Ref.TH.2067-CERN

ON THE RELATION BETWEEN INCLUSIVE AND EXCLUSIVE $p_\perp$ DISTRIBUTIONS

F. Elvekjaer *)

CERN -- Geneva

A B S T R A C T

Inclusive and exclusive scattering is studied within an impact parameter framework. It is shown that the inclusive $p_\perp$ distribution at a fixed value of centre-of-mass longitudinal momentum, $p_\parallel$, is governed only by the impact parameter structure of the observed particle. Next we introduce to each of the produced particles a radius, which is assumed to be independent of all the other particles, including those in the initial state. This enables us at a fixed value of $p_\parallel$ to relate the elastic πN, KN and NN data, at laboratory momenta less than 20 GeV, to the pion, kaon, proton and antiproton inclusive $p_\perp$ distributions at ISR energies.

----

*) Address after 1 September 1975 :
II. Institut für Theoretische Physik,
Universität Hamburg.

Ref.TH.2067-CERN
25 August 1975

1. - INTRODUCTION

The interest in large $p_\perp$ data, both for exclusive two-to-two and for inclusive processes, has been very much motivated by the hope that one would in this kinematic region be able to see the effect of point-like structures within the hadrons. In the inclusive 90^o spectra, especially of π's and K's, at ISR energies one has indeed seen evidence [1] in favour of the power behaviour at large $p_\perp$ of the form $p_\perp^{-N}$ which is characteristically associated with point interactions [2]. For exclusive two-to-two processes the situation is less clear. Although attempts in fitting fixed angle data at large $p_\perp$ with power laws have been partly successful it is clear that the arbitrariness involved in the phenomenology has made firm conclusions difficult [3]. Furthermore it was argued in Ref. 4) that the large angle pp elastic data is somehow controlled by a geometrical length $R \simeq 1$ fm. In Ref. 5), the data were studied within an impact parameter framework. Here the impact parameter b was defined as the variable conjugate to transverse centre-of-mass momentum $p_\perp$. As a result one is led to study data as functions of $p_\perp$ at fixed values of longitudinal momentum, $p_\parallel$, and not as is usually done at fixed energy or angle. The reason for this is that the impact parameter as defined this way commutes with $p_\parallel$ whereas it does not commute with energy or angle. It is found that exponential behaviour of the type $e^{-2Rp_\perp}$ gives a very good description of $(d\sigma/dt)(pp \to pp)$ for $p_\perp \lesssim 3$ GeV and also it gives a reasonable description of $(d\sigma/dt)(\pi p \to \pi p)$ for $p_\perp \lesssim 1.5 - 2.0$ GeV at fixed values of $p_\parallel$ in the range $0 \leq p_\parallel \lesssim 1.4$ GeV. That behaviour can be nicely explained in this impact parameter formalism by associating a radius R to the interaction or more precisely by a singularity in the b plane at $\pm iR$.

It is the purpose of this paper to show that the $p_\perp$ distributions of both exclusive two-body processes and of inclusive processes can be consistently described and related within the impact parameter formalism.

For each of the produced particles we define the (two-dimensional) impact parameter $\vec{b}$ as the variable conjugate to the centre-of-mass transverse momentum $\vec{p}_\perp$. It is emphasized that the usual small angle approximations [see, for example, Ref. 6)] are avoided. As discussed in Ref. 5), it is of fundamental importance to go to large $p_\perp$ in order to study the impact parameter structure of the scattering. In Section 2 we show, under certain smoothness assumptions on the impact parameter amplitude, that the inclusive $p_\perp$ spectrum is governed only by the $\vec{b}$ structure of the observed particle.

Here it is important that there are many more particles produced than the observed one, which, because of the averaging over all the other produced particles, in the transverse plane scatters as if off a point-like scattering centre. Instead of a radius of the interaction we then introduce in Section 3 a radius of each of the produced particles. This radius is assumed to be independent of the particles in the initial state and of all the other particles. Without any detailed model for the shape of the particles, i.e., about the sharpness of the edge of a particle, we are then, at a given value of $p_\parallel$, able to relate the exclusive $p_\perp$ distributions to the inclusive $p_\perp$ distributions.

The $p_\perp$ distribution for the process $a+b \to c+d$ is governed by the sum $R = R_c + R_d$ of the radii of the produced particles and the $p_\perp$ distribution for the inclusive process $a+b \to c+X$ is governed by the radius R_c. In Section 4, we study at $p_\parallel = 0$ $p_\perp$ distributions of data for πN, KN and pp elastic cross-sections as compared with inclusive π, K, p and $\bar{p}$ spectra also at $p_\parallel = 0$. The comparison works quite well for $p_\perp \lesssim 2.3$ GeV in inclusive p and $\bar{p}$ production and for $p_\perp \lesssim 1.0$ - - 1.2 GeV in the pion and kaon cases. Beyond these $p_\perp$ values it seems in all cases that the data show some excess relative to the exponential fall-off in $p_\perp$ given by the radii. This excess is possibly due to point interactions, which give rise to power behaviour in $p_\perp$, and we argue that the reason why this possible power behaviour sets in for smaller $p_\perp$ in the pion than in the proton spectra might be due to the fact that $R_\pi > R_p$.

2. - INCLUSIVE $p_\perp$ DISTRIBUTIONS FROM IMPACT PARAMETER AMPLITUDES

First we discuss the inclusive amplitude. Consider the process

$$a + b \to c + x_1 + x_2 + \cdots + x_n \tag{1}$$

where in the centre-of-mass system the particle c has momentum $\vec{p} = (p_\parallel, p_\perp)$ and particle x_i has momentum $\vec{Q}_i = (k_i, \vec{q}_i)$. $p_\parallel$ and k_i are the longitudinal components and $\vec{p}_\perp$ and $\vec{q}_i$ are the two-dimensional projections onto the transverse plane.

