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Decay of correlations for slowly decreasing potentials

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When the interaction potential of a ferromagnet decreases like r^{-s}, we prove that two-point correlations (i) do not decay faster than r^{-s} and (ii) decay at least like r^{-s} at large magnetic field and, moreover, at least like $r^{-(s-2\nu)}$, where ν is the space dimension, for any nonzero magnetic field and arbitrary temperature. Extensions of the latter result to n-point correlations and to other systems are indicated, and a central-limit theorem, or Gaussian limit of block-spin distribution, is mentioned for slowly decreasing ferromagnetic interactions at any nonzero magnetic field.

I. INTRODUCTION

The main results on the decay of correlations in classical statistical mechanics have been obtained so far for finite-range or exponentially decreasing potentials. It has been proved in a number of cases that the correlations decay exponentially with distance in pure phases, and it is reasonably believed that they decay exponentially arbitrarily close to critical points with a rate of exponential falloff that tends to zero near these points.

Much less information is known in the case of potentials that decrease only like an inverse power r^{-s} of the distance. It is the purpose of this paper to present various rigorous results in this domain.

We shall consider lattice spin-$\frac{1}{2}$ systems in a ν-dimensional space ($\nu \geq 1$), or possibly the corresponding lattice gases obtained by the Lee-Yang isomorphism. Given a finite box Λ, the Hamiltonian H_Λ, the partition function Q_Λ, and the n-point correlation functions $\sigma_\Lambda(X)$ (where $X = x_1, \ldots, x_n$, $n \geq 1$, and where each point x_i, $i = 1, \ldots, n$, is a point of the lattice) are defined, respectively, for free boundary conditions by the relations

$$H_\Lambda(\sigma) = -\frac{1}{2} \sum_{\substack{y_i, y_j \in \Lambda \\ i \neq j}} J(y_i - y_j)\sigma_{y_i}\sigma_{y_j} - \sum_{y \in \Lambda} h_y \sigma_y ,$$

$$Q_\Lambda = \sum_{\sigma}{}_{\Lambda} \exp[-\beta H_\Lambda(\sigma)] ,$$

$$\sigma_\Lambda(X) = Q_\Lambda^{-1} \sum_{\sigma}{}_{\Lambda} \left(\prod_{x \in X} \sigma_x \right) \exp[-\beta H_\Lambda(\sigma)] .$$

In these equations, σ is a configuration of spins σ_y at each point y of the lattice in the box Λ, $\sigma_y = \pm\frac{1}{2}$, h_y is a magnetic field at each point y in Λ which in usual cases is independent of y and is denoted by h, β is the reciprocal temperature, and the interaction J satisfies the inequality

$$\sum_{\substack{y \in Z^\nu \\ y \neq 0}} |J(y)| < \infty ,$$

where Z^ν denotes the ν-dimensional lattice, each point y of the lattice being labeled by ν integer coordinates.

A *ferromagnetic* system is characterized by the further condition

$$J(y) \geq 0 , \quad \text{for every } y \in Z^\nu , \ y \neq 0 .$$

The pair, or two-point, connected correlation function $\sigma_\Lambda^T(x_1, x_2)$ is defined by the relation

$$\sigma_\Lambda^T(x_1, x_2) = \sigma_\Lambda(x_1, x_2) - \sigma_\Lambda(x_1)\sigma_\Lambda(x_2).$$

The n-point connected correlations are defined more generally by induction through the formulas

$$\sigma_\Lambda^T(X) = \sigma_\Lambda(X) - \sum_{\substack{\{X_1, \ldots, X_k\} \\ k > 1}} \prod_{i=1}^{k} \sigma_\Lambda^T(X_i) ,$$

where the sum on the right-hand side runs over all nontrivial partitions of $X = x_1, \ldots, x_n$ in subsets X_i.

The physical correlations are obtained, for possibly different boundary conditions, in the limit when the box Λ is made infinitely large. This limit is well defined and is independent of the boundary conditions if, being given the interaction J, the values of the magnetic field and of the reciprocal temperature do not correspond to a point in the phase-transition region. This region is known in the case of ferromagnets to be of the form $h = 0$, $\beta > \beta_c$, where β_c is some critical value.

