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A MODIFIED LIPPMANN-SCHWINGER EQUATION FOR COULOMB - LIKE INTERACTIONS

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A B S T R A C T

It is known that amplitudes which differ from the Coulomb one by an over-all phase factor and by a distribution with a support at zero scattering angle, describe the same scattering process. We utilize this fact to derive new partial wave expansions, which have finite expansion coefficients, for amplitudes of Coulomb-like interactions. A modified form of the Lippmann-Schwinger equation is derived, which is free of infra-red divergences. For the case of the Coulomb interaction this equation leads to a different amplitude from the Coulomb one but equivalent to it as both describe the same scattering process. The method can be extended to derive free of infinities partial wave expansions of some field theoretical amplitudes.

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1. INTRODUCTION

In recent years several papers appeared with a rigorous treatment of the Coulomb scattering[1]. The anomalies of the Coulomb scattering are well known[1-3]. One way of avoiding some of the difficulties has been the approach of Dollard and others[1,4] to replace the asymptotic condition by a more general one that is satisfied by the Coulomb potential. This approach leads to a well-defined theory, but its practical applications are still rather limited due to complicated calculations. In practice all Coulomb potentials are screened ones, and this fact suggests that a treatment based on screened potentials should be possible. Such an approach was adopted by Taylor[3], who investigated the Coulomb amplitude as the limit of a screened one. One of his main conclusions is that the amplitude of the screened Coulomb potential converges as a distribution to the Coulomb amplitude times an over-all phase factor. The "screened" amplitude may differ from the Coulomb amplitude by an over-all phase factor and by a distribution with a support at zero angle only.

In this work we present new partial wave expansions for amplitudes of Coulomb-like interactions by modifying the over-all phase and by adding a distribution with a support at zero angle only. All these changes are not affecting the observed scattering quantities. On the other hand, this procedure will allow us to obtain partial wave equations without the usual infrared divergences. In this way the Lippmann-Schwinger equation will be modified to a form free of infrared divergences. A similar procedure can be applied to other equations, such as the Bethe-Salpeter equation and its approximations or to some perturbation expansions. The modified Lippmann-Schwinger equation is derived in Section 5; Sections 2, 3, and 4 contain introductory results which are needed for the derivation and its interpretation.

2. AMBIGUITIES OF COULOMB PHASE SHIFTS

The ambiguities of the Coulomb amplitude were mentioned in Section 1. In this section we will demonstrate that the Coulomb phase shifts have an ambiguity of an over-all arbitrary constant common to all phase shifts. The Coulomb amplitude is presented in a form

$$f_C(\theta)=\frac{1}{2ik}\sum_{\ell=0}^{\infty}(2\ell+1)\left(e^{2i\sigma_\ell}-1\right)P_\ell(\cos\theta), \tag{2.1}$$

where σ_ℓ are the Coulomb phase shifts and k is the momentum. Let us consider the amplitude obtained from (2.1) by adding a constant ε to all σ_ℓ's:

$$f_\varepsilon(\theta) = \frac{1}{2ik}\sum_{\ell=0}^{\infty}(2\ell+1)\left[e^{2i(\sigma_\ell+\varepsilon)}-1\right]P_\ell(\cos\theta) \tag{2.2}$$

$$= \frac{1}{2ik}\sum_{\ell=0}^{\infty}(2\ell+1)\left[(e^{2i\sigma_\ell}-1)e^{2i\varepsilon}+(e^{2i\varepsilon}-1)\right]P_\ell(\cos\theta)$$

$$= e^{2i\varepsilon}f_C(\theta) + \frac{e^{2i\varepsilon}-1}{2ik}\sum_{\ell=0}^{\infty}(2\ell+1)P_\ell(\cos\theta) \tag{2.3}$$

The distribution $\frac{1}{2}\Sigma(2\ell+1)\ P_\ell(\cos\theta)$, for $0 \le \theta \le \pi$, has the same properties as the Dirac delta function $\delta(1-\cos\theta)$. Thus $f_\varepsilon(\theta)$ differs from the Coulomb amplitude $f_C(\theta)$ by an over-all phase factor and by a distribution with a support at $\theta = 0$; therefore the two amplitudes describe the same scattering process.