The energy is given by (neglect masses)

$$\begin{aligned}\sqrt{s} &= \sqrt{p_{\parallel}^2 + \vec{p}_{\perp}^{\,2}} + \sum_{i=1}^{n} \sqrt{k_i^2 + \vec{q}_i^{\,2}} \\ &\simeq \sqrt{p_{\parallel}^2 + \vec{p}_{\perp}^{\,2}} + \sum_{i=1}^{n} |k_i|\end{aligned} \qquad (2)$$

since on the average, phenomenologically, $|\vec{q}_i| \ll k_i$ [see, e.g., Ref. 7].

In the following we want to consider the inclusive process $a + b \to c + X$ only in the transverse plane and it follows from Eq. (2) that energy conservation can be disregarded during integration over the $\vec{q}_i$'s. Then the inclusive cross-section is given by (suppressing longitudinal variables)

$$\frac{d\sigma}{d\vec{p}_{\perp}}(a + b \to c + \underline{X}) = \sum_n \sigma_n(p_{\perp}) \qquad (3)$$

where σ_n in terms of the amplitude F_n for the process (1) is given by

$$\sigma_n(p_{\perp}) = \int |F_n(\vec{p}_{\perp}, \vec{q}_1, \vec{q}_2, \cdots, \vec{q}_n)|^2 \, \delta^{(2)}(\vec{p}_{\perp} + \sum_i \vec{q}_i) \prod_i d^2\vec{q}_i \qquad (4)$$

The impact parameter $\vec{b}_i$ is defined as the variable conjugate to the transverse momentum $\vec{q}_i$. The amplitude F_n can then be written in terms of its impact parameter transform A_n as [6]

$$F_n(\vec{p}_{\perp}, \vec{q}_1, \cdots, \vec{q}_n) = \left(\frac{1}{2\pi}\right)^{2n+2} \int d^2\vec{b}_c \prod_i d^2\vec{b}_i \; e^{-i\vec{b}_c \vec{p}_{\perp}} \; e^{-i\sum_i \vec{b}_i \vec{q}_i}$$
$$\delta^{(2)}(\vec{b}_c + \sum_i \vec{b}_i - \vec{B}) \, A_n(\vec{b}_c, \vec{b}_1, \cdots, \vec{b}_n) \; . \qquad (5)$$

$\vec{b}_c$ is the impact parameter of the observed particle c. The δ function expresses the fact that the scattering is a function only of differences of $\vec{b}$'s. Due to the translational invariance in the transverse plane $\vec{B}$ is arbitrary and we choose $\vec{B} = \vec{0}$.

Inserting Eq. (5) into Eq. (4) we arrive at (see Appendix)

$$\sigma_n(p_\perp) = -(2\pi)^{-4}\int d^2\vec{b}_1 \cdots d^2\vec{b}_n\, d^2\Delta\vec{b}_1\; e^{-i(n+1)\Delta\vec{b}_1\vec{p}_\perp}\, A_n\Big(\vec{b}_c = -\sum_{i=1}^{n}\vec{b}_i, \vec{b}_1, \vec{b}_2, \cdots, \vec{b}_n\Big)$$
$$A_n^*\big(\vec{b}_c + n\Delta\vec{b}_1, \vec{b}_1 + \Delta\vec{b}_1, \cdots, \vec{b}_n + \Delta\vec{b}_1\big). \tag{6}$$

Changing now integration variables

$$\big(\vec{b}_1, \cdots, \vec{b}_n, \Delta\vec{b}_1\big) \longrightarrow \Big(\vec{b}_c = -\sum_{i=1}^{n}\vec{b}_i, \vec{b}_2, \vec{b}_3, \cdots, \vec{b}_n, \vec{b}_c' = \vec{b}_c + n\Delta\vec{b}_1\Big)$$

we get

$$\sigma_n(p_\perp) \simeq \int d^2\vec{b}_c\, d^2\vec{b}_c'\; e^{i(\vec{b}_c - \vec{b}_c')\vec{p}_\perp} f_n(\vec{b}_c, \vec{b}_c') \tag{7}$$

where

$$f_n(\vec{b}_c, \vec{b}_c') = \frac{1}{(2\pi)^4 n}\int d^2\vec{b}_2\, d^2\vec{b}_3 \cdots d^2\vec{b}_n\; A_n\Big(\vec{b}_c, \vec{b}_1 = \vec{b}_c - \sum_{i=2}^{n}\vec{b}_i, \vec{b}_2, \cdots, \vec{b}_n\Big)$$
$$A_n^*\big(\vec{b}_c', \vec{b}_1 + \Delta\vec{b}_1, \vec{b}_2 + \Delta\vec{b}_1, \cdots, \vec{b}_n + \Delta\vec{b}_1\big) \tag{8}$$

with $\Delta\vec{b}_1 = (\vec{b}_c' - \vec{b}_c)/n$. In Eq. (7) we have assumed n large enough to justify the approximation $\vec{b}_c' - \vec{b}_c = n\Delta\vec{b}_1 \approx (n+1)\Delta\vec{b}_1$.