The organization and contents of the paper are as follows.

In Sec. II, we are interested in lower bounds on the two-point connected functions. It is sometimes believed on a phenomenological basis that the correlations still decay exponentially away from critical points, even in the case of potentials that decrease only like an inverse power of the distance. It has been proved[1] however, in various situations in which analyticity properties are known, that the correlations cannot decay exponentially if the potential does not. If we restrict our attention to ferromagnetic systems, we prove here moreover, by using the Griffiths-Hurst-Sherman (GHS) inequality,[2] that the two-point connected correlations cannot, as a matter of fact, decay faster than the potential.

We next consider the converse problem, namely, the determination of upper bounds on the connected correlations.

On the one hand, it is known,[3] for general lattice (or continuous) gases, or also by the Lee-Yang isomorphism for spin-$\frac{1}{2}$ systems, that the two-point connected correlations decay at least like $r^{-(s-\nu)}$ at low activity or at large magnetic field, respectively. In the Appendix of the present paper, it is shown that the power of decrease in these situations is actually s and not $s-\nu$.

This result combined with the previous one shows, as stated at the end of Sec. II, that the two-point connected correlation of a ferromagnetic system decays exactly like r^{-s} at large magnetic field if the potential decays like r^{-s}.

In a number of other situations, for instance for ferromagnetic spin systems at any nonzero magnetic field h and arbitrary temperature, it has also been proved,[4] by using analyticity properties with respect to the magnetic field, that the connected correlations decay at least like $r^{-s(\beta,h)}$, $s(\beta,h)>0$. However, the power $s(\beta,h)$ obtained in Ref. 4 tends to zero when $h\to 0$.

By using analyticity properties with respect to the interaction potential and magnetic field, instead of using only analyticity with respect to the magnetic field, we show in Sec. III that the two-point connected correlations decay at least like $r^{-(s-2\nu-\alpha)}$, where ν is, as above, the space dimension and α is arbitrarily small, for arbitrary values of h ($\neq 0$) and arbitrary values of β. The result holds again here for ferromagnetic systems, in which case the needed analyticity properties with respect to the potential can be established on the basis of the results of Ref. 5 on the zeros of the partition function.

The fixed loss of power, i.e., 2ν, is due to technical reasons,[6] and it seems safe to conjecture that the correlations decay as a matter of fact like r^{-s} arbitrarily close to critical points.

The above result of Sec. III on the decay of correlations is extended to n-point connected correlations in Sec. IV. As a corollary, a central-limit theorem, or Gaussian limit for the block-spin distribution, is also given for slowly decreasing ferromagnetic interactions at any nonzero magnetic field.

We conclude with the following remark. The methods and results of Secs. III and IV can apply to various other systems as soon as one can have similar analyticity properties with respect to the potential. An example is the case of lattice gases with arbitrary potentials and arbitrary activities: The methods of the present paper, together with the results of Ref. 5 on the zeros of the partition function, then lead, at high temperature, to analogous results on the decay of correlations and on central-limit theorems.

II. LOWER BOUNDS ON TWO-POINT CONNECTED CORRELATIONS OF FERROMAGNETIC SYSTEMS, AND RESULTS AT LARGE MAGNETIC FIELD

Theorem 1 below applies to ferromagnetic spin-$\frac{1}{2}$ systems and $\sigma^T(x_1,x_2)$ is there the unique two-point connected correlation obtained (independently of boundary conditions) in the $\Lambda\to\infty$ limit when $h\neq 0$, or $h=0$, $\beta<\beta_c$, or is alternatively anyone of the two functions obtained in the $\Lambda\to\infty$ limit in each pure phase at $h=0$, $\beta>\beta_c$, when the boundary conditions correspond to all spins $\frac{1}{2}$, or all spins $-\frac{1}{2}$ outside Λ. The theorem has been obtained in collaboration with H. Kunz.