3. THE BORN APPROXIMATION

The Coulomb amplitude is given by

$$f_C(\theta) = -\frac{\eta}{2k}e^{2i\sigma_0}\left(\frac{1-\cos\theta}{2}\right)^{-1-i\eta}, \tag{3.1}$$

where

$$\eta = \frac{Z_1 Z_2 e^2 m}{k}$$

is the Coulomb parameter.

Expanding (3.1) in powers of η we obtain

$$f_C(\theta) = -\frac{\eta}{2k\Delta} + i\,\frac{2\gamma+\ln\Delta}{2k\Delta}\,\eta^2 + \frac{(2\gamma+\ln\Delta)^2}{4k\Delta}\,\eta^3 + \dots, \tag{3.2}$$

where $\Delta = \frac{1}{2}(1-\cos\theta)$ and γ is the Euler constant. On the other hand, for the Coulomb phase shifts one has

$$\sigma_\ell = \eta\,\psi(\ell+1) + O(\eta^2), \tag{3.3}$$

where ψ is the digamma function and

$$\psi(\ell) = -\gamma + 1 + \frac{1}{2} + \dots + \frac{1}{\ell-1}. \tag{3.4}$$

Let us substitute Eq. (3.3) in (2.2) and expand in powers of η. By comparing equal powers with Eq. (3.2) we obtain for the Born approximation

$$-\frac{\eta}{k(1-\cos\theta)} = \frac{\eta}{k}\sum_{\ell=0}^{\infty}(2\ell+1)\psi(\ell+1)P_\ell(\cos\theta) \tag{3.5}$$

It is impossible to obtain this expansion by applying the usual procedure of finding the expansion coefficients a_n of a Legendre expansion

$$a_n = -\frac{\eta}{2k}\int_{-1}^{1}\frac{P_n(\cos\theta)}{1-\cos\theta}\,d\cos\theta \tag{3.6}$$

as all these integrals diverge. In Appendix A we show that Eq. (3.5) really holds. We derive there a more general expression

$$-\frac{\eta}{k(1-\cos\theta)} = \frac{\eta}{k}\sum_{\ell=0}^{\infty}(2\ell+1)\left[\psi(\ell+1)+\varepsilon\right]P_\ell(\cos\theta), \tag{3.7}$$

where ε is an arbitrary constant number. The equality should be understood in terms of distributions. One should note that the inclusion of the number ε changes the Born approximation by a distribution with a support at $\theta = 0$ only, thus this has no effect on observed scattering quantities.

4. THE YUKAWA POTENTIAL

In Section 5 we will derive a modified Lippmann-Schwinger equation by screening the Coulomb potential. For this purpose we will use the Yukawa potential. The matrix elements of the Yukawa potential

$$V(r) = -\alpha\exp(-\mu r)/r \tag{4.1}$$

in momentum space, are given by

$$\langle\vec{p}\,|V|\,\vec{q}\rangle = -\frac{\alpha}{2\pi^2}\,\frac{1}{(\vec{q}-\vec{p})^2+\mu^2} \tag{4.2}$$

$$= -\frac{\alpha}{4\pi^2 pq}\sum_{\ell=0}^{\infty}(2\ell+1)Q_\ell(X_{pq})P_\ell(\cos\theta), \tag{4.3}$$

where

$$X_{pq} = (p^2+q^2+\mu^2)/(2pq) \tag{4.4}$$

P_ℓ and Q_ℓ are Legendre polynomials and functions, respectively, and μ is the screening mass; $\vec{p}$ and $\vec{q}$ are momenta, and the normalization used throughout this paper is $\langle\vec{p}/\vec{q}\rangle = \delta^{(3)}(\vec{p}-\vec{q})$. The Legendre functions satisfy the following relation[5]:

$$Q_\ell(X) = P_\ell(X)\,Q_0(X) - W_{\ell-1}(X) \tag{4.5}$$

where $W_{\ell-1}(X)$ are polynomials of degree $\ell - 1$; their properties are given in Appendix B. Let us substitute Eq. (4.5) into (4.3)

$$\begin{aligned} \langle \vec{p}\,|V|\,\vec{q}\,\rangle = -\frac{\alpha}{4\pi^2 pq}\Big[\, Q_0(X_{pq})\sum_{\ell=0}^{\infty}(2\ell+1)P_\ell(\cos\theta)P_\ell(X_{pq}) \\ - \sum_{\ell=1}^{\infty}(2\ell+1)W_{\ell-1}(X_{pq})P_\ell(\cos\theta)\Big]. \end{aligned} \tag{4.6}$$

The first sum on the r.h.s. of Eq. (4.6) is a distribution such that for $\mu \to 0$ and for on-the-energy-shell momenta $p = q$, it has a support only at $\theta = 0$. Let us denote this sum by

$$\langle \vec{p}\,|R|\,\vec{q}\,\rangle = -\frac{\alpha}{4\pi^2 pq}\,Q_0(X_{pq})\sum_{\ell=0}^{\infty}(2\ell+1)P_\ell(X_{pq})P_\ell(\cos\theta), \tag{4.7}$$

We can subtract this sum from the scattering amplitude for $\mu \to 0$, without distorting the scattering process. In the first approximation for the T-matrix elements we will have

$$\begin{aligned} \langle \vec{p}\,|\tilde{V}|\,\vec{q}\,\rangle &= \langle \vec{p}\,|V|\,\vec{q}\,\rangle - \langle \vec{p}\,|R|\,\vec{q}\,\rangle \\ &= \frac{\alpha}{4\pi^2 pq}\sum_{\ell=1}^{\infty}(2\ell+1)W_{\ell-1}(X_{pq})P_\ell(\cos\theta)\,. \end{aligned} \tag{4.8}$$

for $|\vec{p}| = |\vec{q}|$ and for $\mu \to 0$, $X_{pq} = 1$ and we obtain

$$\begin{aligned} \langle \vec{p}\,|\tilde{V}|\,\vec{q}\,\rangle \xrightarrow[\mu\to 0]{} \frac{\alpha}{4\pi p^2}\sum_{\ell=1}^{\infty}(2\ell+1)W_{\ell-1}(1)P_\ell(\cos\theta) \\ = \frac{\alpha}{4\pi p^2}\sum_{\ell=1}^{\infty}(2\ell+1)\big[\psi(\ell+1)-\psi(1)\big]P_\ell(\cos\theta)\,, \end{aligned} \tag{4.9}$$

without infinities. Equation (4.9) coincides with Eq. (3.9) for the particular case $\varepsilon = -\psi(1) = -\gamma$.

5. THE MODIFIED LIPPMANN-SCHWINGER EQUATION

Let us treat first the Coulomb scattering; the generalization to Coulomb-like potentials is straightforward. We start with the Lippmann-Schwinger equation for the Yukawa potential, which is our screened Coulomb potential. The equation for the T-matrix elements is

$$\langle\vec{p}|T|\vec{q}\rangle = \langle\vec{p}|V|\vec{q}\rangle + \int \langle\vec{p}|V|\vec{k}\rangle \frac{d^3k}{E(q)-E(k)+i\epsilon} \langle\vec{k}|T|\vec{q}\rangle, \tag{5.1}$$

where E is the energy, $\vec{q}$ is on the energy-shell momentum, $\vec{p}$ and $\vec{k}$ are off-the-energy-shell momenta. Let us define