Now we come to our smoothness assumption on the impact parameter amplitude. Since hadrons are assumed to be extended objects with spatial extension of the order of 1 fm, we also assume that A_n is significantly different from zero only in a limited region in $\vec{b}$ space with extension $\approx$1 fm. We furthermore assume that n is big enough so that A_n, as a function of $\vec{b}_i$, is approximately constant over a range of the order of (1 fm)/n. It then follows that we can put $\Delta\vec{b}_1 = \vec{0}$ in Eq. (8). Having done that we write after a change in integration variables $(\vec{b}_i \rightarrow [\vec{b}_i - 1/n\ \vec{b}_c])$ $f_n(\vec{b}_c, \vec{b}_c')$ as

$$f_n(\vec{b}_c,\vec{b}_c') \simeq \frac{1}{(2\pi)^4 n}\int d^2\vec{b}_2\, d^2\vec{b}_3 \cdots d^2\vec{b}_n\, A_n\Big(\vec{b}_c, -\sum_2^n \vec{b}_i + \frac{1}{n}\vec{b}_c, \vec{b}_2 + \frac{1}{n}\vec{b}_c, \ldots, \vec{b}_n + \frac{1}{n}\vec{b}_c\Big)$$
$$A_n^*\Big(\vec{b}_c', -\sum_2^n \vec{b}_i + \frac{1}{n}\vec{b}_c', \vec{b}_2 + \frac{1}{n}\vec{b}_c', \ldots, \vec{b}_n + \frac{1}{n}\vec{b}_c'\Big) \tag{9}$$

Under our smoothness assumption we neglect the terms $1/n\ \vec{b}_c$ and we get by Eq. (7)

$$\sigma_n(p_\perp) \approx \frac{1}{(2\pi)^4 n}\int d^2\vec{b}_2 \cdots d^2\vec{b}_n \left| \int d^2\vec{b}_c\, e^{i\vec{b}_c\vec{p}_\perp} A_n\Big(\vec{b}_c, -\sum_2^n \vec{b}_i, \vec{b}_2, \ldots, \vec{b}_n\Big)\right|^2 . \tag{10}$$

It is clear from Eq. (10) that A_n can depend only on the modulus of $\vec{b}_c$ and we finally get

$$\sigma_n(p_\perp) \approx \frac{1}{(2\pi)^2 n}\int d^2\vec{b}_2 \cdots d^2\vec{b}_n \left| \int_0^\infty b_c\, db_c\, J_0(p_\perp b_c)\, A_n\Big(b_c, -\sum_2^n \vec{b}_i, \vec{b}_2, \ldots, \vec{b}_n\Big)\right|^2 \tag{11}$$

where J_0 is the Bessel function of order 0.

This constitutes our general result for the inclusive $p_\perp$ spectrum. We started off by Eq. (5) where it is explicit that the scattering is a function only of differences of $\vec{b}$'s. Equation (11) shows that effectively, under our rather weak smoothness assumption, we can calculate the $p_\perp$ behaviour of particle c while keeping the impact parameters of $x_1, x_2, \ldots, x_n$ fixed if n is large. From the discussion in Ref. 5), we know that the large $p_\perp$ behaviour of the Fourier-Bessel transform appearing in Eq. (11) is completely determined by the singularities in the complex b_c plane. It therefore follows that also the large $p_\perp$ behaviour of $\sigma_n(p_\perp)$ is determined by the analytic structure of A_n as a function of b_c. One may say that the many unobserved particles serve to produce a scattering centre at the origin $\vec{b} = \vec{0}$ against which the observed particle, c, scatters.

The region of large $p_\perp$ to which we have referred in the above discussion is, in the following, taken to exclude only $p_\perp \approx 0$. This is justified both by the actual behaviour of cross-sections and by mathematical examples *).

3. - ANALYTIC IMPACT PARAMETER STRUCTURE

For the purpose of unifying the description of exclusive and inclusive $p_\perp$ distributions we first briefly recall the result of Ref. 5) for the case with two particles in the final state.

For the process $a+b \to c+d$ we write, by Eq. (5), the amplitude as a function of the centre-of-mass transverse momentum $\vec{p}_\perp = \vec{p}_{c\perp} - \vec{p}_{d\perp}$ at a fixed value of longitudinal momentum $p_\parallel$:

$$F(p_\perp) = (2\pi)^{-4} \int d^2\vec{b}_c \, e^{-i(\vec{b}_c - \vec{b}_d)\vec{p}_\perp} A(\vec{b}_c - \vec{b}_d) \tag{12}$$

Clearly only the relative impact parameter $\vec{b} = \vec{b}_c - \vec{b}_d$ enters and since A is a function only of the modulus of $\vec{b}$ we get by Eq. (12)

$$F(p_\perp) = (2\pi)^{-3} \int_0^\infty b\,db\, J_0(p_\perp b)\, A(b) \tag{13}$$

The b_c appearing in Eq. (11) is the distance in the b plane from particle c to the origin. Writing the b of Eq. (13) as composed of such b's we find the b's pertaining to inclusive and exclusive scattering given in the following way (see Fig. 1) :

$$\left.\begin{array}{lll} a+b \to c+\bar{X} & : & b = b_c \\ a+b \to c+d & : & b = b_c - b_d \end{array}\right\} \tag{14}$$

---

*) For example, a pole at $b^2 = (1\text{ fm})^2$ gives the asymptotic $p_\perp$ dependence $\sqrt{p_\perp}\, e^{-(1\text{ fm})p_\perp}$ to within a few per cent already for $p_\perp \gtrsim 0.2$ GeV.

To understand what the impact parameter structure might be, we summarize the phenomenological result of Ref. 5). Here it was found that the dominant component in the $p_\perp$ distributions of $d\sigma/dt$ for πN and pp elastic scattering at fixed $p_\parallel$ is of the form

$$\frac{d\sigma}{dt}(c+d \to c+d) \sim e^{-2R_{cd}p_\perp} \tag{15}$$

in the $p_\perp$ regions where experimental data are available, i.e., in the range $0 < p_\perp \lesssim 2.0$ GeV for πN and in $0 < p_\perp \lesssim 3.0$ GeV for pp. Also there is at most a slight dependence of R_{cd} upon $p_\parallel$ for $0 \le p_\parallel \lesssim 1.4$ GeV. We neglect in this discussion factors of the form $p_\perp^{-\alpha}$ with α small ($|\alpha| \lesssim 1$). Such factors are presently phenomenologically unobservable except for $p_\perp \approx 0$.

