INCLUSIVE PROCESSES AT HIGH ENERGIES

V.V. Ezhela, A.A. Logunov, M.A. Mestvirishvili and V.A. Petrov
Institute for High Energy Physics, Serpukhov, USSR.
(Presented by A.A. Logunov).

Introduction

For the first time in papers [I,2] the behaviour of some physical characteristics of inelastic processes

$$a + b \rightarrow c + \cdots$$

with a detected particle " C " were investigated. Later on [3] these processes were called inclusive ones. Unitarity properties and analytic conditions in angular variables allowed one to draw definite conclusions [4] on asymptotic behaviour of differential cross-section $d^2\sigma_{ab\to cd}/d\cos\theta\, d\varphi$ for the process:

$$a + b \rightarrow c + d + \cdots$$

Basing on the results obtained in paper [4], it was shown (see [5]) that pionization, if it does exist, is accompanied with a very large mean multiplicity.

New physical characteristics (see [5]) for inclusive processes are being discussed in the present paper. They are as follows: average multiplicity of particles with bounded momenta in the c.m.s., at large angles, at small angles, etc.

The paper presents an analysis of the hypothesis on pionization [6], scale invariance [7,3,8] and limited fragmentation [9] in terms of analyticity and unitarity. It is shown, that if the hypothesis of scale invariance takes place, then analyticity and unitarity conditions lead to a very large (maximum) mean multiplicity at large angles θ and φ. In this the differential cross-section cannot decrease with energy increase more rapidly than $\frac{1}{s}$.

Analyticity results in the fact, that either there exists a diffraction domain in asymuthal angle φ, shrinking with energy increase, or if there is no diffraction domain, then scale invariance is violated at large angles θ and φ and pionization does not take place.

If for any kind of particles the multiplicity for large angles slowly grows with energy, then scale invariance is being violated for this kind of particles.

In Sec. II, proceeding from analyticity, we have improved the bound for differential cross-section of an inclusive process, that was earlier obtained in paper/4/. It should be noted, that this bound is obtained in weaker initial assumptions on analyticity, as compared with paper/4/. It is also shown, that the bound obtained cannot be improved under such conditions in the sense of power dependence on energy.

PART I

PARTICLE PRODUCTION AT LARGE ANGLES AND HYPOTHESIS ON SCALE INVARIANCE

§ I. Physical Characteristics for Inclusive Processes

To describe inclusive processes:

$$a + b \to c_1 + \cdots + c_m + \cdots \qquad \mathrm{I}$$

it is convenient to introduce the following characteristics:

The total cross-section for an inclusive process

$$\sigma_{ab\to c_1\cdots c_m}(s) = \sum_j \sigma^j_{ab\to c_1\cdots c_m\cdots}(s) \qquad (\mathrm{I})$$

where summing spreads over all the channels of reaction I,which contain particles C_I, C_2 ... C_m in the final state, and the differential cross-section for process I

$$\frac{d\sigma_{ab\to c_1\cdots c_m}}{d^3k_{c_1}\cdots d^3k_{c_m}} = \sum_j \frac{d\sigma^j_{ab\to c_1\cdots c_m\cdots}}{d^3k_{c_1}\cdots d^3k_{c_m}} \qquad (2)$$

In (I) and (2)

$$\sigma^j_{ab\to c_1\cdots c_m\cdots}(s) \quad \text{and} \quad \frac{d\sigma^j_{ab\to c_1\cdots c_m\cdots}}{d^3k_{c_1}\cdots d^3k_{c_m}}$$

are the total and differential cross-section for the j-channel of reaction I, in this

$$\sigma^j_{ab\to c_1\cdots c_m\cdots}(s) = \int d^3k_{c_1}\cdots d^3k_{c_m}\frac{d\sigma^j_{ab\to c_1\cdots c_m\cdots}}{d^3k_{c_1}\cdots d^3k_{c_m}}(s) \quad (3)$$

Here

$$d^3k_{c_i} = \frac{d\vec{k}_{c_i}}{(2\pi)^3 2E_{c_i}}$$

We will also introduce the particle momentum distribution

$$f_{ab\to c_1\cdots c_m}(s,\vec{k}_{c_1},\dots,\vec{k}_{c_m}) = \sum_j n^j_{c_1}\cdots n^j_{c_m}\frac{d\sigma^j_{ab\to c_1\cdots c_m\cdots}}{d^3k_{c_1}\cdots d^3k_{c_m}} \qquad (4)$$

where $n^j_{c_i}$ is the number of the particles of c_i kind, produced in the j-channel of reaction I.

The momentum distribution is normalized in the following way:

$$\int f_{ab\to c_1\cdots c_m}(s,\vec{k}_{c_1},\dots,\vec{k}_{c_m})d^3k_{c_1}\cdots d^3k_{c_m} = \langle n_{c_1}\cdots n_{c_m}\rangle \sigma_{ab\to c_1\cdots c_m}(s) \quad (5)$$

where $\langle n_{c_1}\cdots n_{c_m}\rangle$ is the mean value of the product of multiplicities for the particles of $c_1, c_2,\dots,c_m$ kinds, produced in reaction I. From here, in particular

$$\int f_{ab\to c}(s,\vec{k}_c)d^3k_c = \langle n_c\rangle \sigma_{ab\to c}(s) \qquad (6)$$

where $\langle n_c\rangle$ is mean multiplicity of the particles of "C" kind.

$$\int f_{ab\to cd}(s,\vec{k}_c,\vec{k}_d) = \langle n_c n_d\rangle \sigma_{ab\to cd}(s) \qquad (7)$$

Generally speaking, generation of the particles $c_1\cdots c_m$ in reaction I may be different in various parts of the phase space (e.g. with different production mechanisms - pionization, fireballs, etc.) besides in different parts of the phase space the momentum distribution may be of different asymptotic behaviour.

When studying such a characteristic as mean multiplicity, where integration is performed in the whole phase space, the effects connected with different particle generation mechanisms are fully smeared off. However, if we treat mean multiplicity in a certain subdomain of phase space, then even this mean multiplicity alone can supply us with a more detailed information on the production mechanism of detected particles in inclusive process (I).

Let us give a definition of mean multiplicity in some domain V of the phase space:

Let momentum distribution (4) be expressed in some independent variables s , $\xi_1, \ldots, \xi_\nu$ and let V be some subdomain of the phase space, than it is easy to see, that:

$$\int\limits_V f_{ab\to c_1\cdots c_m}(s,\xi_1,\ldots,\xi_\nu)\,dv = \langle n_{c_1}\cdots n_{c_m}\rangle_V\, \sigma_{ab\to c_1\ldots c_m}(s,V) \quad (8)$$

where

$$\sigma_{ab\to c_1\cdots c_m}(s,V) = \int\limits_V \frac{d\sigma_{ab\to c_1\cdots c_m}}{d^3k_{c_1}\cdots d^3k_{c_m}}\,dv\,, \quad (9)$$

$$dv = J(\xi_1,\ldots,\xi_\nu;s)\,d\xi_1\ldots d\xi_\nu$$

$J(\xi_1,\ldots,\xi_\nu;s)$ is a Jacobian of transition from variables $\vec{k}_{c_i}$ to ξ_α. If the whole phase volume is devided in some way into R subdomains V_r , then:

$$\int f_{ab\to c_1\cdots c_m}(s,\xi_1,\ldots,\xi_\nu)\,dv = \sum_{r=1}^{R}\langle n_{c_1}\cdots n_{c_m}\rangle_{V_r}\,\sigma_{ab\to c_1\cdots c_m}(s,V_r) \quad (IO)$$

In conclusion we will drow out some general relations (sum rules) for inclusive reactions.

Let us consider some definite reaction channel:

$$a+b\to c+d+e+f+\cdots$$

then the energy conservation law for momentum will be:

$$p_a+p_b=\sum_{\nu=1}^{N_j}k_\nu$$

where N_j is the number of particles in the final state. The total cross-section for this reaction channel will be written in the form:

$$\sigma^j(s) = \frac{(2\pi)^4}{4\sqrt{s}\,|\vec{p}_a|}\int\prod_{i=1}^{n_c^j} d^3k_c^i \prod_{i=1}^{n_d^j} d^3k_d^i \cdots |\langle ab|T^j|cd\cdots\rangle|^2 \delta^4(p_a+p_b-\sum_\nu k_\nu)$$

Let us consider the magnitude:

$$(p_a+p_b)_\mu \sigma^j(s) = \int\Big(\sum_\nu k_\nu\Big)\prod_{i=1}^{n_c^j} d^3k_c^i \prod_{i=1}^{n_d^j} d^3k_d^i \cdots |\langle ab|T^j|cd\cdots\rangle|^2 \delta^4(p_a+p_b-\sum_\nu k_\nu)\frac{(2\pi)^4}{4\sqrt{s}\,|\vec{p}_a|}$$

We will devide the sum of the momenta of the particles in the final state into the momentum sums according to the kinds of particles:

$$\sum_{\nu=1}^{N_j} k_\nu = \sum_{i=1}^{n_c^j} k_c^i + \sum_{i=1}^{n_d^j} k_d^i + \cdots$$

