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EIGHTH-ORDER CONTRIBUTION TO THE CALLAN-SYMANZIK FUNCTION IN QUANTUM ELECTRODYNAMICS FROM FOURTH-ORDER VACUUM POLARIZATION INSERTIONS

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A B S T R A C T

We have calculated the contribution to the Callan-Symanzik function $\beta(\alpha)$ in quantum electrodynamics from the Feynman diagrams corresponding to fourth-order vacuum polarization corrections to the fourth-order vacuum polarization (see Figs. 3b and 4). The result is

$$\left(\frac{\alpha}{\pi}\right)^4\left[\frac{-61}{1296}+\frac{2}{27}\pi^2+\frac{1}{2}\zeta(3)\right] = 1.285\left(\frac{\alpha}{\pi}\right)^4.$$

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The Callan-Symanzik function [1),2)] $\beta(\alpha)$ in quantum electrodynamics (QED) has been calculated in perturbation theory [3)] up to sixth-order,

$$\beta(\alpha) = \frac{2}{3}\frac{\alpha}{\pi} + \frac{1}{2}\left(\frac{\alpha}{\pi}\right)^2 - \frac{121}{144}\left(\frac{\alpha}{\pi}\right)^3 + \mathcal{O}\left[\left(\frac{\alpha}{\pi}\right)^4\right]$$

The simplicity of the sixth-order coefficient is only apparent. In the intermediate steps of the calculation there appear higher transcendentals which finally cancel when the total is evaluated. The motivation for higher order calculations of $\beta(\alpha)$ in QED is twofold. On the one hand we search for regularity patterns which may help in getting a deeper understanding of the perturbation series ; in particular the nature of the fixed point structure of the renormalization group [4)]. On the other hand, the knowledge of higher orders in $\beta(\alpha)$ is useful for high accuracy calculations of observables governed by the short-distance behaviour of QED. One example is the anomalous magnetic moment of the muon [5)].

A classification of the possible Feynman diagrams which, in principle, contribute to the eighth-order photon proper self-energy, and hence to $\beta(\alpha)$, is as follows [6)] : genuine one-fermion loop type diagrams, Fig. 1 ; genuine two-fermion loop type diagrams, Fig. 2 ; two-fermion loop type diagrams which after shrinkage of internal photon self-energies become one-fermion loop type diagrams, Fig. 3a,b ; three-fermion loop type diagrams which after shrinkage become one-fermion loop type, Fig. 4. Let us call their respective contributions to $\beta(\alpha)$

$$\beta_4^{[1]}, \quad \beta_4^{[2]}, \quad \beta_4^{[2,1]}, \quad \beta_4^{[3,1]} .$$

Within $\beta_4^{[2,1]}$ we shall further distinguish between second-order vacuum polarization corrections, Fig. 3a, and fourth-order vacuum polarization corrections, Fig. 3b :

$$\beta_4^{[2,1]} = \beta_4^{[2,1]}(2^{nd}\text{ v.p.}) + \beta_4^{[2,1]}(4^{th}\text{ v.p.}) .$$

We have calculated, analytically, $\beta_4^{[2,1]}(4^{th}\text{v.p.})$ and $\beta^{[3,1]}$; i.e., the full effect of the fourth-order vacuum polarization correction. The method follows closely the sixth-order calculation described in detail in Ref. 3). The battle-horse of the calculation is the function $\Xi(m^2/-k^2)$ related to the lowest order light-by-light forward scattering amplitude $\Pi^{\mu\nu\rho\sigma}(q,k,m^2)$ as follows

$$\frac{1}{24}\, g_{\mu\nu}\, g_{\rho\sigma}\, g_{\alpha\beta}\, \frac{\partial^2}{\partial q_\alpha \partial q_\beta}\, \Pi^{\mu\nu\rho\sigma}(q,k,m^2) = \frac{\alpha^2}{-k^2}\, \Xi\left(m^2/-k^2\right) .$$

A useful parametrization of this function is 7) ($z \equiv m^2/-k^2$; $0 \leq z \leq \infty$)

$$I(z) = -\frac{4}{3}\int_0^1 dx \sum_{n=1}^{4} \frac{x^{n-1}(1-x)^{n-1}}{[z+x(1-x)]^n}\left[B_n x^2(1-x)^2 + E_n x(1-x)^3 + D_n (1-x)^4\right]$$

with

$$\begin{aligned} B_1 &= 42 , & B_2 &= -170 , & B_3 &= 190 , & B_4 &= -69 \\ E_1 &= -124 , & E_2 &= 289 , & E_3 &= -274 , & E_4 &= 90 \\ D_1 &= 24 , & D_2 &= -65 , & D_3 &= 62 , & D_4 &= -21 \end{aligned}$$

Note that $\sum_{i=1}^{4} D_i = 0$ and that $E_4 + B_4 + D_4 = 0$.