$$\langle\vec{p}|\tilde{T}|\vec{q}\rangle = \langle\vec{p}|T|\vec{q}\rangle - \langle\vec{p}|R|\vec{q}\rangle \quad , \tag{5.2}$$

where $\langle\vec{p}|R|\vec{q}\rangle$ is defined in (4.7). Let us now modify Eq. (5.1) in the following way:

$$\begin{aligned}\langle\vec{p}|\tilde{T}|\vec{q}\rangle = \langle\vec{p}|\tilde{V}|\vec{q}\rangle &+ \int \langle\vec{p}|V|\vec{k}\rangle \frac{d^3k}{E(q)-E(k)+i\epsilon} \langle\vec{k}|R|\vec{q}\rangle \\ &+ \int \langle\vec{p}|V|\vec{k}\rangle \frac{d^3k}{E(q)-E(k)+i\epsilon} \langle\vec{k}|\tilde{T}|\vec{q}\rangle ,\end{aligned} \tag{5.3}$$

where $\langle\vec{p}|\tilde{V}|\vec{q}\rangle$ is defined in (4.8). Now the limit $\mu \to 0$ can be performed on Eqs. (5.3) and (4.9) without infinities. This can be seen explicitly by going to the partial wave equations

$$\begin{aligned}\langle p\ell|\tilde{T}|q\ell\rangle = \langle p\ell|\tilde{V}|q\ell\rangle &+ \int \langle p\ell|V|k\ell\rangle \frac{k^2dk}{E(q)-E(k)+i\epsilon} \langle k\ell|R|q\ell\rangle \\ &+ \int \langle p\ell|V|k\ell\rangle \frac{k^2dk}{E(q)-E(k)+i\epsilon} \langle k\ell|\tilde{T}|q\ell\rangle ,\end{aligned} \tag{5.4}$$

where

$$\langle p\ell|\tilde{V}|q\ell\rangle = \frac{\alpha}{\pi pq} W_{\ell-1}(X_{pq}) , \tag{5.5a}$$

$$\langle p\ell|V|q\ell\rangle = -\frac{\alpha}{\pi pq} Q_\ell(X_{pq}) \tag{5.5b}$$

and

$$\langle p\ell|R|q\ell\rangle = -\frac{\alpha}{\pi pq} Q_0(X_{pq}) P_\ell(X_{pq}) . \tag{5.6}$$

Equation (5.4) is now free of infrared divergencies. Equations (5.3) or (5.4) can be written in the following operational form:

$$\tilde{T} = \tilde{V} + VgR + Vg\tilde{T}, \tag{5.7}$$

while the Lippmann-Schwinger equation has the form

$$\tilde{T} = W + Wg\,\tilde{T}. \tag{5.8}$$

One can find W by solving formally Eqs. (5.7) and (5.8). From Eq. (5.8) we obtain

$$\tilde{T} = (1 - Vg)^{-1}(\tilde{V} + Vg\,R), \tag{5.9}$$

and from Eq. (5.8)

$$\tilde{T} = (1 - Wg)^{-1}W = W(1 - gW)^{-1}. \tag{5.10}$$

Comparing Eq. (5.9) with (5.10) we obtain

$$(\tilde{V} + Vg\,R)(1 - gW) = (1 - Vg)W, \tag{5.11}$$

from which, taking into account Eq. (4.8), we obtain the integral equation for W

$$W = \tilde{V} + Vg\,R + (R - Vg\,R)\,g\,W. \tag{5.12}$$

In practice there is no need to solve Eq. (5.12) as one can proceed directly with Eq. (5.3).