In the b plane the behaviour (15) is associated with singularities at $b = \pm b_0$, where

$$b_0 = i R_{cd} \tag{16}$$

The type of the interaction determines the exponent α. For the behaviour (15) only the position of the singularity is relevant. Physically this means that the detailed shape of the interaction does not enter. R_{cd} is interpreted as a radius of the interaction and the displacement of the b singularity onto the imaginary axis means that the b profile of the amplitude for real b is a smooth function, not, for example, a step function.

Appealing now to Eqs. (14), it seems plausible that R_{cd} should be composed of two terms

$$R_{cd} = R_c + R_d \tag{17}$$

where R_c and R_d are some characteristic lengths (radii) associated with particles c and d. We have here already implicitly made the assumption that $R_{c,d}$ are independent of the particles a and b in the initial state.

The result (17) is obtained if we associate singularities to the impact parameter b_j at $\pm iR_j$. From Eqs. (14), we then get the singularity b_0 of Eqs. (16), (17). We also get singularities at $b = \pm b_0'$ with

$b_o' = i(R_c - R_d)$. If $R_c \approx R_d$ this interaction will, however, be very central (almost point-like) and we expect such terms to show up at much larger values of $p_\perp$ than the contribution given by the singularity b_o.

It is now clear how, within this scheme, one is able to relate the $p_\perp$ distributions of inclusive and exclusive processes. From Eqs. (14) the impact parameter, b, for the process $a + b \to c + X$ is given by the impact parameter b_c of the observed particle, c. b_c has singularities at $\pm iR_c$ and these produce a $p_\perp$ spectrum of the form

$$\frac{d\sigma}{d\vec{p}_\perp}(a + b \to c + \underline{\bar{X}}) \sim e^{-2R_c p_\perp} \quad . \tag{18}$$

In the whole discussion we have considered only the transverse variables. In addition to the impact parameter b (or the centre-of-mass transverse momentum $p_\perp$) we choose the centre-of-mass longitudinal momentum, $p_\parallel$, and the missing mass, M, as the variables specifying the inclusive amplitude. This we do because b commutes with $p_\parallel$ and M and not, for example, with the more conventional variables $\sqrt{s}$ and $x = 2p_\parallel/\sqrt{s}$ or the scattering angle. R_c is therefore in principle a function of $p_\parallel$ and M :

$$R_c = R_c(p_\parallel, M) \quad . \tag{19}$$

Since we have already assumed that R_c is independent of the type of particles a and b in the initial state it seems natural also to assume that R_c is independent of the total energy or M :

$$R_c \simeq R_c(p_\parallel) \; . \tag{20}$$

One could also turn the problem around and say that the M dependence should be determined phenomenologically. As it turns out, Eq. (20) is, however, consistent with data thus supporting the idea that the impact parameter structure of particle c is something characteristic to that particle and independent of the initial state particles and of what else is produced in the process.

The preceding discussion has been concerned exclusively with the singularities associated with the finite size of the interacting hadrons. There is, however, nothing in the framework which forbids singularities near or at the origin $b = 0$. These singularities would correspond to point interactions and they produce [5] the familiar power behaviour at very large $p_\perp$, which has been studied in the quark parton models [2),3)].

4. - EXCLUSIVE AND INCLUSIVE $p_\perp$ DISTRIBUTIONS. COMPARISON WITH DATA

Equations (15), (17), (18), (20) allow us at a given value of $p_\parallel$ to relate the inclusive to the exclusive $p_\perp$ distributions. In the following we will study elastic data at $p_\parallel = 0$ (90^o) for πp, Kp and pp scattering compared with the ISR data on inclusive π, K, p and $\bar{p}$ production at $p_\parallel = 0$.

From the observed inclusive $p_\perp$ distributions at $p_\parallel = 0$ (90^o) at the ISR [1] it is clear that the form (18) works only up to a certain $p_\perp$ value, which for the K and π spectra lies at $p_\perp = 1.0 - 1.5$ GeV and for the p and $\bar{p}$ spectra at $p_\perp = 2.0 - 2.5$ GeV. We therefore calculate the slopes from the data below this critical $p_\perp$ value. The results are given in the Table. The precise choice of range of $p_\perp$ values used in the fits is made in each case such that an extension of this range would increase the χ^2 considerably. On Fig. 2 we compare our best fit with the ISR data at $\sqrt{s} = 53$ GeV.

Before we go to the comparison with the exclusive slopes we notice from the Table that the data comply with the rule (20) and furthermore we see that

$$R_c \approx R_{\bar{c}} \quad . \tag{21}$$

From the study of elastic πN and pp cross-sections of Ref. 5), we extract, by Eq. (17), the pion and proton radii

$$R_\pi = 3.25 \pm 0.25 \text{ GeV}^{-1} \; ; \; R_p = 2.05 \pm 0.05 \text{ GeV}^{-1} \tag{22}$$

for $0 \leq p_\parallel \lesssim 1.4$ GeV.

These radii are on Fig. 3 compared with those determined from the ISR inclusive cross-sections.

The inclusive kaon slope R_K is over the ISR energies consistent with

$$R_K \approx 2.5 \ \mathrm{GeV}^{-1} \tag{23}$$

This taken together with the R_p of (22) gives us a value for $R_{Kp} = R_K + R_p$. This is, on Fig. 4, confronted with 90° cross-section for the process $K^+p \rightarrow K^+p$. The data for this process are not by themselves good enough for a determination of a slope but the comparison is satisfactory.

In conclusion the comparison of slopes in inclusive and exclusive $p_\perp$ distributions works well over the whole range of ISR energies thus confirming the simple factorization property of the contributions from the radius singularities expressed by Eqs. (15), (17), (18), (20).

The comparison with data is in this paper done at $p_\parallel = 0$. It was found in Ref. 5) that $R_{\pi N}$ and R_{pp} are almost independent of $p_\parallel$ in the range $0 \leq p_\parallel \lesssim 1.4$ GeV. In the present scheme this of course immediately carries over to the inclusive case. We should perhaps in this place emphasize that the determination of $R_{\pi N}$ and R_{pp} phenomenologically is strongly dependent on the fixed $p_\parallel$ data for $p_\perp \gtrsim 1.0$ GeV and this is the explanation for the somewhat unconventional result that $R_\pi > R_p$.

