NOTES ON PHASE SPACE

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I INTRODUCTION

In the study of strong interactions of elementary particles a great variety of different final state interactions have been detected by the observation that relative yields, momentum and effective mass distributions of the particles deviate from the expectations from phase space. We can mention such discoveries as the hyperon isobars, the pion resonances etc. The calculation of phase space predictions for the particular interaction under investigation is, therefore, often useful and even necessary in order to extract information about the interaction between the particles in the final state.

The purpose of these notes will be to show the derivation of the general phase space formula for a system of n particles in the final state and demonstrate the use of a recursion relation in the calculation of the Lorentz invariant phase space. We will discuss the momentum spectrum of a single random particle and the angular distribution between any two particles in the centre of mass of the n particles. The effective mass distribution of any number of particles can also be calculated with the help of a recursion relation. For some special cases we will also discuss the effect of a resonance between two particles on the effective mass distribution of any two of the particles in the final state. We will also write down some of the special properties of the 3-body phase space first used by Dalitz in his special representation (Dalitz plot). Finally we will make some comparison between the predictions from phase space and the experimental data on hyperon resonances.

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The concept of "phase space" is closely connected to the calculation of transition rates, and to introduce and define the concept we will therefore consider the example of multipion production in a nucleon nucleon collision

$$N + N \rightarrow N + N + n\pi$$

which was first discussed by Fermi (Progr. Theoret. Phys. (Japan) 5, 570 (1950)) in his theory for pion production.

The probability per unit time that the above reaction will take place (the transition rate W) can be expressed by Fermi's well-known golden rule No. 2

$$W = \frac{2\pi}{\hbar}|\langle\psi_f|H'|\psi_i\rangle|^2 \quad F \qquad (1)$$

where ψ_i and ψ_f are initial and final state wave functions resepctively, and $\langle\psi_f|H'|\psi_i\rangle$ is the matrix element (M) for the transition from i to f caused by a perturbation of the Hamiltonian H'. The multiplication factor F is what we call the phase space or density of state factor. F is a function of the total energy of the system (E) and of the masses of the individual particles in the final state.

We can express Eq. (1) in two formally different ways as:

$$W = \frac{2\pi}{\hbar}|M'|^2 \quad \rho(E) \qquad (1a)$$

and

$$W = \frac{2\pi}{\hbar}|M''|^2 \quad R(E) \qquad (1b)$$

where in Eq. (1a) the matrix element is expressed in a form which is not invariant under Lorentz transformations, (as is the case for matrix elements calculated in non relativistic quantum theory). In Eq. (1b) M'' is Lorentz invariant, (as is the case for the so-called Feynman amplitudes, i.e. the matrix elements calculated in relativistic quantum field theory by application of the Feynman rules.) Since W should be the same in both cases, it follows that ρ(E) and R(E) are different; ρ(E) is not invariant under Lorentz transformations whereas R(E) is an invariant.

The matrix elements M′ and M″ are in general completely unknown, especially for transitions caused by strong interactions. The purpose of the statistical theory introduced by Fermi for calculation of transition rates, is therefore to make certain assumptions about the behaviour of the matrix element. The simplest assumption one can make, is that the matrix element is a constant, independent of the individual momenta of the particles in the final state (but not necessarily independent of E). In this case, for a constant matrix element, the transition rate as well as individual particle momenta in the final state, are determined by the phase space factor alone.

It may be worth while here to stress that the approximation of a constant matrix element may well be a crude one, and it is not a priori given which of the approximations:

M′ = constant (phase space not invariant)

or

M″ = constant (phase space invariant)

is the best one. In his original treatment Fermi used a non invariant phase space. Lately it has become the fashion to use a Lorentz invariant phase space. The best, and perhaps only, justification for this is that the invariant phase space is the easiest to calculate. We will in these notes mainly discuss the Lorentz invariant phase space.

II NON INVARIANT PHASE SPACE

a) Definition and general formula

For one particle a definite state of motion (i.e. specified position (x,y,z) and momentum (p_x, p_y, p_z)) can be represented as a point in a 6 dimensional phase space. Conversely each point in phase space corresponds to a definite state of motion of the particle.

Classical mechanics places no limitations on the density of the representation points. A given value of x can be combined with any value of p_x, etc. It is in principle possible to make simultaneous measurements of x and p_x with infinite accuracy and then localize a point in phase space. Thus classically there will be an infinite number of points available in phase space for a particle confined to a certain region in space and with a certain energy.

Quantum mechanics on the other hand requires the representation points to be separated by finite distances. The uncertainty principle states that it is impossible to measure position and momentum simultaneously with infinite precision. For each pair of canonical variables

$$\Delta x_j \, \Delta p_j \geq 2\pi\hbar \quad \text{etc.}$$

A state of motion can only be given with this indefiniteness and corresponds in phase space to a finite volume, or an elementary cell, of size $(2\pi\hbar)^3$.

The number of final states N_1 available to one particle will therefore be finite and equal to the total volume of the phase space, divided by the size of the elementary cell,

$$N_1 = \frac{1}{(2\pi\hbar)^3} \int dx\, dy\, dz\, dp_x\, dp_y\, dp_z \equiv \frac{1}{(2\pi\hbar)^3} \int d^3x\, d^3p \ .$$

If the particle is confined to a geometrical volume V we can write

$$N_1 = \frac{V}{(2\pi\hbar)^3} \int d^3 p \ .$$

For a particle of total energy less than or equal to E and mass m, N_1 will be the number of cells in a volume enclosed in momentum space by the sphere

$$p_x^2 + p_y^2 + p_z^2 = E^2 - m^2 \ .$$

Given the total number of states we now <u>define</u> the density in phase space as the number of states per unit energy interval

$$\rho_1(E) = \frac{dN_1}{dE} \ . \qquad (2)$$

This is for short, called "phase space" for one particle.

The extension to a system of n particles with energy $\leq$ E is quite simple. The number of final states N_n will be the product of the number of final states for each particle, thus

$$N_n = \left[\frac{1}{(2\pi\hbar)^3}\right]^n \int_{x_i p_i} \prod_{i=1}^{n} d^3x_i \; d^3p_i \; .$$

Since in practice all the particles are confined to the same geometrical volume $V = V_i = \int d^3x_i$ we can write

$$N_n = \left[\frac{V}{(2\pi\hbar)^3}\right]^n \int_{p_i} \prod_{i=1}^{n} d^3p_i \; . \qquad (3)$$

For instance in our example of multipion production, the interaction is assumed to take place in a sphere with as radius the Compton wavelength of the pion.

This formula gives the number of available cells in the final state for a system of n spinless particles. If the particle i has spin S_i the above expression should be multiplied by

$$\prod_{i=1}^{n} (2S_i + 1) \; .$$

Now, since the geometrical volume factor, spin factor etc. can be included in the final normalization of the phase space integral, we will neglect these factors here and simply put:

$$\rho_n(E) = \frac{dN_n}{dE} = \frac{d}{dE} \int \prod_{i=1}^{n} d^3p_i \; .$$

As already said this integral should be extended over all possible values of p_i. Now, in order to conserve total momentum ($\vec{P}$), the n particle momenta are not independent but constrained by the equation

$$\sum_{i=1}^{n} \vec{p}_i - \vec{P} = 0 \; . \qquad (4)$$

It is usual to indicate this restriction by putting

$$\rho_n(E) = \frac{d}{dE} \int \prod_{i=1}^{n-1} d^3 p_i$$

where one does not include particle n since the momentum of this particle is already given by Eq. (4).

We will, however, include the restrictions Eq. (4) by the introduction of the Dirac δ function. We use the fact that from the definition of the δ function

$$\int d^3 p_n \, \delta^3 (\vec{p}_n - (\vec{P} - \sum_{i=1}^{n-1} \vec{p}_i)) = 1$$

for all integrations including

$$\vec{p}_n = \vec{P} - \sum_{i=1}^{n-1} \vec{p}_i$$

(that is for momentum balance).

We also want to introduce the requirement of energy conservation

$$\sum_{i=1}^{n} E_i - E = 0 .$$

Since

$$\int \delta \left(\sum_{i=1}^{n} E_i - E \right) dE = 1$$

and

$$\frac{d}{dE} \int \delta\left(\sum_{i=1}^{n} E_i - E \right) dE = \delta\left(\sum_{i=1}^{n} E_i - E \right)$$

we can replace d/dE by

$$\delta\left(\sum_{i=1}^{n} E_i - E \right).$$

We then get the general formula for the non invariant phase space.

$$\rho_n(E) = \int \prod_{i=1}^{n} d^3p_i \; \delta^3(\sum_{i=1}^{n} \vec{p}_i - \vec{P}) \; \delta(\sum_{i=1}^{n} E_i - E). \qquad (5)$$

This formula is symmetrical in all the n particles, and the requirements of energy and momentum conservation are explicitly given.

To illustrate the explicit calculation of $\rho(E)$ for one particular reaction we will consider one simple example, namely that of two particles in the final state. The formulae for 3 and 4-body non invariant phase space are given by M. Block in Phys. Rev. 101, 796 (1956).

b) Two-body non invariant phase space

Let the masses of the two particles in the final state be m_1 and m_2 and their momenta $\vec{p}_1$ and $\vec{p}_2$ in the centre of mass. From Eq. (5) we have

$$\rho_2(E) = \int d^3p_1 \; d^3p_2 \; \delta^3(\vec{p}_1 + \vec{p}_2) \; \delta(E_1 + E_2 - E)$$

$$= \int d^3p_1 \; d^3p_2 \; \delta^3(\vec{p}_1 + \vec{p}_2) \; \delta(\sqrt{m_1^2 + p_1^2} + \sqrt{m_2^2 + p_2^2} - E).$$

In the following we will make use of some general rules for integration of δ functions given in Appendix A.

Integration over p_2 gives

$$\rho_2(E) = \int d^3p_1 \; \delta(\sqrt{m_1^2 + p_1^2} + \sqrt{m_2^2 + p_1^2} - E).$$

In polar coordinates we can write

$$d^3p_1 = p_1^2 \; dp_1 \; d\Omega_1 = p_1^2 \; dp_1 \; d(\cos\Theta_1) \; d\Phi_1 \quad .$$

The integrations over $\cos\Theta$ and Φ give a factor 4π, thus

$$\rho_2(E) = \int 4\pi \; p_1^2 \; dp_1 \; \delta\underbrace{(\sqrt{m_1^2 + p_1^2} + \sqrt{m_2^2 + p_1^2} - E)}_{A}$$

Since

$$\frac{dA}{dp_1} = \frac{p_1}{E_1} + \frac{p_1}{E_2} = \frac{p_1}{E_1}\frac{E}{E_2}$$

and A = 0 for

$$p_1 = \frac{\{[E^2 - (m_2 + m_1)^2]\ [E^2 - (m_2 - m_1)^2]\}^{1/2}}{2E}$$

we get by integration over p_1

$$\rho_2(E) = \frac{4\pi\ p_1 E_1 E_2}{E} \tag{6}$$

$$= \frac{4\pi}{E}\ \frac{\{[E^2 - (m_2 - m_1)^2]\ [E^2 - (m_2 + m_1)^2]\}^{1/2}}{2E}\ \frac{E^4 - (m_2^2 - m_1^2)^2}{4E^2}$$

This is the expression for non invariant 2-body phase space.

III LORENTZ INVARIANT PHASE SPACE

a) General formula

The phase space formula Eq. (5) is not symmetrical in E and p and therefore clearly not invariant under Lorentz transformations. The simplest way to find an invariant expression for the phase space formula is to replace d^3p_i in formula (5) by $d^3p_i/2E_i$. This corresponds to the relation between non relativistic matrix elements M′, Eq. (1a) and Feynman amplitude M″, Eq. (1b). We then get

$$R_n(E) = \int \prod_{i=1}^{n} \frac{d^3p_i}{2E_i}\ \delta^3(\sum_{i=1}^{n} \vec{p}_i - \vec{P})\ \delta(\sum_{i=1}^{n} E_i - E) \tag{7a}$$

which is invariant under Lorentz transformations. The factor $2E_i$ enters from a normalization of the wave function in field theory. We may qualitatively understand the meaning of this normalization factor as follows. In non relativistic quantum mechanics we express the probability density of, say one particle, as $|\psi|^2$, where ψ is the wave function describing the particle and where $\int |\psi|^2 \, dx\, dy\, dz = 1$ when integrated over total space. This expression for the probability density is not a relativistic invariant. When applied to a relativistic case, the density (probability per cm^3) observed from a moving system appears greater by a factor $\gamma = (1 - v^2/c^2)^{-1/2}$ because of Lorentz contraction of the volume element. It is important to note, howev-.', that the total energy of the particle has also changed by the same factor γ. If we therefore use as a probability density the expression $|\sqrt{2E}\,\psi|^2$, the density will be relativistically invariant. (The factor 2 is a convention). Introducing this normalization into formula Eq. (1a) we can write

$$W = \frac{2\pi}{\hbar} |M'|^2 \rho(E) = \frac{2\pi}{\hbar} |M'|^2 \left(\prod_{i=1}^{n} 2E_i\right) \left(\prod_{i=1}^{n} \frac{1}{2E_i}\right) \rho(E)$$

$$= \text{constant } |M'|^2 \; R(E)$$

where now

$$R(E) = \rho(E) \prod_{i=1}^{n} \frac{1}{2E_i} \; .$$

This formula which is also expressed in Eq. (7a) can be written in a more symmetrical form by introducing the four vector

$$q_i \equiv (\vec{p}_i, E_i) = (p_{ix}, p_{iy}, p_{iz}, E_i) \text{ of length } q_i^2 = E_i^2 - p_i^2 \; .$$

Using the rules for integration over a δ function given in the Appendix we find that

$$\int d^4q \; \delta(q^2 - m^2) \equiv \int d^3p \; dE \; \delta[E^2 - (p^2 + m^2)] \quad (\text{for } E > 0)$$

$$= \int \frac{d^3p}{2E} \quad \text{for } E^2 = p^2 + m^2 \; .$$

We must, however, eliminate the negative root of $p^2 + m^2$, and do this by making the convention that all integrations over E, E_i are limited to positive values.

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Similarly we introduce $Q = (\vec{P}, E)$ and write

$$\delta^4 \left(\sum_{i=1}^{n} q_i - Q \right) \equiv \delta^3 \left(\sum_{i=1}^{n} \vec{p}_i - \vec{P} \right) \delta \left(\sum_{i=1}^{n} E_i - E \right) .$$

Introducing these expressions in Eq. (7a) gives

$$R_n(E) = \int \prod_{i=1}^{n} \left[d^4 q_i \ \delta(q_i^2 - m_i^2) \right] \delta^4 \left(\sum_{i=1}^{n} q_i - Q \right) \qquad (7b)$$

Expression (7b) is in most literature presented as the definition of Lorentz invariant phase space.

We can qualitatively understand the meaning of the term $\delta(q_i^2 - m_i^2)$, since $q_i^2 - m_i^2 = E_i^2 - p_i^2 - m_i^2$ will be zero for q_i^2 evaluated on the mass shell of the particle. This term therefore essentially represents the constraint that for all i's we must have

$$E_i^2 = m_i^2 + p_i^2 .$$

(By convention we have limited E_i to positive values).

The Lorentz invariant expression given by Eq. (7) gives the total volume in momentum space available for n particles of given masses and total energy E. Clearly this volume R_n for a given energy E is just a number. The knowledge of this number is necessary for estimations of cross-sections and relative yields.

Before performing the integrations over all the momenta, p_i, we may, however, (at least in practice) regard R_n as a function of all p_i, thus $R_n(E) = R_n(E, p_1 \ldots p_n)$. This expression can now provide us with the differential momentum spectrum of any of the particles, k, simply by evaluation of dR_n/dp_k. Clearly, dR_n/dp_k is just the expression one obtains from formula Eq. (7) by omitting the integration over the k'th particle.

b) Lorentz invariant two-body phase space

For two particles of masses m_1 and m_2 with momenta $\vec{p}_1$ and $\vec{p}_2$ in the centre of mass we get from Eq. (7b)

$$R_2 = \int d^4 q_1 \; d^4 q_2 \; \delta(q_1^2 - m_1^2) \; \delta(q_2^2 - m_2^2) \; \delta^4 (q_1 + q_2 - Q)$$

$$= \int d^3 p_1 \; d^3 p_2 \; dE_1 \; dE_2 \; \delta[E_1^2 - (m_1^2 + p_1^2)] \; \delta[E_2^2 - (m_2^2 + p_2^2)]$$

$$\times \; \delta^3 (\vec{p}_1 + \vec{p}_2) \; \delta(E_1 + E_2 - E) \; .$$

By using the integration rules for a δ function we obtain

$$R_2 = \int \frac{d^3 p_1}{2E_1} \frac{d^3 p_2}{2E_2} \; \delta^3 (\vec{p}_1 + \vec{p}_2) \; \delta(E_1 + E_2 - E)$$

which we could have written down directly from Eq. (7a). We further get by integration over p_2 ,

$$R_2 = \int \frac{1}{4E_1 \; E_2(p_1)} \; d^3 p_1 \; \delta(E_1 + E_2(p_1) - E)$$

$$= \int \frac{1}{4E_1 \; E_2(p_1)} \; d\Omega_1 \; p_1^2 \; dp_1 \; \delta(E_1 + E_2(p_1) - E) \; .$$

Since the two particles are emitted isotropically in space the integration over $d\Omega_1$ gives a factor 4π. Thus integration over p_1 gives

$$R_2 = \frac{\pi \; p_1^2}{E_1 E_2} \; \frac{1}{\frac{p_1}{E_1} + \frac{p_1}{E_2}}$$

and finally

$$R_2 = \frac{\pi \; p_1}{E} = \frac{\pi}{E} \; \frac{\{[E^2 - (m_2 - m_1)^2] \; [E^2 - (m_2 + m_1)^2]\}^{1/2}}{2E} \; . \qquad (8)$$

We note that this Lorentz invariant expression for 2-body phase space R_2 is different from the non invariant expression ρ_2 evaluated earlier. For this special case (2-body) we also see that

$$R_2 = \rho_2 \cdot \frac{1}{2E_1} \cdot \frac{1}{2E_2} \; .$$

We can proceed similarly to calculate R_3 using the general formula Eq. (7). Then we will soon find however, that the expressions cannot be integrated straightforwardly. Instead of attempting to derive the 3-body phase space directly from Eq. (7) we will first evaluate a useful recursion relation for phase space, first given by Srivastava and Sudarshan (Phys. Rev. 110, 765 (1958)).

c) Recursion relation

We re-write formula (7) for the Lorentz invariant phase space for n particles with initial state four vector $Q = (\vec{P}, E)$ as

$$R_n(\vec{P},E) = \int \prod_{i=1}^{n} [d^4 q_i \; \delta(q_i^2 - m_i^2)] \; \delta^4 \; (\sum_{i=1}^{n} q_i - Q) \; . \qquad (9)$$

In the centre of mass of the n particles we can write (compare our preceding example for 2-body phase space)

$$R_n(0,E) = \int \prod_{i=1}^{n} \frac{d^3 p_i}{2E_i} \, \delta^3 \; (\sum_{i=1}^{n} \vec{p}_i) \; \delta(\sum_{i=1}^{n} E_i - E)$$

$$= \int \frac{d^3 p_n}{2E_n} \int \prod_{i=1}^{n-1} \frac{d^3 p_i}{2E_i} \, \delta^3[\sum_{i=1}^{n-1} \vec{p}_i - (-\vec{p}_n)] \; \delta[\sum_{i=1}^{n-1} E_i - (E - E_n)] \, .$$

We see that the last integral is the phase space for n - 1 particles with total momentum $(-\vec{p}_n)$ and total energy $(E - E_n)$.