Theorem 1. The two-point connected correlation of a ferromagnetic system does not decay faster than the potential J. Namely, for any $\beta>0$ and any real h, there exists a strictly positive constant $C(\beta,h)$, such that

$$\sigma^T(x_1,x_2;\beta,h)>C(\beta,h)J(x_1-x_2)\,. \qquad (1)$$

Proof. The Griffiths-Hurst-Sherman (GHS) inequality[2] ensures that

$$\sigma_\Lambda^T(x_1,x_2;\beta,\{h_x\})\geq\sigma_\Lambda^T(x_1,x_2;\beta,\{h'_x\})\geq 0 \qquad (2)$$

whenever $0\leq h_x\leq h'_x$, where Λ is a box of arbitrary size and $\{h_x\}$ and $\{h'_x\}$ are two systems of magnetic fields at each point x. On the other hand, the correlation $\sigma_\Lambda^T(x_1,x_2;h_{x_1},h_{x_2},\ \{h_x=+\infty,x\neq x_1,x_2\})$ can be computed explicitly. It is given in the $\Lambda\to\infty$ limit by the formula

$$\sigma^T(x_1,x_2;h_{x_1},h_{x_2},\{h_x=+\infty,x\neq x_1,x_2\})$$
$$=4z_1z_2\frac{e^{4\beta J(x_1,x_2)}-1}{[1+z_1+z_2+z_1z_2e^{4\beta J(x_1,x_2)}]^2}, \qquad (3)$$

where

$$z_i=\exp\left[-2h_{x_i}+2\sum_{y\in Z^\nu;\,y\neq x_i}J(x_i,y)\right],\quad i=1,2$$

and J is the (two-body) potential.

To prove Theorem 1 for given $h>0$, β arbitrary, or $h=0$, $\beta<\beta_c$, it is sufficient to use formulas (2) and (3) and to choose, in formula (3), h_{x_1},h_{x_2} sufficiently large. The result is obtained similarly by symmetry for any given $h<0$. Finally, it is obtained at $h=0, \beta>\beta_c$ for the plus and minus phases by using the fact that the corresponding correlation functions in each pure phase can be obtained as limits when $h\to 0$ of the correlation functions at $h>0$ or $h<0$, respectively. Q.E.D.

If we next restrict our attention to the case of large magnetic field, Theorem 1, together with the results of the Appendix, yields moreover the following result.

Theorem 2. If the potential J of a lattice ferromagnetic system satisfies the inequalities

$$J_0/r^s < J(x_1-x_2) < J_0'/r^s, \quad r\neq 0 \tag{4}$$

where $r=|x_1-x_2|$, $s>\nu$, and J_0, J_0' are given strictly positive constants, then being given any $\beta>0$, there exists an $h_0>0$ such that the following inequalities hold for any real h satisfying $|h|>h_0$:

$$C(\beta,h)/r^s < \sigma^T(x_1,x_2;\beta,h) < C'(\beta,h)/r^s, \quad r\neq 0 \tag{5}$$

where $C(\beta,h)$ and $C'(\beta,h)$ are strictly positive constants (that may depend on β and h).

III. DECAY OF TWO-POINT CORRELATIONS OF FERROMAGNETIC SYSTEMS AT ARBITRARY NONZERO MAGNETIC FIELD

Being given a ferromagnetic potential that decreases like r^{-s}, Theorem 1 of Sec. II shows that the two-point connected correlations cannot decrease faster than r^{-s}. We prove here the following converse result for any nonzero magnetic field.

Theorem 3. Being given a ferromagnetic two-body interaction $J\geq 0$ satisfying the bound

$$J(x_1-x_2) < J_0(1+|x_1-x_2|)^{-s}, \tag{6}$$

there exists, for any nonzero magnetic field h_0, any reciprocal temperature β, and any $\alpha>0$, a constant $C(\beta,h_0,\alpha)$ such that

$$|\sigma^T(x_1,x_2;h_0,\beta,J)| < C(\beta,h_0,\alpha)(1+|x_1-x_2|)^{-(s-2\nu-\alpha)}. \tag{7}$$

The proof given below uses, as already mentioned in Sec. I, analyticity with respect to the potential and magnetic field instead of using only analyticity with respect to the magnetic field, which alone cannot lead to a fixed power of decrease for the correlations. It is based on the fact that the correlation function $\sigma^T(x_1,x_2;h_0,\beta,J)$ can be obtained, by analytic continuation, from the correlation function of a system with an interaction decreasing much quicker than J and with large magnetic field.

