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DECAY OF CORRELATIONS FOR SLOWLY DECREASING POTENTIALS

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ABSTRACT

When the interaction potential of a ferromagnet decreases like r^{-s}, we prove that two-point correlations :

- do not decay faster than r^{-s} ;
- decay at least like r^{-s} at large magnetic field and moreover at least like $r^{-(s-2\nu)}$, where ν is the space dimension, for any non-zero magnetic field and arbitrary temperature.

Extensions of this latter result to n point correlations and to other systems are indicated and a central limit theorem (or gaussian limit of block-spin distribution) for slowly decreasing ferromagnetic interactions at any non-zero magnetic field is mentioned.

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The main results on the decay of correlations in classical statistical mechanics have been obtained so far for finite range or exponentially decreasing potentials. It has then been proved in a number of cases that the correlations decay exponentially with distance (in pure phases) and it is reasonably believed that they decay exponentially arbitrarily close to critical points (with a rate of exponential fall-off that tends to zero near these points).

The situation is different in the case of potentials that decrease only like an inverse power r^{-s} of the distance. On the one hand, it is sometimes believed on phenomenological bases that the correlations still decay exponentially away from critical points. However, it is proved [1], in a number of cases, that the correlations cannot decay exponentially if the potential does not. If we restrict our attention to ferromagnetic systems, we prove here the following stronger result, established in collaboration with H. Kunz :

<u>Theorem 1</u> The two-point (connected) correlation of a ferromagnetic system [i.e., $J(x_1-x_2) \geq 0$] does not decay faster than the potential J :

$$\sigma^T(x_1,x_2;\beta,h) > C(\beta,h)\, J(x_1-x_2) \quad (1)$$

<u>Proof</u>

The Griffiths-Hurst-Sherman (GHS) inequality [2] ensures that :

$$\sigma_\Lambda^T(x_1,x_2;\beta,\{h_x\}) \geq \sigma_\Lambda^T(x_1,x_2;\beta,\{h'_x\}) \geq 0 \quad (2)$$

whenever $0 \leq h_x \leq h'_x$, where Λ is an arbitrary box and $\{h_x\}$, $\{h'_x\}$ are two systems of magnetic fields at each point x. On the other hand, the correlation $\sigma^T(x_1,x_2;h_{x_1},h_{x_2},\{h_x=+\infty, x\neq x_1,x_2\}$ can be computed. It is given by the formula :

$$\sigma^T(x_1,x_2;h_{x_1},h_{x_2},\{h_x=+\infty,x\neq x_1,x_2\})$$
$$= 4 z_1 z_2 \frac{(e^{4\beta J(x_1,x_2)}-1)}{[1+z_1+z_2+z_1 z_2 e^{4\beta J(x_1,x_2)}]^2} \quad (3)$$

where

$$z_i = e^{-2h_{x_i} + 2\sum_{y \in Z^\nu \setminus x_i} J(x_i, y)}, \quad i = 1,2$$

and J is the (two-body) potential.

To prove Theorem 1 for a given value h of the magnetic field, it is sufficient to use formulae (2) and (3) and to choose, in formula (3), h_{x_1}, h_{x_2} sufficiently large. Q.E.D.

We now consider the converse problem, namely the determination of upper bounds on the decay of correlations. On the one hand, it is known [3], for general (discrete or continuous) lattice gases, or also (by the Lee-Yang isomorphism) for spin systems, that the two-point correlations decay at least like $r^{-(s-\nu)}$ at low activity (resp. at large magnetic field). In the Appendix, we prove that the power of decrease of the correlations at low activity is actually s and not $s-\nu$.

In a number of other situations, for instance for ferromagnetic spin systems at any non-zero magnetic field h and arbitrary temperature, it has also been proved [4] that the correlations decay at least like $r^{-s(\beta,h)}$, $s(\beta,h) > 0$ (where β is the reciprocal temperature). However, the power $s(\beta,h)$ obtained in Ref. 4) tends to zero when $h \to 0$.

We show below that the correlations decay at least like $r^{-(s-2\nu-\alpha)}$, where α is arbitrarily small and ν is the space dimension, for arbitrarily small values of $h (\neq 0)$ and arbitrary values of β. The fixed loss of power, i.e., 2ν, is due to technical reasons [5], and it seems safe to conjecture that the correlations decay like r^{-s} arbitrarily close to critical points.