Thus

$$R_n(0,E) = \int \frac{d^3 p_n}{2E_n} R_{n-1} [(-\vec{p}_n), \; (E - E_n)] \; .$$

Moreover, since R is Lorentz invariant, R_{n-1} must also be the same in a system with zero total momentum where the total energy is

$$\epsilon = \sqrt{(E - E_n)^2 - (-p_n)^2}$$

so that

$$R_{n-1}\ [(-\vec{p}_n),\ (E-E_n)] = R_{n-1}\ (0,\epsilon) .$$

This gives the following recursion relation between n and n- 1 body Lorentz invariant phase space.

$$R_n(0,E) = \int \frac{d^3 p_n}{2E_n} R_{n-1}\ (0,\epsilon) \qquad (10)$$

where

$$\epsilon^2 = (E-E_n)^2 - p_n^2 .$$

We will next use this recursion relation to derive the 3-body phase space.

d) Three-body phase space

For three particles with masses m_1, m_2, m_3 and momenta in centre of mass p_1, p_2, p_3 formula Eq. (10) gives

$$R_3(0,E) = \int \frac{d^3 p_3}{2E_3} R_2(0,\epsilon)$$

where

$$\epsilon^2 = (E-E_3)^2 - p_3^2 .$$

From the determination of R_2 in the preceding paragraph we get

$$R_3(0,E) = \int \frac{d^3 p_3}{2E_3} \left(\frac{\pi p'}{E'}\right)$$

where p' is the momentum of each particle in the 2-body system with energy E' in their centre of mass.

Thus

$$E' = E_1' + E_2' = \sqrt{m_1^2 + p'^2} + \sqrt{m_2^2 + p'^2} .$$

Since obviously $\epsilon = E'$ we get by solving for p'

$$p' = \frac{\{[\epsilon^2 - (m_1 + m_2)^2]\,[\epsilon^2 - (m_1 - m_2)^2]\}^{1/2}}{2\epsilon} .$$

Substituting this expression for p' into the above formula for $R_3(0,E)$ one gets

$$R_3(0,E) = \int \frac{4\pi p_3^2\ dp_3}{2E_3}\ \pi\ \frac{\{[E^2 + m_3^2 - 2EE_3 - (m_1 - m_2)^2]\,[E^2 + m_3^2 - 2EE_3 - (m_1 + m_2)^2]\}^{1/2}}{2(E^2 + m_3^2 - 2EE_3)} \qquad (11)$$

This integral should be taken between the minimum $p_3(\min)$ and maximum $p_3(\max)$ values of p_3. Obviously $p_3(\min) = 0$, which takes place when particles 1 and 2 are emitted antiparallel with equal momenta. The maximum momentum of the third particle will be obtained when the other two are emitted parallel and with the same velocity (opposite to $\vec{p}_3$). In this case we have

$$E = \sqrt{m_3^2 + p_3^2(\max)} + \sqrt{(m_1 + m_2)^2 + p_3^2(\max)}$$

which when solved for $p_3(\max)$ gives

$$p_3(\max) = \frac{\{[E^2 - (m_1 + m_2 - m_3)^2]\,[E^2 - (m_1 + m_2 + m_3)^2]\}^{1/2}}{2E} . \qquad (12)$$

For the general case where the three particles have different masses, Eq. (11) is an elliptical integral.

Finally, we want to point out that the differential momentum spectrum of a particle from 3-body final state is given by dR_3/dp_3 which is explicitly given in Eq. (11). We see that the momentum spectrum is a function of the total energy E of the system and the masses of the three particles m_1, m_2 and m_3 only.

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IV ANGULAR DISTRIBUTION

a) General formula

The general invariant phase space integral Eq. (7) can be used to find an expression for the angular distribution between any two particles among a system of n particles. The distribution will be derived in the centre of mass of the n particles.

The angle Θ between the two particles, say n and n - 1, can be found from

$$\cos \Theta = \frac{\vec{p}_n \cdot \vec{p}_{n-1}}{p_n \, p_{n-1}} .$$

We will in the following, interpret R_n (before performing the integration) as a function of all the momenta $p_1, p_2 \ldots p_n$; thus we can also express R_n as a function of $\cos \Theta$, that is $R_n = R_n\,(0, E, \cos \Theta)$. The angular distribution function between the two particles is then given by

$$\frac{dR_n}{d(\cos \Theta)} .$$

Applying the same procedure as for the derivation of the recursion relation Eq. (10), we get

$$R_n(0,E, \cos \Theta) = \int \frac{d^3 p_n}{2E_n} \frac{d^3 p_{n-1}}{2E_{n-1}} R_{n-2} \,(0, \epsilon_{n-2}) \qquad (13a)$$

where

$$\epsilon^2_{n-2} = (E - E_n - E_{n-1})^2 - (\vec{p}_n + \vec{p}_{n-1})^2$$

$$= E^2 + m^2_n + m^2_{n-1} - 2\,(EE_n + EE_{n-1} - E_n E_{n-1} + p_n p_{n-1} \cos \Theta).$$

In polar coordinates, with Φ_n and Θ_n giving the direction of $\vec{p}_n$ and Φ and Θ the direction of $\vec{p}_{n-1}$ with respect to $\vec{p}_n$ (see Fig. 1), we can write

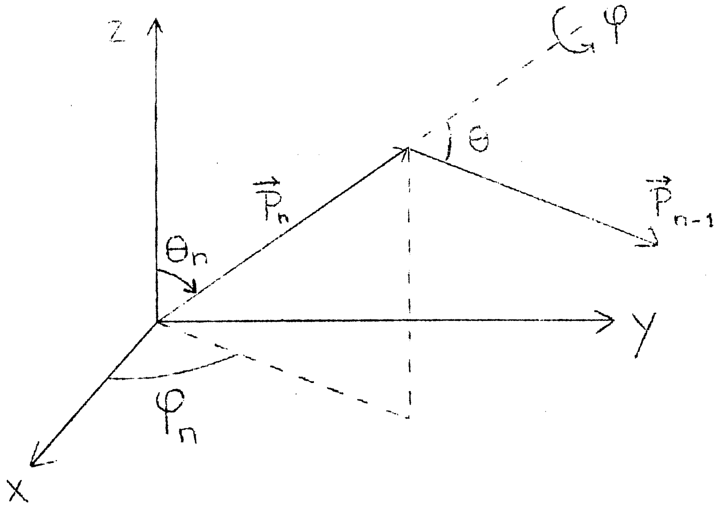


Fig. 1

$$d^3 p_n = p_n^2 \, dp_n \, d\Omega_n = p_n^2 \, dp_n \sin \Theta_n \, d\Theta_n \, d\Phi_n$$

$$d^3 p_{n-1} = p_{n-1}^2 \, dp_{n-1} \, d\Omega_{n-1} = p_{n-1}^2 \, dp_{n-1} \sin \Theta \, d\Theta \, d\Phi \, .$$

The integration over $d\Omega_n = \sin \Theta_n \, d\Theta_n \, d\Phi_n$ gives a factor 4π and the integration over $d\Phi$ a factor 2π, so that Eq. (13a) can be expressed as

$$\frac{dR_n(0,E, \cos \Theta)}{d(\cos \Theta)} = 2\pi^2 \int \frac{p_n^2 \, p_{n-1}^2}{E_n E_{n-1}} \, dp_n \, dp_{n-1} \, R_{n-2} \, (0, \epsilon_{n-2}) \, . \qquad (13b)$$

The integration limits of dp_{n-1} and dp_n will depend on $\cos \Theta$. In general for all possible values of $\cos \Theta$ the integration limits of p_n have to be equal to or greater than $p_n(\min)$ and equal to or less than $p_n(\max)$ where

$$p_n(\min) = 0$$

and $$p_n(\max) = \frac{\{[E^2 - (m_1 + \ldots + m_{n-1} + m_n)^2] \, [E^2 - (m_1 + \ldots + m_{n-1} - m_n)^2]\}^{1/2}}{2E} \, .$$

The corresponding limits for p_{n-1} are found by replacing n by n - 1 in the above equation. The value for $p_n(\max)$ represents the configuration where all the first n - 1 particles are emitted with the same velocity (that is as one body) opposite to the particle n.

Let us now look in more detail at the integration limits as a function of cos Θ. If we first integrate over dp_{n-1}, we want the limits for fixed p_n and fixed cos Θ.

It is easily verified that p_{n-1} will now have its maximum value if the momentum $(\vec{p}_n + \vec{p}_{n-1})$ is compensated by one body of mass

$$M = \sum_{i=1}^{n-2} m_i .$$

This determines the maximum value of p_{n-1} for fixed p_n and cos Θ to be the positive root in the equation:

$$E_n + (m_{n-1}^2 + p_{n-1}^2)^{1/2} + \left[\left(\sum_{i=1}^{n-2} m_i\right)^2 + p_n^2 + p_{n-1}^2 + 2p_n p_{n-1} \cos\Theta \right]^{1/2} = E . \tag{14}$$

The positive solution of this equation is

$$p_{n-1} = \frac{-ap_n \cos\Theta + (E - E_n)\sqrt{a^2 - 4m_{n-1}^2\, b}}{2b} \tag{15}$$

where

$$a = (E - E_n)^2 - \left(\sum_{i=1}^{n-2} m_i\right)^2 + m_{n-1}^2 - p_n^2$$

$$b = (E - E_n)^2 - p_n^2 \cos^2\Theta .$$

The upper limit of p_n for fixed cos Θ can, in principle, be found if we interchange p_n and p_{n-1} in Eq. (15) and determine the value of p_{n-1} for which $\partial p_n / \partial p_{n-1} = 0$.

b) <u>Three-body angular distribution</u>

Using formula Eq. (13b) from the preceding paragraph, we find for the angular distribution between particles 2 and 3 from a 3-body phase space

$$\frac{dR_3(0,E,\ \cos\Theta)}{d(\cos\Theta)} \propto \int \frac{p_2^2\ p_3^2}{E_2\ E_3}\, dp_2\ dp_3\ R_1(0,\epsilon) \tag{16}$$

where

$$\epsilon^2 = [E - (E_2 + E_3)]^2 - (\vec{p}_2 + \vec{p}_3)^2 \ .$$

We first find

$$R_1(0,\epsilon) = \int d^3p_1\ dE_1\ \delta^3(\vec{p}_1)\ \delta(E_1 - \epsilon)\ \delta[E_1^2 - (p_1^2 + m_1^2)]$$

which upon integration over p_1 and E_1 gives

$$R_1(0,\epsilon) = \delta(\epsilon^2 - m_1^2)\ .$$

Thus

$$\frac{dR_3(0,E,\ \cos\Theta)}{d(\cos\Theta)} = \int \frac{p_2^2\ p_3^2}{E_2\ E_3}\, dp_2\ dp_3\ \delta(\epsilon^2 - m_1^2)$$

$$= \int \frac{p_2^2\ p_3^2}{E_2\ E_3}\, dp_2\ dp_3\ \delta\ \underbrace{[(E - E_2 - E_3)^2 - p_2^2 - p_3^2 - 2p_2 p_3 \cos\Theta - m_1^2]}_{A}$$

Integration over p_2 can be done easily here because of the δ function. We calculate

$$\frac{\partial A}{\partial p_2} = -2(E - E_3)\cdot\frac{p_2}{E_2} - 2p_3\ \cos\Theta$$

for the value of p_2 for which $A = 0$. This value of p_2 can be found from formula Eq. (15) if we put $n = 3$.

The integration over p_2 therefore gives

$$\frac{dR_3(0,E,\ \cos\Theta)}{d(\cos\Theta)} = \int \frac{p_2^2\ p_3^2}{E_3}\, dp_3\ \frac{1}{2p_2(E - E_3) + 2p_3\ E_2\ \cos\Theta} \tag{17}$$

where p_2 is given by Eq. (15) as a function of p_3. The final integration over p_3 cannot be performed exactly in the general case where all three particle masses are different and all different from zero.

V EFFECTIVE MASS

a) Definition and special case

The effective mass of two particles with masses m_1 and m_2 is defined as

$$M_{12}^2 = (E_1 + E_2)^2 - (\vec{p}_1 + \vec{p}_2)^2 \tag{18}$$

We will in this Chapter discuss the effective mass of a system of particles with momentum distributions determined from phase space. To illustrate a possible straightforward method, we will first calculate the effective mass of two particles from 3-body phase space.

In the preceding Chapter we evaluated the differential momentum spectrum of a random particle in 3-body phase space

$$\frac{dR_3}{dp_3} = \frac{\pi^2 p_3^2}{E_3} \frac{\{[E^2 - 2EE_3 + m_3^2 - (m_1 + m_2)^2]\ [E^2 - 2EE_3 + m_3^2 - (m_1 - m_2)^2]\}^{1/2}}{E^2 - 2EE_3 + m_3^2} \tag{19}$$

We are interested here in the differential distributiob dR_3/dM_{12} for the effective mass of particles 1 and 2.

Introducing energy and momentum conservation in Eq. (18) one gets

$$\begin{aligned} M_{12}^2 &= (E - E_3)^2 - p_3^2 \\ &= E^2 - 2EE_3 + m_3^2 \ . \end{aligned} \tag{20}$$

From this follows by differentiation

$$M_{12}\ dM_{12} = -E\ dE_3 = -\frac{Ep_3}{E_3}\ dp_3 \ .$$

Now, p_3 can be found from Eq. (20) as an explicit function of M_{12}

$$p_3 = \frac{\{[E^2 - (M_{12} + m_3)^2]\ [E^2 - (M_{12} - m_3)^2]\}^{1/2}}{2E} \ .$$

We then get

$$\frac{dR_3}{dM_{12}} = \frac{dR_3}{dp_3}\frac{dp_3}{dM_{12}} = \frac{M_{12}E_3}{Ep_3}\frac{dR_3}{dp_3} \quad \text{(dropping minus sign)}$$

and finally by the use of Eq. (20) and (19)

$$\frac{dR_3}{dM_{12}} = \frac{\pi^2 \left\{\left[M_{12}^2-(m_1+m_2)^2\right]\left[M_{12}^2-(m_1-m_2)^2\right]\left[E^2-(m_3+M_{12})^2\right]\left[E^2-(m_3-M_{12})^2\right]\right\}^{1/2}}{2E^2 M_{12}} \tag{21}$$

which is the differential effective mass distribution of two particles from 3-body phase space. The lower and upper limits of M_{12} are easily found from the limits on p_3 given in Eq. (12). We get

$$M_{12}(\text{min}) = m_1 + m_2$$

$$M_{12}(\text{max}) = E - m_3 \quad .$$

With reference to the next paragraph which gives a general formula for the effective mass distribution, we want to re-write Eq. (21) in a form which reveals the features of the general formula.

We put

$$\frac{dR_3}{dM_{12}} = (2M_{12})\,\frac{\pi}{M_{12}}\,\frac{\left\{\left[M_{12}^2-(m_1+m_2)^2\right]\left[M_{12}^2-(m_1-m_2)^2\right]\right\}^{1/2}}{2M_{12}}$$

$$\times\,\frac{\pi}{E}\,\frac{\left\{\left[E^2-(m_3+M_{12})^2\right]\left[E^2-(m_3-M_{12})^2\right]\right\}^{1/2}}{2E} \; .$$

Recalling formula Eq. (8) for the 2-body phase space we see that the expression above is a product of two different R_2's, thus

$$\frac{dR_3}{dM_{12}} = (2M_{12})\; R_2\,(0,M_{12},m_1,m_2)\; R_2\,(0,E,m_3,M_{12})$$

where the first factor R_2 is the 2-body phase space for particles of masses m_1 and m_2 and total energy M_{12}. The last factor represents a system of masses m_3 and M_{12} with total energy E.

b) General formula

The effective mass of k particles selected from n body phase space ${}^{n}_{k}M$ is given by

$$ {}^{n}_{k}M^2 = (\sum_{i=1}^{k} E_i)^2 - (\sum_{i=1}^{k} \vec{p}_i)^2 = (\sum_{i=1}^{k} q_i)^2 \qquad (22) $$

or from energy and momentum conservation

$$ {}^{n}_{k}M^2 = (E - \sum_{i=k+1}^{n} E_i)^2 - (\vec{P} - \sum_{i=k+1}^{n} \vec{p}_i)^2 $$

$$ = (Q - \sum_{i=k+1}^{n} q_i)^2 \; . $$

We have chosen here the k particles to be the first k when numbering the n particles in order 1,2 ... k, k+1 .. n. This convention does not restrict the generality of our evaluation. We would now like to find an expression

$$ f(\mu^2) = \frac{dR_n}{d({}^{n}_{k}M^2)} = \frac{d}{d({}^{n}_{k}M^2)} R_n \; (P,E,m_1, \; \dots \; m_n) $$

where $f(\mu^2)$ is the probability that the effective mass of the first k particles has the value μ. We have explicitly written for clarity that R_n is a function of all the masses of the n particles. Using the expression for R_n from formula Eq. (7b) we can write $f(\mu^2)$ as

$$ f(\mu^2) = \int [\prod_{i=1}^{n} d^4 q_i \; \delta(q_i^2 - m_i^2)] \, \delta^4 \, (\sum_{i=1}^{n} q_i - Q) \; \delta({}^{n}_{k}M^2 - \mu^2) \, . \qquad (23) $$

The δ function $\delta({}^{n}_{k}M^2 - \mu^2)$ makes all contributions from phase space vanish except for the cases where ${}^{n}_{k}M^2 = \mu^2$.

We will now try to transform Eq. (23) to a form where we can make use of the earlier developed formula for R_n and its recursion relation*. This will be more convenient for practical calculations.