Substituting this expression into the foregoing formula we will obtain:

$$(p_a+p_b)^\mu \sigma^j(s) = n_c^j \int d^3k_c \frac{d\sigma^j}{d^3k_c} k_c^\mu + n_d^j \int d^3k_d \frac{d\sigma^j}{d^3k_d} k_d^\mu + \cdots$$

From here summing in all the reaction channels and taking into account (4) we find the sum rule in the form:

$$(p_a+p_b)^\mu \sigma_{tot}(s) = \int d^3k_c\, k_c^\mu f_{ab\to c}(s,\vec{k}_c) + \int d^3k_d\, k_d^\mu f_{ab\to d}(s,\vec{k}_d) + \cdots \qquad \text{(II)}$$

For the furthergoing it is useful to write down the sum rule in some other form. Let E be the energy of an incident hadron, M_t be the target mass, then in the lab system, using energy conservation law, we will write down the equality:

$$\sum_{\nu=1}^{N_j} (E_k - k_{\parallel})_\nu = M_t + E - |\vec{p}_a|$$

At high energies this expression is of the following form:

$$\sum_{\nu=1}^{N_j} M_t^{-1} (E_k - k_{\parallel})_\nu = 1$$

Repeating all the foregoing reasonings, carried out when obtaining relation (II) we will find:

$$\sigma_{tot}(s) = \int d^3k_c \frac{E_c - k_{\parallel}^c}{M_t} f_{ab\to c}(s,\vec{k}_c) + \int d^3k_d \frac{E_d - k_{\parallel}^d}{M_t} f_{ab\to d}(s,\vec{k}_d) + \cdots \qquad (12)$$

Here summation is performed over all the kinds of particles, which may be produced in process (I).

Sometimes the distribution function $f(s,\vec{k}_c)$ is normalized as follows:

$$\int d^3k_c\, f(s,\vec{k}_c) = \langle n_c \rangle_t\, \sigma_{tot}(s) \qquad (13)$$

$\sigma_{tot}(s)$ – total cross-section for interaction

$\langle n_c \rangle_t$ – mean multiplicity for the particles of "c" kind, taken for all the opened reaction channels.

This normalization differs from the one, presented before (see formula 6). The mean multiplicity $\langle n_c \rangle$, derived above, was taken for all the channels of the reaction, that contain the particle "c" in the final state. If using normalization (13) then relation (II) can be presented in the form

$$(p_a + p_b)^\mu = \langle n_c \rangle_t \overline{k_c^\mu} + \langle n_d \rangle_t \overline{k_d^\mu} + \cdots$$

where $\overline{k_c^\mu}$ is the mean value for the projection of the " C " type particle momentum.

If we had not included the elastic channel, then relation (II) would have been of the form:

$$(p_a+p_b)^\mu \sigma_{inel}(s) = \sum_\nu \int d^3k_\nu \, k_\nu^\mu f(s,\vec{k}_\nu)$$

From here in the c.m.s. we will have:

$$\sqrt{s}\,\sigma_{inel}(s) = \sum_\nu \int d^3k_\nu \, E_\nu f(s,\vec{k}_\nu) \qquad \text{(IIa)}$$

If we introduce mean multiplicity for the particles of " ν " kind in all the inelastic channels:

$$\langle n_\nu \rangle_i = \frac{1}{\sigma_{inel}} \int d^3k_\nu \, f(s,\vec{k}_\nu)$$

and determine the energy mean value for the particles of " ν " kind:

$$\overline{E}_\nu = \left[\int d^3k_\nu f(s,\vec{k}_\nu) \right]^{-1} \int d^3k_\nu E_\nu f(s,\vec{k}_\nu)$$

then relation (IIa) will be written in the form:

$$\sqrt{s} = \sum_\nu \overline{E}_\nu \langle n_\nu \rangle_i \qquad \text{(IIb)}$$

The magnitude equal to

$$\mathcal{K}_\nu = \frac{1}{\sqrt{s}\,\sigma_{inel}} \int d^3k_\nu E_\nu f(s,\vec{k}_\nu) = \frac{\overline{E}_\nu}{\sqrt{s}} \langle n_\nu \rangle_i$$

defines the fraction of the total energy, carried away by all the particles of " ν " kind and is called partial inelasticity.

§2. Hypothesis on Limiting Fragmentation (HLF)

In paper[9] it is assumed, that at high energies there takes place the following particle production mechanism in collision due to meza-like continuous structure of hadrons.

In the lab system (where one of hadrons is at rest) in the collision process there are produced particles with bounded momenta, independent of energy (target fragments) and particles, whose momentum increases alongside with the total energy (incident hadron fragments).

It is assumed, that at $s \to \infty$ the momentum distribution of the target fragments tend to some limiting distributions different from 0 and independent of the total energy of the system. It should be noted, that the fragments of an incident particle are not included into any limiting distribution. Study of incident particle fragments requires a consideration of a projectile coordinate system (in which an incident hadron is at rest).

Similar to the foregoing it is assumed, that in the projectile system there exist limiting distributions for fragments of incident hadrons at $s \to \infty$.

For example, in the lab system

$$\lim_{s\to\infty} W_1(\vec{p}_c; s)\, d^3p_c = f(\vec{p}_c)\, d^3p_c$$

$$\lim_{s\to\infty} W_n(\vec{p}_1, \dots, \vec{p}_n; s)\, d^3p_1 \cdots d^3p_n = f(\vec{p}_1, \dots, \vec{p}_n)\, d^3p_1 \cdots d^3p_n \tag{I4}$$

in accordance with the HLF the momentum distribution, for example, for one or two particles are the functions in the form:

$$f(p_{\parallel}^c, \vec{p}_{\perp}^{\,c}) \ , \ f(p_{\parallel}^c, \vec{p}_{\perp}^{\,c}; p_{\parallel}^d, \vec{p}_{\perp}^{\,d})$$

where $\vec{p}^{\,c}$ and $\vec{p}^{\,d}$ are momenta for the particles of "c" and "d" kind in the lab system. For the sake of convinience we will introduce the variable

$$x = \frac{2p_{\parallel}^{*}}{\sqrt{s}}$$

where $p_{\parallel}^{*}$ is a longitudinal momentum in the c.m.s. Using the Lorentz transformation this variable can easily be expressed via a lonitudinal momentum $p_{\parallel}$ in the lab system:

$$x = \frac{2}{\sqrt{s}}\left[p_{\parallel} ch u - E_p sh u \right]$$

Here

$$sh u = \frac{p_b^{*}}{M_t} \simeq \frac{\sqrt{s}}{2M_t}, \quad E_p = \sqrt{p_{\parallel}^2 + \vec{p}_{\perp}^{\,2} + m^2}$$

If s is large, and $p_{\parallel}$ is fixed, then:

$$x \simeq M_t^{-1}(p_{\parallel} - E_p)$$

M_t is the mass target. Going from the variable $p_{\parallel}$ to x, the limiting distribution for the functions, e.g. of one or two particles may be written down in the form

$$f(x_c, \vec{p}_{\perp}^{\,c}), \quad f(x_c, \vec{p}_{\perp}^{\,c}; x_d, \vec{p}_{\perp}^{\,d})$$

Thus, we easily see, that from the HLF there quite naturally follows the scale invariance in longitudinal momentum, i.e. the momentum distributions depend on the incident particle energy via the variable x. In our consideration $x < 0$ in the lab system. In transition to the projectile system the variable x will be positive. We would like to underline, that the HLF does not contain the value $x = 0$.

§ 3. Hypothesis on Scale Invariance

In papers[7,3] the hypothesis on Scale Invariance has been put forward on the basis of model treatments. The essence of this hypothesis lies in the fact, that the momentum distribution, e.g. for one or two particles at tending to infinity is of the form:

$$f(x,\vec{p}_\perp^{\,c}),\ f(x_c,\vec{p}_\perp^{\,c};\,x_d,\vec{p}_\perp^{\,d}) \tag{I5}$$

i.e. momentum distributions do not explicitly depend on energy. This hypothesis includes the point $x=0$ as well. Various aspects of this hypothesis have been considered in papers[8]. Carrying out a comparison between the hypothesis on scale invariance and the HLF we can see that they lead to similar results, if $x<0$ or $x>0$. The point $x=0$ does not correspond to any bounded momentum either in the lab system or in the projectile system, it corresponds in particular to the bounded momenta in the c.m.s.

Let us express the mean multiplicity via the momentum distribution $f(x,\vec{p}_\perp)$

$$2(2\pi)^3\sigma_{ab\to c}\cdot\langle n_c\rangle = \mathcal{J}(s) \tag{I6}$$

where

$$\mathcal{J}(s)=\int d\vec{p}_\perp dx\,\frac{f(x,\vec{p}_\perp)}{\sqrt{x^2+4s^{-1}(m_c^2+\vec{p}_\perp^{\,2})}}\,. \tag{I7}$$

As the function $f(x,\vec{p}_\perp)$ is bounded, the integral $\mathcal{J}(s)$ in the domain asymptotic in S increases with energy not more rapidly than:

$$\mathcal{J}(s) \leqslant const \, \ln \frac{s}{m_c^2} \tag{18}$$

If the function $f(x,\vec{p}_\perp)$ at $x \to 0$ decreases so that the integral

$$\int \frac{dx}{x} f(x,\vec{p}_\perp)$$

converges, then the function $\mathcal{J}(S)$ is independent of energy i.e. constant.