Proof. We consider the analytic mapping $t\to(z(t), J(r,t))$, where t is a complex variable, $|t|<1$, defined by

$$z(t)=(t/t_0)^n z_0, \tag{8}$$

$$J(r,t)=\begin{cases} J(r) & \text{if } r<r_0 \\ J(r)(1+r)^{-s(t)} & \text{if } r\geq r_0, \end{cases} \tag{9}$$

where

$$z(t)=e^{-2\beta h(t)}, \quad z_0=e^{-2\beta h_0}, \quad s(t)=s_0(t_0-t),$$

$$t_0=1-(s-2\nu-\alpha)/s_0, \quad s_0>0,$$

and where the choice of s_0, n, r_0 is given below. We note that

$$\sum_{x\in Z^\nu x\neq\{0\}} |J(x,t)| < \sum_{x\in Z^\nu x\neq\{0\}} J(x)(1+|x|)^{s-2\nu-\alpha} < \infty. \tag{10}$$

Let a be a given number such that $0<a<h_0/2$. If n satisfies the bound

$$t_0^{-n} z_0 < e^{-4\beta a}, \tag{11}$$

and if r_0 is chosen (independently of s_0, n) such that for every $r\geq r_0$

$$\tanh[\beta J_0(1+r)^{-(2\nu+\alpha)}] < \frac{[2\beta a C(\alpha)]^2}{\{\exp[2\beta a C(\alpha)]+1\}^2} \times (1+r)^{-(2\nu+\alpha/2)} \tag{12}$$

with

$$C(\alpha)^{-1}=\sum_{x\in Z^\nu x\neq\{0\}} (1+|x|)^{-(\nu+\alpha/2)},$$

then Proposition 4.2 of Ref. 5 allows one to show that $Q_\Lambda(\{h(t)+\lambda_x\}, \beta, J(.;t))\neq 0$ for arbitrary Λ, whenever $|t|<1$ and $|\lambda_x|<a$, for every $x\in Z^\nu$ (Q_Λ is the partition function for the finite box Λ and λ_x is an additional magnetic field at each point x).

By the methods of Ref. 7, this result in turn ensures that $\sigma^T(x_1,x_2;t)=\sigma^T(x_1,x_2;h(t),\beta,J(.;t))$ is analytic with respect to t in the domain $|t|<1$ and satisfies there the bound

$$|\sigma^T(x_1,x_2;t)| < C_1, \tag{13}$$

where the constant C_1 is independent of x_1, x_2, t.

On the other hand, let $z_2>0$ be chosen (independently of s_0) such that $(z_2, \beta J)$ is in the low-activity region, i.e., the analyticity domain obtained from the Kirkwood-Salzburg domain by the Lee-Yang isomorphism. If $\eta>0$ is such that

$$\left(\frac{t_0-\eta}{t_0}\right)^n z_0 < z_2, \tag{14}$$

then the low-activity methods of Ref. 4 (part II) allow one to derive, in the region $|t|<t_0-\eta$, the bound

$$|\sigma^T(x_1,x_2;t)| < C_2(1+|x_1-x_2|)^{-[(\eta s_0+s)-\nu-\epsilon]}, \tag{15}$$

where $\epsilon>0$ is any given number and C_2 is independent of x_1, x_2, and t.

The analyticity for $|t|<1$ then leads to a simple interpolation between the bounds (13) and (15) by a method analogous to that used in the proof of Theorem 3 in Ref. 4, or alternatively by more general methods based on the study of holomorphy envelopes.[8] It yields

$$|\sigma^T(x_1,x_2;t_0)| < C_3(1+|x_1-x_2|)^{-(\eta s_0+s-\nu-\epsilon)|\ln t_0/\ln(t_0-\eta)|}, \quad (16)$$

where C_3 is independent of x_1, x_2.