Finally we make some remarks on the deviations from the predicted $p_\perp$ spectra, Eqs. (15), (18). It is clear from the discussion in Section 3 that one should expect contributions from other than the radius singularities. Also Fig. 2 shows clear phenomenological deviations at large $p_\perp$. One therefore avoids the contradiction caused by time-reversal invariance in a case with only radius singularities. This problem is most clearly illustrated by the process $\pi\pi \rightarrow N\bar{N}$ and its time-reversed partner because of the relatively big difference between R_π and R_p. In this case the radius singularity is dominating in different kinematic regions in the two processes because of the big mass difference between the pion and the nucleon.

Let us now discuss the deviation of the inclusive $p_\perp$ spectra from the form (18). It is a possibility that the excess relative to (18) seen on Fig. 2 is due to point-like interactions. These would in our framework, as earlier remarked, be described by a singularity in the b plane at $b=0$ and this in turn corresponds to power behaviour at large $p_\perp$ of the form $p_\perp^{-N}$ 5). No matter how weak such a singularity is it will eventually dominate for $p_\perp$ large enough. If now the relative strength of the singularity at $b=0$ to that at $b=iR$ is the same in pion and in proton inclusive production we get that power behaviour takes over at lower $p_\perp$ in the pion case than in the proton case simply because $R_\pi > R_p$. We observe on Fig. 2 that the possible power behaviour takes over in both cases when the cross-section has dropped by approximately three orders of magnitude. This is obviously only a rough argument also because the power, N, might be different in the two cases [2].

5. - CONCLUSIONS

We have studied the relation between $p_\perp$ distributions of cross-sections for inclusive processes with one observed particle in the final state and for exclusive processes with two particles in the final state within an impact parameter framework. The other problem of relating the various inclusive $p_\perp$ distributions has been treated, for example, in Ref. 9) in a different framework.

The first ingredient in our analysis is a smoothness assumption on the multiparticle impact parameter amplitude which states that the amplitude as a function of the impact parameter $\vec{b}_i$ of the i^{th} particle is approximately constant over a range of the order of 1 fm/n, where n is the number of particles in the final state. This taken together with the assumption that the hadrons are extended objects with spatial extension of the order of 1 fm leads to the conclusion that the inclusive $p_\perp$ distributions at ISR energies are governed only by the $\vec{b}$ structure of the observed particle.

Secondly we generalize the result of Ref. 5) by introducing a radius of each particle. This radius is a characteristic of that particle and it is therefore the same in inclusive and exclusive processes. In the impact parameter language the radius R is related to the position of singularities in the impact parameter plane at $b=\pm iR$. The contributions from these radius singularities to the $p_\perp$ distributions of elastic cross-sections and of inclusive cross-sections are then related in the range up to $p_\perp$ values where the more central interactions (e.g., parton interactions) become important.

In this paper we have studied data only at longitudinal centre-of-mass momentum $p_\parallel = 0$. Obviously data at $p_\parallel \neq 0$ should be explored. This, however, is somewhat complicated by the fact that inclusive data usually are presented at fixed values of $\sqrt{s}$ and of $x = 2p_\parallel/\sqrt{s}$ or scattering angle instead of, as we would need it, at fixed values of $p_\parallel$ and missing mass M. Another direction of further study is given by the inclusive processes with more than one observed particle. An interesting question to ask here is whether the correlation between large $p_\perp$ particles might have a simple explanation in terms of impact parameter singularities.

ACKNOWLEDGEMENTS

It is a pleasure to acknowledge helpful discussions with H. Miettinen, J.L. Petersen and F. Schrempp. I also thank R. Hagedorn for his critical reading of the manuscript.

APPENDIX

In this Appendix we indicate how to reduce Eqs. (4), (5) to Eq. (6).

From Eq. (5) we write the amplitude F_n as

$$F_n(\vec{p}_\perp,\vec{q}_1,\dots,\vec{q}_n)=\left(\frac{1}{2\pi}\right)^{2n+2}\int\prod_{i=1}^{n}d^2\vec{b}_i\,e^{-i\vec{b}_c\vec{p}_\perp}\,e^{-i\sum_i\vec{b}_i\vec{q}_i}\,A_n(\vec{b}_c,\vec{b}_1,\dots,\vec{b}_n) \tag{A.1}$$

where

$$\vec{b}_c=-\sum_{i=1}^{n}\vec{b}_i\ . \tag{A.2}$$

By Eq. (4) the cross-section σ_n then is

$$\sigma_n(p_\perp)=\int\prod_i d^2\vec{b}_i\prod_j d^2\vec{b}_j{}'\prod_k d^2\vec{q}_k\; e^{i(\vec{b}_c-\vec{b}_c{}')\vec{p}_\perp}\; e^{i\sum_j(\vec{b}_j-\vec{b}_j{}')\vec{q}_j}$$

$$A_n(\vec{b}_c,\vec{b}_1,\dots,\vec{b}_n)\,A_n^*(\vec{b}_c{}',\vec{b}_1{}',\dots,\vec{b}_n{}')\,\delta^{(2)}(\vec{p}_\perp+\sum_j\vec{q}_j) \tag{A.3}$$

with

$$\vec{b}_c{}'=-\sum_{j=1}^{n}\vec{b}_j{}'\ . \tag{A.4}$$

Perform now the $\vec{q}_1$ integration :

$$\sigma_n(p_\perp)=\int\prod_i d^2\vec{b}_i\prod_j d^2\vec{b}_j{}'\prod_{k=2}^{n}d^2\vec{q}_k\; e^{i[(\vec{b}_c-\vec{b}_c{}')-(\vec{b}_1-\vec{b}_1{}')]\vec{p}_\perp}$$

$$e^{i\sum_{j=2}^{n}[(\vec{b}_j-\vec{b}_j{}')-(\vec{b}_1-\vec{b}_1{}')]\vec{q}_j}\,A_n(\vec{b}_c,\vec{b}_1,\dots,\vec{b}_n)\,A_n^*(\vec{b}_c{}',\vec{b}_1{}',\dots,\vec{b}_n{}') \tag{A.5}$$