E.g. the integral $\mathcal{J}(s)$ will be a constant, for the function $f(x,\vec{p}_\perp)$ that at $\varepsilon > 0$ satisfies condition:

$$f(x,\vec{p}_\perp) \leqslant \frac{\varphi(\vec{p}_\perp)}{[\ln(1/|x|)]^{1+\varepsilon}}$$

If at $x \to 0$ the function $f(x,\vec{p}_\perp)$ decreases slowly enough or tends to a constant value, then, generally speaking, the integral $\mathcal{J}(s)$ will increase alongside with S. If e.g.

$$f(x,\vec{p}_\perp) = \frac{\varphi(\vec{p}_\perp)}{|\ln\ln(1/|x|)| + C}, \quad x \to 0,$$

then

$$\mathcal{J}(S) \geqslant const \, \ln \frac{s}{m_c^2} \cdot \frac{1}{\ln\ln \frac{S}{m_c^2}}$$

If $f(0,\vec{p}_\perp) \neq 0$ and at $x \to 0$ the function $f(x,\vec{p}_\perp)$ tends to the value $f(0,\vec{p}_\perp)$ rapidly enough, then the function $\mathcal{J}(s)$ equals:

$$\mathcal{J}(s) = a_c \ln \frac{S}{m_c^2} + b_c \tag{19}$$

where a_c and b_c are some constants.

For example:

$$a_c = \int d\vec{p}_\perp \, f(0,\vec{p}_\perp)$$

Thus, the integral $\mathcal{J}(s)$ is bounded from top and bottom with inequalities

$$0 < const \leqslant \mathcal{J}(s) \leqslant const \, \ln \frac{s}{m_c^2} \tag{20}$$

From (I6) and inequality (20) it follows, that inclusive cross-section $\sigma_{ab\to c}(s)$ cannot increase with energy more rapidly than $\ln \frac{s}{m_c^2}$. Due to energy conservation law mean multiplicity cannot increase more rapidly than $\frac{\sqrt{s}}{m_c}$, whereof it follows, that on the basis of (20) inclusive cross-section cannot decreases with s more rapidly than

$$\sigma_{ab\to c}(s) \geqslant \frac{const}{\sqrt{s}} \tag{2I}$$

§ 4. Production of Particles with Bounded Momenta and Pionization

Lately a wide discussion is being held on the so-called pionization phenomenon. Paper[6] e.g. gives the following description of pionization effects:

If in the high energy limits there are produced particles with bounded momenta (independent of initial energy) in the c.m.s. and their momentum distribution differs from zero at s, tending to infinity, i.e.

$$\lim_{s\to\infty} f_{ab\to c_1\cdots c_m}(s,\vec{k}_{c_1},\dots,\vec{k}_{c_m}) = f_{ab\to c_1\cdots c_m}(\vec{k}_{c_1},\dots,\vec{k}_{c_m}) \neq 0 \tag{22}$$

then such a phenomenon is called pionization.

Here, using two-particle momentum distribution $f_{ab\to cd}(s,\vec{k}_c,\vec{k}_d)$ we will tackle the problem of the consistency of hypothesis on existence of pionization with bounds for the inclusive cross-section, established from unitarity and analyticity.

In Sec. 2 (see 68) for an inclusive process

$$a+b\to c+d+\cdots \tag{23}$$

there has been obtained the following inequality

$$\frac{d^2\sigma_{ab\to cd}}{d\cos\theta\, d\varphi}\leqslant const\,\frac{\ln^{\gamma}(s/s_0)}{s\,\sin^{\alpha}\theta\,|\sin\varphi|^{\beta}} \tag{24}$$

where $\alpha=4$, $\beta=5{,}5$, $\gamma=9$.

Here θ is the angle between momenta $\vec{p}_a$ and $\vec{k}_c$, and φ is the angle between planes ($\vec{k}_c$, $\vec{p}_a$) and ($\vec{k}_c$, $\vec{k}_d$). Everything is treated in the c.m.s. When deducing inequality (24) use was made of an assumption, that the amplitudes for processes (23) are analytic functions in angular variables $\cos\theta$ and $e^{i\varphi}$ in the neighbourhood of physical points (for details refer to Sec.2).

Integrating (24) over the angular interval

$$(\theta,\varphi)\in V_0=\left\{0<\theta_0\leqslant\theta\leqslant\pi-\theta_0\,,\;0<\varphi_0\leqslant\varphi\leqslant\pi-\varphi_0\,,\;\pi+\varphi_0\leqslant\varphi\leqslant 2\pi-\varphi_0\right\} \tag{25}$$

we will obtain:

$$\sigma_{ab\to cd}(s,V_0)\leqslant\frac{\ln^{\gamma}(s/s_0)}{s} \tag{26}$$

The domain V_0 will be called the domain of large angles. The cross-section $\sigma_{ab\to cd}(s, V_0)$ is connected with the differential cross-section

$$\frac{d\sigma_{ab\to cd}}{d^3k_c d^3k_d} \tag{27}$$

via the formula:

$$\sigma_{ab\to cd}(s,V_0) = \iint\limits_{V_0} d\cos\theta\, d\varphi\, d\upsilon \frac{d\sigma_{ab\to cd}}{d^3k_c\, d^3k_d} \tag{28}$$

$$d\upsilon = (2\pi)^{-5} \frac{k_c^2}{2E_c} \frac{k_d^2}{2E_d} dk_c dk_d d\cos\psi \tag{29}$$

ψ is the angle between momenta $\vec{k}_c$ and $\vec{k}_d$.

In accordance with formula (8) we have

$$\int\limits_{V_0} d\cos\theta\, d\varphi \int d\upsilon\, f_{ab\to cd}(s,\vec{k}_c,\vec{k}_d) = \langle n_c \cdot n_d \rangle_{V_0} \sigma_{ab\to cd}(s,V_0) \tag{30}$$

Here $\langle n_c n_d \rangle_{V_0}$ is the mean multiplicity of particles in the angular interval V_0 .

Taking into account inequalities (26) and (30) we have

$$\iint\limits_{V_0} d\cos\theta\, d\varphi\, d\upsilon\, f_{ab\to cd}(s,\vec{k}_c,\vec{k}_d) \leqslant \langle n_c n_d \rangle_{V_0} \frac{\ln^{\gamma}(s/s_0)}{s} \tag{31}$$

Because of energy conservation law the magnitude $\langle n_c n_d \rangle_{V_0}$ may increase with energy not more rapidly than $s/m_c m_d$, symbolically:

$$\langle n_c n_d \rangle_{V_0} \leqslant \frac{s}{m_c m_d} \tag{32}$$

If pionization does exist, then at $s \to \infty$ due to (22):

$$\int\int_{V_0} d\cos\theta\, d\varphi\, dv\, f_{ab\to cd}(\vec{k}_c, \vec{k}_d) \geqslant const > 0 \qquad (33)$$

Making comparison between (3I) and (33) we find:

$$\langle n_c n_d \rangle_{V_0} \geqslant \frac{s}{m_c m_d \ln^{\gamma}(s/s_0)} \qquad (34)$$

Thus it is necessary for the existence of pionization, that the magnitude $\langle n_c n_d \rangle_{V_0}$ at large s should be equal to

$$\langle n_c n_d \rangle_{V_0} = \frac{s\, O(s)}{m_c m_d \ln^{\gamma}(s/s_0)} \qquad (35)$$

The function $O(s)$ is bounded with inequality

$$const \leqslant O(s) \leqslant \ln^{\gamma}(s/s_0) \qquad (36)$$

As it follows from our inequalities (3I) and (33) the existence of pionization requires, that mean multiplicity reaches value of (35) that is close to the limiting value of (32).

As our conclusions are based on upper estimation (26) which, generally speaking, does not give precise asymptotic behaviour of the magnitude $\sigma_{ab\to cd}(s, V_0)$, then it is quite possible to assume, that pionization leads to limiting multiplicity (in the sense of dependence upon s), i.e.

$$\langle n_c n_d \rangle_{V_0} \sim \frac{s}{m_c m_d} \qquad (37)$$

The cross-section $\sigma_{ab\to cd}(s, V_0)$ should decrease with the energy increase not more rapidly, than:

$$\sigma_{ab\to cd}(s, V_0) \geqslant \frac{\ln^{\gamma}(s/s_0)}{s\, O(s)} \qquad (38)$$

this requirement is necessary for existence of pionization. Or if we take for $\mathcal{O}(s)$ the upper value from inequality (36) we will have:

$$\sigma_{ab\to cd}(s,V_0) \geqslant \frac{\ln^{\gamma}(s/s_0)}{s} \tag{39}$$

Using inequalities (26) and (38) we find[5]:

$$\frac{\ln^{\gamma}(s/s_0)}{s\ \mathcal{O}(s)} \leqslant \sigma_{ab\to cd}(s,V_0) \leqslant \frac{\ln^{\gamma}(s/s_0)}{s} \tag{40}$$

In addition to mean multiplicity of particles $\langle n_c n_d \rangle_{V_0}$ one can introduce mean multiplicity $\langle n_c n_d \rangle_{V_1}$ for particles "c" and "d" of bounded momenta

$$V_1 = \left\{ 0 \leqslant k_c \leqslant p_c\ ,\ 0 \leqslant k_d \leqslant p_d\ ,\ |\cos\psi| \leqslant 1\ ,\ (\theta,\varphi) \in V_0 \right\} \tag{4I}$$

For this physical characteristic we can make conclusions similar to those, we have obtained for $\langle n_c n_d \rangle_{V_0}$.