<u>Theorem 2</u> Being given a ferromagnetic two-body interaction $J \geq 0$ satisfying the bound

$$J(x_1 - x_2) < J_0 (1 + |x_1 - x_2|)^{-s} \tag{4}$$

there exists, for any non-zero magnetic field h_0, any reciprocal temperature β and any $\alpha > 0$, a constant $C(\beta, h_0, \alpha)$ such that :

$$|\sigma^T(x_1,x_2;h_o,\beta,J)| < C(\beta,h_o,\alpha)\,(1+|x_1-x_2|)^{-(s-2\nu-\alpha)} \tag{5}$$

The proof, given below, uses analyticity with respect to the potential and magnetic field, instead of using only analyticity with respect to the magnetic field (which, alone, cannot lead to a fixed power of decrease for the correlations). It is based on the fact that the correlation function $\sigma^T(x_1,y_2;h_o,\beta,J)$ can be obtained, by analytic continuation, from the correlation function of a system with an interaction decreasing much quicker than J and with large magnetic field.

Proof

We consider the analytic mapping $t \to (z(t),J(r,t))$, $t \in \mathbb{C}$, $|t| < 1$, defined by :

$$z(t) = (t/t_o)^n\, z_o \tag{6}$$

$$J(r,t) = \begin{cases} J(r) & \text{if } r < r_o \\ J(r)\,(1+r)^{-s(t)} & \text{if } r \geq r_o \end{cases} \tag{7}$$

where

$$z(t) = e^{-2\beta h(t)},\ z_o = e^{-2\beta h_o},\ s(t) = s_o(t_o-t),\ t_o = 1-\frac{s-2\nu-\alpha}{s_o},$$
$$s_o > 0,$$

and where the choice of s_o, n, r_o is given below. We note that

$$\sum_{x\in Z^\nu\setminus\{0\}} |J(x,t)| < \sum_{x\in Z^\nu\setminus\{0\}} J(x)(1+|x|)^{s-2\nu-\alpha} < \infty \tag{8}$$

Let a be a given number such that $0 < a < h_o/2$. If n satisfies the bound :

$$t_0^{-n} z_0 \quad < \quad e^{-4\beta a} \tag{9}$$

and if r_0 is chosen (independently of s_0, n) such that $\forall r \geq r_0$:

$$\begin{aligned} &\operatorname{th}[\beta J_0 (1+r)^{-(2\nu+\alpha)}] \\ &< \frac{[2\beta a C(\alpha)]^2}{[\exp(2\beta a C(\alpha)) + 1]^2} (1+r)^{-(2\nu+\alpha/2)} \end{aligned} \tag{10}$$

with

$$C(\alpha)^{-1} = \sum_{x \in Z^\nu \setminus \{0\}} (1+|x|)^{-(\nu+\alpha/2)}$$

then Proposition 4-2 of Ref. 6) allows one to show that $Z_\Lambda(\{h(t)+\lambda_x\}, \beta, J(.;t)) \neq 0$ for arbitrary Λ, whenever $|t| < 1$ and $|\lambda_x| < a$, $\forall x \in Z^\nu$ (Z_Λ is the partition function for the finite box Λ and λ_x is an additional magnetic field at each point x).

By the methods of Ref. 7), this result in turn ensures that $\sigma^T(x_1,x_2;t) = \sigma^T(x_1,x_2;h(t),\beta,J(.;t))$ is analytic with respect to t in the domain $|t| < 1$ and satisfies there the bound :

$$|\sigma^T(x_1,x_2;t)| < C_1 \tag{11}$$

where the constant C_1 is independent of x_1,x_2,t.

On the other hand, let $z_2 > 0$ be chosen (independently of s_0) such that $(z_2,\beta J)$ is in the low activity region (i.e., the analyticity domain obtained from the Kirkwood-Salzburg domain by the Lee-Yang isomorphism). If $\eta > 0$ is such that :

$$\left(\frac{t_0-\eta}{t_0}\right)^n z_0 < z_2 \tag{12}$$

then the low activity methods of Ref. 4) (Part II) allow one to derive, in the region $|t| < t_0 - \eta$, the bound :

$$|\sigma^T(x_1, x_2; t)| < C_2 \, (1 + |x_1 - x_2|)^{-[(\eta s_0 + s) - \nu - \varepsilon]} \tag{13}$$

where $\epsilon > 0$ is any given number and C_2 is independent of x_1, x_2 and t.

