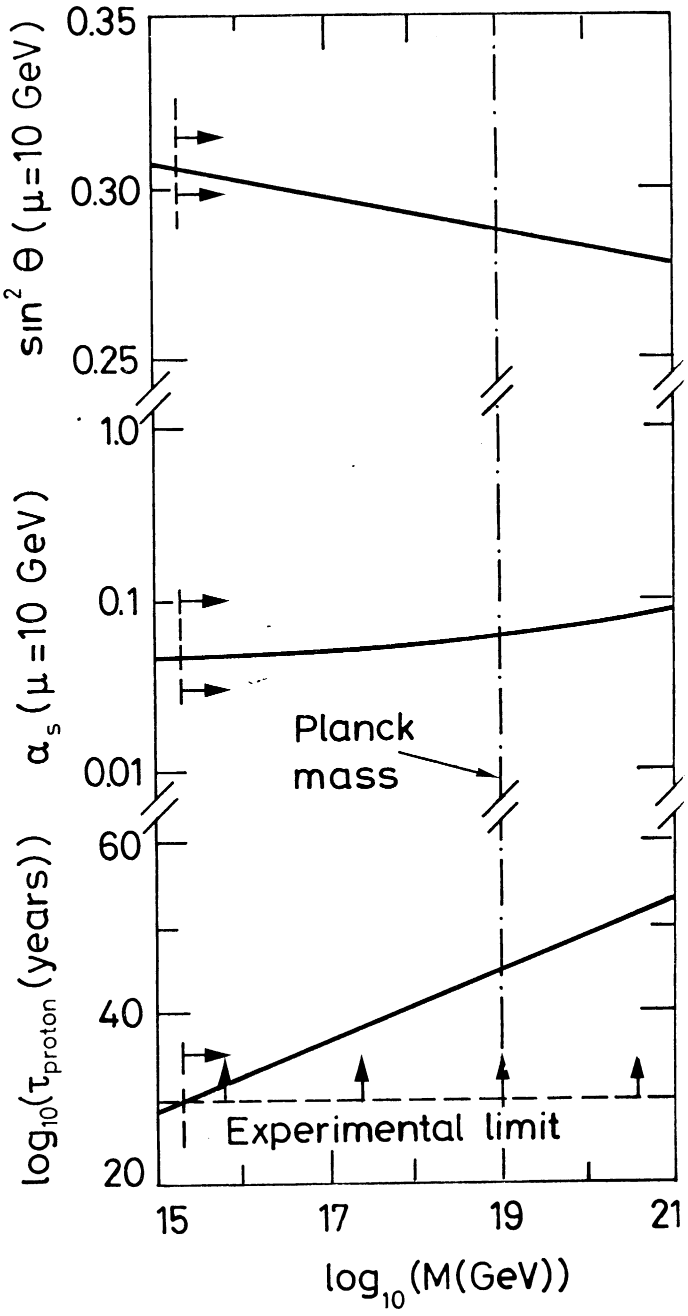
0.35
0.30
0.25
$\sin^2\theta$ (μ=10 GeV)
1.0
0.1
0.01
α_s (μ=10 GeV)
Planck mass
60
40
20
$\log_{10}(\tau_{proton}$(years))
Experimental limit
15
17
19
21
$\log_{10}$(M(GeV))

FIG.10