For experimental studies the characteristic of mean, taken for particles with bounded momenta

$$V_2 = \left\{ 0 \leqslant k_c \leqslant p_c\ ,\ 0 \leqslant k_d \leqslant p_d\ ,\ |\cos\psi| \leqslant 1\ ,\ 0 \leqslant \theta \leqslant \pi\ ,\ 0 \leqslant \varphi \leqslant 2\pi \right\}$$

is also very useful.

The mean multiplicity may be introduced into the diffraction domains.

Let us introduce the angular region, close to 0 and π.

$$\widetilde{V}_0 = \begin{cases} 0 \leqslant \theta \leqslant \theta_0\ ,\ \pi - \theta_0 \leqslant \theta \leqslant \pi\ ; \\ 0 \leqslant \varphi \leqslant \varphi_0\ ,\ \pi - \varphi_0 \leqslant \varphi \leqslant \pi + \varphi_0\ ,\ 2\pi - \varphi_0 \leqslant \varphi \leqslant 2\pi \end{cases}$$

If angular variables θ and φ belong to the domain $V_o + \widetilde{V_o}$, then the argument of the momentum distribution changes in the whole phase volume.

For a two particle momentum distribution formula (IO) for the given devision of the phase volume, can be written in the form:

$$\int f_{ab\to cd}(s,\vec{k}_c,\vec{k}_d)\,d^3k_c\,d^3k_d = \langle n_c n_d\rangle_{V_o}\sigma_{ab\to cd}(s,V_o) + \langle n_c n_d\rangle_{\widetilde{V}_o}\sigma_{ab\to cd}(s,\widetilde{V}_o)$$

Here $\langle n_c n_d\rangle_{\widetilde{V}_o}$ is the mean multiplicity in the angular range $\widetilde{V}_o$.

Summurizing the given paragraph one may say, that pionization (if it does exist) is accompanied with a large (limiting) multiplicity and weak (power) dropping of the cross-section $\sigma_{ab\to cd}(s, V_o)$ for an inclusive process.

If at large s the cross-section $\sigma_{ab\to cd}(s)$ for an inclusive process behaves like $(\ln s/s_o)^{\beta}$, where β may be a negative, as well as a positive number, then the main contribution to the total cross-section is made by fast particles, produced in the diffraction domain, that narrows with energy increase. Quite possible, that the contribution of fast particles " c " and " d " at large angles (independent of s) decreases very rapidly with energy increase, however if pionization exists, the value of the cross-section at large angles will be determined by contribution of particles with bounded momenta (independent of s), and consequently basing on the foregoing it cannot decrease more rapidly than $1/s$.

§ 5. Scale Invariance and Mean Multiplicity of Particles, Produced at Large Angles

According to the Hypothesis on scale invariance /7,3/ the momentum distribution of the particles " c " and " d " depends on the variables x_c , $\vec{p}_\perp^{\,c}$, x_d , $\vec{p}_\perp^{\,d}$ only, i.e.

$$f(x_c, \vec{p}_\perp^{\,c} ; x_d, \vec{p}_\perp^{\,d})$$

Mean multiplicity of particles $\langle n_c n_d \rangle_{V_0}$ in the angular interval, determined before, (see formula 30) is expressed via f in the following way:

$$4(2\pi)^6 \langle n_c n_d \rangle_{V_0} \sigma_{ab\to cd}(s, V_0) = I(s) \qquad \text{where} \qquad (43)$$

$$I(s) = \int d\vec{p}_\perp^{\,c} \int d\vec{p}_\perp^{\,d} \int_{-1+\Delta_c}^{1-\Delta_c} dx_c \int_{-1+\Delta_d}^{1-\Delta_d} dx_d \frac{f(x_c, \vec{p}_\perp^{\,c}; x_d, \vec{p}_\perp^{\,d})}{\sqrt{(x_c^2+\mu_c^2)(x_d^2+\mu_d^2)}} \qquad (44)$$

Here

$$\mu_c^2 = 4s^{-1}[m_c^2 + (\vec{p}_\perp^{\,c})^2] \;, \; \mu_d^2 = 4s^{-1}[m_d^2 + (\vec{p}_\perp^{\,d})^2] \qquad (45)$$

Δ_c and Δ_d are some constants close to zero, they are expressed through the angles θ_0 and φ_0 , introduced before in formula (25) when determining the domain V_0 . Repeating the reasonings of §4 with respect to the function $\mathcal{J}(s)$ with some obvious changes, we will find

$$0 < const \leq I(s) \leq const \, \ln^2 \frac{s}{m_c m_d} . \qquad (46)$$

From formulas (43) and (46) it is clear, that if the scale invariance takes place, than due to energy conservation law the cross-section for particle production "c" and "d" in the angular range V_0 cannot decrease more rapidly than:

$$\sigma_{ab\to cd}(s, V_0) \geqslant \frac{const}{s} \tag{47}$$

From formulas (43) and (46) and inequality (26) we find, that in the domain asymptotic in s mean multiplicity $\langle n_c n_d \rangle_{V_0}$ increases with s more rapidly than:

$$\langle n_c n_d \rangle_{V_0} \geqslant \frac{s}{m_c m_d \ln^{\gamma}(s/m_c m_d)} \tag{48}$$

i.e. the scale invariance leads to the fact, that mean multiplicity of particles "c" and "d" in the angular interval almost reaches its limiting value (in the sense of dependence upon s). Even if one of inequalities (47) and (48) is not fulfilled, then scale invariance will be violated in the range of large angles, i.e. the momentum distribution will quite explicitly depend on energy. Large mean multiplicity leads to the fact, that particles of bounded momenta (independent of s) can carry away a considerable fraction of the system total energy.

Let us consider an inclusive process of the form:

$$a + b \to c + \cdots$$

and select the cases of particle "c" production, when the angle θ (angle between momenta $\vec{p}_a$ and $\vec{k}_c$) is within the interval:

$$\tau_0 = \{ 0 < \theta_0 \leq \theta \leq \pi - \theta_0 \} \tag{49}$$

The mean multiplicity for the particles of "c" kind, in the angular interval τ_0 is expressed via the momentum distribution in the following way:

$$2(2\pi)^3\langle n_c\rangle_{\tau_0} = \frac{1}{\sigma_{ab\to c}(s,\tau_0)} \int d\vec{p}_\perp^{\,c} \int_{-1+\Delta_c}^{1-\Delta_c} \frac{dx_c}{\sqrt{x_c^2+\mu_c^2}} f(x_c, \vec{p}_\perp^{\,c}) \tag{50}$$

where Δ_c is some constant close to zero. Δ_c is a function θ_0 .

From here it is to obtain

$$\langle n_c\rangle_{\tau_0} \geqslant \frac{const}{\sigma_{ab\to c}(s,\tau_0)} \tag{5I}$$

Thus, if the scale invariance is fulfilled, then inclusive cross-section cannot decrease more rapidly than:

$$\sigma_{ab\to c}(s,\tau_0) \geqslant \frac{const}{\sqrt{s}} \tag{52}$$

In conclusion we would make an important remark on behaviour of differential cross-section (24).

Let at large s inclusive cross-section $\sigma_{ab\to cd}(s)$ behaves as $ln^{\beta}(s/s_0)$ where β may be either negative, or positive.

At fixed angles θ and φ not equal to 0 and π momentum transfers $|t_{ac}|$ and $|t_{ad}|$ are large and increase with energy as s .

Decrease of differential cross-section (24) with energy increase gives an evidence, that probability of the processes with a large momentum transferred is very small as compared with that of the processes going on with small momentum transferred. This fact is in accord with the notion that hadrons are of mesa-like structure[IO,II].

It is worth noticing, that if momenta of particles "c" and "d" are bounded, then momentum transfers $|t_{ac}|$ and $|t_{ad}|$ are also large and increase with energy as $\sqrt{s}$ for any angles θ and φ, including values 0 and π.

Therefore one might expect, the differential cross-section for "c" and "d" particle production of final momenta (independent of s) at the angles 0 and π to decrease with energy increase. This phenomenon has been discussed in paper /I2/.