Another useful piece of information for the calculation is the knowledge of the second and fourth order asymptotic form of the photon proper self energy

$$\alpha\, \Pi_R^{\infty}\left(\frac{k^2}{m^2}, \alpha\right) = \frac{\alpha}{\pi}\left(a_1 + b_1 \log\frac{-k^2}{m^2}\right) + \left(\frac{\alpha}{\pi}\right)^2\left(a_2 + b_2 \log\frac{-k^2}{m^2}\right) + \mathcal{O}\left[\left(\frac{\alpha}{\pi}\right)^3\right]$$

where

$$a_1 = \frac{5}{9} , \quad b_1 = -\frac{1}{3} , \quad a_2 = \frac{5}{24} - \zeta(3) , \quad b_2 = -\frac{1}{4} .$$

The contribution to $\beta_4^{[2,1]}(4^{th}$ v.p.) is then separated into two parts : the contribution from the light-by-light kernel $\Xi(z)$; and a remainder from the mass renormalization counter terms [3] (see Fig. 5). We find

$$\beta_4^{[2,1]}(4^{th.}\text{ v.p. ; light-by-light}) =$$

$$\left(\frac{\alpha}{\pi}\right)^4\left(-\frac{1}{8}\right)\left\{a_2\,\Xi(0)+b_2\int_0^\infty dz\,\log z\,\frac{d}{dz}\,\Xi(z)\right\}=\left(-\frac{7}{48}-\frac{1}{2}\,\zeta(3)\right)\left(\frac{\alpha}{\pi}\right)^4;$$

and

$$\beta_4^{[2,1]}(4^{th.}\text{ v.p. ; mass counterterm}) =$$

$$\left(\frac{\alpha}{\pi}\right)^4\frac{2}{3}\int_0^1 dx\,(x-2)\left[a_2+b_2\log\frac{(1-x)^2}{x}\right]=\left(-\frac{1}{3}+\zeta(3)\right)\left(\frac{\alpha}{\pi}\right)^4;$$

i.e.,

$$\beta_4^{[2,1]}(4^{th.}\text{ v.p.})=\left(-\frac{23}{48}+\frac{1}{2}\,\zeta(3)\right)\left(\frac{\alpha}{\pi}\right)^4=0.122\left(\frac{\alpha}{\pi}\right)^4$$

Correspondingly, from the improper fourth-order vacuum polarization corrections (see Fig. 4), we find

$$\beta_4^{[3,1]}(\text{light-by-light}) =$$

$$-\frac{1}{8}\int_0^\infty\frac{dz}{z}\left(a_1^2-2a_1b_1\log z+b_1^2\log^2 z\right)\frac{d}{dz}\,\Xi(z)=-\frac{19}{162}\left(\frac{\alpha}{\pi}\right)^4;$$

and

$$\beta_4^{[3,1]}(\text{mass counterterm}) =$$

$$-\frac{2}{3}\int_0^\infty dx\,(x-2)\left[a_1^2+2a_1b_1\log\frac{(1-x)^2}{x}+b_1^2\log^2\frac{(1-x)^2}{x}\right]$$

$$=\left(89/162+\frac{2\pi^2}{27}\right)\left(\frac{\alpha}{\pi}\right)^4,$$

i.e.,

$$\beta_4^{[3,1]} = \left(\frac{35}{81} + \frac{2}{27}\pi^2\right)\left(\frac{\alpha}{\pi}\right)^4 = 1.163 \left(\frac{\alpha}{\pi}\right)^4$$

The complete contribution to $\beta(\alpha)$ from fourth-order vacuum polarization corrections to the fourth-order vacuum polarization (Figs. 3b and 4) is then

$$\beta_4^{[2,1]}(4^{th}\ \mathrm{v.p.}) + \beta_4^{[3,1]} =$$

$$\left(\frac{-61}{1296} + \frac{2}{27}\pi^2 + \frac{1}{2}\zeta(3)\right)\left(\frac{\alpha}{\pi}\right)^4 = 1.285 \left(\frac{\alpha}{\pi}\right)^4$$