The bound (7) of Theorem 2 follows by choosing s_0 sufficiently large and, for instance, $\eta = s_0^{-1/3}$, $n = s_0^{2/3}$. Q.E.D.

Remarks. (i) The methods of the Appendix allow one to obtain, at sufficiently low activity, bounds of the type (15) in which the power of decrease $(\eta s_0 + s - \nu - \epsilon)$ is replaced by $\eta s_0 + s$. The power of decrease in the bound (16) is correspondingly replaced by $(\eta s_0 + s)(|\ln t_0|/|\ln(t_0-\eta)|)$. This method involves, however, a few technical complications and does not lead to a better uniform bound than $s - 2\nu - \alpha$ on the power of decrease in Theorem 2.

(ii) The potential J can be considered as the sum of a finite range potential and of a small perturbation decreasing like r^{-s}. This remark, together with the methods and results of Ref. 9, leads to an alternative proof of Theorem 2. The power of decrease obtained is not better than in Theorem 2.

IV. EXTENSION TO n-POINT CORRELATIONS AND CENTRAL-LIMIT THEOREM FOR SLOWLY DECREASING POTENTIALS

We now present the following extension of Theorem 3 to n-point functions.

Theorem 4. Being given a ferromagnetic interaction $J \geq 0$ satisfying the bound

$$J(x_1 - x_2) < J_0(1+|x_1-x_2|)^{-s},$$

there exists, for any nonzero magnetic field h_0, any reciprocal temperature β, and any $\alpha > 0$, a constant $C(\beta, h_0, \alpha)$ such that the following "strong cluster property" holds:

$$|\sigma^T(x_1,\ldots,x_n;h_0,\beta,J)| \leq C(\beta,h_0,\alpha)^n \times \sum_{\mathcal{T}(x_1,\ldots,x_n)} \prod_{(x_i,x_j)\in\mathcal{T}} (1+|x_i-x_j|)^{-s-2\nu-\alpha}, \quad (17)$$

where the sum runs over all trees-$\mathcal{T}$ on the points $x_1, \ldots, x_n$ (i.e., connected graphs without closed loops).

Proof. The bound (13) in the proof of Theorem 2 is adapted to the case of n-point functions by methods analogous to those of Theorem 4 in Ref. 10. The bounds (15) and consequently (16) are adapted by the methods of Ref. 4. The bound (16) is replaced by

$$|\sigma^T(x_1,\ldots,x_n;t_0)| < C^n n_1!\ldots n_p! \times \exp\left(-L(X)(\eta s_0 + s - \nu - \epsilon) \times \frac{|\ln t_0|}{|\ln(t_0-\eta)|}\right), \quad (18)$$

where $L(X)$ is the minimal length of all trees constructed on the points $x_1, \ldots, x_n$ of X, and possibly on other supplementary vertices, with respect to the distance $d(x,y) = \ln(1+|x-y|)$ (the factor $n_1!\ldots n_p!$ appears when the points $x_1,\ldots,x_n$ are not all distinct and occupy only p positions $x_1',\ldots,x_p'$ occurring respectively $n_1,\ldots,n_p$ times).

The bound (17) of Theorem 3 follows from the bound (18) and the result of the Appendix of Ref. 11. Q.E.D.

The bounds (17) of Theorem 3 are an extension to the case of slowly decreasing potentials of the results previously proved in Ref. 10 for finite-range potentials. As the latter, they are adapted in a straightforward way to systems of complex magnetic fields $\{h_x\}$ with nonzero real parts, and they ensure when $s > 3\nu$ that

$$\sum_{x_2,\ldots,x_n \in Z^\nu} |\sigma^T(x_1,\ldots,x_n)| < C^n n! < \infty, \quad (19)$$

where C is independent of $x_1,\ldots,x_n$ and of n.