At the angles $\theta = 0, \pi$ there should take place changes in the behaviour regime of differential cross-section with increase of s. In this case the particle "d" may be produced at small angles (or at the angles close to π) with respect to the momentum of incident particle and the momentum transfer $|t_{ad}|$ will be very small. In this there exists a diffraction domain in angle θ (and angle $\pi - \theta$) that shrinks with energy increase. These diffraction domains make the main contribution to the inclusive cross-section $\sigma_{ab\to cd}(s)$. When the angle θ is in the interval:

$$\tau_0 = \{ 0 < \theta_0 \le \theta \le \pi - \theta_0 \} \tag{53}$$

and the angle $\varphi = 0, \pi$, then there also take place changes in differential cross-section behaviour with increase of s, i.e. in angle φ (and angle $\pi - \varphi$) there exists a diffraction domain, narrowing with energy increase. The circumstance, that at $\varphi = 0, \pi$ production of "d" particle at small angles (and angles close to π) with respect to the incident particle momentum is in favour of this conclusion. In this the momentum transfer $|t_{ad}|$

reaching the possible smallest value. If there were no changes in the behaviour mode of differential cross-section in the domain τ_0 at $\varphi = 0, \pi$, then at large angles θ the scale invariance should be violated and there should not be pionization, as in this case

$$\sigma_{ab\to c}(s,\tau_0) \leq \frac{\ln^{\gamma} s/s_0}{s} \tag{54}$$

this contrudicts inequality (52),that comes out of hypothesis on scale invariance.

§ 6. Behaviour of Inelastic Cross-Section for Production of Particles with Bounded Momenta with Energy Increase

Let us consider an inclusive process:

$$a + b \to \nu c + \cdots$$

when there are ν particles of " c " kind in the final state with momenta in the region:

$$0 \leq |\vec{k}_i| < p \quad , \quad i = 1, \ldots, \nu \tag{56}$$

where p is the quantity independent of energy.

We will introduce the cross-section for the process, in which in the final state of the reaction there are produced n_c particles of " c " kind, out of which ν particles have momenta within interval (56), and the production angles in the interval:

$$V_c = \begin{cases} 0 < \theta_0 \le \theta_i \le \pi - \theta_0 ; \\ 0 < \varphi_0 \le \varphi_i \le \pi - \varphi_0 ; \; \pi + \varphi_0 \le \varphi_i \le 2\pi - \varphi_0 ; \; i = 1, \dots, \nu \end{cases} \quad (57)$$

Using expression (4) for the momentum distribution for the case of similar particles we have:

$$\sum_{n_c=\nu}^{\sqrt{s}/m_c} n_c (n_c - 1) \cdots (n_c - \nu + 1) \sigma_{ab \to \nu c} (s, p, V_c ; n_c) = const > 0 \quad (58)$$

Due to inequality (26) we have:

$$\sum_{n_c=2}^{\sqrt{s}/m_c} \sigma_{ab \to 2c} (s, p, V_c ; n_c) \le \frac{\ln^{\gamma}(s/s_0)}{s} \quad (59)$$

These are only the reactions, in which multiplicity of the particles of " c " kind is within the interval:

$$\frac{\sqrt{s}}{m_c \ln^{\frac{\gamma}{2}}(s/s_0)} \lesssim n_c \lesssim \frac{\sqrt{s}}{m_c} \quad (60)$$

that make contribution to sum

$$\sum_{n_c=2}^{\sqrt{s}/m_c} n_c (n_c - 1) \sigma_{ab \to 2c} (s, p, V_c ; n_c) = const > 0 \quad (61)$$

on the basis of inequality (59).

Thus, pionization, if it only exists, is for certain accompanied by large multiplicity. From here it follows that a reaction with the multiplicity smaller than in (60) do not make any contribution to two-particle function of the distribution, and consequently to pionization as well.

If we assume, the processes with multiplicity smaller than in (60) neither give a contribution to other distribution function at $s \to \infty$, then we may come to a conclusion that if pionization exists, then:

$$const \frac{1}{(\sqrt{s})^{\nu}} \lesssim \sigma_{ab\to\nu c}(s,p,V_c) \lesssim const \left(\frac{\ln^{\frac{\gamma}{2}}(s/s_0)}{\sqrt{s}}\right)^{\nu} \qquad (62)$$

where $\sigma_{ab\to\nu c}(s,p,V_c)$ is the production cross-section of ν particles in domains (56) and (57).

PART II

ANALYTICITY, UNITARITY AND UPPER BOUND OF DECREASE OF DIFFERENTIAL CROSS-SECTION FOR INCLUSIVE PROCESS.

In the given part the upper bound for decrease of differential cross-section for an inclusive process with energy increase has been found proceeding from analyticity in the neighbourhood of physical points and unitarity conditions. As compared with the results, obtained earlier in paper /4/, the bound in the sense of logarithmic dependence upon energy has been improved, and, what is more essential, this bound was obtained at weaker assumptions on analyticity, than those in paper /4/.

§ 1. Analyticity of inelastic process amplitude in angular variables.

In this section we will give a summary of the results on analytic properties of the inelastic process amplitude, resulting from the basic principles of theory.

Let us consider the reaction:

$$a + b \to c + d + A_j \qquad /II/$$

where A_j - is a hadron group.

We will treat the following variables in the c.m.s.:

$s = (p_a + p_b)^2$ - the squared total energy of the system;

$\cos\theta$ - where θ is the angle between momenta $\vec{p}_a$ and $\vec{k}_c$;

$e^{i\varphi}$ - where φ is the angle between the planes $/\vec{p}_a, \vec{k}_c/$ and $/\vec{k}_c, \vec{k}_d/$;

ξ - is a set of variables, that gives the configuration of particle momenta in the final state.

The Dyson representation for the amplitude $T^j_{ab\to cd}(s, \cos\theta, e^{i\varphi}, \xi)$ may be written in the form:

$$T^{j}_{ab\to cd}(s,\vec{n},\xi) = \int_{x_L(s)}^{\infty} dx \int d\vec{u}\, \frac{\Psi(x,\vec{u},s,\xi)}{x-(\vec{u}\cdot\vec{n})} \qquad /1/$$

where $\vec{n}$ is a unit vector along the momentum $\vec{p}_a$;
$\vec{u}$ is an arbitrary unit vector;
$x_L(s)$ is a semi-major of the Lehmann ellips.

From here it follows /see/13/ / that the amplitude $T^{j}_{ab\to cd}$ is analytic in variables $z = \cos\theta$ and $\omega = e^{i\varphi}$ in the domain, determined by the condition:

$$\left(|1+z||\omega| + |1-z|\frac{1}{|\omega|}\right)\left(|1+z|\frac{1}{|\omega|} + |1-z||\omega|\right) < 4x_L^2(s) \qquad /2/$$

excluding points z, belonging to the segments $/-x_L(s),-1/$ and $/1, x_L(s)/$. As the Dyson representation is true for $\overset{*}{T}{}^{j}_{ab\to cd}(s, \cos\theta, e^{i\varphi}, \xi)$ as well, then the amplitude $\overset{*}{T}{}^{j}_{ab\to cd}$ will be analytic function in variables z and ω in domain /2/.

From condition /2/ it follows, that at physical values of the variable ω the functions $T^{j}_{ab\to cd}$ and $\overset{*}{T}{}^{j}_{ab\to cd}$ will be analytic in z in the domain

$$|1+z| + |1-z| < 2x_L(s) \qquad /3/$$

excluding the points z, belonging to the segments $[-x_L(s),-1]$ and $[1, x_L(s)]$.

For physical values of the variable $z = \cos\theta$ the analyticity domain of the functions $T^{j}_{ab\to cd}$ and $\overset{*}{T}{}^{j}_{ab\to cd}$ in the variable ω depends on the angle θ.

In this case inequality /2/ is of the form

$$r_L^-(s,\theta) < |\omega| < r_L^+(s,\theta) \qquad /4/$$

where

$$r_L^{\pm}(s,\theta) = \frac{1}{\sin\theta}\left[\sqrt{x_L^2(s)-\cos^2\theta} \pm \sqrt{x_L^2(s)-1}\right]$$

In domain /2/ the functions $T^{j}_{ab\to cd}$ and $\overset{*}{T}{}^{j}_{ab\to cd}$ are decomposed into the Hartogs - Laurent series, e.g.

$$T^{j}_{ab\to cd}(s,z,\omega,\xi) = \sum_{m=-\infty}^{\infty} \omega^m F^{j}_{m}(s,z,\xi) \qquad /5/$$

The functions $F_m^{j}(s,z,\xi)$ and $\overset{*}{F}{}_m^{j}(s,z,\xi)$ will be analytic in z in domain /3/. They are decomposed in series in Wigner functions, e.g.

$$F_m^{j}(s,z,\xi)=\left[\frac{2\sqrt{s}}{|\vec{p}_a|}\right]^{\frac{1}{2}}\sum_{\ell=|m|}^{\infty}(2\ell+1)T_\ell^{m}(s,\xi;j)\,d_{|m|}^{\ell}(z) \qquad /6/$$

It should be noted, that unitary condition has the form:

$$\mathrm{Im}\, f_\ell(s)=|f_\ell(s)|^2+\sum_{m=-\ell}^{\ell}\sum_{j}\int d\Gamma_j\,|T_\ell^{m}(s,\xi;j)|^2+\cdots \qquad /7/$$

Here $\sum_j \cdots$ denots summation in all the channels, that give a contribution to reaction/II/, and $d\Gamma_j$ is an element of the phase space; $f_\ell(s)$ is a partial amplitude for the elastic scattering.