Since

$$\int \delta^4 \left(\sum_{i=1}^{k} q_i - {}_k^n M \right) d^4\, {}_k^n M = 1$$

for integrations including

$$\sum_{i=1}^{k} q_i = {}_k^n M$$

we can write

$$\delta^4 \left(\sum_{i=1}^{n} q_i - Q \right) = \delta^4 \left(\sum_{i=1}^{k} q_i + \sum_{i=k+1}^{n} q_i - Q \right)$$

$$= \int \delta^4 \left(\sum_{i=1}^{k} q_i - {}_k^n M \right) \delta^4 \left({}_k^n M + \sum_{i=k+1}^{n} q_i - Q \right) d^4\, {}_k^n M \,.$$

When we put this expression into Eq. (23) we get

$$f(\mu^2) = \int \left[\prod_{i=1}^{k} d^4 q_i\, \delta(q_i^2 - m_i^2) \right] \left[\prod_{i=k+1}^{n} d^4 q_i\, \delta(q_i^2 - m_i^2) \right] \times$$

$$\delta^4 \left(\sum_{i=1}^{k} q_i - {}_k^n M \right) \delta^4 \left({}_k^n M + \sum_{i=k+1}^{n} q_i - Q \right) \delta\left({}_k^n M^2 - \mu^2 \right) d^4\, {}_k^n M$$

$$= \int \left\{ \left[\prod_{i=1}^{k} d^4 q_i\, \delta(q_i^2 - m_i^2) \right] \delta^4 \left(\sum_{i=1}^{k} q_i - {}_k^n M \right) \right\}$$

$$\times \left\{ \left[\prod_{i=k+1}^{n} d^4 q_i\, \delta(q_i^2 - m_i^2) \right] d^4\, {}_k^n M\, \delta\left({}_k^n M^2 - \mu^2 \right) \delta^4 \left({}_k^n M + \sum_{i=k+1}^{n} q_i - Q \right) \right\} .$$

Since with the introduced nomenclature we can write

$$R_k(0, {}_k^n M, m_1 \ldots m_k) = \int \left[\prod_{i=1}^{k} d^4 q_i\, \delta(q_i^2 - m_i^2) \right] \delta^4 \left(\sum_{i=1}^{k} q_i - {}_k^n M \right)$$

---

*The following use of the recursion relation to calculate the effective mass distribution was, according to the author's knowledge, first made by A. Muller and A. Verglas in an internal report (1962) at Centre d'Etudes Nucleaires de Saclay, Paris.

and

$$R_{n-k+1}(0,E,m_{k+1} \ldots m_n, \mu) = \int \prod_{i=k+1}^{n} d^4 q_i \, \delta(q_i^2 - m_i^2) \, d^4 \, {}^n_k M \, \delta({}^n_k M^2 - \mu^2) \times$$

$$\delta^4({}^n_k M + \sum_{i=k+1}^{n} q_i - Q)$$

we finally get

$$f(\mu^2) = R_k(0,\mu,m_1 \ldots m_k) \cdot R_{n-k+1}(0,E,m_{k+1} \ldots m_n, \mu) \, . \qquad (24)$$

This is a general formula for the effective mass of k particles from n body phase space.

We see that $f(\mu^2)$ is a product of two functions. The first, R_k, gives the probability that the first k particles have total energy μ in their centre of mass. The second factor R_{n-k+1} is the probability that the n-k particles plus one other particle with mass μ (equal to the effective mass of the first k particles) have total energy E in the centre of mass of the original n particles. In other words, $f(\mu^2)$ expresses the probability that all n particles have energy E and simultaneously the k first particles have energy μ.

Exercise

Make use of the rule

$$\frac{dR}{d\mu^2} = \frac{1}{2\mu} \frac{dR}{d\mu}$$

and derive from Eq. (24) the formula for effective mass distribution of two particles from 3-body phase space given in Eq. (21).

Compare this result with the curve for the (=Kπ) system given in Chapter VIII.

c) "Shape" of effective mass distributions

The purpose of this paragraph is simply to show that all effective mass distributions (the probability to find the effective mass μ between μ and $\mu + d\mu$) may be classified into a few groups of distributions; each group has a characteristic shape determined by the values of n and k. The general appearance of each group may be used in a qualitative check of a particular calculation of an effective mass distribution. A comparison of the different shapes for different final state configurations may also give an indication of, for instance, in which configuration it will be most fruitful to look for a predicted resonance, in order to minimize the background expected from phase space.

In general the effective mass distribution has zero probability for two values of the effective mass ${}^{n}_{k}M$

$$ {}^{n}_{k}M(\min) = m_1 + m_2 + \dots + m_k $$

$$ {}^{n}_{k}M(\max) = E - (m_{k+1} + \dots + m_n) \ . $$

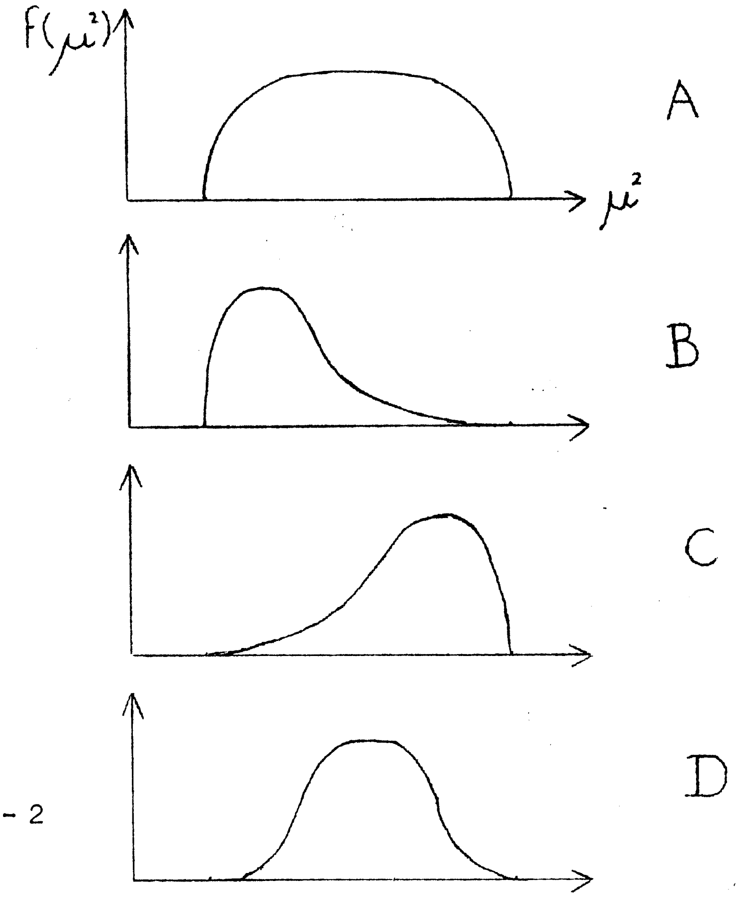
The tangents to the curve at these points (minimum and maximum) will to a large degree determine the shape of the distribution, and we have chosen to characterize the curve according to whether these tangents are horizontal or vertical. The following drawings illustrate the four different shapes one can have

$n = 3 \quad k = 2$

$n \geq 4 \quad k = 2$

$n \geq 4 \quad k = n - 1$

$n \geq 5 \quad 3 \leq k \leq n - 2$



Exercise

Show that the distribution of type A has vertical tangents at both the minimum and maximum value of ${}^{3}_{2}M$. To do this calculate $df(\mu^2)/d\mu$ from formula Eq. (21) and (24).

Show by the same method that the distribution of type B has vertical tangent at the minimum value of ${}^{n}_{2}M$.

VI EFFECT OF A RESONANCE ON THE EFFECTIVE MASS DISTRIBUTION

a) Statement of problem

We will in this Chapter consider a system where we observe in total n particles in the final state but where we also observe, or know there exists, a resonance between the k first particles so that ${}^{n}_{k}M = M^*$. For this system we ask the question: What is the effective mass distribution between any number of random particles in the final state? It is clear that the presence of a resonance between some of the particles will influence the effective mass distribution between all pairs or groups of particles in the system.

We will assume in the following discussion that only (n-k+ 1) particles, with masses $M^*, m_{k+1} \ldots m_n$, are produced in the original production process which can be well separated from the decay process $M^* \rightarrow m_1 + \ldots m_k$. This description may be meaningful in particular if the resonance has narrow width. To simplify the calculations we will also make the extreme assumption that the resonance has zero width and can be described by a δ function, $\delta({}^{n}_{k}M - M^*)$. It is clear that this assumption is in most cases quantitatively not correct, but it will, even for a broad resonance, describe in a qualitative way, the effect which the resonance has on the effective mass distribution of the particles.

Our problem can be separated into three cases according to the method of calculation of the effective mass.

(i) Calculation of effective mass of a group of particles none of which participate in the resonance. In this case we can use directly the earlier developed formulae applied to a (n- k+ 1) body final state, where one particle has mass M^*, that is

$$R_{n-k+1}(0,E,M^*,m_{k+1} \ldots m_n).$$

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(ii) Effective mass of a group of particles which all participate in the resonance. This can also be calculated from the formulae in Chapter V using a k body phase space with total energy M*,

$$R_n(0, M^*, m_1, m_2 \ldots m_k) .$$

(iii) Effective mass of a group of particles, some of which participate in the resonance and some do not. This problem is more complex than (i) and (ii) and cannot in the general case be performed with the help of a recursion relation. We will in the following paragraphs discuss some very simple special cases. In these calculations we have again assumed a resonance of zero width. In principle the calculations might as well be performed with a resonance of finite width as for instance with a simple Breit Wigner shape

$$f({}^n_k M) = \text{const} \left[({}^n_k M - M^*)^2 + (\Gamma/2)^2 \right]^{-1} .$$

b) Three-body

We start with the invariant 3-body phase space formula

$$R_3 \; \alpha \int \frac{d^3 p_1}{E_1} \frac{d^3 p_2}{E_2} \frac{d^3 p_3}{E_3} \, \delta^3 (\vec{p}_1 + \vec{p}_2 + \vec{p}_3) \; \delta(E_1 + E_2 + E_3 - E) .$$

We want to find the distribution of effective mass of particles 1 and 2 (dR_3/dM_{12}) for the case where particles 2 and 3 are already in a resonant state. We will first assume the resonance has zero width and mass M*. In that case we can write

$$\frac{dP_3}{dM_{12}} = \frac{d}{dM_{12}} \int \frac{d^3 p_1}{E_1} \frac{d^3 p_2}{E_2} \frac{d^3 p_3}{E_3} \, \delta(p) \; \delta(E) \; \delta(M_{23} - M^*) .$$

Integration over p_3 gives

$$\frac{dR_3}{dM_{12}} = \frac{d}{dM_{12}} \int \frac{d^3 p_1}{E_1} \frac{d^3 p_2}{E_2} \frac{1}{E_3 (p_1 p_2)} \, \delta(E_1 + E_2 + E_3 (p_1 p_2) - E) \; \delta(M_{23} - M^*)$$

Integration over angles gives (as demonstrated several times earlier)

$$\frac{dR_3}{dM_{12}} = \frac{d}{dM_{12}} \int \frac{p_1 dp_1}{E_1} \frac{p_2 dp_2}{E_2} \frac{p_1 p_2 d(\cos\Theta)}{E_3 (p_1 p_2)} \delta(E)\ \delta(M) \ .$$

Now for p_1 and p_2 constant we have from momentum conservation

$$p_3 dp_3 = p_1 p_2 d(\cos\Theta)$$

and since

$$p_3 dp_3 = E_3 d E_3$$

we get by substitution and integration over E_3

$$\frac{dR_3}{dM_{12}} = \frac{d}{dM_{12}} \int d E_1 d E_2 \ \delta(M_{23} - M^*)$$

where momentum and energy conservation require that

$$E_2 = E - E_1 - E_3 = E - E_1 - \frac{E^2 - M_{12}^2 + m_3^2}{2E} = \frac{E^2 + M_{12}^2 - m_3^2}{2E} - E_1 \ .$$

For constant E_1 it follows that

$$dE_2 = \frac{M_{12}}{E} dM_{12} \ .$$

Substituting into the expression above and recalling that

$$M_{23} = \sqrt{E^2 + m_1^2 - 2E\ E_1}$$

we get

$$\frac{dR_3}{dM_{12}} = \int dE_1 \frac{M_{12}}{E} \delta(\sqrt{E^2 + m_1^2 - 2EE_1} - M^*) \ .$$

Integration over E_1 gives

$$\frac{dR_3}{dM_{12}} = \frac{M_{12}}{E} \frac{\sqrt{E^2 + m_1^2 - 2E\ E_1}}{E}$$

for $$E_1 = \frac{E^2 - M^{*2} + m_1^2}{2E} \ .$$

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Finally

$$\frac{dR}{dM_{12}} \propto \frac{M_{12}}{E} \cdot \frac{M^*}{E} \quad . \tag{25}$$

We need the maximum and minimum values of M_{12}. Due to the resonance $M_{23} = M^*$ particle one always has a fixed energy

$$E_1 = \frac{E^2 + m_1^2 - M^{*2}}{2E} \quad .$$

We see that

$$M_{12}^2 = m_1^2 + m_2^2 + 2E_1 E_2 - 2p_1 p_2 \cos \Theta$$

is a minimum for $\cos \Theta = 1$. To find the value of E_2 for which this takes place we put

$$M_{12} \frac{\partial M_{12}}{\partial E_2} = E_1 - p_1 \cdot \frac{E_2}{p_2} = 0$$

and find

$$p_2 = \frac{m_2}{m_1} p_1$$

which gives

$$M_{12}^2(\min) = m_1^2 + m_2^2 + 2 \frac{m_2}{m_1} (E_1^2 - p_1^2)$$

$$= (m_1 + m_2)^2 \quad .$$

Similarly for $\cos \Theta = -1$ we find

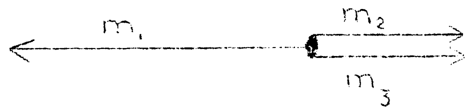
$$M_{12}^2(\max) = m_1^2 + m_2^2 + 2 \frac{m_2}{m_1} (E_1^2 + p_1^2)$$

$$= m_1^2 + m_2^2 - 2m_1 m_2 + 4 \frac{m_2}{m_1} E_1^2$$

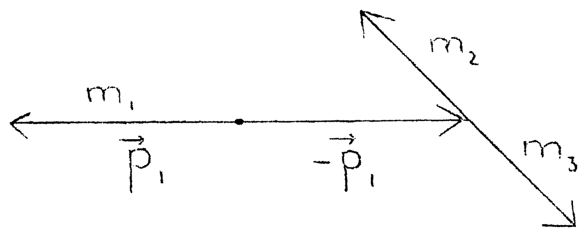
$$= (m_1 - m_2)^2 + \frac{m_2}{m_1} \frac{\left(E^2 + m_1^2 - M^{*2}\right)^2}{E^2} \quad .$$

Exercise

A 3-body final state with a resonance between two of the particles ($M_{23} = M^*$) can be thought of as an original 2-body decay



followed by a decay of the resonance



into two particles which in the M^* centre of mass are emitted with fixed energy. Make a Lorentz transformation of particle 1 to the M^* centre of mass, and from the fact that m_2 is emitted isotropically in this system, derive the effective mass distribution of M_{12} given in formula Eq. (25).

c) Four-body

We want to find the distribution dR_4/dM_{12} for the case where we have a resonance between particles 2 and 3. Assuming again that the resonance $M_{23} = M^*$ has zero width, we can write

$$\frac{dR_4}{dM_{12}} \propto \frac{d}{dM_{12}} \int \frac{d^3p_1}{E_1} \frac{d^3p_2}{E_2} \frac{d^3p_3}{E_3} \frac{d^3p_4}{E_4} \delta^3(\vec{p}_1 + \vec{p}_2 + \vec{p}_3 + \vec{p}_4)$$

$$\times\ \delta(E_1 + E_2 + E_3 + E_4 - E)\ \delta(M_{23} - M^*)\ .$$

Integration over p_4 gives

$$\frac{dR_4}{dM_{12}} = \frac{d}{dM_{12}} \int \frac{d^3p_1}{E_1} \frac{d^3p_2}{E_2} \frac{d^3p_3}{E_3} \frac{1}{E_4}\ \delta\left[E_1 + E_2 + E_3 + E_4(p_1\, p_2 p_3) - E\right]\ \delta(M_{23} - M^*)$$

for momentum balance.

We introduce polar coordinates as illustrated in Fig. 2.

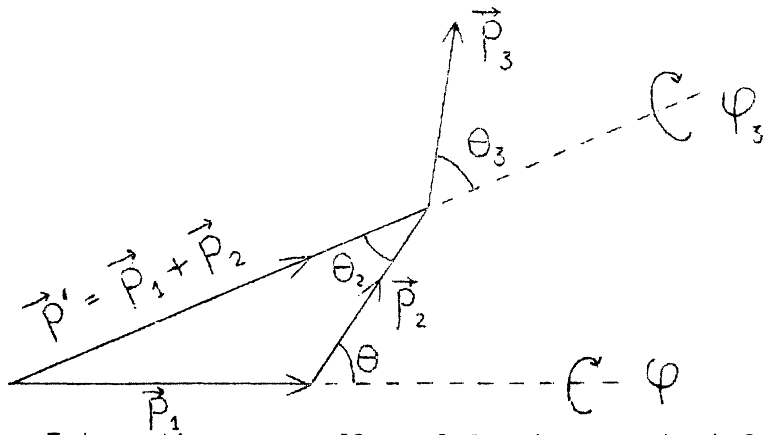


Fig. 2

Integrations over $d\Omega_1$ and $d\varphi$ give constant factors only; therefore

$$\frac{dR_4}{dM_{12}} \propto \frac{d}{dM_{12}} \int \frac{p_1^2 dp_1}{E_1} \frac{p_2^2 dp_2 \; d(\cos\Theta)}{E_2} \frac{p_3^2 dp_3 \; d(\cos\Theta_3)}{E_3} \frac{d\Phi_3 \quad 1}{E_4 \, (p_1 \, p_2 \, p_3)}$$

$$\times \; \delta(E_1 + E_2 + E_3 + E_4 - E) \; \delta(M_{23} - M^*) \, .$$

From momentum conservation we have

$$p_4^2 = (\vec{p}_1 + \vec{p}_2 + \vec{p}_3)^2 = p'^2 + p_3^2 + 2p' p_3 \cos\Theta_3$$

so that for constant p', p_3 and $\cos\Theta$

$$p_4 \; dp_4 = E_4 \; dE_4 = p' p_3 \; d(\cos\Theta_3) \; .$$

Substituting and integrating over E_4 one gets

$$\frac{dR_4}{dM_{12}} = \frac{d}{dM_{12}} \int \frac{p_1^2 dp_1}{E_1} \frac{p_2^2 dp_2 \; d(\cos\Theta)}{E_2} \frac{dE_3 \, d\Phi_3}{p'} \, \delta(M_{23} - M^*)$$

for $E_4 = E - E_1 - E_2 - E_3$ (energy conservation). If we express E_4 by p_4 from the expression above we get

$$p'^2 + p_3^2 + 2p_3 p' \cos\Theta_3 = (E - E_1 - E_2 - E_3)^2 - m_4^2$$

or when solved with respect to $\cos \Theta_3$

$$\cos \Theta_3 = \frac{(E - E_1 - E_2 - E_3)^2 - m_4^2 - p'^2 - p_3^2}{2p' p_3} . \qquad (26)$$

Formula Eq. (26) will be used later to find the integration limits of E_3. Since

$$M_{12}^2 = (E_1 + E_2)^2 - (\vec{p}_1 + \vec{p}_2)^2$$

$$= m_1^2 + m_2^2 + 2E_1 E_2 - 2p_1 p_2 \cos \Theta$$

we can for constant p_1 and p_2 put

$$M_{12} \, dM_{12} = p_1 p_2 d (\cos \Theta) .$$

If we also put $dp_1 = \frac{E_1}{p_1} dE_1$ etc. we get

$$\frac{dR_4}{dM_{12}} = M_{12} \int \frac{dE_1 \, dE_2 \, dE_3 \, d\Phi_3}{p'} \delta (M_{23} - M^*) .$$