Unlike what happens in previous lower order calculations we see two transcendental numbers appearing in this result : π^2 and $\zeta(3)$, the Riemann zeta function of argument three. The quantity π^2 appears in the calculation of the mass renormalization counter-term from the improper fourth-order vacuum polarization (Fig. 5b) ; $\zeta(3)$ appears because of the dependence of $\beta_4^{[2,1]}(4^{th}\mathrm{v.p.})$ on the fourth-order vacuum polarization constant a_2. A priori, it seems unlikely that these transcendentals should cancel with similar contributions from other eighth-order sources. Indeed the diagrams which contribute to $\beta^{[1]}$, $\beta^{[2]}$ and $\beta^{[2,1]}(2^{nd}\mathrm{v.p.})$ have very different topological structure.

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6) This is a convenient classification already made in Ref. 3).

7) See Ref. 3), Appendix A. An alternative parametrization is also given in the Appendix C of this reference.

FIGURE CAPTIONS

Figure 1 Genuine one-fermion loop diagrams.

Figure 2 Genuine two-fermion loop diagrams.

Figure 3 Two-fermion loop diagrams which after shrinkage of internal photon self-energies become one-fermion loop type. Figure 3a corresponds to second-order vacuum polarization corrections ; figure 3b to fourth-order proper vacuum polarization corrections.

Figure 4 Three-fermion loop diagrams which after shrinkage of internal photon self-energies become one-fermion loop type.

Figure 5 Mass-renormalization counter-terms due to fourth-order vacuum polarization insertions which give eighth-order contributions to $\beta(\alpha)$; a) from the proper fourth-order vacuum polarization ; b) from the improper graphs.