The analyticity for $|t| < 1$ then leads to a simple interpolation between the bounds (11) and (13) by a method analogous to that used in the proof of Theorem 3 in Ref. 4), or alternatively by more general methods based on the study of holomorphy envelopes [8]. It yields

$$|\sigma^T(x_1, x_2; t_0)| < C_3 \, (1 + |x_1 - x_2|)^{-[\eta s_0 + s - \nu - \varepsilon] \times \left|\frac{\log t_0}{\log(t_0 - \eta)}\right|} \tag{14}$$

where C_3 is independent of x_1, x_2.

The bound (5) of Theorem 2 follows by choosing s_0 sufficiently large and, for instance, $\eta = (s_0)^{-1/3}$, $n = (s_0)^{+2/3}$. Q.E.D.

Remarks

1) The methods of the Appendix allow one to obtain, at sufficiently low activity, bounds of the type (13) in which the power of decrease $(\eta s_0 + s - \nu - \epsilon)$ is replaced by $\eta s_0 + s$. The power of decrease in the bound (14) is correspondingly replaced by $(\eta s_0 + s)(|\log t_0|/|\log(t_0 - \eta)|)$. This method involves, however, a few technical complications and does not lead to a better uniform bound than $s - 2\nu - \alpha$ on the power of decrease in Theorem 2.

2) The potential J can be considered as the sum of a finite range potential and of a small perturbation decreasing like r^{-s}. This remark, together with methods and results of Ref. 9) leads to an alternative proof of Theorem 2. For related technical reasons, the power of decrease obtained is not better than in Theorem 2. However, this method confirms the conjecture that the correlations decay as a matter of fact like r^{-s}.

We now present the following extension of Theorem 2 to n point functions :

<u>Theorem 3</u> Being given a ferromagnetic interaction $J \geq 0$ satisfying the bound

$$J(x_1 - x_2) < J_0 (1 + |x_1 - x_2|)^{-s}$$

there exists, for any non-zero magnetic field h_0, any reciprocal temperature β and any $\alpha > 0$ a constant $C(\beta, h_0, \alpha)$ such that the following "strong cluster property" holds :

$$\begin{aligned} &|\sigma^T(x_1, \ldots, x_n ; h_0, \beta J)| \\ &\leq C(\beta, h_0, \alpha)^n \sum_{\mathcal{T}(x_1, \ldots, x_n)} \\ &\qquad \prod_{(x_i, x_j) \in \mathcal{T}} (1 + |x_i - x_j|)^{-(s - 2\gamma - \alpha)} \end{aligned} \tag{15}$$

where the sum Σ runs over all trees on the points $x_1, \ldots, x_n$ (i.e., connected graphs without closed loops).

<u>Proof</u>

The bound (11) in the proof of Theorem 2 is adapted to the case of n point functions by methods analogous to those of Theorem 4 in Ref. 10). The bounds (13) and consequently (14) are adapted by the methods of Ref. 4). The bound (14) is replaced by :

$$|\sigma^T(x_1, \ldots, x_n ; t_0)| < C^n n_1! \ldots n_p! \, e^{-\left[(\eta s_0 + s - \gamma - \varepsilon) \times \frac{|\log t_0|}{|\log(t_0 - \eta)|}\right] \times L(X)} \tag{16}$$

where $L(X)$ is the minimal length of all trees constructed on the points $x_1, \ldots, x_n$ of X, and possibly on other supplementary vertices, with respect to the distance $d(x,y) = \log(1 + |x-y|)$ (the factor $n_1! \ldots n_p!$ appears when the points $x_1, \ldots, x_n$ are not all distinct and occupy only p positions $x'_1, \ldots, x'_n$ occurring respectively $n_1, \ldots, n_p$ times).

The bound (15) of Theorem 3 follows from the bound (16) and the result of the Appendix of Ref. 11). Q.E.D.