Interpreting the argument of the δ function as a function of Φ_3 we get by performing the integration over this variable

$$\frac{dR_4}{dM_{12}} = M_{12} \int \frac{dE_1 \, dE_2 \, dE_3}{p'} \frac{1}{\left| \frac{\partial (M_{23} - M^*)}{\partial \Phi_3} \right|} \quad (M_{23} - M^*) = 0$$

Here M_{23} is given by

$$M_{23}^2 = (E_2 + E_3)^2 - (\vec{p}_2 + \vec{p}_3)^2 = m_2^2 + m_3^2 + 2E_2 E_3 - 2p_2 p_3 \cos \Theta_{23}$$

where Θ_{23} (the angle in space between $\vec{p}_2$ and $\vec{p}_3$) is a function of Θ_2, Θ_3 and Φ_3. From spherical geometry

$$\cos \Theta_{23} = \cos \Theta_2 \cos \Theta_3 + \sin \Theta_2 \sin \Theta_3 \cos \Phi_3$$

Θ_3 is given by Eq. (26) and Θ_2 can be found from

$$\cos\Theta_2 = \frac{\vec{p}'\,\vec{p}_2}{p'\,p_2} = \frac{(\vec{p}_1+\vec{p}_2)\,\vec{p}_2}{p'\,p_2} = \frac{p_1 p_2 \cos\Theta + p_2^2}{p'\,p_2} \qquad (27)$$

$$= \frac{m_1^2 + m_2^2 - M_{12}^2 + 2E_1 E_2 + 2p_2^2}{2p'\,p_2} .$$

By proper substitution it is therefore possible to express explicitly M_{23} as a function of Φ_3 and determine $\partial(M_{23}-M^*)/\partial\Phi_3$. This expression should then be evaluated for the Φ_3 that makes $M_{23} - M^* = 0$. We find

$$\cos\Phi_3 = \frac{m_2^2 + m_3^2 + 2E_2 E_3 - 2p_2 p_3 \cos\Theta_2 \cos\Theta_3 - M^{*2}}{2p_2 p_3 \sin\Theta_2 \sin\Theta_3} .$$

Recalling that Θ_3 and Θ_2 by Eq. (26) and (27) are expressed as functions of E_1, E_2, E_3 and M_{12}, we see that we are left with a triple integral in E_1, E_2 and E_3. This integral has to be solved numerically. The integration limits can be found as follows. For fixed E_1 and E_2 are E_3(min) and E_3(max) determined by $M_{23} = M^*$ for $\cos\Theta_{23} = -1$ and $\cos\Theta_{23} = +1$ respectively, together with the possible restriction that from Eq. (26) $-1 \leq \cos\Theta_3 \leq +1$. Similarly the limits for E_2 with fixed E_1 are determined from $M_{12} = m_1^2 + m_2^2 + 2E_1 E_2 - 2p_1 p_2 \cos\Theta$ by putting $\cos\Theta = \pm 1$ if for these values Eq. (27) satisfies $-1 \leq \cos\Theta_2 \leq +1$.

The lower and upper limits of E_1 are given by

$$E_1(\text{min}) \geq m_1$$

and

$$E_1(\text{max}) \leq \frac{\{[E^2 - (M^* + m_4 + m_1)^2]\,[E^2 - (M^* + m_4 - m_1)^2]\}^{1/2}}{2E} .$$

VII DALITZ PLOT

a) Contours of the Dalitz plot

We will discuss here in some more detail the 3-body phase space, and especially the properties of the Dalitz representation.

First we take the general case of three particles with different rest masses m_1, m_2 and m_3. In their centre of mass the particles are emitted with kinetic energies T_1, T_2 and T_3. The Dalitz representation is a scatter plot of the kinetic energies of any two of the particles, say T_1 and T_2, along the x and y axes of a Cartesian coordinate system. The kinematical limits of the reaction imposed by energy and momentum conservation will now confine the points to the area within a closed curve which touches the two axes, see Fig. 3.

The kinematical constraints are

$$E_3 = E - (E_1 + E_2) \qquad (E_1 = T_1 + m_1 \text{ etc.})$$
$$p_3^2 = p_1^2 + p_2^2 + 2p_1 p_2 \cos\Theta_{12} \quad (\Theta_{12} \text{ angle between } \vec{p}_1 \text{ and } \vec{p}_2) . \tag{28}$$

It is clear that for given T_1 and T_2 (or equivalent p_1 and p_2) we have also a fixed T_3. From the second equation (28) we then have determined Θ_{12}. This means that for given T_1 and T_2 we have a uniquely specified situation.

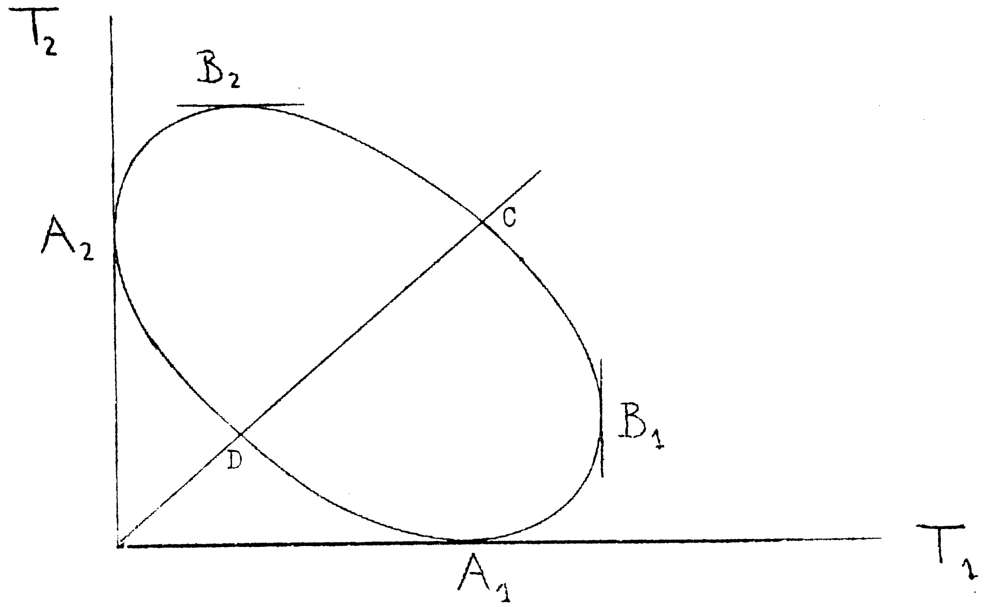


Fig. 3

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In other words, each point on the Dalitz plot corresponds to a well-defined configuration in the final state.

The contour of the Dalitz plot represents the configurations where particles 1 and 2 are emitted parallel or antiparallel, that is for

$$\cos \Theta = a = \pm 1$$

where $a = +1$ for $\vec{p}_1$ and $\vec{p}_2$ parallel and $a = -1$ for $\vec{p}_1$ and $\vec{p}_2$ antiparallel, that is at the lower (A_1 D A_2) and upper (A_1 B_1 B_2 A_2) half of the contour of the Dalitz plot respectively.

From equation (28) above we can eliminate E_3 and P_3 and get

$$T_1 = \frac{\nu + a\sqrt{\nu^2 - uw}}{u} \qquad (29)$$

where

$$u = B^2 - 2ET_2$$
$$\nu = BC - (AB + C - 2m_1 m_2)\, T_2 + ET_2^2$$
$$w = (C - AT_2)^2$$
$$A = E - m_1$$
$$B = E - m_2$$
$$C = \tfrac{1}{2}[(E - m_1 - m_2)^2 - m_3^2]$$

Formula Eq. (29) gives the general relation between T_1 and T_2 along the contour of the Dalitz plot. We will look in more detail at some special cases along the contour and see what they mean physically.

First let the curve touch the T_1 and T_2 axes at points A_1 and A_2 respectively (Fig. 3). Point A_1 corresponds to the case where particle 2 has zero momentum, that is $\vec{p}_1 = -\vec{p}_3$. The value of T_1 at A is

$$T_1(A) = \frac{(E - m_1 - m_2)^2 - m_3^2}{2(E - m_2)} .$$

Similarly point A_2 corresponds to the case where particle 1 has zero momentum and $\vec{p}_2 = -\vec{p}_3$. $T_2(A)$ can be found by interchanging particles 1 and 2 above.

The straight line in Fig. 3 represents states where $|\vec{p}_1| = |\vec{p}_2|$. This line crosses the contour at points C and D. Point C corresponds to the state where $p_3 = 0$ and a value of T_1 given by

$$T_1(C) = \frac{(E - m_3 - m_1)^2 - m_2^2}{2(E - m_3)} .$$

We denote the maximum values of T_1 and T_2 by B_1 and B_2 respectively. Point B_1 corresponds to the situation where particles 2 and 3 are emitted parallel and with equal velocity, that is

$$T_1(\max) = \sqrt{m_1^2 + p_1^2(\max)}$$

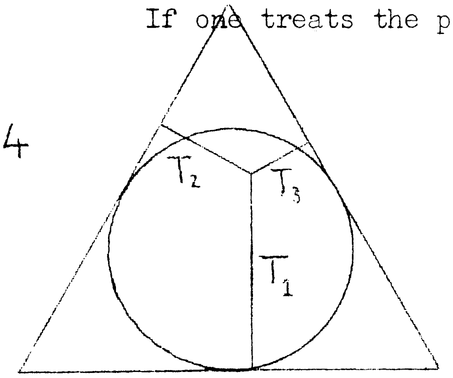
where $p_1(\max)$ is given by equation (12) in Chapter III.d.

Point B_2 is found in the same way by interchanging particles 1 and 2 above, and corresponds to the case where particles 2 and 3 are emitted parallel and with equal velocity but opposite to particle 1.

In the case where the three particles have equal masses as in the decay of the τ meson, the ω resonance etc. another coordinate system is mostly used. This representation is based on the fact, that from any point inside an equilateral triangle (Fig. 4) the sum of the distances to the sides is equal to the height of the triangle. One therefore plots the kinetic energies of the particles along the normals to the sides; then $T_1 + T_2 + T_3$ = Q value = height of triangle. Not all points inside the triangle are available because of momentum conservation.

If one treats the pions non-relativistically, one sees easily that the points have to be within the inscribed circle of the triangle. For more details about this special kind of Dalitz plot we refer to the original work of R.H. Dalitz (Proc. Phys. Soc. A69, 527 (1956), and Reports in Prog. in Phys. 20, 163 (1957)). For the relativistic case, see Fabri, Nuovo Cimento 11, 479 (1954).

Fig. 4



Exercise

Place a Cartesian coordinate system with origin in the centre of the inscribed circle in Fig. 4 with y axis along T_1.

Show that

$$x = \frac{T_2 - T_3}{\sqrt{3}}$$

$$y = T_3 - \frac{Q}{3} \text{ (Q value = height of triangle)}$$

and prove that for non-relativistic particles (of equal mass) the constraint equations

$$T_1 + T_2 + T_3 = Q \quad \text{(Energy conservation)}$$

$$\begin{cases} \vec{p}_1 + \vec{p}_2 + \vec{p}_3 = 0 \\ \cos \Theta_{12} = \pm 1 \end{cases} \quad \text{(Momentum conservation for collinear particles)}$$

lead to

$$x^2 + y^2 = \left(\frac{Q}{3}\right)^2$$

which is the equation for the inscribed circle of the triangle.

Exercise

A resonance between particles m_1 and m_2 will give points clustering along a 45° line crossing the T_1 and T_2 axes on a T_1 T_2 plot. Derive the equation for this line for a given resonance mass $M_{12} = M^*$.

b) Distribution of points on Dalitz plot

We will show here one of the special advantages of the Dalitz representation, namely: equal areas on the Dalitz plot correspond to equal probabilities in the Lorentz invariant phase space. In other words, phase space predicts uniform population of points on a Dalitz plot.

The importance of this fact appears in practice when one plots the kinetic energies (for 3-body states) on a Dalitz plot to see if the points are equally distributed throughout the plot. If the points are clustered in certain regions on the plot or along certain lines, this indicates that (apart from experimental bias and statistical fluctuation) some final state interaction has affected the distribution. The density of points is proportional to the square of the invariant matrix element of the reaction.

We will now prove that phase space predicts uniform population of points. From formula Eq. (7) we get

$$R_3 = \int \frac{d^3 p_1}{2E_1} \frac{d^3 p_2}{2E_2} \frac{d^3 p_3}{2E_3} \delta(E_1 + E_2 + E_3 - E) \; \delta^3(\vec{p}_1 + \vec{p}_2 + \vec{p}_3)$$

which by integration over p_3 gives

$$R_3 = \int \frac{1}{8E_1 \; E_2 \; E_3} d\Omega_1 \; p_1^2 \; dp_1 \; d\Omega_2 \; p_2^2 \; dp_2 \; \delta(E_1 + E_2 + E_3 - E) .$$

Integrations over space angles other than Θ_{12} gives

$$R_3 = \int \frac{1}{8E_1 \; E_2 \; E_3} 4\pi \; p_1^2 \; dp_1 \; 2\pi \; d(\cos \Theta_{12}) p_2^2 \; dp_2 \; \delta(E_1 + E_2 + E_3 - E) .$$

Now, the space angle between $\vec{p}_1$ and $\vec{p}_2$ is for fixed p_1 and p_2 determined from momentum conservation

$$p_3^2 = p_1^2 + p_2^2 + 2p_1 \; p_2 \; \cos \Theta_{12}$$

so that

$$p_3 \; dp_3 = p_1 \; p_2 \; d(\cos \Theta_{12}) .$$

Further: from

$$p^2 = E^2 - m^2$$

we find

$$p \; dp = E\,dE = E \; dT$$

which when substituted into the expression for R_3 above, gives

$$R_3 \propto \int dT_1 \; dT_2 \; dT_3 \; \delta(E_1 + E_2 + E_3 - E) .$$

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Integration over T_3 finally gives

$$R_3 \propto \int dT_1 \; dT_2$$

which expresses the fact that the density of final states is proportional to the area in a T_1 T_2 plot.

Exercise

Show that for non-relativistic particles, that is for

$$T = \frac{p^2}{2m} ,$$

it is the non-invariant phase space ρ and not the invariant phase space R which is proportional to the area in a T_1 T_2 plot.

c) Effective mass plot

We showed in the preceding paragraph that equal areas on a Dalitz plot correspond to equal probabilities in phase space. We will now introduce another much used form of Dalitz plot, namely one where the effective mass squared of any two particles from 3-body final state is plotted along the x and y axes in a Cartesian coordinate system*.

We will now show that phase space predicts a uniform population of points on this plot as well.

We have

$$\begin{aligned} M_{12}^2 &= (E_1 + E_2)^2 - (\vec{p}_1 + \vec{p}_2)^2 \\ &= (E - E_3)^2 - p_3^2 \\ &= E^2 + m_3^2 - 2EE_3 \\ &= E^2 + m_3^2 - 2Em_3 - 2ET_3 \end{aligned}$$

* A useful representation for the masses of a pair of particles from a 4-body final state has recently been given by Goldhaber et al. in Phys. Rev. Letters 9, 330 (1962) and Physics Letters 6, 62 (1963).

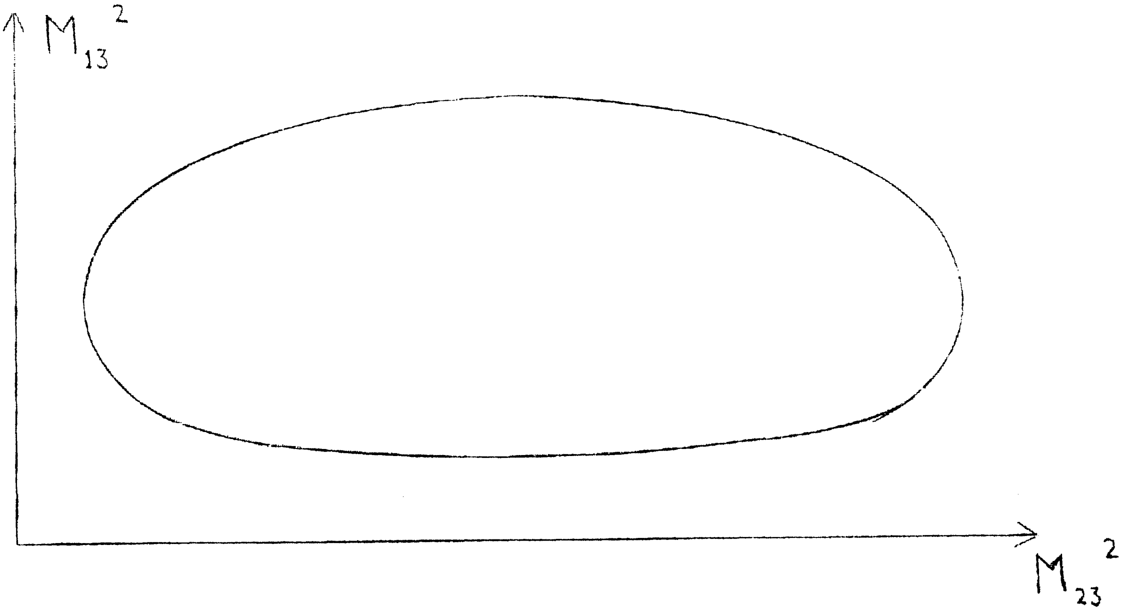

Fig. 5

We see that M_{12}^2 is a linear function of T_3, which by differentiation gives

$$d(M_{12}^2) \propto dT_3 .$$

This means that

$$d(M_{23}^2)\ d(M_{13}^2) \propto dT_1\ dT_2 \quad ,$$

or put in words: Areas on the M_{23}^2 M_{13}^2 plot are proportional to areas on the T_1 T_2 plot. From this follows that equal areas on the M^2 plot correspond to equal probabilities in phase space.

We may note that the minimum and maximum values of the invariant masses are given by

$$M_{12}(\min) = m_1 + m_2$$

$$M_{12}(\max) = E - m_3 \quad .$$

Similarly for M_{13}, by interchanging particle 2 and 3. The contours on the M^2 plot can in general be found by using the relation between T_1 and T_2 given by equation (29). For each value of T_2 we have $p_2 = \sqrt{T_2^2 + 2m_2 T_2}$ and

$M_{13}^2 = (E - E_2)^2 - p_2^2$. From Eq. (29) we find the corresponding T_1 and can then calculate

$$M_{12}^2 = (E_1 + E_2)^2 - (p_1^2 + p_2^2 \pm 2p_1\ p_2) \ .$$

Exercise

Show that if a resonance $M_{13} = M^*$ decays isotropically in space with respect to particle 2 (that is $dR/d\cos\Theta_{12}$ = constant) the points will be evenly distributed along the M_{12}^2 axes on an $M_{12}^2\ M_{13}^2$ plot.

We see also that formula Eq. (25)

$$\frac{dR}{dM_{12}} \propto M_{12} \text{ for } M_{13} = M^*$$

can be directly verified from the fact that the points are evenly distributed along the M_{12} axis on a $M_{12}^2\ M_{13}^2$ plot.