Integration of the $\vec{q}_k$'s $(k = 2,3,\ldots,n)$ give δ functions

$$\prod_{i=2}^{n} \delta^{(2)}\left((\vec{b}_i - \vec{b}_i') - \Delta\vec{b}_1 \right)$$

with

$$\Delta\vec{b}_1 = \vec{b}_1 - \vec{b}_1' \tag{A.6}$$

and Eq. (A.5) is reduced to

$$\sigma_n(p_\perp) = -(2\pi)^{-4}\int d^2\vec{b}_1 \cdots d^2\vec{b}_n\, d^2\Delta\vec{b}_1\, e^{i[(\vec{b}_c - \vec{b}_c') - \Delta\vec{b}_1]\vec{p}_\perp}$$

$$A_n(\vec{b}_c, \vec{b}_1, \ldots, \vec{b}_n)\, A_n^*(\vec{b}_c', \vec{b}_1 + \Delta\vec{b}_1, \vec{b}_2 + \Delta\vec{b}_1, \ldots, \vec{b}_n + \Delta\vec{b}_1) \tag{A.7}$$

where, by Eqs. (A.2), (A.4)

$$\begin{aligned} \vec{b}_c - \vec{b}_c' &= -(\vec{b}_1 - \vec{b}_1') - \sum_{i=2}^{n} (\vec{b}_i - \vec{b}_i') \\ &= -n\,\Delta\vec{b}_1 \quad . \end{aligned} \tag{A.8}$$

Equation (A.8) inserted into Eq. (A.7) gives the desired Eq. (6).

$\sqrt{s}$ (GeV)	π^+	π^-	K^+	K^-	p	$\bar{p}$
23	3.04 ± 0.04	3.22 ± 0.05	2.55 ± 0.07	2.69 ± 0.08	2.09 ± 0.03	2.11 ± 0.04
31	2.99 ± 0.03	3.07 ± 0.03	2.38 ± 0.07	2.65 ± 0.09	2.02 ± 0.02	2.06 ± 0.02
45	3.03 ± 0.04	3.00 ± 0.04	2.56 ± 0.07	2.49 ± 0.09	2.07 ± 0.03	1.96 ± 0.02
53	2.96 ± 0.03	2.99 ± 0.03	2.40 ± 0.06	2.33 ± 0.07	1.95 ± 0.02	1.89 ± 0.02
63	3.12 ± 0.04	3.13 ± 0.04	2.42 ± 0.14	2.40 ± 0.15	1.92 ± 0.03	1.95 ± 0.03

Table: Estimates of R(GeV^{-1}), Eq. (18), from 90^o inclusive $p_\perp$- spectra.

R E F E R E N C E S

1) B. Alper et al., (British-Scandinavian Collaboration), Phys.Letters 47B, 75 (1973); Phys.Letters 47B, 275 (1973); Nuclear Phys. B87, 19 (1975); CERN preprint (July, 1975) - to be published in Nuclear Phys. B.

2) V. Matveev, R. Muradyan and A. Tavkhelidze, Lettere at Nuovo Cimento 7, 719 (1973);

S.J. Brodsky and G.R. Farrar, Phys.Rev.Letters 31, 1153 (1973);

R. Blankenbecler and S.J. Brodsky, Phys.Rev. D10, 2973 (1974).

3) D. Sivers, SLAC preprint SLAC-PUB-1457(T/E) (1974).

4) B. Schrempp and F. Schrempp, Phys.Letters 55B, 303 (1975).

5) F. Elvekjaer and J.L. Petersen, Nuclear Phys. B94, 100 (1975).

6) F.S. Henyey, Phys.Letters 45B, 363 (1973).

7) W.R. Frazer et al., Revs.Modern Phys. 44, 284 (1970).

8) J.A. Danysz et al., Nuclear Phys. B14, 161 (1969);

W. De Baere et al., Nuovo Cimento 45A, 885 (1966);

J. Whitmore et al., Phys.Rev. D3, 1092 (1971);

J.N. MacNaughton et al., Nuclear Phys. B14, 237 (1969);

A. Eide et al., Nuclear Phys. B60, 173 (1973);

C. Baglin et al., CERN preprint (1975).

9) T. Inami, DESY preprint 75-15 (1975).

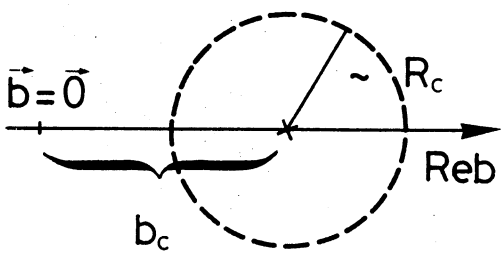
FIGURE CAPTIONS

Figure 1 : Impact parameter for inclusive and exclusive processes.

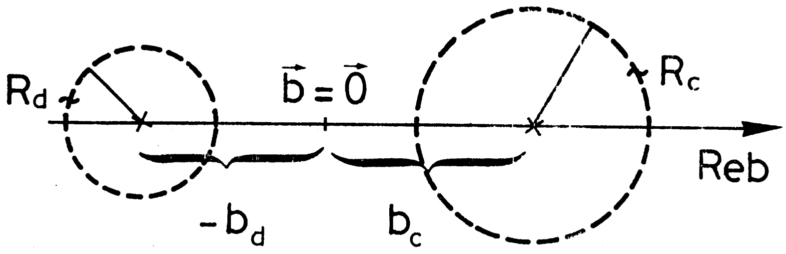
Figure 2 : The π, K, p and $\bar{p}$ spectra at $\sqrt{s} = 53$ GeV and $\theta_{cm} = 90^{\circ}$ compared with the contribution from the radius singularity, Eq. (18).