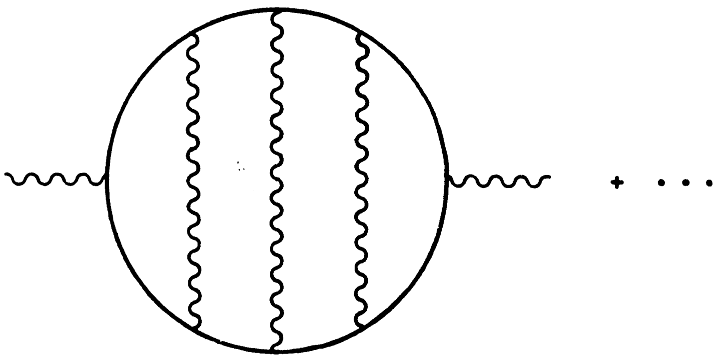

FIG. 1

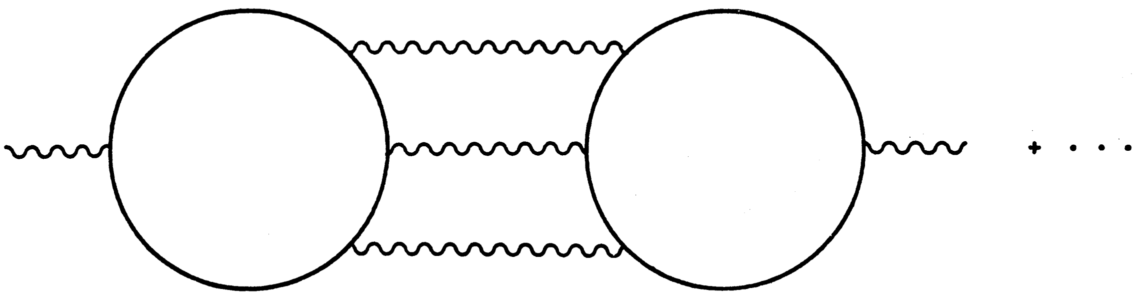

FIG. 2

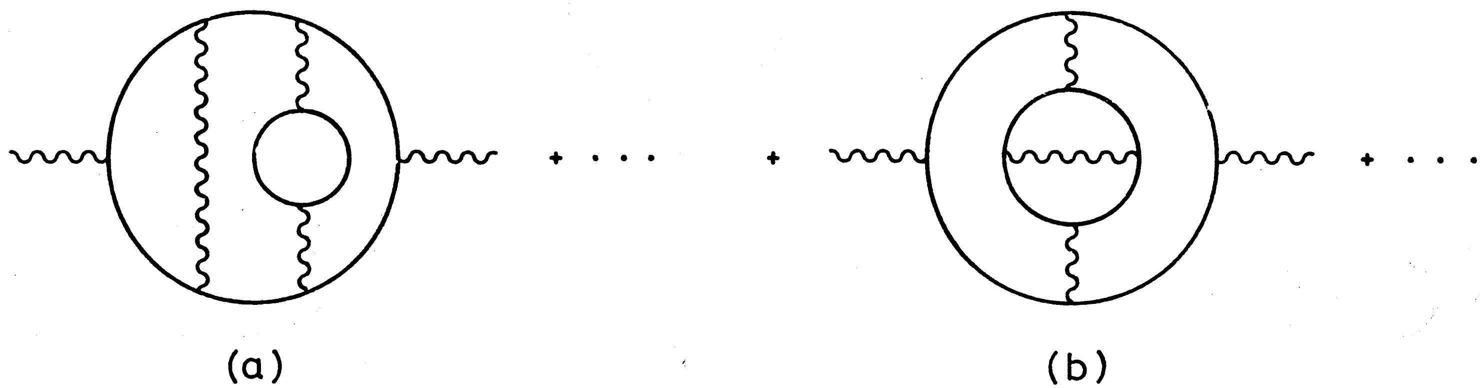

FIG. 3

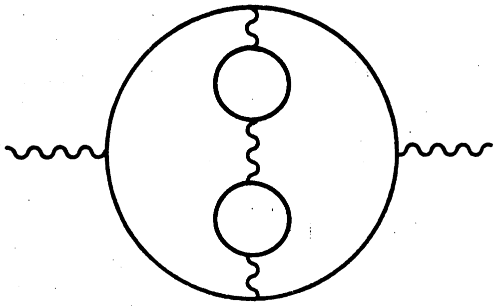

FIG. 4

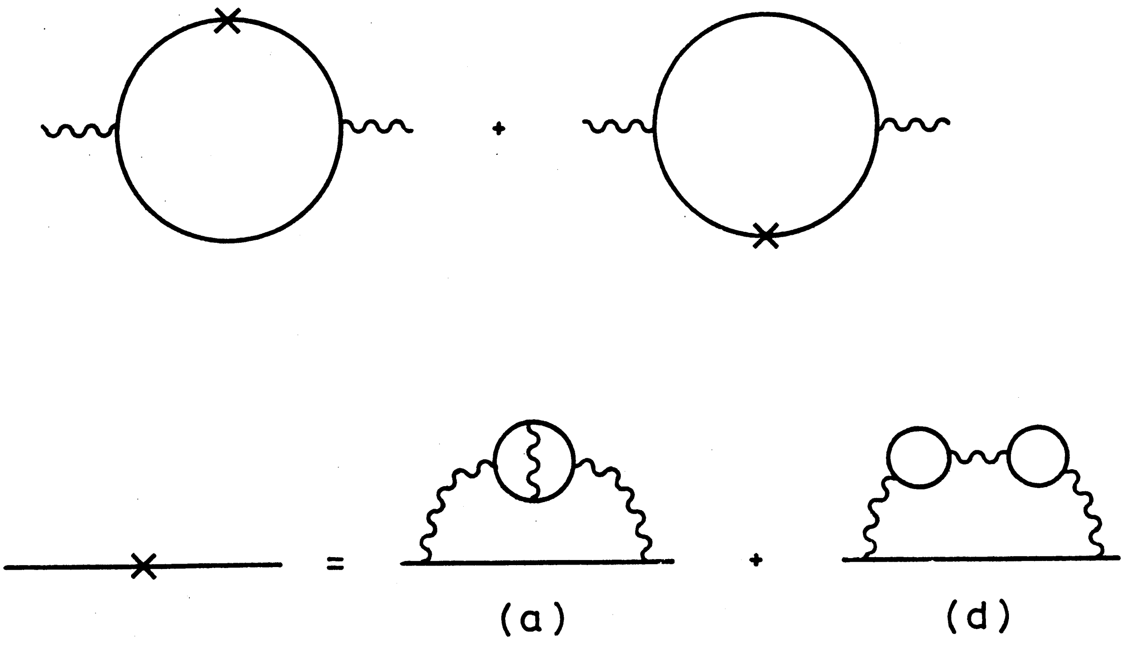

FIG. 5