Show that the three possible effective mass combinations are related to each other by the equation

$$M_{12}^2 + M_{13}^2 + M_{23}^2 = E^2 + m_1^2 + m_2^2 + m_3^2 \ .$$

d) Effects of angular momentum conservation

We have stated in the preceding paragraphs that phase space predicts uniform population of points on a Dalitz plot. This statement is true, however, only to the extent to which one can neglect the influence of constraint equations imposed by angular momentum conservation. In production processes, for example, where one has a Q value high enough to expect some contribution from production amplitudes with angular momentum states greater than zero, one might expect these amplitudes to contribute differently to different final states, since not all final states otherwise allowed by phase space, will conserve angular momentum. The areas on the Dalitz plot corresponding to such states will then be depopulated. Since one expects the contribution from higher partial waves to increase with the Q value of the production process, the relative depopulation of points on the Dalitz plot will probably vary with the Q value in the production process. This makes it important in the study of particle resonances that the presence of an apparent "bump" on the phase space plot for one particular incident momentum is verified for other incident particle momenta.

The effects of angular momentum conservation in connection with the production of the Y_1^* (1385) in the reaction $K_2^0 + p \to \Lambda + \pi^+ + \pi^0$, has been discussed by R.K. Adair in Rev.Mod.Phys. 33, 406 (1961). A detailed quantitative estimate of the effect depends on the type of particles involved in the reaction, such as the values of the particle spins, isospins etc. We will give here a simple qualitative description of the angular momentum effect with reference to the Dalitz plot for particles of different masses.

The configuration of three particles in their centre of mass system can be specified in terms of two momenta; $\vec{p}$ the momentum of say, the third particle in the three particle rest system, and $\vec{q}$ the momentum of particle one or two in the centre of mass system of these two particles (see Fig. 6).

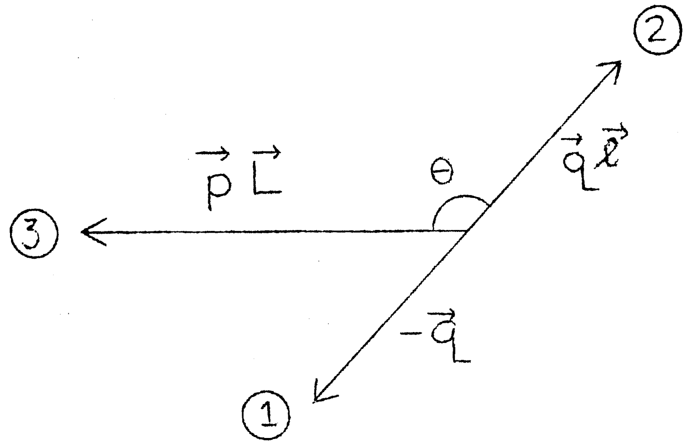

Fig. 6

Using this description of the three particle state we can express the total angular momentum ($\vec{J}$) of the three particles as the vector sum of two independent angular momenta $\vec{\ell}$ and $\vec{L}$ defined as follows: Particles 1 and 2 revolve about their mutual centre of mass with orbital angular momentum $\vec{\ell}$, while particles 1 and 2 together with particle 3 revolve around the 3-body centre of mass with orbital angular momentum $\vec{L}$. The total angular momentum of the 3-body system is then $\vec{J} = \vec{\ell} + \vec{L}$. For simplicity we will in the following assume all particles in initial and final states to have spin zero.

For a given total energy of the system, the total production amplitude (A) of a particular angular momentum state $\vec{J}$ can be expressed as a sum of partial amplitudes of different $\vec{\ell}$ and $\vec{L}$.

$$A = \Sigma\, a_{J\ell L} \,.$$

The summation is to be extended over all possible values of ℓ and L satisfying $\vec{\ell} + \vec{L} = \vec{J}$. The complex partial production amplitudes, a, are functions of the individual particle momenta, but are already integrated over space angles i.e. the total production intensity is given by $|A|^2 = \Sigma\, |a_{J\ell L}|^2$. The intensity of a specific partial wave can be expressed as

$$|a_{J\ell L}|^2 = K\, R_3(0,E)\; P_L P_\ell$$

where K contains the matrix element and is a function of the same variables as a; $R_3(0,E)$ is the invariant 3-body phase space integral, and P_L and P_ℓ are the angular momentum barrier factors for the orbital angular momentum of particle 3, and the two particle system 1 and 2 respectively. (The labelling of the particles as 1,2 or 3 is of course in arbitrary order and the labelling should be permuted in the summation above for the total production amplitude).

P_L and P_ℓ can be calculated for different values of linear momenta and orbital angular momentum of the system. For a simple qualitative estimate it suffices to remember that for fixed angular momentum J, P_J decreases with decreasing linear momentum of the particle so that for $J > 0$ is $P_J = 0$ for zero momentum. For fixed momentum, P_J decreases with increasing values of angular momentum. From this follows that contributions from partial waves with L or $\ell > 0$ diminish whenever one of the particle momenta is zero or whenever two particles touch in momentum space, e.g. have zero relative momentum. The points in phase space corresponding to these configurations can easily be found on the Dalitz plot.

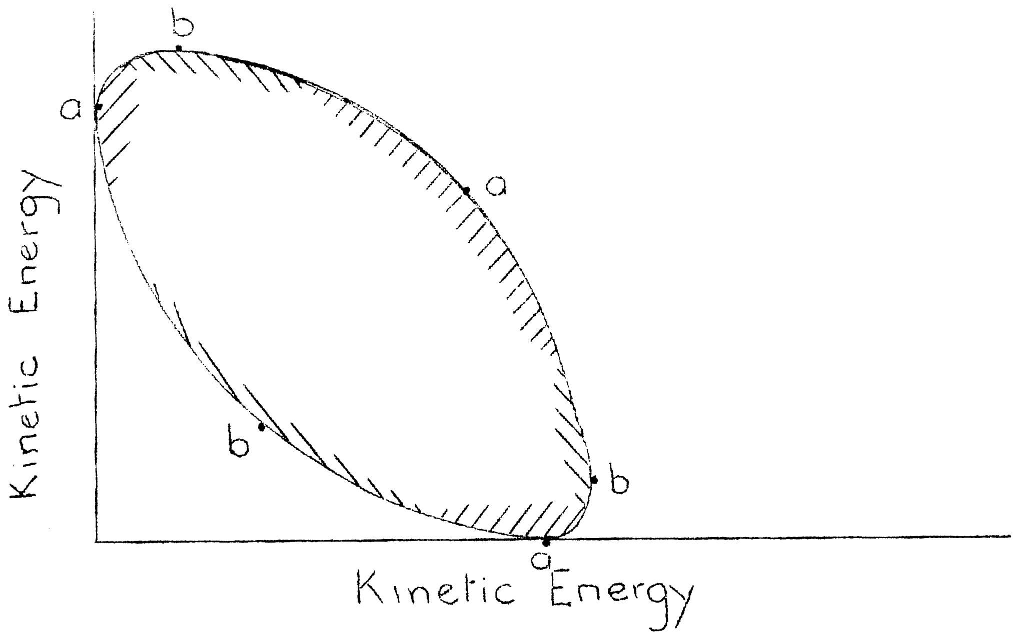

Fig. 7

With reference to Fig. 7 we schematically indicate the physical interpretation of a few special points on the Dalitz plot. We have not indicated the kinetic energy of which particle is plotted along the axes on Fig. 7, since the choice is arbitrary.

Points a represent states where one of the particles has zero momentum.

Points b represent states where two particles have equal momenta.

It follows therefore that partial amplitudes where L > 0, which we indicate by $a_{J\ell L>0}$, give no contribution around the points marked a, and partial amplitudes with $\ell > 0$ $(a_{J\ell>0\,L})$ are zero around points b in Fig. 7. The size of the region which in this way will be depopulated, obviously depends on the absolute magnitude of the angular momentum vectors $\vec{\ell}$ and $\vec{L}$. For instance around points a we expect P_L to be small for

$$|\vec{p}| < \frac{L\cdot\hbar}{r}$$

where r is an effective radius, which we can take as the Compton wavelength of the pion. We find that partial amplitudes with L=1 are strongly reduced for $|\vec{p}| < 150$ MeV/c.

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Similarly we expect p_ℓ to be small around the points b for $|\vec{q}| < 150 \cdot \ell$ MeV/c. Thus, to summarize, in the presence of angular momentum states $J \geq 1$ one expects some depletion of events in the region of the points a and b in Fig. 7. These areas have been shaded to give a qualitative illustration of the effect.

So far we have seen that the requirement of angular momentum conservation gives rise to a centrifugal barrier effect which tends to depopulate the Dalitz plot in certain regions. The arguments are valid irrespective of the direction in space of the angular momentum vector.

We will now consider another effect of angular momentum conservation, mainly pertinent to production processes. This effect arises from the fact that the orbital angular momentum vector in the initial state is not arbitrarily distributed in space but confined to a plane perpendicular to the direction of the incoming particle. Even though the Dalitz plot representation does not contain any information regarding the orientation of the particles in final state with respect to the beam direction or production plane, we nevertheless find that the requirement of the angular momentum vector perpendicular to the beam will influence the distribution of the points on the Dalitz plot.

A convenient description of the angular distribution of the particles in the three particle system (see Fig. 6) is in terms of the distribution function $dN/d\cos\Theta$ where the angle Θ is given by

$$\cos\Theta = \frac{\vec{p}\cdot\vec{q}}{pq} .$$

One particular advantage of this description is the following. Equal intervals along lines parallel to the axes on the Dalitz plot correspond to equal intervals of $\cos\Theta$. This means that if $\vec{q}$ is isotropically distributed in space with respect of $\vec{p}$ the Dalitz plot will be evenly populated with points along lines parallel to the axes and along lines at 45° with respect to the axes. (This can easily be proved by performing a Lorentz transformation of $\vec{q}$ and Θ to the three particle centre of mass system. (Cf. also exercise in Chapter VII.c).

Since $\vec{\ell}$ and $\vec{L}$ are randomly orientated in planes perpendicular to $\vec{q}$ and $\vec{p}$ respectively, an isotropic distribution of $\vec{q}$ with respect to $\vec{p}$ results in a uniform distribution of the space angle between $\vec{\ell}$ and $\vec{L}$. We will now show that the restriction on the direction (given by the vector sum of $\vec{J}$ = $\vec{\ell} + \vec{L}$ perpendicular to the beam) leads to a non-uniform distribution in space between $\vec{q}$ and $\vec{p}$.

We introduce a Cartesian coordinate system with z axis in the beam direction and assume the physical system to exhibit rotational symmetry about this direction. With no loss of generality the y axis can therefore be taken in the direction of the total angular momentum $\vec{J}$. Angular momentum conservation now requires $\vec{L}$ and $\vec{\ell}$ to lie in a plane through the y axis. For all combinations of $\vec{L}$ and $\vec{\ell}$ satisfying

$$\vec{J} = \vec{L} + \vec{\ell}$$

obviously

$$j_z = L_z + \ell_z = 0 \ .$$

All plane angles β between $\vec{L}$ and $\vec{\ell}$ are equally probable. We will now seek the angular distribution between $\vec{p}$ and $\vec{q}$ for a fixed and arbitrary value of β, that is, we want the distribution function $dN(\beta)/d\cos\Theta$. We recall that $\vec{p}$ lies randomly in a plane (S in Fig. 8) perpendicular to $\vec{L}$ and $\vec{q}$ randomly in a plane (T) perpendicular to $\vec{\ell}$. We arbitrarily fix $\vec{p}$ to be perpendicular to the intersection line between the two planes. The direction of $\vec{q}$ is given by its angle γ with a plane containing $\vec{p}$ and $\vec{L}$, see Fig. 8. It then follows that the angle Θ in space between $\vec{q}$ and $\vec{p}$ is given by

$$\cos\Theta = \cos\gamma \cos(\pi - \beta)$$

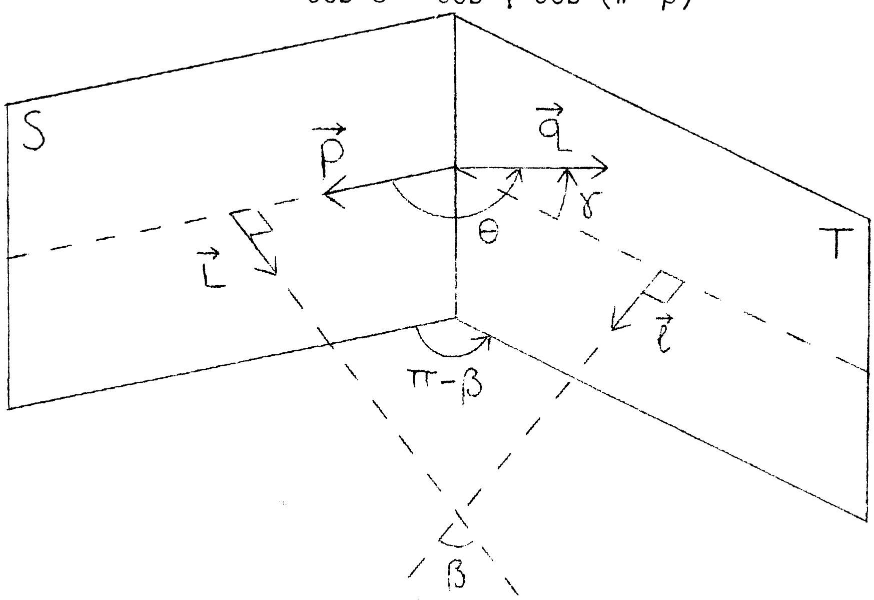


Fig. 8

Since $\vec{q}$ can have any direction in T, $\frac{dN(\beta)}{d\gamma}$ = const, or equivalently

$$\frac{dN(\beta)}{d\cos\gamma} \propto \frac{1}{\sin\gamma}$$

we find

$$\frac{dN(\beta)}{d\cos\Theta} = \frac{dN(\beta)}{d\cos\gamma} \cdot \frac{d\cos\gamma}{d\cos\Theta} \propto \frac{1}{\sin\gamma} \cdot \frac{1}{\cos(\pi-\beta)} \quad .$$

Eliminating γ we get the distribution function for fixed β

$$\frac{dN(\beta)}{d\cos\Theta} \propto (\cos^2\beta - \cos^2\Theta)^{-1/2} \quad .$$

Finally, since all plane angles β have equal weights we find by integration over β

$$\frac{dN}{d\cos\Theta} \propto \int_0^{+\Theta} (\cos^2\beta - \cos^2\Theta)^{-1/2}\, d\beta \quad .$$

For numerical calculation this expression can be given a more convenient form on substituting

$$\sin\Theta = k \qquad \sin\beta = kt \; .$$

We then get a standard form of an elliptical integral of the first kind

$$\frac{dN}{d\cos\Theta} = f(k) \propto \int_0^1 \frac{dt}{\sqrt{1-t^2}\,\sqrt{1-k^2t^2}}$$

which is tabulated in, for instance, Jahnke and Emde "Tables of Functions".

The distribution function is shown in Fig. 9.

We also see from Fig. 9 that dN/d cos Θ, which is symmetrical about cos Θ = 0, is strongly peaked forwards and backwards. This results in an uneven population of points on the Dalitz plot, i.e. there will be more points concentrated near the boundary than in the middle of the plot.

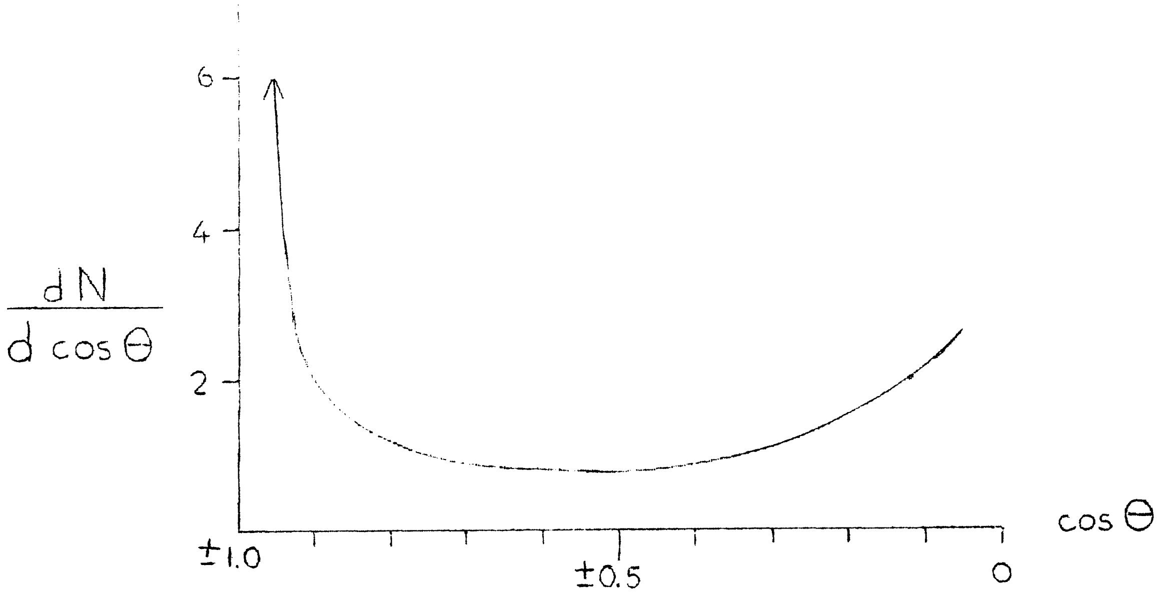


Fig. 9

In summing up we conclude: The effect of angular momentum barrier is to decrease the point density in special regions near the boundary of the Dalitz plot. The uneven distribution in space of the angular momentum vector will lead to decreasing point density around the centre of the plot. We stress, however, that these remarks are qualitative only. In a specific case, the effects will depend strongly on the particular type of particles in question due to the conservation of isospin and parity.

e) Dalitz-Stevenson plot

From a very detailed study of the distribution of points on a Dalitz plot Maglic and Stevenson et al. (Phys.Rev.Letters 7, 178 (1961), Phys.Rev. 125, 687 (1962) were able to determine the spin and parity for the three particle decay mode $\omega^0 \rightarrow \pi^+ + \pi^- + \pi^0$. Their elegant treatment is also applicable to other three particle decay modes proceeding via strong interactions. Stevenson et al. made some special assumptions about the form of the matrix element. This enabled them to perform a quantitative comparison between prediction and experimental data. Before reviewing their arguments, however, we will state some more general and qualitative arguments given by Dalitz which are valid irrespective of the particular form of the transition matrix element and which are valid for spin values less than three. [R.H. Dalitz: "Three Lectures on Elementary Particles" BNL 735 (1961) and "Strange Particles and Strong Interactions, Oxford University Press (1962)].

The foundation of the following discussion is the experimental fact that no charged state of the ω has been found. Consequently ω has isospin I = 0, i.e. the isospin wavefunction Φ_ω is a scalar.