The generalization of the above procedure to Coulomb-like potentials is straightforward. If (in operational form) the potential is given by

$$V = V_C + V_s, \tag{5.13}$$

where V_C is the Coulomb potential and V_s a short-range potential, then the modified Lippmann-Schwinger equation is of the form of Eq. (5.3) or (5.7), where now

$$\tilde{V} = V_C + V_s - R. \tag{5.14}$$

It should be noted that Eq. (5.3) will not reproduce exactly the Coulomb amplitude (3.1). We see the differences already in the lowest approximations (3.7) and (4.9). It is not yet clear what modifications are needed in Eq. (5.3) in order to reproduce exactly the Coulomb amplitude (3.1). Nevertheless the two amplitudes should be equivalent in the sense discussed above, and they should describe the same scattering process.

6. SUMMARY AND DISCUSSION

Equation (5.3) is our modified Lippmann-Schwinger equation. We have derived this equation by screening the Coulomb potential using for this purpose the

Yukawa potential with a mass μ. The limit $\mu \to 0$ becomes well defined after removing from the amplitude the distribution (4.7), and the remaining part is free of infrared divergences. The amplitude resulting from Eq. (5.3) is equivalent to the usual Coulomb amplitude (3.1) except for an over-all phase and a distribution with a support at zero scattering angle. In this work a partial wave expansion for the Born approximation of the Coulomb amplitude [Eq. (3.9)] was found. This expansion, although pointwise divergent, is well defined as a distribution. The Born approximation corresponding to our modified equation is given by Eq. (4.9) and it differs from the Born approximation of the Coulomb amplitude Eq. (3.7) by a distribution with a support at zero scattering angle. The procedure used in this paper can easily be extended to other cases. The matrix element (4.2) appears in quantum field theoretical perturbation expansions multiplied by some factors. So the separation given in Eq. (4.6), with the limiting procedure $\mu \to 0$, can be applied to all cases where propagators proportional to Eq. (4.2) appear. A natural place to apply this procedure will be the Bethe-Salpeter equation in a Lippmann-Schwinger-like form.

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APPENDIX A

THE PARTIAL WAVE EXPANSION OF $(1 - \cos\theta)^{-1}$

In this appendix we will derive Eq. (3.9). For finding a partial wave expansion one cannot use in this case Eq. (3.8), as all the coefficients are infinite. We will use three different methods to derive Eq. (3.9). First let us assume that the expansion

$$1/(1-z) = \sum_{n=0}^{\infty} (2n+1) a_n P_n(z) \tag{A.1}$$

exists. Let us multiply both sides by (1 - z) and compare coefficients of the same Legendre polynomials $P_n(z)$. Using the relation

$$z P_n(z) = [n P_{n-1}(z) + (n+1) P_{n+1}(z)] / (2n+1) \tag{A.2}$$

we obtain

$$\begin{aligned} 1 &= \sum (2n+1) a_n (1-z) P_n(z) \\ &= (a_0 - a_1) P_0(z) + \sum_{n=1}^{\infty} [(2n+1) a_n - (n+1) a_{n+1} - n a_{n-1}] P_n(z), \end{aligned} \tag{A.3}$$

from which the following recursion relation holds

$$a_0 - a_1 = 1 \tag{A.4}$$

$$(n+1) a_{n+1} = (2n+1) a_n - n a_{n-1} \tag{A.5a}$$

or

$$(n+1)(a_{n+1} - a_n) = n (a_n - a_{n-1}) . \tag{A.5b}$$

One can see that

$$(n+1)(a_{n+1} - a_n) = \text{const}. \tag{A.6}$$

is a solution of (A.5b). The constant in (A.6) must be equal to (-1) in order to satisfy Eq. (A.4), thus

$$a_{n+1} = a_n - 1/(n+1) \tag{A.7}$$

with an arbitrary a_0. From (A.7) we obtain

$$a_n = a_0 - 1 - \frac{1}{2} - \ldots - \frac{1}{n} = a_0 - \psi(n+1) + \psi(1), \tag{A.8}$$

where ψ is the digamma function. The relation (A.7) can also be obtained by calculating the difference between two successive coefficients

$$a_{n+1} - a_n = \frac{1}{2}\int_{-1}^{1} \frac{P_{n+1}(z) - P_n(z)}{1-z}\, dz \quad , \tag{A.9}$$