Let us consider the function

$$\Phi(z_1\omega_1;z_2\omega_2;s)=\sum_j\int d\Gamma_j\, T^{j}_{ab\to cd}(s,z_1,\omega_1,\xi)\overset{*}{T}{}^{j}_{ab\to cd}(s,z_2,\omega_2,\xi) \qquad /8/$$

The differential cross-section $d^2\sigma_{ab\to cd}/d\cos\theta\, d\varphi$ can easily be expressed via the values of the function Φ in the points $z_1=z_2=\cos\theta$, $\omega_1=\omega_2=e^{i\varphi}$:

$$\frac{d^2\sigma_{ab\to cd}}{d\cos\theta\, d\varphi}=\sum_j\frac{d^2\sigma^{j}_{ab\to cd}}{d\cos\theta\, d\varphi}=\frac{1}{2|\vec{p}_a|\sqrt{s}}\Phi(\cos\theta,e^{i\varphi};\cos\theta,e^{i\varphi};s) \qquad /9/$$

To end this Section we will tackle the analytic properties of the function Φ, following from the general principles of theory. From definition /8/ it is obvious, that the function Φ is analytic in z_1, z_2, ω_1, ω_2 in the product of domains /2/,

Substituting decompositions /5/ and /6/ into /8/ we will obtain:

$$\Phi(z_1,\omega_1;z_2,\omega_2;s)=\frac{2\sqrt{s}}{|\vec{p}_a|}\sum_{m_1=-\infty}^{\infty}\omega_1^{m_1}\sum_{m_2=-\infty}^{\infty}\omega_2^{-m_2}\,\Phi_{m_1m_2}(z_1,z_2;s) \qquad /10/$$

The function $\Phi_{m_1 m_2}(z_1, z_2; s)$ is defined by

$$\Phi_{m_1 m_2}(z_1,z_2;s)=\sum_{\ell_1=|m_1|}^{\infty}(2\ell_1+1)\sum_{\ell_2=|m_2|}^{\infty}(2\ell_2+1)\, C^{\ell_1\ell_2}_{m_1 m_2}(s)\, d^{\ell_1}_{|m_1|}(z_1)\, d^{\ell_2}_{|m_2|}(z_2) \qquad /\ 11\ /$$

The coefficients $C^{\ell_1 \ell_2}_{m_1 m_2}(s)$ are related to the partial amplitude T^m_ℓ through

$$C^{\ell_1\ell_2}_{m_1 m_2}(s)=\sum_j \int d\Gamma_j\, T^{m_1}_{\ell_1}(s,\xi;j)\, \overset{*}{T}{}^{m_2}_{\ell_2}(s,\xi;j) \qquad /\ 12\ /$$

From unitary condition and Bunyakovsky - Schwartz inequality it follows[14] that

$$\left|C^{\ell_1\ell_2}_{m_1 m_2}(s)\right| \leqslant \sqrt{Im f_{\ell_1}(s)\cdot Im f_{\ell_2}(s)} \qquad /\ 13\ /$$

Taking into account inequality /13/, as well as analyticity of the imaginary part of the amplitude for elastic scattering in $\cos\theta$ in the Martin ellips we will find, that the function $\Phi(z_1,\omega_1; z_2,\omega_2; s)$ is analytic in variables z_1, ω_1, z_2, ω_2 in the product of domains:

$$\frac{1}{2}\left(|\omega_i| + \frac{1}{|\omega_i|}\right)\left(|1-z_i| + |1+z_i|\right) < 2x_0(s), \quad i=1,2. \qquad /\ 14\ /$$

excluding the points z_i $(i=1,2)$ in the segments $[-x_0(s), -1]$ and $[1, x_0(s)]$.
Here $x_0(s)=\sqrt{\frac{x_M(s)+1}{2}}$, $x_M(s)$ is the semi-major of the Martin ellips.

Thus, the function $\Phi(z_1,\omega_1; z_2,\omega_2; s)$ is analytic in the union of the product of domains /2/ and the that of domains /14/.

It should be noted, that at physical values ω_i, the function Φ is analytic in z_1 and z_2 in the product of the domains defined by condition

$$d_i = \left\{ \left(|1-z_i| + |1+z_i| \right) < 2\, x_o(s) \right\} , \quad i = 1,2 , \qquad / 15 /$$

excluding the points z_i $(i=1,2)$ belonging to the segments $[-x_o(s), -1]$ and $[1, x_o(s)]$.

If the variables are $z_1 = z_2 = \cos\theta$, $\theta \in [0,\pi]$, then the function Φ will be analytic in the variables ω_1 and ω_2 in the product of the rings

$$r^- = \min\left\{ r_L^-(s,\theta), r_o^-(s) \right\} < |\omega_i| < \max\left\{ r_L^+(s,\theta), r_o^+(s) \right\} = r^+ \qquad / 16 /$$
$$i = 1,2,$$

where $r_o^{\pm}(s) = x_o(s) \pm \sqrt{x_o^2(s) - 1}$

As

$$\Phi_{m_1 m_2}(z_1, z_2; s) = \frac{1}{(2\pi i)^2} \oint\limits_{|\omega_1|=1} \frac{d\omega_1}{\omega_1^{m_1+1}} \oint\limits_{|\omega_2|=1} \frac{d\omega_2}{\omega_2^{-m_2+1}} \Phi(z_1,\omega_1; z_2,\omega_2; s) \qquad / 17 /$$

then the function $\Phi_{m_1 m_2}(z_1, z_2; s)$ is analytic in z_1 and z_2 in domain /15/.

§ 2. Integral representation.

Let us consider the function, defined by decomposition

$$G_m^j(s,z,\xi;t) = \left[\frac{2\sqrt{s}}{|\vec{p}_a|} \right]^{\frac{1}{2}} \sum_{\ell=|m|}^{\infty} (2\ell+1) T_\ell^m(s,\xi;j) d_{|m|}^{\ell}(z) t^{\ell-|m|} \qquad / 18 /$$

As series /6/ converges for any values z in domain /3/, then series /18/ will converge in t in the circle of R radius for physical values $z = \cos\theta$

$$R = x_L(s) + \sqrt{x_L^2(s) - 1} \qquad / 19 /$$

As in decomposition of the functions F_m^j and G_m^j there are present identical coefficients T_ℓ^m, then G_m^j may be expressed via F_m^j. Indeed as the function

$$(1-z^2)^{-\frac{|m|}{2}} F_m^j(s,z,\xi)$$

is analytic in variable z in the Lehmann ellips, then using the Cauchy theorem and decomposition of the form:

$$\frac{1}{z'-z}\left(\frac{1-z^2}{1-z'^2}\right)^{\frac{|m|}{2}} = \sum_{\ell=|m|}^{\infty}(2\ell+1)\, d^{\ell}_{|m|}(z)\, e^{\ell}_{|m|}(z') \qquad /\ 20\ /$$

where $e^{\ell}_{|m|}(z')$ is the Vigner function of the second kind[15] we will obtain

$$\left[\frac{2\sqrt{s}}{|\vec{p}_a|}\right]^{\frac{1}{2}} T^m_\ell(s,\xi;j) = \frac{1}{2\pi i}\oint_{\partial E_L} d\zeta\, F_m^j(s,\zeta,\xi)\, e^{\ell}_{|m|}(\zeta) \qquad /\ 21\ /$$

Substituting /21/ into decomposition /18/ we will find

$$G_m^j(s,z,\xi;t) = \frac{1}{2\pi i}\oint_{\partial E_L} d\zeta\, H_m(t,\cos\theta,\zeta)\, F_m^j(s,\zeta,\xi), \qquad /\ 22\ /$$

where

$$H_m(t,\cos\theta,\zeta) = \sum_{\ell=|m|}^{\infty}(2\ell+1)\, t^{\ell-|m|}\, d^{\ell}_{|m|}(\theta)\, e^{\ell}_{|m|}(\zeta) \qquad /\ 23\ /$$

Substituting into /23/ integral representations for the functions of[16] form

$$d^{\ell}_{|m|}(\theta) = [N(\ell,m)]^{-1}\cdot\frac{1}{2\pi i}\oint_{|\tau|=1}\frac{d\tau}{\tau} K_m(\theta,\tau)\left[\cos\theta+\frac{i}{2}\left(\tau+\frac{1}{\tau}\right)\sin\theta\right]^{\ell-|m|} \qquad /\ 24\ /$$

$$e^{\ell}_{|m|}(\zeta) = N(\ell,m)\cdot\frac{1}{2\pi i}\oint_{|w|=a}\frac{dw}{w} L_m(\zeta,w)\frac{1}{w^{\ell-|m|}(1-\zeta^2)^{\frac{|m|}{2}}}$$

we will obtain

$$H_m(t, \cos\theta, \zeta) = (1-\zeta^2)^{-\frac{|m|}{2}} \mathcal{H}_m(t, \cos\theta, \zeta) \qquad /\ 25\ /$$

where

$$\mathcal{H}_m(t, \cos\theta, \zeta) = \frac{1}{(2\pi i)^2} \oint_{|\tau|=1} \frac{d\tau}{\tau} K_m(\theta,\tau) \oint_{|w|=a} dw\, L_m(\zeta, w) \times$$
$$\times \frac{(2|m|-1)t\left[\cos\theta + \frac{i}{2}(\tau+\frac{1}{\tau})\sin\theta\right] + (2|m|+1)w}{\left\{w - t\left[\cos\theta + \frac{i}{2}(\tau+\frac{1}{\tau})\sin\theta\right]\right\}^2} \qquad /\ 26\ /$$