The bounds (15) of Theorem 3 are an extension to the case of slowly decreasing potentials of the results previously proved in 10) for finite range potentials. As the latter, they are adapted in a straightforward way to systems of complex magnetic fields $\{h_x\}$ (with non-zero real parts). As a consequence, they yield, by using the same methods as in Ref. 12) [see Eq. (3.6)], the following central limit theorem previously proved for finite range potentials or exponentially decreasing pair correlations [13)] :

<u>Corrolary</u> For any ferromagnetic system with interaction $J(r) < C/(1+r)^{3\nu+\epsilon}$ and non-zero magnetic field, the characteristic function of the block spin distribution tends to that of a Gaussian as the size of the block spin tends to infinity.

The observations [12),14)] concerning corrections for large size L of block-spins and their interpretation also apply to the present situation.

We conclude this note with the following remarks. The methods and results presented above can apply to various other systems as soon as one can have similar analyticity properties with respect to the potential. In particular in the case of systems with arbitrary potentials and arbitrary activities the results of Ref. 6) on the location of the zeros of the partition function together with the methods of the present paper and of Ref. 12) lead to analogous results on the decay of two-point and n point correlations, and on central limit theorems at high temperature.

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APPENDIX : LOW ACTIVITY RESULTS

We indicate here a method which is slightly different from that of 3) for deriving decay properties at low activity. It does not use the Fourier transformation and then avoids loosing a factor $r^{-\nu}$ in the decay of the correlations [the method is also valid for continuous systems. The bound assumed on Φ is then to be replaced by a similar bound on $f(r) = e^{\Phi(r)}-1$.] The connected pair correlation is denoted below by ρ^T.

Theorem Being given a potential Φ such that :

$$|\Phi(r)| < \Phi_0 \, (1+\gamma r)^{-\chi} \quad (r \neq 0)$$

where Φ_0, $\chi > \nu$, $\gamma > 0$ are given numbers, there exist constants $R > 0$ and C such that for $|z| < R$:

$$|\rho^T(x_1, x_2 ; z, \Phi)| \leqslant C \, (1+\gamma |x_1 - x_2|)^{-\chi}$$

Proof

It follows from Ref. 3) that there exist $R_1 > 0$ and K_1 such that the following bounds hold for $|z| < R_1$:

$$|\rho^T(x_1, x_2, z, \Phi)| \leqslant \sum_{n \geqslant 0} (z K_1)^n \sum_{y_1, \dots y_n \in Z^{n\nu}} |f(x_1 - y_1)| \times |f(y_1 - y_2)| \times \dots \times |f(y_n - x_2)|$$

where $f(r) = e^{\Phi(r)}-1$.

The theorem is then a consequence of the following lemma together with the remark that :

$$|f(r)| < (1+\Phi_0 e^{\Phi_0}) \, (1+\gamma r)^{-\chi} \; , \forall r \in Z^\nu$$

Lemma

$$\sum_{y_1,\dots,y_n \in Z^{n\nu}} \Big[(1+\gamma|x_1-y_1|)^{-\chi} \times (1+\gamma|y_1-y_2|)^{-\chi} \cdots \times \cdots \times (1+\gamma|y_n-x_2|)^{-\chi} \Big]$$

$$< K(\gamma,\chi)^n \, (1+\gamma|x_1-x_2|)^{-\chi}$$

Proof of the Lemma

This result is proved by induction on n, by using the following inequality :

$$\sum_{y_1 \in Z^\nu} (1+\gamma|x_1-y_1|)^{-\chi} (1+\gamma|y_1-y_2|)^{-\chi}$$

$$\leq \Big(2 \sum_{y \in Z^\nu} (1+\gamma^2 |y|/2)^{-\chi} \Big) \times (1+\gamma|x_1-y_2|)^{-\chi}$$

which is obtained by dividing the y_1 space into the two regions

$$|x_1-y_1| \geq |y_1-y_2| \,,\quad |x_1-y_1| \leq |y_1-y_2|$$

[In the case of continuous systems the sums are replaced by integrals, and a further factor $(1+\gamma)^\chi$ appears in the right-hand side of the last inequality.]

Note

At the moment of publishing this work, we learn that low activity results related to those of the present Appendix have been recently obtained independently by M. Lavaud (to be published).

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