Using the relations[5)]

$$\sum_{\ell=0}^{n} (2\ell+1) P_\ell(z) = (n+1)[P_n(z) - P_{n+1}(z)]/(1-z) \tag{A.10a}$$

$$= \frac{d}{dz}[P_n(z) + P_{n+1}(z)] \tag{A.10b}$$

in (A.9), the result in the form of Eq. (A.7) is immediately obtained. Equation (A.8) can also be obtained by the limiting procedure of Eq. (4.9). A change in the over-all constant a_0 in Eq. (A.8) is proportional to a distribution

$$\sum_{n=0}^{\infty} (2n+1) P_n(z) \tag{A.11}$$

with a support at $z = 1$. Thus

$$1/(1-z) = -\sum_{n=0}^{\infty} (2n+1)[\psi(n+1) + \mathrm{const}]\, P_n(z) \,, \tag{A.12}$$

which is defined up to a distribution with a support at $z = 1$. The series (A.12) is a divergent one but it represents the l.h.s. of (A.12) as a distribution. All this can be shown by repeating the procedure of Ref. 3. The series (A.12) can be summed also be using special summation methods of divergent series. By checking numerically, we found that using the Wynn ε algorithm[6)] on the r.h.s. of (A.12) the l.h.s. value was reproduced. We have also checked the following new summation method suitable for partial waves determined up to a distribution with a support at $z = 1$ only:

$$S_N = \sum_{n=0}^{N} (2n+1)(a_n - a_{N+1})\, P_n(z) \quad . \tag{A.13}$$

The $\lim_{N\to\infty} S_N$ seems to converge (slowly) to the l.h.s. of Eq. (A.12). The idea behind this method is that an over-all constant (a_{N+1}) can be subtracted (a distribution with a support at $z = 1$) in such a way that the coefficients $a_N - a_{N+1}$ are decreasing for $n < N + 1$, resembling an asymptotic expansion. The series (A.12) can be summed exactly to $(1 - z)^{-1}$ using Legendre-Padé approximants. In Appendix B this is proved for a more general case. The particular case of Eq. (3.5) was treated in Ref. 7.

APPENDIX B

THE PARTIAL WAVE EXPANSIONS OF $(y - z)^{-1}$

In this appendix we shall discuss the expansion properties of $(y - z)^{-1}$. If $y > 1$ it is possible to use

$$a_n = \frac{1}{2}\int_{-1}^{1} \frac{P_n(z)}{y-z}\,dz = Q_n(y) \tag{B.1}$$

and to obtain the expansion

$$1/(y-z) = \sum_{n=0}^{\infty} (2n+1)\, Q_n(y)\, P_n(z)\,. \tag{B.2}$$

But let us follow the procedure used in Appendix A. Let us assume

$$1/(y-z) = \sum_{n=0}^{\infty} (2n+1) a_n\, P_n(z)\,. \tag{B.3}$$

Let us multiply both sides by $(y - z)$ and equate the coefficients of the same Legendre polynomials. We obtain, using Eqs. (A.2) and (B.3)

$$\begin{aligned} 1 &= \sum_{n=0}^{\infty} (2n+1) a_n\, (y-z)\, P_n(z) \\ &= (a_0 y - a_1) P_0(z) + \sum_{n=1}^{\infty} \left[(2n+1) y a_n - (n+1) a_{n+1} - n\, a_{n-1}\right] P_n(z)\,. \end{aligned} \tag{B.4}$$