Figure 3 : The π, K and N radii. The bands give R_{π} and R_{p} as calculated by Eq. (17) from πN and pp elastic scattering. The points are the radii as estimated from the ISR 90° spectra for π^{+}(o), π^{-}(•), K^{+}(□), K^{-}(■), p(Δ) and $\bar{p}$(▲) (Table).

Figure 4 : $K^{+}p$ elastic cross-sections at 90° compared with the contribution from the radius singularity as predicted by inclusive spectra. Data from Ref. 8).



$b = b_c$



$b = b_c - b_d$

FIG 1

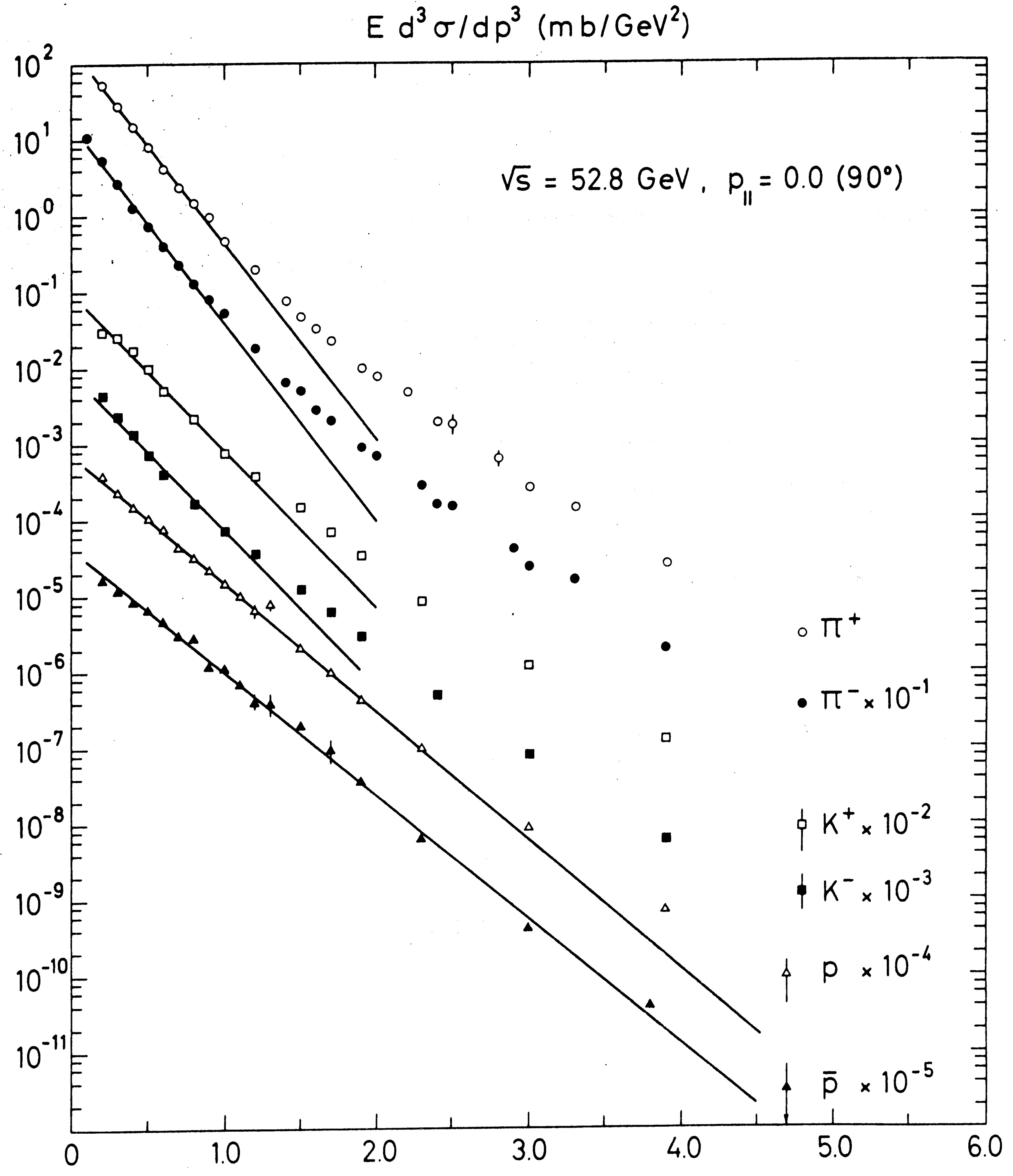

E d³σ/dp³ (mb/GeV²)
√s = 52.8 GeV , p∥ = 0.0 (90°)
10²
10¹
10⁰
10⁻¹
10⁻²
10⁻³
10⁻⁴
10⁻⁵
10⁻⁶
10⁻⁷
10⁻⁸
10⁻⁹
10⁻¹⁰
10⁻¹¹
π⁺
π⁻ × 10⁻¹
K⁺ × 10⁻²
K⁻ × 10⁻³
p × 10⁻⁴
p̄ × 10⁻⁵
0
1.0
2.0
3.0
4.0
5.0
6.0
p⊥ (GeV)