Here

$$K_m(\theta,\tau) = \frac{\left(\cos\frac{\theta}{2} + i\tau\sin\frac{\theta}{2}\right)^{2|m|}}{(-i\tau)^{|m|}}, \quad 1 < a < |\zeta + \sqrt{\zeta^2-1}|,$$

$$L_m(\zeta, w) = \frac{1}{\sqrt{1-2\zeta w + w^2}} \ln \frac{\zeta - w + \sqrt{1-2\zeta w+w^2}}{\sqrt{\zeta^2-1}} \times$$
$$\times \frac{1}{2}\left\{\left[\frac{\zeta w - 1 + \sqrt{1-2\zeta w+w^2}}{w^2}\right]^{|m|} + \left[\frac{\zeta w - 1 - \sqrt{1-2\zeta w+w^2}}{w^2}\right]^{|m|}\right\}$$

Such a choice of " a " gave a possibility to make summation in ℓ under the integral sign.

So, the integral representation for the function G_m^j at physical values Z is of the form

$$G_m^j(s, \cos\theta, \xi; t) = \frac{1}{2\pi i} \oint_{\partial E_L} \frac{d\zeta}{(1-\zeta^2)^{\frac{|m|}{2}}} F_m^j(s,\zeta,\xi) \mathcal{H}_m(t, \cos\theta, \zeta) \qquad /\ 27\ /$$

Here integration is carried out in the Lehmann ellips.

We will introduce the function

$$G_{m_1 m_2}(z_1, z_2; t_1, t_2; s) = \frac{|\vec{p}_a|}{2\sqrt{s}} \sum_j \int d\Gamma_j\, G_{m_1}^j(s, z_1, \xi; t_1) \overset{*}{G}{}_{m_2}^j(s, z_2, \xi; t_2) \qquad /\ 28\ /$$

It should be noted, that at $t_1 = t_2 = 1$, we have

$$G_{m_1 m_2}(z_1, z_2; 1, 1; s) = \Phi_{m_1 m_2}(z_1, z_2; s) \qquad /\ 29\ /$$

Substituting into /28/ integral representations of type /27/ instead of the function $G^{j}_{m_1}$ and $\overset{*}{G}{}^{j}_{m_2}$ we will find

$$G_{m_1 m_2}(\cos\theta, \cos\theta; t_1, t_2; s) = \frac{1}{(2\pi i)^2} \oint\limits_{\partial E_L} \frac{d\zeta_1}{(1-\zeta_1^2)^{\frac{|m_1|}{2}}} \oint\limits_{\partial E_L} \frac{d\zeta_2}{(1-\zeta^2)^{\frac{|m_2|}{2}}} \times \qquad /\ 30\ /$$

$$\times\, \Phi_{m_1 m_2}(\zeta_1, \zeta_2; s)\, \mathcal{H}_{m_1}(t_1, \cos\theta, \zeta_1)\, \mathcal{H}_{m_2}(t_2, \cos\theta, \zeta_2).$$

Here we have taken into account, that

$$\Phi_{m_1 m_2}(\zeta_1, \zeta_2; s) = \frac{|\vec{p}_a|}{2\sqrt{s}} \sum_j \int d\Gamma_j\, F^{j}_{m_1}(s, \zeta_1, \xi)\, \overset{*}{F}{}^{j}_{m_2}(s, \zeta_2, \xi) \qquad /\ 31\ /$$

The function $G_{m_1 m_2}(z_1, z_2; t_1, t_2; s)$ may be presented in the form of a double series.For this purpose we will substitute into / 28/ the decomposition of form /18/ instead of G^{j}_{m}, and we will have

$$G_{m_1 m_2}(z_1, z_2; t_1, t_2; s) = \sum_{\ell_1 = |m_1|}^{\infty} \sum_{\ell_2 = |m_2|}^{\infty} (2\ell_1+1)(2\ell_2+1)\, C^{\ell_1 \ell_2}_{m_1 m_2}(s)\, d^{\ell_1}_{|m_1|}(z_1)\, d^{\ell_2}_{|m_2|}(z_2)\, t_1^{\ell_1 - |m_1|} \cdot t_2^{\ell_2 - |m_2|} \qquad /\ 32\ /$$

§ 3. Analytic properties of the Function $G_{m_1m_2}(\cos\theta, \cos\theta; t_1, t_2; s)$ in variables t_1 and t_2 .

On the basis of representation /30/, taking into account analyticity of the function $\Phi_{m_1m_2}(\zeta_1, \zeta_2; s)$ in variables ζ_1 and ζ_2 in the domains of /15/-type, one can show, that the function $G_{m_1m_2}(\cos\theta, \cos\theta; t_1, t_2; s)$ is analytic in variables t_1 and t_2 in circles of radius $r_o^+(s)$ .It should be noted, that this analyticity may directly be established from decomposition /32/.Indeed, for physical values $\cos\theta$ due to unitary limitations / 13/ series /32/ converges absolutely in the variables t_1 and t_2 in the circles with radius $r_o^+(s)$ .This analyticity is a consequence of general principles of theory.Let us make some additional assumptions.

Assumption A

As it was exhibited before the function $\Phi_{m_1m_2}(\zeta_1, \zeta_2; s)$ is analytic in the variables ζ_1 and ζ_2 in the product of the domains $d_1 \times d_2$ .Let $O_1(\theta_o, \tilde{\eta})$ and $O_2(\theta_o, \tilde{\eta})$ be some fixed neighbourhood of the physical points $\zeta_1 = \cos\theta_o$ and $\zeta_2 = \cos\theta_o$, $\theta_o \in [0, \pi]$ $\tilde{\eta}$ characterises the size of the neighbourhood.The union of the domains d_i and O_i will be dezignated as $\mathcal{D}_i$ .Now, we will assume, that the function $\Phi_{m_1m_2}(\zeta_1, \zeta_2; s)$ is analytic in the product of the domains $\mathcal{D}_1 \times \mathcal{D}_2$.

As usual we will consider the function $\Phi_{m_1m_2}(\zeta_1, \zeta_2; s)$ in the domain of analyticity in the variables ζ_1 and ζ_2 be polynomially bounded in s i.e.

$$|\Phi_{m_1m_2}(\zeta_1, \zeta_2; s)| \leq \left(\frac{s}{s_o}\right)^n \qquad / 33 /$$

Now we will show, that from the assumption A it follows that the function $G_{m_1m_2}(\cos\theta_o, \cos\theta_o; t_1, t_2; s)$ is analytic in variables t_1 and t_2 in some fixed neighbourhood of the points $t_1 = 1$ and $t_2 = 1$ .Firstly we will study analytic properties of the function $G_{m_1 m_2}$ in variable t_1 .Let us write down /30/ in the form

$$G_{m_1 m_2}(\cos\theta_0, \cos\theta_0; t_1, t_2; s) = \frac{1}{2\pi i} \oint_{\partial E_L} \frac{d\zeta_1}{(1-\zeta_1^2)^{\frac{|m_1|}{2}}} \Psi_{m_1 m_2}(\zeta_1, \cos\theta_0, t_2; s) \mathcal{H}_{m_1}(t_1, \cos\theta_0, \zeta_1) \qquad /\ 34\ /$$

where

$$\Psi_{m_1 m_2}(\zeta_1, \cos\theta_0, t_2; s) = \frac{1}{2\pi i} \oint_{\partial E_L} \frac{d\zeta_2}{(1-\zeta_2^2)^{\frac{|m_2|}{2}}} \Phi_{m_1 m_2}(\zeta_1, \zeta_2; s) \mathcal{H}_{m_2}(t_2, \cos\theta_0, \zeta_2) \qquad /\ 35\ /$$

Considering representation /26/ for the function $\mathcal{H}_{m_1}(t_1, \cos\theta_0, \zeta_1)$ we see, that if

$$|t_1| \cdot \left| \cos\theta_0 + \frac{i}{2}\left(\tau + \frac{1}{\tau}\right) \sin\theta_0 \right| < 1$$

then integration over ζ_1 and W may be interchanged and after substituting /26/, /34/ will be changed for

$$G_{m_1 m_2}(\cos\theta_0, \cos\theta_0; t_1, t_2; s) = \frac{1}{2\pi i} \oint \frac{d\tau}{\tau} K_{m_1}(\theta, \tau) \times \qquad /36/$$

$$\times \oint_{|w|=a} dw \frac{(2|m_1|-1) t_1 \left[\cos\theta_0 + \frac{i}{2}\left(\tau + \frac{1}{\tau}\right)\sin\theta_0\right] + (2|m_1|+1) w}{\left\{ w - t_1 \left[\cos\theta_0 + \frac{i}{2}\left(\tau + \frac{1}{\tau}\right) \sin\theta_0\right]\right\}^2} \mathcal{L}_{m_1 m_2}(w, \cos\theta_0, t_2; s)$$

where

$$\mathcal{L}_{m_1 m_2}(w, \cos\theta_0, t_2; s) = \frac{1}{2\pi i} \oint_{\partial E_L} \frac{d\zeta_1}{(1-\zeta_1^2)^{\frac{|m_1|}{2}}} \Psi_{m_1 m_2}(\zeta_1, \cos\theta_0, t_2; s) L_{m_1}(\zeta_1, w) \qquad /\ 37\ /$$