We assume that the decay

$$\omega \rightarrow \pi^+ + \pi^- + \pi^0$$

proceeds via a strong interaction, i.e. isospin and parity are conserved. Therefore, the isospin wavefunction of the final state must also be a scalar. The only way to obtain a scalar quantity from the three isovector wavefunctions Φ_1, Φ_2 and Φ_3 of the pions, is to form a triple product

$$\Phi_\omega = \Phi_1 \cdot (\Phi_2 \times \Phi_3)$$

$\Phi_\omega = 0$ and the transition matrix element will vanish if two or more of the particles are equal. Thus, the $3\pi^0$ decay mode of the ω^0 is forbidden.

The triple product Φ_ω is antisymmetric in any pair of pions. Since the pions are Bose particles their total wavefunction (which can be expressed as a product of a space function and isospin function) must be symmetric. Thus, the space wavefunction of the ω must also be antisymmetric in any pair of pions. This fact has important consequences on the symmetry of the points on a Dalitz plot representing the decay configurations of the pions.

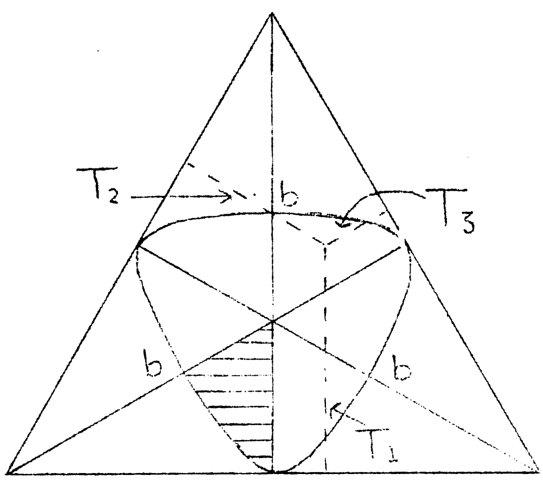


Fig. 10.

We consider in Fig. 10 a triangular plot (cf. Fig. 4 in Chapter VII.a) of the kinetic energies of the three pions. The contour deviates slightly from a circle because of the relativistic energies of the pions. From the antisymmetry of the space wavefunction in all pairs of pions it follows that the distribution of events is unchanged by a reflection across any one of the three symmetry axes of the triangle. This means that the distribution of points should be the same in all six sectors of the Dalitz plot; one of these sectors is shaded in Fig. 10. In a study of the statistical distribution one can, therefore, very conveniently, concentrate the points in a so called 6-fold Dalitz plot.

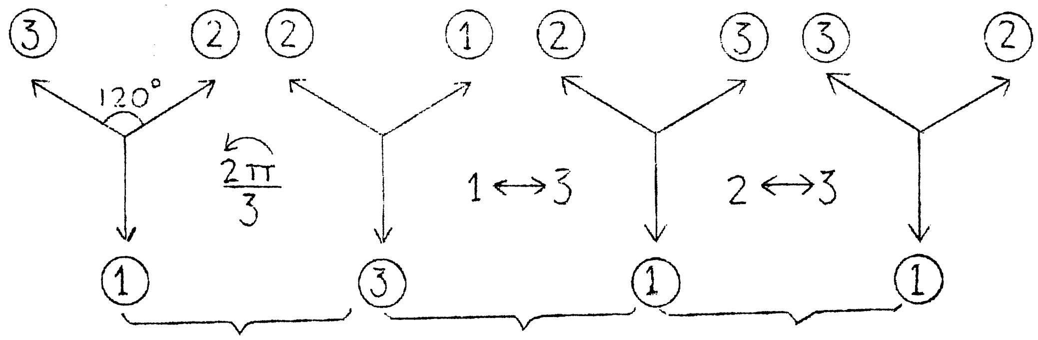
We will next consider some general features of the Dalitz plot distributions expected for different spin (J) and parity (w) assignments of the ω meson. (Sometimes we will denote the sign of the parity by a superscript to the spin value, e.g. J^+, J^-).

(i) The density of points will vanish whenever one of the pions has its maximum kinetic energy. This follows from the antisymmetry of the space wavefunction. An interchange of any two pions makes the transition matrix element vanish if two pions "touch" in momentum space. This happens when the third pion has its maximum energy which is indicated by the points b in Fig. 10. The above statement holds for all spin and parity assignments of the ω to be considered.

(ii) The density of points will vanish on the boundary of the plot for ω parity $w = (-1)^J$. The contour represents configurations where the three pions are emitted collinearly and their directions can therefore be specified by a single vector. The space wavefunction will be a spherical harmonic $Y_J^m(\cos\Theta)$, which has parity $(-1)^J$. Since the intrinsic parity of each pion is (-1), the total parity of the collinear configuration is $(-1)^3$. $(-1)^J = -(-1)^J$. Hence, if the ω parity is $(-1)^J$, the matrix element must vanish on the boundary of the Dalitz plot.

(iii) The density of points will vanish at the centre of the plot if the ω parity is even. The centre of the plot represents configurations where the pions have equal energy and their directions of motion make angles of 120° with each other. Therefore, if the three particle system is first rotated 120° around an axis (N) normal to its plane it can be returned to its initial position by interchanging the particles 1 → 3 and 2 → 3.

The operations are schematically illustrated below.



$$\exp\left(\frac{2\pi}{3}\,mi\right) \qquad (-1) \qquad (-1)$$

If the angular momentum is quantized along N with a magnetic quantum number m, the rotation corresponds to an operation

$$\exp\left(\frac{2\pi}{3}\,mi\right) .$$

Each of the two successive interchanges corresponds to multiplication of the space wavefunction with a factor (-1). Since the system is restored to its initial position we must have

$$\exp\left(\frac{2\pi}{3}\,mi\right) \cdot (-1)(-1) = +1 .$$

For $J < 3$ (recall that $m \leq J$) this equation can only be satisfied for $m = 0$. This means that $m = 0$ is the only possibility for a non-vanishing matrix element of the symmetrical configuration. This information will now be used when we perform the following operations.

We reflect the symmetrical configuration with respect to the origin and return it to the initial state by rotating the system 180° about N (the normal to the plane of the system). The effect of the first operation is to multiply the space wavefunction by $(-1)^3 \cdot w$ (w = intrinsic parity of the ω) whereas the second operation for $m = 0$ leaves the wavefunction unchanged. Thus, we require

$$(-1)^3\, w(+1) = +1 .$$

This means that the matrix element must vanish at the symmetry point if the ω parity is even. This statement is valid for all $J < 3$.

(iv) The density of points will vanish at the centre of the plot if J is even. To show this we rotate the configuration at the symmetry point about an axis along the direction of motion of one pion, say pion 3, and return the system to the initial state by the interchange $1 \leftrightarrow 2$.

The first operation corresponds to an operator

$$\exp(i\pi J_y)$$

where the y axis is oriented along the direction of motion of particle 3. Now, since $m = 0$, J_y can also be quantized. Therefore the rotation multiplies the wavefunction by $(-1)^J$. The second operation (interchange $1 \leftrightarrow 2$) changes the wavefunction by a factor (-1). In total we must have

$$(-1)^J(-1) = +1 \ .$$

Thus, the matrix element has to be equal to zero at the centre of the plot if J is even. This statement is also valid for all $J < 3$.

The above qualitative but general considerations are sufficient to distinguish between the possible spin and parity assignments of the ω meson if $J < 3$. A 0^+ state of the ω decaying via the mode $\omega \rightarrow \pi^+ + \pi^- + \pi^0$ violates the conservation of parity. The other possible states are $0^-, 1^-, 1^+, 2^-$ and 2^+. From (iii) and (iv) follows that the states 1^+ and $0^-, 2^+, 2^-$ respectively, would all demand zero point density at the centre of the plot. The experimental distribution on the 6-folded Dalitz plot in Fig. 11 clearly reveals that the density of points does not vanish near the centre of the plot. Thus, of all the admissible spin parity assignments for $J < 3$ we are left with only one possibility, 1^-, which from (ii) should require a depopulation of events near the contour. This is in agreement with the observed distribution in Fig. 11.

We conclude therefore, that the ω meson probably has $J = 1$ and $w = -1$. This was already shown by Stevenson et al. in their original paper, and we now proceed to discuss their assumptions about the transition matrix element and the quantitative comparison they made on the basis of the assumptions.

The transition matrix element of the ω decay is most conveniently analysed in terms of the vectors $\vec{p}$ and $\vec{q}$ introduced in Fig. 6. We recall that $\vec{p}$ is the momentum of one pion (3) in the 3-body centre of mass system and $\vec{q}$ the momentum of one of the other pions (2) in the centre of mass of these two pions (1 and 2).

No rigorous derivation of the different matrix elements will be attempted here. Our aim is simply to present some physical arguments from which one can understand the qualitative form of the simplest matrix elements derived from different tentative spin and parity assignments of the ω. The implicit assumption being that the wavelengths of the decay pions are large compared to the interaction dimension of the decay which presumably is of the order of the Compton wavelength of the ω. Consequently the momentum dependence of the transition matrix element will be determined mainly by the coefficients for the centrifugal barrier penetration and will for L and ℓ greater or equal to one be of the form

$$M = \langle\psi_f|H_{int}|\psi_i\rangle \propto \frac{(pR)^L}{(2L+1)!!}\frac{(qR)^\ell}{(2\ell+1)!!}$$

where R is the radius of interaction. We note that the momentum dependence of M is independent of R, so that in order to study the variation of M from the Dalitz plot it suffices to write

$$M \propto p^L q^\ell .$$

One general remark can be made about the possible values of ℓ. A reversal of $\vec{q}$ which multiplies the wavefunction by $(-1)^\ell$ corresponds to an interchange of the two particles (1 ↔ 2 in Fig. 6) which changes the wavefunction by (-1). Thus we must have

$$(-1)^\ell . (-1) = +1 .$$

It follows that only odd values of ℓ are allowed. Therefore only $\ell = 1$ will be considered in the following discussion of the three possible matrix elements for ω spin less or equal to one.

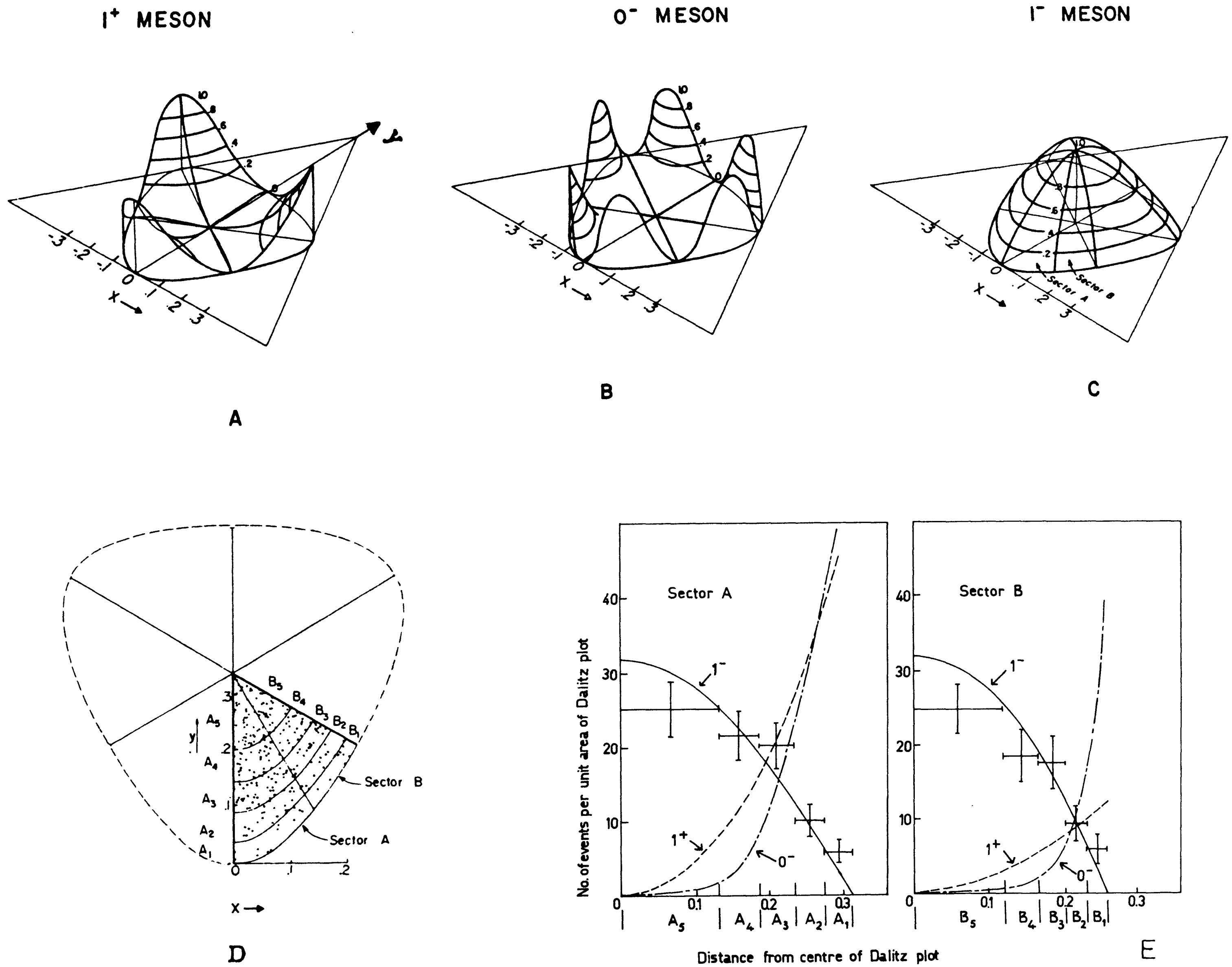


Fig. 11

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0^- meson ($L = \ell = 1$). Since the intrinsic parity of the three pions is $(-1)^3 = -1$ the 0^- meson should require a scalar transition matrix element. The only scalar quantity which is odd with respect to an interchange of any two pions (say 1 and 2) is a scalar product of the form $\vec{p} \cdot \vec{q}$ (interchange of particles 1 and 2 corresponds to a reversal of $\vec{q}$).

We will now express this scalar product in terms of the centre of mass energies of the pions. In the non-relativistic limit we can write

$$E_1 = \frac{\left(-\frac{\vec{p}}{2} - \vec{q}\right)^2}{2m} + m$$

$$E_2 = \frac{\left(-\frac{\vec{p}}{2} + \vec{q}\right)^2}{2m} + m$$

where E_1 and E_2 are the total energies of pions 1 and 2 in the 3-body centre of mass system. From these two equations follows

$$\vec{p} \cdot \vec{q} = m(E_1 - E_2) \ .$$

Now, since the matrix element (M) should be symmetric in the labels of all three pions, it will, for a 0^- meson be of the form

$$M(0^-) = (E_1 - E_2)\ (E_2 - E_3)\ (E_3 - E_1) \ .$$

We see that $M(0^-)$ vanishes whenever two of the pions have equal energies, that is along all three symmetry lines of the Dalitz plot. In particular the density of points will be very low where the symmetry lines intersect, that is, in the centre of the plot.

1^- meson. The matrix element describing the transition from a vector meson state 1^- to the three pion state must have the properties of an axial vector (pseudovector). For the decay with $L = \ell = 1$, $M(1^-)$ must therefore be of the form $\vec{p} \times \vec{q}$. We note that this matrix element is also odd for an interchange of two particles (reversal of $\vec{q}$). Since $\vec{p}_3 = \vec{p}$ and $\vec{p}_2 = -\vec{p}/2 + \vec{q}$ etc.
M can be expressed in terms of the momenta of the particles in their overall centre of mass system as

$$M(1^-) = \vec{p}_1 \times \vec{p}_2 + \vec{p}_2 \times \vec{p}_3 + \vec{p}_3 \times \vec{p}_1$$

where M is made symmetrical in the labels of all three pions.

We see that $M(1^-) = 0$ whenever the pions are emitted collinearly; that is the density of points vanishes on the boundary of the Dalitz plot.

<u>1^+ meson</u>. The transition matrix element must in this case have the properties of a vector. The simplest decay of a 1^+ ω meson is by the emission of one s-wave pion (L = 0) and two pions in a relative p-state ($\ell = 1$). Then the matrix element will be of the form $E_3 \vec{q}$. Since

$$\vec{q} = \frac{\vec{p}_3}{2} + \vec{p}_2$$

we have from momentum conservation

$$\vec{q} = - \tfrac{1}{2}(\vec{p}_1 - \vec{p}_2) \ .$$

Thus, a matrix element symmetrical in the labels of all pions and odd for the interchange of any two pions will be proportional to

$$M(1^+) = E_3(\vec{p}_1 - \vec{p}_2) + E_1(\vec{p}_2 - \vec{p}_3) + E_2(\vec{p}_3 - \vec{p}_1) \ .$$

This matrix element vanishes whenever any two pions have the same momentum, that is at the points b in Fig. 10. It also vanishes at the symmetry point of the Dalitz plot where $E_1 = E_2 = E_3$.

The variation in the point density on the Dalitz plot for the different matrix elements can be illustrated on a three dimensional plot - referred to as Dalitz-Stevenson plot (see Fig. 11) - where the height above the Dalitz plot is proportional to the square of the matrix element. Due to the finite width of the resonances (i.e. the variation of the Q value from event to event) it is most meaningful to make the Dalitz plot in terms of the normalized variables T_1/Q, T_2/Q and T_3/Q. In a Cartesian coordinate system with x axis and origin as indicated in Fig. 11 we now have

$$x = \frac{T_2 - T_3}{\sqrt{3}\ Q}$$

$$y = \frac{T_3}{Q} \ .$$

Figures 11 A,B, and C show isometric graphs of $|M(1^+)|^2$, $|M(0^-)|^2$ and $|M(1^-)|^2$ respectively. The maximum height above the plane is arbitrarily chosen as unity and the contours have been drawn at 0.2 intervals. For comparison with experimental data the contours are projected onto the plane of the Dalitz plot. Because of the symmetry of the plot referred to earlier, it suffices to make the projection in one sixth of the Dalitz plot. Such a projection is shown for the 1^- meson in Fig. 11 D.

The matrix elements for the 1^+ and 0^- meson show considerable and different azimuthal variation. For instance $M(0^-) = 0$ for x=y=0 whereas $M(1^+)$ is large at the same point. The 6-folded Dalitz plot has therefore more or less arbitrarily been divided into two sectors A and B. The contours (for constant $|M|^2$) in turn divide each sector into sub-areas A_1 ... A_5 and B_1 ... B_5 as illustrated in the 1^- case in Fig. 11 D. Finally, Fig. 11 E displays the number of events found within each area A_1, A_2 etc., as well as the theoretical curves of $|M|^2$ (obtained by azimuthal integration from Fig. 11 A, Band C) versus the distance from the centre of the Dalitz plot. (This plot is also generally referred to as Stevenson plot). It is evident that the 1^- assignment to the ω meson is in perfect agreement with the experimental result. On the other hand, a 0^- or a 1^+ assignment would contradict the experimental data for both sectors A and B.

With reference to the general remarks made in the preceding paragraph about the effect of angular momentum barrier on the distribution of points on the Dalitz plot, one comment should be made. It was shown that the contribution from partial waves with $L > 0$ would diminish whenever one of the particles in final state had zero linear momentum. In the case of the 1^+ ω meson, however, $|M(1^+)|^2$ has its maximum value at $y = 0$ (Fig. 11 A). The reason for this, is, that the matrix element is calculated for $L = 0$ only. Inclusion of higher L values would still give finite values of $M(1^+)$ for $y = 0$, but M would show a decrease for $T_3 \to 0$ due to smaller contribution from $L > 0$ states.