These bounds yield in turn, by the same methods as in Ref. 12 [see Eq. (3.6)], the following central-limit theorem previously proved for finite-range potentials or exponentially decreasing pair correlations.[12-14]

Corollary. For any ferromagnetic system with interaction $J(r) < C/(1+r)^{3\nu+\epsilon}$ and nonzero magnetic field, the characteristic function of the block-spin distribution tends to that of a Gaussian as the size of the block spin tends to infinity. The observations[12,14] concerning corrections for large size L of block spins and their interpretation also apply to the present situation.

We finally note that in contrast to the case of exponentially decreasing potentials, one needs here the tree-decay bounds of Eq. (17) to show that the sum in the left-hand side of Eq. (19) is finite and hence to obtain the central-limit theorem. Weaker bounds derived from bounds on the two-point functions would not be sufficient here.

APPENDIX: LOW-ACTIVITY RESULTS

The results of this Appendix are presented here for convenience for lattice gases at low activity. They hold also for lattice spin-$\frac{1}{2}$ systems at large magnetic field, in view of the Lee-Yang isomorphism. They apply without conditions on the sign of the potential and extend also to continuous gases.

The method used is slightly different from that of

Ref. 3. As a matter of fact, it does not use the Fourier transformation and hence avoids losing a factor $r^{-\nu}$ in the decay of correlations. The method and results are due to a collaboration with M. Duneau and P. Renouard. Related results have been obtained independently in Ref. 15.

We denote here by ϕ, z, ρ^T, respectively, the potential, activity, and connected correlation functions of the lattice gas considered. In the case of continuous gases, the bound assumed on ϕ is to be replaced by a similar bound on $f(r) = e^{-\beta\Phi(r)} - 1$. The proof and results are then completely analogous (see remark at the end).

Theorem. Being given a potential Φ such that

$$|\phi(r)| < \phi_0(1+\gamma r)^{-\chi}, \quad r \neq 0$$

where ϕ_0, $\chi > \nu$, $\gamma > 0$ are given numbers, there exist constants $R > 0$ and C such that for $|z| < R$,

$$|\rho^T(x_1, x_2; z, \phi)| < C(1+\gamma|x_1 - x_2|)^{-\chi}. \quad (20)$$

Proof. It follows from Ref. 3 that there exist $R_1 > 0$ and K_1 such that the following bounds hold for $|z| < R_1$:

$$|\rho^T(x_1, x_2; z, \phi)| \leq \sum_{n\geq 0} (zK_1)^n \sum_{y_1, \ldots, y_n \in Z^{n\nu}} |f(x_1 - y_1)||f(y_1 - y_2)| \cdots |f(y_n - x_2)|, \quad (21)$$

where $f(r) = e^{-\beta\Phi(r)} - 1$.

The theorem is then a consequence of the following lemma together with the remark that

$$|f(r)| < (1 + \phi_0 e^{\phi_0})(1+\gamma r)^{-\chi}, \text{ for every } r \in Z^\nu. \quad (22)$$

Lemma.

$$\sum_{y_1, \ldots, y_n \in Z^{n\nu}} [(1+\gamma|x_1 - y_1|)^{-\chi}(1+\gamma|y_1 - y_2|)^{-\chi} \times \cdots \times (1+\gamma|y_n - x_2|)^{-\chi}] < K(\gamma, \chi)^n (1+\gamma|x_1 - x_2|)^{-\chi}. \quad (23)$$

Proof of the lemma. This result is proved by induction on n, by using the following inequality:

$$\sum_{y_1 \in Z^\nu} (1+\gamma|x_1 - y_1|)^{-\chi}(1+\gamma|y_1 - y_2|)^{-\chi} \leq \left(2\sum_{y \in Z^\nu}(1+\tfrac{1}{2}\gamma^2|y|)^{-\chi}\right)(1+\gamma|x_1 - y_2|)^{-\chi}, \quad (24)$$

which is obtained by dividing the y_1 space into the two regions

$$|x_1 - y_1| \geq |y_1 - y_2|, \quad |x_1 - y_1| \leq |y_1 - y_2|.$$

In the case of continuous systems, the sums are replaced by integrals, and a further factor $(1+\gamma)^\chi$ appears in the right-hand side of Eq. (24), in order to cover the region $|x_1 - y_2| < 1$. The results are then unchanged.

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