Hence the expansion coefficients satisfy the recursion relation

$$a_0 y - a_1 = 1 \tag{B.5}$$

$$(n+1)\, a_{n+1} = (2n+1) y\, a_n - n\, a_{n-1} \tag{B.6}$$

with an arbitrary a_0. Let us substitute in (B.6) and (B.5)

$$a_n = a_0 P_n(y) + b_n \tag{B.7}$$

Due to Eq. (A.2) we obtain the following recursion relation for the b_n's:

$$\begin{aligned} b_0 &= 0\,, \quad b_1 = -1 \\ (n+1)\, b_{n+1} &= (2n+1) y\, b_n - n\, b_{n-1} \end{aligned} \tag{B.8}$$

Thus

$$1/(y-z) = a_0 \sum_{n=0}^{\infty} (2n+1) P_n(y) P_n(z) + \sum_{n=1}^{\infty} (2n+1) b_n P_n(z)\,. \tag{B.9}$$

Equation (B.2) is a particular case of Eq. (B.9) with

$$a_0 = Q_0(y) \;, \qquad b_n = -W_{n-1}(y) \;, \tag{B.10}$$

where $W_{n-1}(y)$ are polynomials of degree n - 1 and are equal to[5]

$$W_{n-1}(y) = \sum_{k=1}^{n} \frac{1}{k} P_{k-1}(y) P_{n-k}(y) \; . \tag{B.11}$$

They can be also evaluated using (B.8) and (B.10). Thus for the first few polynomials we have

$$W_{-1}(y) = 0 \;, \quad W_0(y) = 1 \;, \quad W_1(y) = \tfrac{3}{2} y \;,$$

$$W_2(y) = \tfrac{5}{2} y^2 - \tfrac{2}{3} \;, \quad \text{etc'} \ldots \quad .$$

The distribution which appears in (B.9)

$$D(y,z) = \frac{1}{2} \sum_{n=0}^{\infty} (2n+1) P_n(y) P_n(z) \tag{B.12}$$

is equal to the Dirac delta function $\delta(y - z)$ for y and z which are in the closed interval $[-1, 1]$ only (the closure relation for Legendre polynomials).

We shall show below that the series (B.9) can be summed exactly, to give $(y - z)^{-1}$, using Legendre-Padé approximants[7,8]. We shall approximate the expansion (B.3) by the $[N/1](z)$ approximant

$$\sum_n (2n+1) a_n P_n(z) \simeq \frac{d_0 P_0(z) + 3 d_1 P_1(z) + \ldots + (2N+1) d_N P_N(z)}{P_0(z) + c_1 P_1(z)} = \tag{B.13}$$

$$= [N/1](z)$$

in such a way that

$$[P_0(z) + c_1 P_1(z)] \sum_{n=0}^{N+2} (2n+1) a_n P_n(z) - \sum_{n=0}^{N} (2n+1) d_n P_n(z) = O(P_{N+2}(z)) \tag{B.14}$$

will differ from zero by a series of Legendre polynomials of order bigger than N + 1. Remembering that $P_0(z) = 1$, $P_1(z) = z$ and using Eqs. (B.14) and (A.2), we find that

$$(2n+1) a_n + [n a_{n-1} + (n+1) a_{n+1}] c_1 = d_n \;, \quad n = 0, 1, \ldots, N \;, \tag{B.15}$$

$$(2N+3) a_{N+1} + [(N+1) a_N + (N+2) a_{N+2}] c_1 = 0 \tag{B.16}$$

Equations (B.5) and (B.6) can be rewritten in the form

$$(2n+1)y\alpha_n - n\alpha_{n-1} - (n+1)\alpha_{n+1} = \delta_{n0} \ . \tag{B.17}$$

From Eqs. (B.17) and (B.16) we find

$$c_1 = -1/y \tag{B.18}$$

and from Eqs. (B.18), (B.17), and (B.15)

$$\alpha_n = -\delta_{n0}/y \ . \tag{B.19}$$

Thus from Eqs. (B.19), (B.18), and (B.13) we find

$$[N/1](z) = 1/(y-z) \tag{B.20}$$

for any N = 0, 1, ... and for any a_0.

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