VIII. EXAMPLES

a) Branching ratio in π decay

One has observed two 2-body decay modes of the π^+ meson

$$\pi^+ \rightarrow \mu^+ + \nu$$

$$\pi^+ \rightarrow e^+ + \nu \ .$$

If the matrix element of the transition is the same in both reactions the branching ratio is determined by phase space. We will calculate here the branching ratio predicted both from Lorentz non-invariant and Lorentz invariant phase space. In the centre of mass of the pion we have

$$\begin{aligned} E &= 139.59 \text{ MeV} \\ E_\mu &= 109.78 \text{ MeV} \\ p_\mu &= 29.81 \text{ MeV/c} \\ E_e &= 69.80 \text{ MeV} \\ p_e &= 69.79 \text{ MeV/c} \ . \end{aligned}$$

Using formulae (6) and (8) these values give the following phase space predictions for the branching ratio

$$\frac{\rho_2(\mu\nu)}{\rho_2(e\nu)} = 0.427 \text{ (non-invariant)}$$

$$\frac{R_2(\mu\nu)}{R_2(e\nu)} = 0.207 \text{ (invariant)}$$

We notice firstly a marked difference between prediction from non-invariant and invariant phase space. Secondly we have a large discrepancy between the predicted values and the experimental result.

Experimental branching ratio

$$\left(\frac{\mu\nu}{e\nu}\right) \approx 10^4 \ .$$

From this discrepancy follows that there is no reason to believe that the matrix element is the same for the two decay modes. On the contrary, we expect from the two component theory of the neutrino that the matrix element will be very sensitive to the velocity of the emitted lepton. Since the positively charged leptons are preferentially emitted with positive helicity, the decay rates of $\pi^+ \rightarrow \mu^+ + \nu$ and $\pi^+ \rightarrow e^+ + \nu$ are reduced by factors $(1-{}^{v}\mu/c)$ and $(1-{}^{v}e/c)$ respectively. Disregarding phase space this gives a branching ratio of the order of 10^4 [see for instance A. Lundby, Progress in Elementary Particle and Cosmic Ray Physics V, 1 (1960)].

b) Effective mass distributions

To illustrate the use of formulae (21) and (24) and to show the appearance of some effective mass distributions, consider the reaction

$$K^- + N \rightarrow K^{\overset{0}{+}} + \Xi^- + n\pi$$

which has recently been studied at CERN (Belliere et al. Physics Letters 6, 316 (1963) and the Sienna Conference 1962).

The reactions were produced in a heavy liquid bubble chamber filled with CF_3Br by a separated K^- beam with an average momentum in the chamber of about 3.4 GeV/c. A K^- interaction with a single nucleon at rest corresponds to a total energy in the centre of mass of 2.75 GeV. Since in the heavy liquid the nucleons are bound in a nucleus, the centre of mass energy will be spread out due to the Fermi momentum of the nucleons. The effect of the Fermi momentum on the phase space distribution of the $\Xi\pi$ mass is illustrated in Fig. 12 for the case where only one pion is produced (3-body final state). We have calculated here from Eq. (21) the $\Xi\pi$ mass distribution for the case of a nucleon at rest (No Fermi) and for the two rather extreme cases that the target nucleon moves parallel to the beam and with a momentum of 200 MeV/c towards or away from the beam particle. These extremes correspond to a centre of mass energy of 3.01 GeV and 2.52 GeV respectively. For the cases with two or three pions (4 or 5-body final state) we have used Eq. (24) to calculate the $\Xi\pi$ mass distribution for a target nucleon at rest only.

The experimental values of the $\Xi\pi$ effective masses are presented as histograms in Fig. 13 and Fig. 14 for events containing 1,2 and 3 pions.

The $\Xi^-\pi^-$ system which has a third component of isotopic spin $I_z = -3/2$ and therefore $I \geq 3/2$ is presented in Fig. 13 and the $\Xi^-\pi^+$ and $\Xi^-\pi^0$ mass values which have $I \geq \frac{1}{2}$ are combined in Fig. 14.

The curves in Fig. 13 and Fig. 14 are phase space curves constructed from Fig. 12 for interactions with target nucleons at rest, and are simply a super position of 3,4 and 5-body phase space weighted according to the respective number of events observed.

There seems to be good agreement between the experimental $\Xi^-\pi^-$ mass distribution and the predicted phase space curve. This is rather remarkable since we have neglected all effects from Fermi motion of the target nucleons, secondary interaction in the target nucleus of primary and secondary particles etc. On the other hand the phase space curve in Fig. 14 seems to fit the experimental histogram rather poorly.

The above qualitative statements should be expressed in a more objective and quantitative way by performing a goodness of fit test of the distributions. To do this we divide the mass values of Fig. 13 into four groups with approximately equal number of events in each group. Recalling the definition of χ^2 of a variable x

$$\chi^2 = \sum_i \frac{[x_i\,(\text{experimental}) - x_i\,(\text{theoretical})]^2}{x_i\,(\text{theoretical})}$$

we find from Fig. 13 $\chi^2 \approx 3$ for three degrees of freedom. This gives a probability of about 40%, so that our experimental mass distribution should have $\chi^2 > 3.0$. In other words, if our sample of events is taken from a universe which follows the laws of phase space there is 40% probability that new measurements performed on an equivalent number of events will give a distribution which deviates even more from the theoretical curve.

Correspondingly we find from Fig. 14 $\chi^2 \approx 10$ for four degrees of freedom, i.e. a probability of about 2%. It is therefore rather unlikely that the $\Xi^-\pi^{0,+}$ mass distribution follows phase space.

We interpret that data in Fig. 14 as a likely production of two $\Xi\pi$ resonances; one is the well-established Ξ_δ with mass 1.53 GeV, the other is a possible resonance at about 1.75 GeV.

The phase space effective mass distributions of the $\Xi\pi\pi$ system from the same 4 and 5-body final states are shown in Fig. 15.

The effect of a resonance on the effective mass distribution is illustrated in Fig. 16 for the following reactions

$$K^- + N \rightarrow K^* + \Xi^-$$
$$\quad\quad\quad\quad \hookrightarrow K + \pi$$

and

$$K^- + N \rightarrow K^* + \Xi^- + \pi^0$$
$$\quad\quad\quad\quad \hookrightarrow K^0 + \pi^+ \quad .$$

Fig. 16 illustrates both the $\Xi^-\pi^0$ and $\Xi^-\pi^+$ effective mass distribution from the last configuration.

c) Angular distribution

To illustrate the use of the formulae evaluated in Chapter IV we have calculated the angular distribution of two pions from the τ-decay. This example is particularly simple since as a good approximation we can do the calculation for three non-relativistic particles of equal mass. The result is shown in Fig. 17.

Exercise

Show that for three non-relativistic particles of equal mass, the integration limits of p_3 in Eq.(17) will be given by

$$p_3(\text{min}) = 0$$
$$p_3(\text{max}) = \left(\frac{4mE}{4 - \cos^2\Theta} \right)^{1/2}$$

where m = mass of any of the three particles. Show that the angular distribution is independent of the mass of the particles and of the total energy of the system E.

d) Dalitz plot

In Fig. 18 is drawn the contour of a $T_{\pi^+} + T_{\pi^-}$ Dalitz plot for the reaction

$$K^- + p \rightarrow \Lambda + \pi^+ + \pi^-$$

with total energy in the centre of mass of about 2.02 GeV. This reaction has been studied at CERN in a 30 cm hydrogen bubble chamber (Cooper et al. 1963 Sienna International Conference on Elementary Particles) using a separated K^- beam with momentum 1.45 GeV/c.

Fig. 18 shows the result from an analysis of 582 events, and reveals an example of a non-uniform distribution of points. There is a marked clustering of points for both a constant T_{π^-} and a constant T_{π^+} value indicating the production of $\Lambda\pi^+$ and $\Lambda\pi^-$ resonances respectively, i.e. $Y_1^{*\pm}(1385)$.

Figure 18 shows also a tendency that the points within a resonance band may not be evenly distributed along the axis. For the Y_1^{*+} there seems to be more points for high T_{π^-} values etc. One explanation of this is some production of the ρ^0 meson. We leave as an exercise the finding of the band on the plot within which one should expect points due to the meson resonance $\rho^0 \rightarrow \pi^+ + \pi^-$.

Acknowledgement

The author would like to thank his colleagues at CERN and Oslo for many valuable comments. In particular cand. real. A.G. Frodesen is acknowledged for many clarifying discussions and suggestions of improvement.

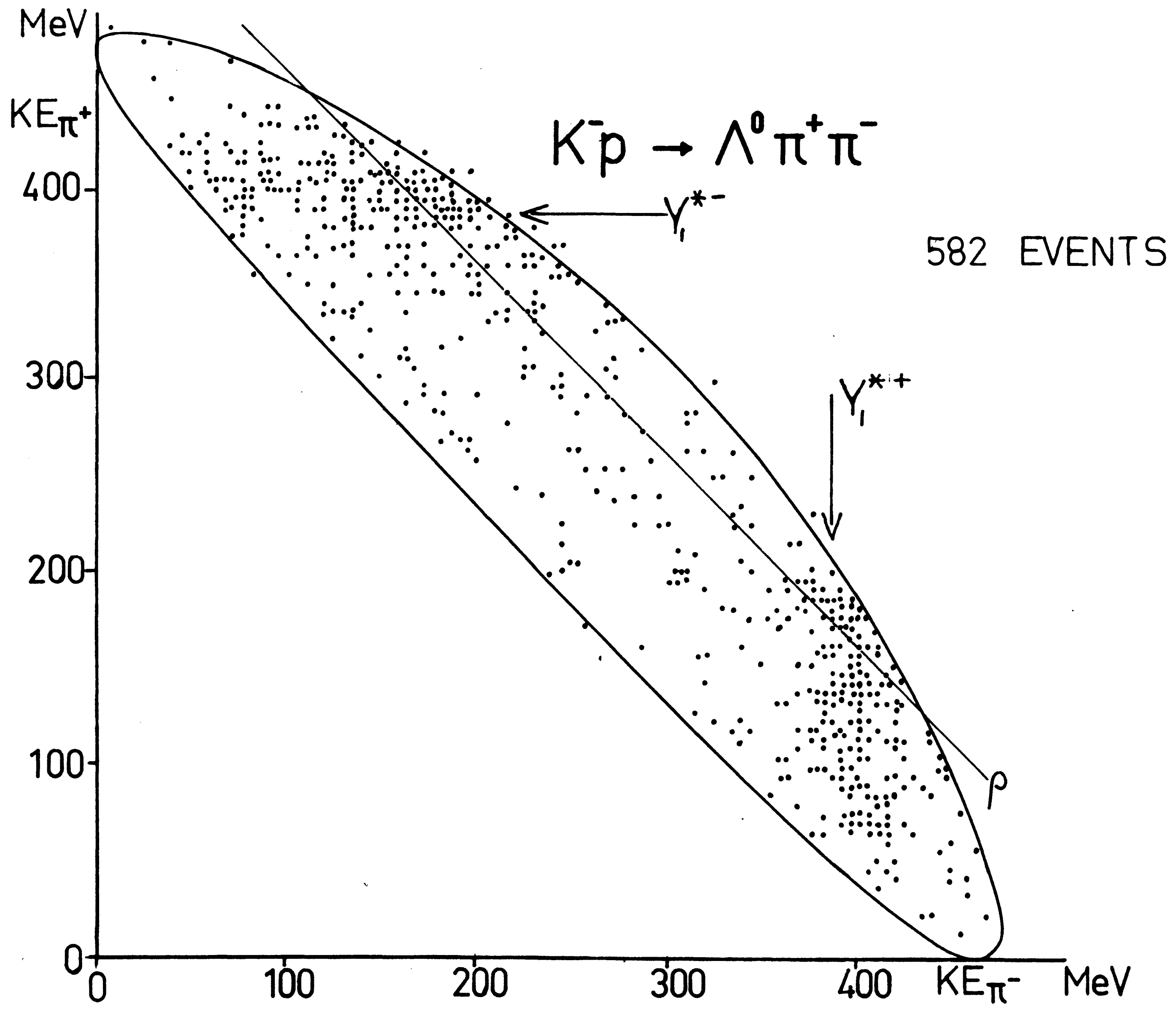

Fig.18

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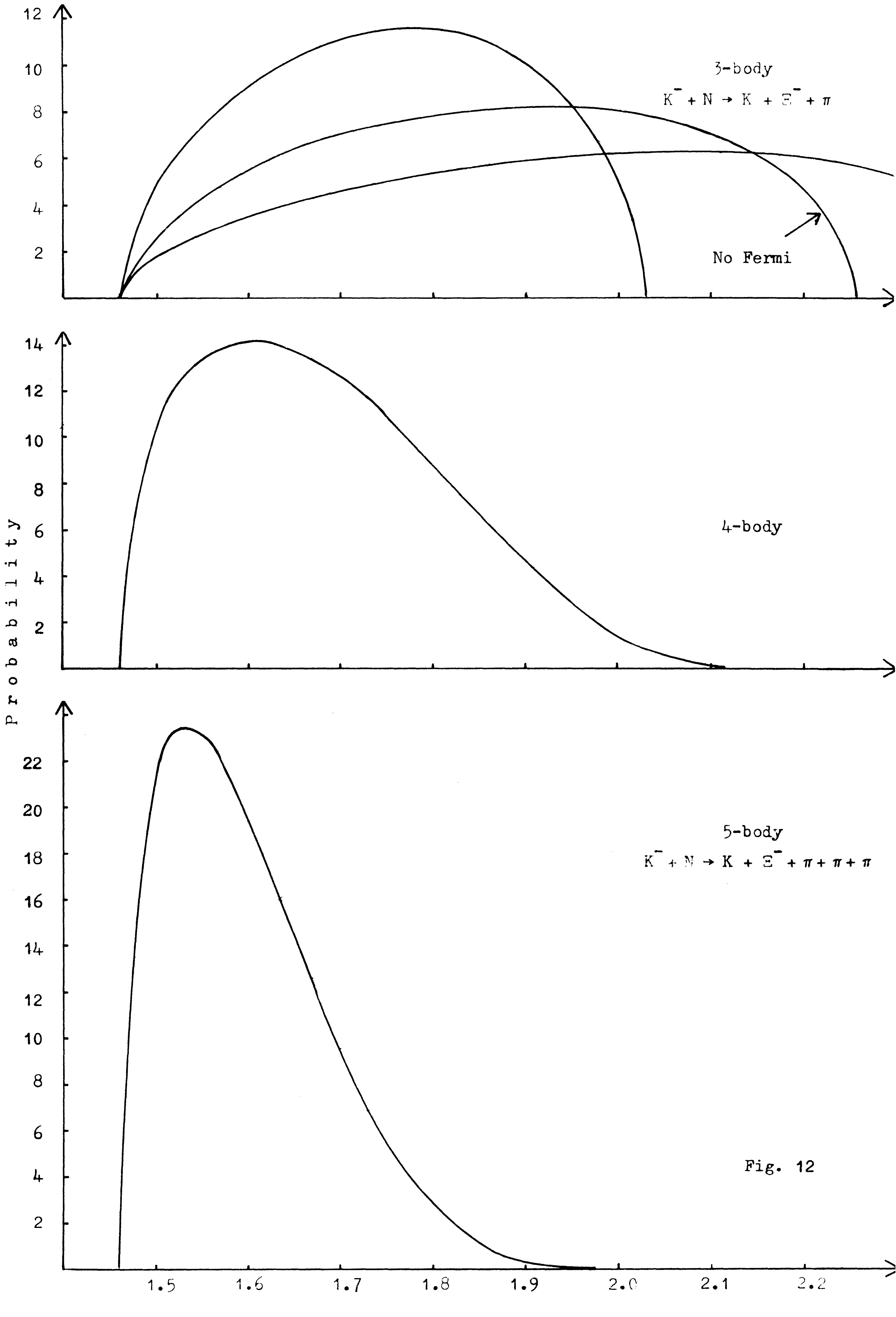
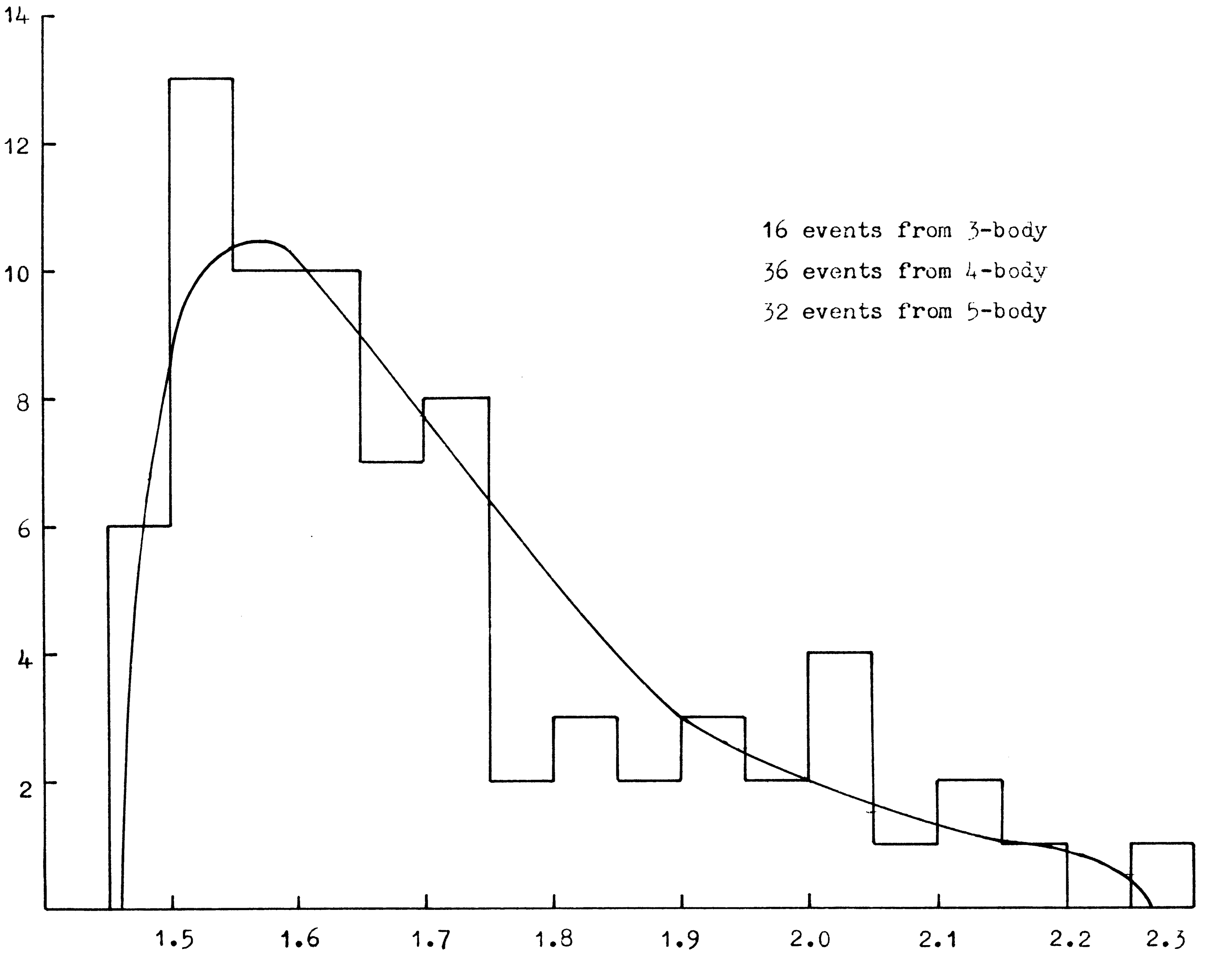