FIG 2

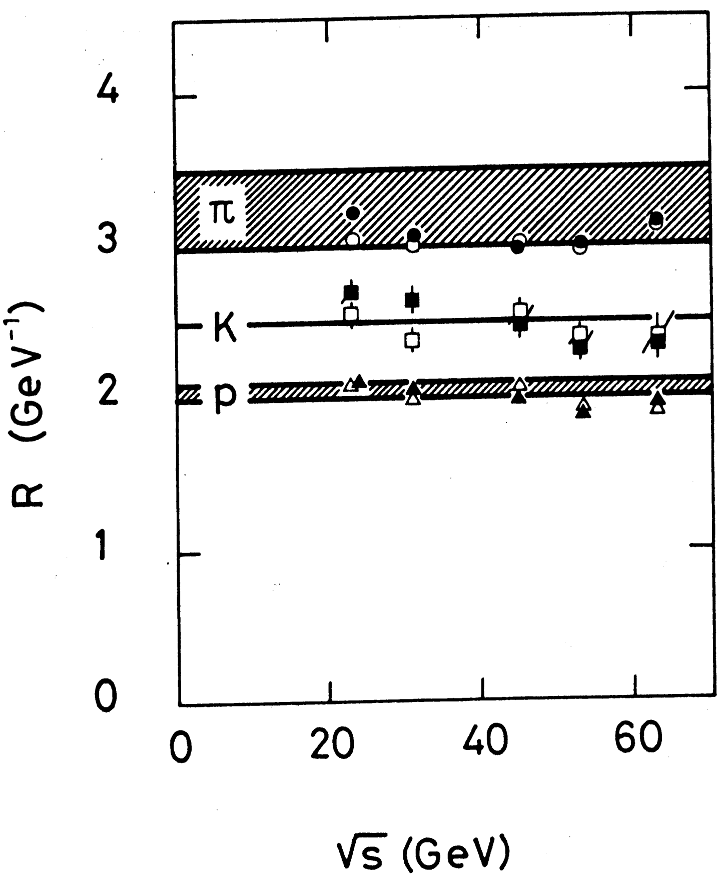


FIG 3

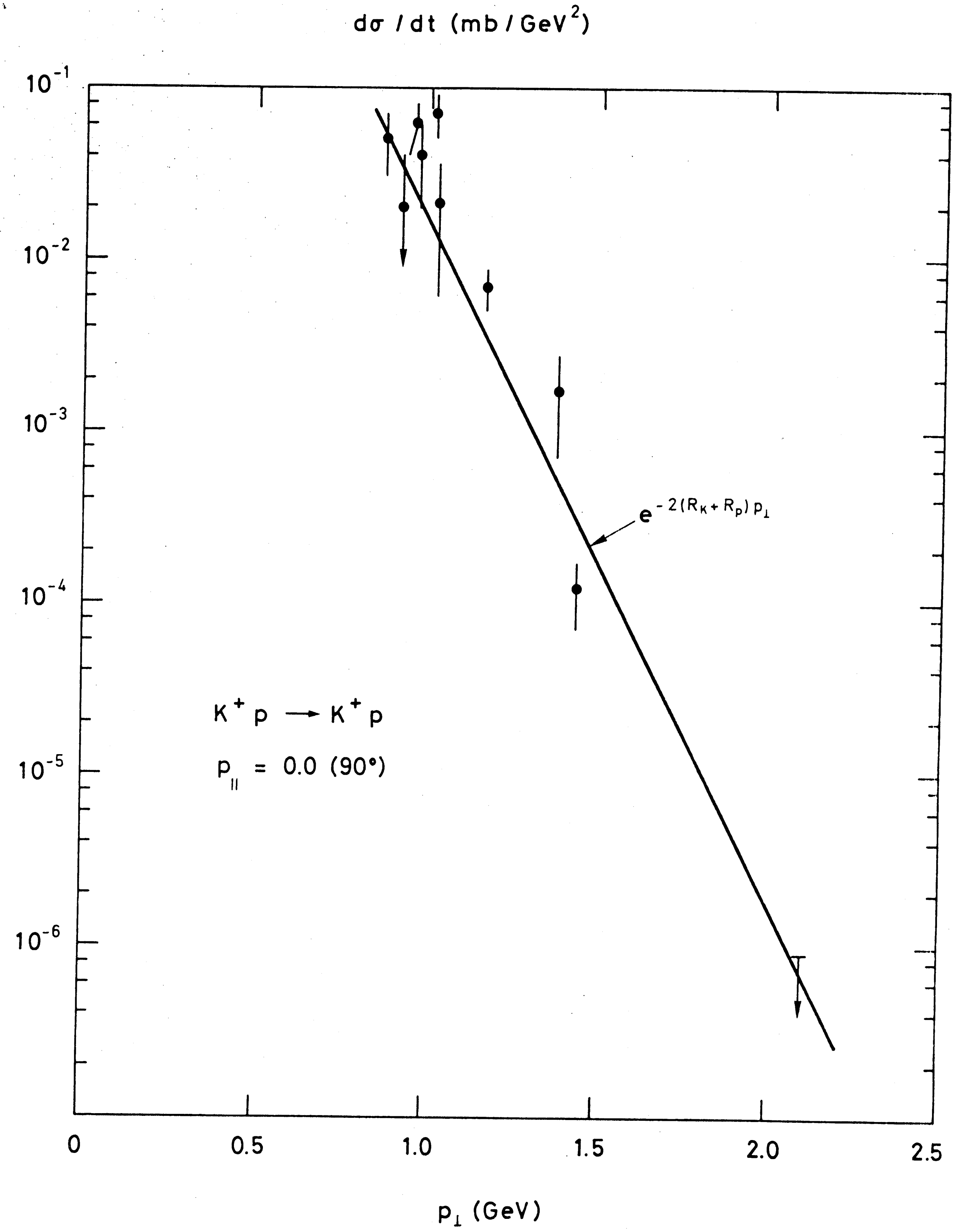

dσ / dt (mb / GeV²)
10⁻¹
10⁻²
10⁻³
10⁻⁴
10⁻⁵
10⁻⁶
$e^{-2(R_K+R_p)p_\perp}$
$K^+ p \rightarrow K^+ p$
$p_\parallel = 0.0$ (90°)
0
0.5
1.0
1.5
2.0
2.5
$p_\perp$ (GeV)

FIG 4