3-body
$K^- + N \rightarrow K + \Xi^- + \pi$
No Fermi
4-body
5-body
$K^- + N \rightarrow K + \Xi^- + \pi + \pi + \pi$
Probability
1.5
1.6
1.7
1.8
1.9
2.0
2.1
2.2

Fig. 12



Effective Mass $\Xi^-\pi^-$ in GeV

Fig. 13

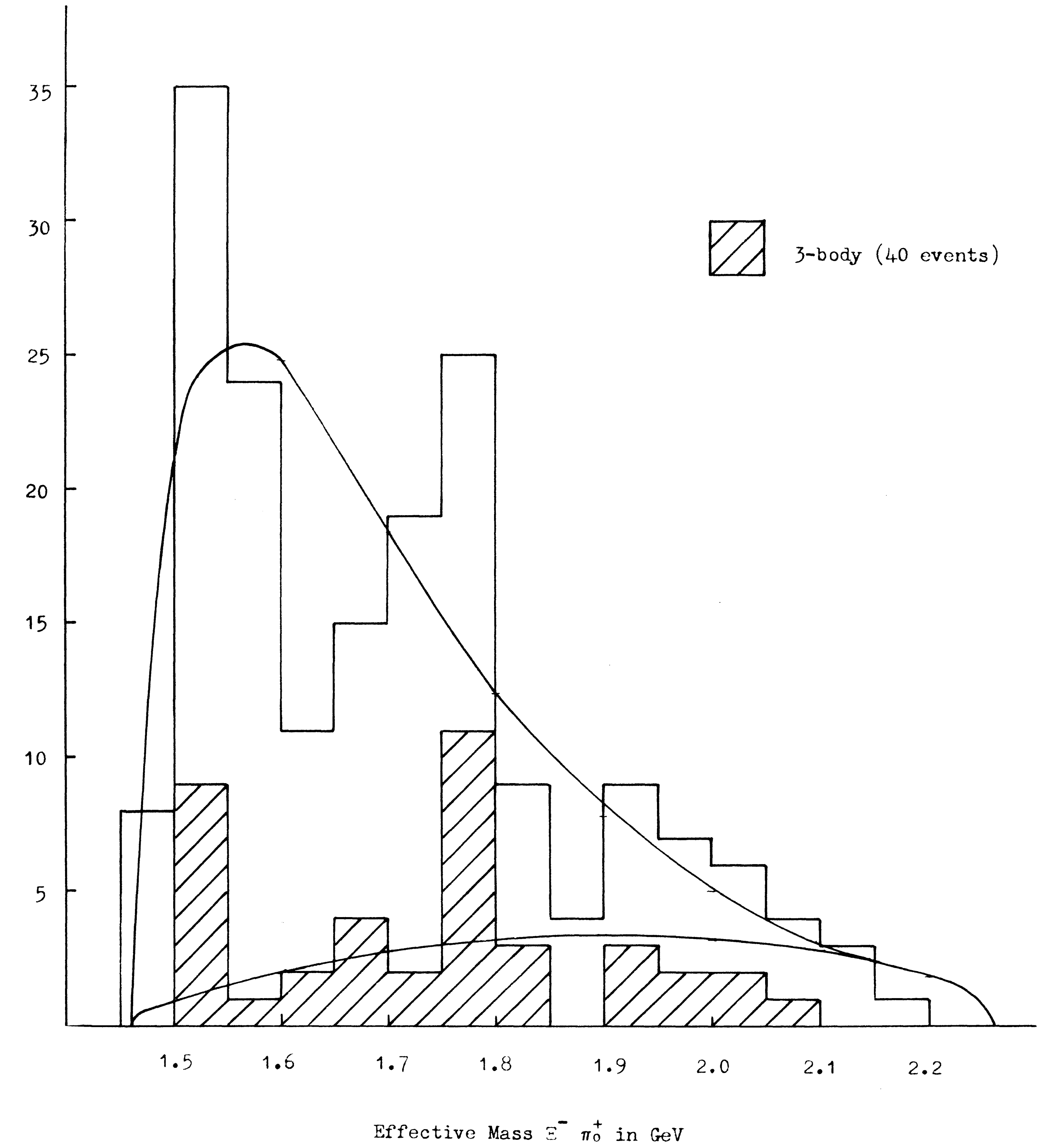

35
30
25
20
15
10
5
3-body (40 events)
1.5
1.6
1.7
1.8
1.9
2.0
2.1
2.2
Effective Mass $\Xi^- \pi_0^+$ in GeV

Fig. 14

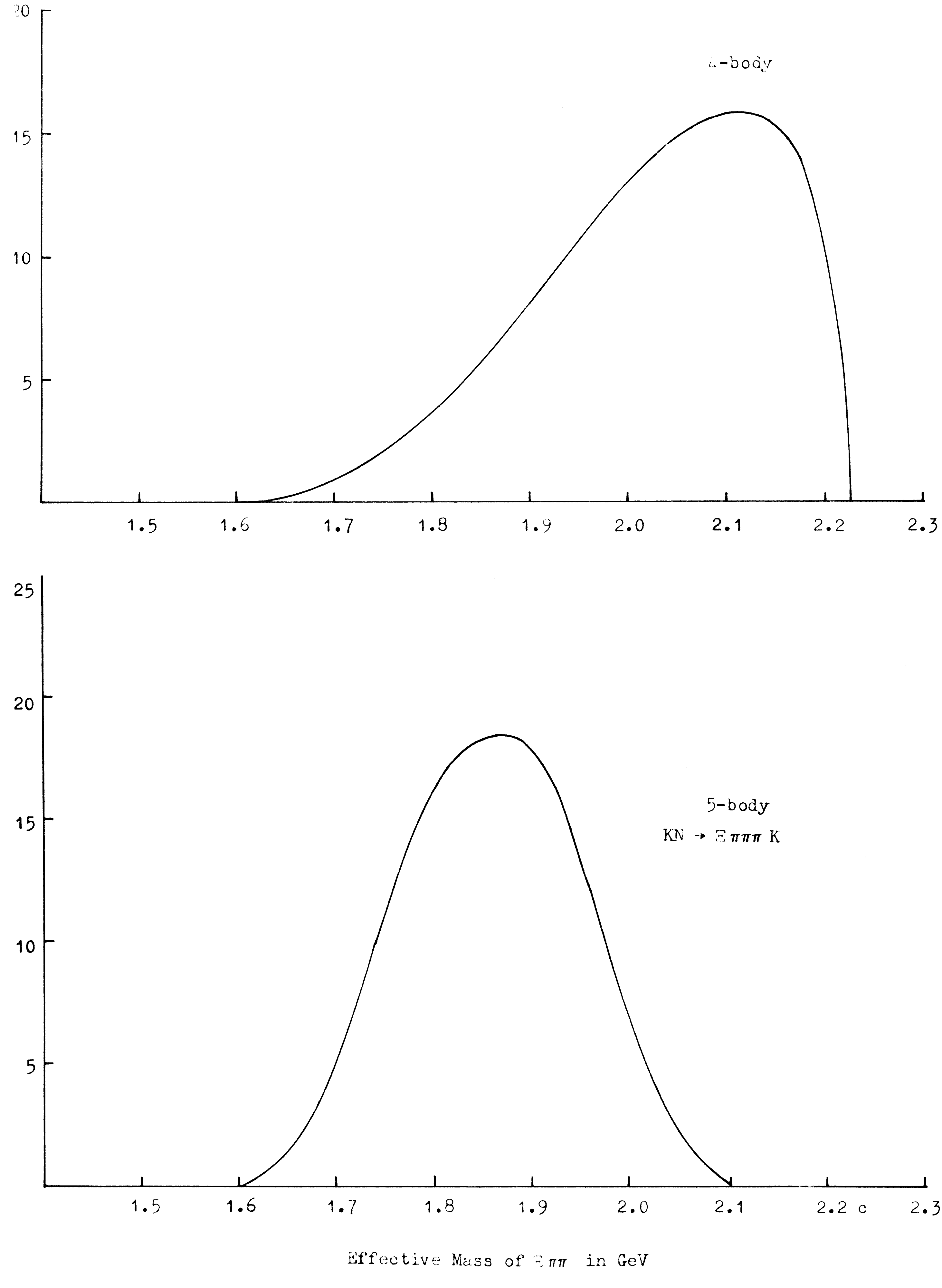
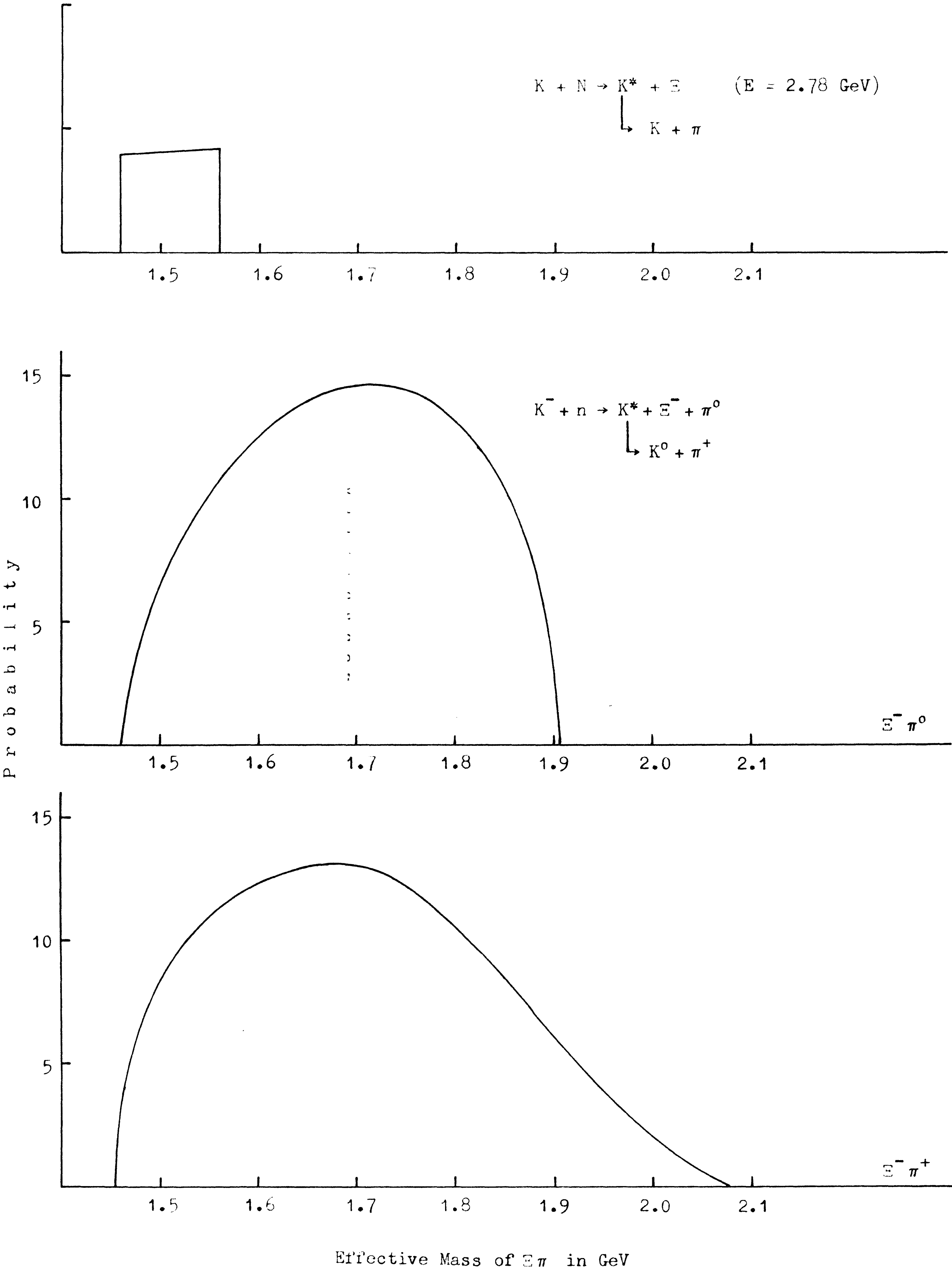
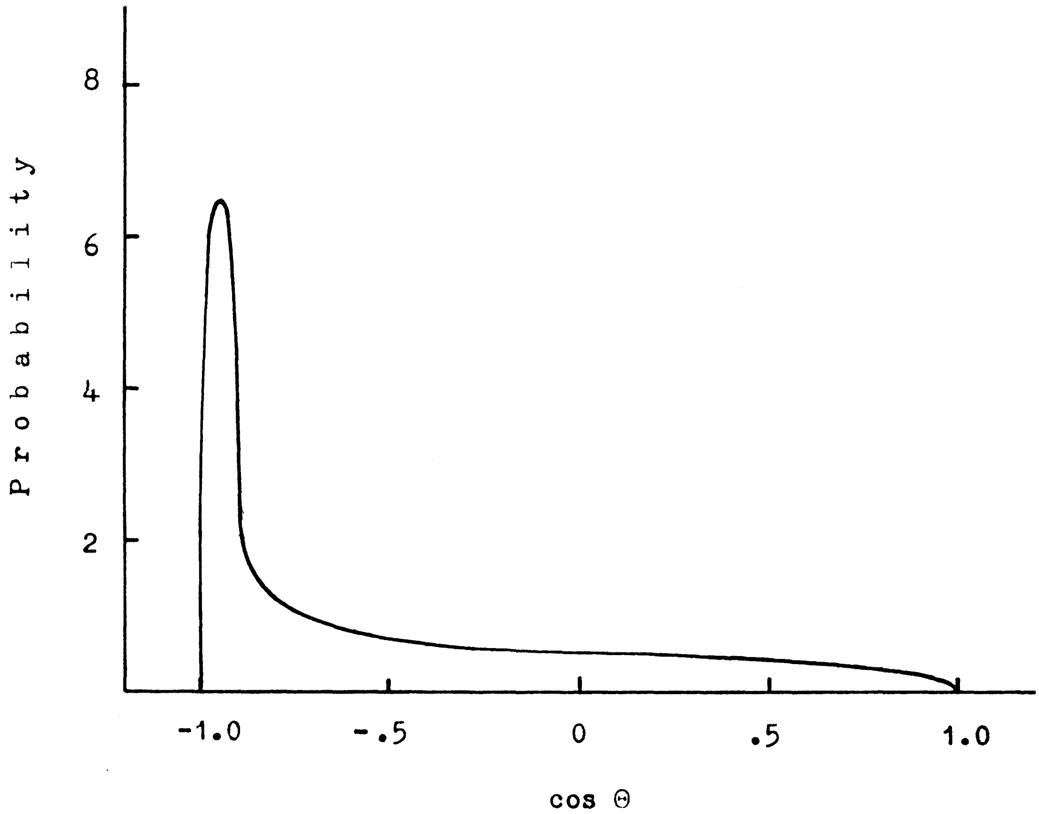

20
15
10
5
4-body
1.5
1.6
1.7
1.8
1.9
2.0
2.1
2.2
2.3
25
20
15
10
5
5-body
KN → Ξ πππ K
1.5
1.6
1.7
1.8
1.9
2.0
2.1
2.2 c
2.3
Effective Mass of Ξ ππ in GeV

Fig. 15



K + N → K* + Ξ (E = 2.78 GeV)
↳ K + π
1.5 1.6 1.7 1.8 1.9 2.0 2.1
15
10
5
K⁻ + n → K* + Ξ⁻ + π⁰
↳ K⁰ + π⁺
Ξ⁻ π⁰
1.5 1.6 1.7 1.8 1.9 2.0 2.1
Probability
15
10
5
Ξ⁻ π⁺
1.5 1.6 1.7 1.8 1.9 2.0 2.1
Effective Mass of Ξπ in GeV

Fig. 16



Angular Distribution in τ decay

Fig. 17

APPENDIX

We define the Dirac δ function by the equations

$$\left.\begin{array}{ll} \delta(x) = 0 & \text{for } x \neq 0 \\ \lim\limits_{x\to 0} \delta(x) \to \infty & \text{in such a way that } \int \delta(x)\, dx = 1 \end{array}\right\} \qquad (i)$$

for all integrations enclosing the origin.

From this definition it follows that

$$\int_a^b f(x)\, \delta(x-c)\, dx = \begin{cases} f(c) & \text{for } a < c < b \\ 0 & \text{for } c < a \text{ or } c > b. \end{cases} \qquad (ii)$$

Correspondingly if we define a three-dimensional δ function $\delta(\vec{r})$ as

$$\delta(\vec{r}) = \delta(x)\, \delta(y)\, \delta(z)$$

the integration over a volume V gives

$$\int_V f(\vec{r})\, \delta(\vec{r}-\vec{r}_0)\, d\vec{r} = \begin{cases} f(\vec{r}_0) & \text{if } \vec{r}_0 \text{ lies inside V} \\ 0 & \text{if } \vec{r}_0 \text{ lies outside V.} \end{cases}$$

In the case where the argument of the δ function is itself a function, like $\delta(\Phi(x))$, we find by substitution

$$\int \delta[\Phi(x)]\, dx = \int \delta(y)\, \frac{dy}{|\Phi'(x)|}$$

where the absolute value is necessary to ensure that $dx = \frac{dy}{|\Phi'(x)|}$ is always positive.

From the above follows

$$\int \delta[\Phi(x)]\ dx = \frac{1}{|\Phi'(x_0)|} \text{ if } \Phi(x_0) = 0 . \qquad \text{(iii)}$$

The equation (iii) can also be generalized, using Eq. (ii), as

$$\int g(x)\ \delta[\Phi(x)]\ dx = \frac{g(x_0)}{|\Phi'(x_0)|} \text{ if } \Phi(x_0) = 0 .$$

This rule is valid for all functions $g(x)$ which are continuous at $x = x_0$.

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SYMBOLS USED

$\vec{P}$ = Momentum of the system in initial state.

E = Total energy of the system in initial state. It will be evident from the context whether this is in the laboratory or centre of mass system.

E_i = Total energy of i'th particle in final state.

m_i = Rest mass of i'th particle.

T_i = Kinetic energy of i'th particle in final state.

$\vec{p}_i$ = Momentum of i'th particle in final state.

M_{ij} = Effective mass of two particles i and j.

${}^{n}_{k}M$ = Effective mass of the first k particles taken from a system of n particles.

q_i = Four vector = $(\vec{p}_i, E_i)$ of length $q_i^2 = E_i^2 - p_i^2$.

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