As the function $\Phi_{m_1 m_2}(\zeta_1, \zeta_2; s)$ is analytic in ζ_1 and ζ_2 in the product of the domains $\mathcal{D}_1 \times \mathcal{D}_2$, then the function $(1-\zeta_1^2)^{-\frac{|m_1|}{2}} \Psi_{m_1 m_2}(\zeta_1, \cos\theta_0, t_2; s)$ will be analytic in ζ_1 in the union of ellips with a semimajor x_0 and the domain O_1 .This union will be dezignated as Q_1 , and its bound as ∂Q_1 .In representation /37/ the integration contour may be deformed into ∂Q_1 i.e.

$$\mathcal{L}_{m_1m_2}(w,\cos\theta_0,t_2;s)=\frac{1}{2\pi i}\oint\limits_{\partial Q_1}\frac{d\zeta_1}{(1-\zeta_1^2)^{\frac{|m_1|}{2}}}\Psi_{m_1m_2}(\zeta_1,\cos\theta_0,t_2;s)L_{m_1}(\zeta_1,w) \qquad /\ 38\ /$$

The function $L_{m_1}(\zeta_1,w)$ is analytic in variable W through the whole plane, with exeption for the point $W=0$ and the logarythmic cut from the point $W=\zeta_1+\sqrt{\zeta_1^2-1}$ to infinity.

When ζ_1 changes through the ellips, the beginning of the cut

$$W=\zeta_1+\sqrt{\zeta_1^2-1} \qquad /\ 39\ /$$

in the plane W changes along the circumference of the radius $r_0^+(s)$ .The part of the bound ∂Q_1 , that envelops the neighbourhood O_1 is reflected with the help of /39/ into the curves in the plane W that are bounds for the neighbourhood of the points $W=e^{\pm i\theta_0}$.

Thus, the function $\mathcal{L}_{m_1m_2}(W,\cos\theta_0,t_2;s)$ will be analytic in W with exception for the point $W=0$ in the domains, whose boundary $\partial\Omega_1$ is obtained from ∂Q_1 by mapping /39/.The region of analyticity in W will be dezignated as Ω_1 .In representation /36/ as the contour of integrating over W one may take the boundary $\partial\Omega_1$ After changing the integrating contour it is obvious from /36/ that the function under integrating will be analytic in t_1 and τ in the domain, where the values of the function

$$W(\tau,t_1,\theta_0)=t_1\left[\cos\theta_0+\frac{i}{2}\left(\tau+\frac{1}{\tau}\right)\sin\theta_0\right] \qquad /\ 40\ /$$

lie inside the domain Ω_1 and τ does not equal zero and infinity.

If the domain Ω_1 were convex, then the analyticity domain of the function $G_{m_1m_2}$ in t_1 would be determined by the condition under which the points $g_\pm=t_1e^{\pm i\theta_0}$ would lie inside domain Ω_1 .

Really, in our choice of the integration contour $|\tau|=1$ the domain of analyticity in t_1 is defined by the requirement that the segment

$$t_1\left[\cos\theta_0 + i\sin\theta_0\cos\alpha\right], \quad 0 \leq \alpha \leq 2\pi,$$

that connects the points g_+ and g_- lies inside the domain Ω_1. If the domain Ω_1 is not convex, then the analytic continuation in t_1 requires deformation of the contour of integrating in τ, as the straight line connecting the points $t_1 e^{i\theta_0}$ and $t_1 e^{-i\theta_0}$ may not wholly lie inside the domain Ω_1.

Easy to show, that the contour in τ - plane from a unit circumference can always be deformed into some contour γ, going through the points $\tau = \pm 1$ and such, that the curve

$$t_1\left[\cos\theta_0 + i\sin\theta_0\,\frac{1}{2}\left(\tau+\frac{1}{\tau}\right)\right], \quad \tau\in\gamma$$

lies inside the domains Ω_1.

Thus, again the domain of analyticity in t_1 is defined by the requirement, that the points $t_1 e^{i\theta_0}$ and $t_1 e^{-i\theta_0}$ are inside the domain Ω_1. The domain Ω_1 is a union of the circle with $r_0^+(s)$ radius and some fixed neighbourhoods of the points $e^{\pm i\theta_0}$, the point $w=0$ excluded. If t_1 lies in some fixed neighbourhood of the point $t_1 = 1$, then the points $g_\pm$ lie in w plane in the neighbourhood of the points $e^{\pm i\theta_0}$. Hence, the function $G_{m_1m_2}(\cos\theta_0, \cos\theta_0; t_1, t_2; s)$ continues in t_1 into some fixed neighbourhood of the point $t_1 = 1$ for any fixed value of $|t_2| < r_0^+(s)$.

Using the reasonings similar to those aforementioned, we will prove that the function $\Psi_{m_1m_2}(\zeta_1, \cos\theta_0, t_2; s)$ is analytic in some fixed neighbourhood of the point $t_2 = 1$.

Thus, $G_{m_1m_2}(\cos\theta_0, \cos\theta_0; t_1, t_2; s)$ is analytic in t_1 and t_2 in the product $\widetilde{\mathcal{D}}_1 \times \widetilde{\mathcal{D}}_2$ too, where $\widetilde{\mathcal{D}}_i$ is the union of the circle with $r_0^+(s)$ radius and some fixed neighbourhood of the point $t_i = 1$.

In conclusion to this Section, we will obtain the upper bound for the function $G_{m_1m_2}(\cos\theta_0, \cos\theta_0; t_1, t_2; s)$ in the analyticity domain in and .

Due to the foregoing reasonings representation /30/ may be written down in the following way

$$G_{m_1m_2}(\cos\theta_0,\cos\theta_0,t_1,t_2;s)=\frac{1}{2\pi i}\oint_{\partial Q_1}\frac{d\zeta_1}{(1-\zeta_1^2)^{\frac{|m_1|}{2}}}\oint_{\partial Q_2}\frac{d\zeta_2}{(1-\zeta_2^2)^{\frac{|m_2|}{2}}}\times \qquad /41/$$

$$\times\,\Phi_{m_1m_2}(\zeta_1,\zeta_2;s)\mathcal{H}_{m_1}(t_1,\cos\theta_0,\zeta_1)\mathcal{H}_{m_2}(t_2,\cos\theta_0,\zeta_2)$$

On the basis of /33/ as well as of inequality

$$\frac{1}{2\pi}\left|\oint_{\partial Q_i}d\zeta_i\mathcal{H}_{m_i}(t_i,\cos\theta_0,\zeta_i)\right|\leqslant(2|m_i|+1)\left(\frac{s}{s_0}\right)^{4k+1}A^{|m_i|}$$

where A and k are some constants, we will have the following upper bound

$$|G_{m_1m_2}(\cos\theta_0,\cos\theta_0;t_1,t_2;s)|\leqslant M, \qquad /42/$$

$$M=\mathrm{const}\left(\frac{s}{s_0}\right)^{N+6+2(|m_1|+|m_2|)},\quad N>n$$

for large $s\gg s_0$ in the analyticity domain in t_1 and t_2.

§ 4. The upper bound for the function

In the previous Section it has been shown, that in assumption A that the function $G_{m_1m_2}(\cos\theta_0,\cos\theta_0;t_1,t_2;s)$ is analytic in t_1 and t_2 in the product of the domains $\widehat{\mathcal{D}}_1\times\widehat{\mathcal{D}}_2$. Let us remined, that the domains $\widehat{\mathcal{D}}_i$ are such, that the points $t_i=1$ have some fixed neighbourhood independent of s.

To obtain the required bound, we will make use of the theorem about two constants. For this purpose we will perform the following costructions in the planes t_1 and t_2 /see fig.1/.

In the plane t_1 we will draw a circumference with the radius $e^{-\zeta}$, $\zeta>0$. Then through the point $t_1=r_0^+(s)+\eta\sin\theta_0$ we will draw a circumference with the centre on the real axis, which is orthogonal with respect to the circumference with $e^{-\zeta}$ radius. At η small enough, this circumference lies wholly in the domain $\widetilde{\mathcal{D}}_1$. Let us designate the crossing points of these circumferences as g and h and their arcs between the points g and h, and going through the points $e^{-\zeta}$, $r_0^+(s)+\eta \sin\theta_0$ as C_0 and C_η, respectively.We will draw a circumference through the points g, h and $t_1 = 1$. A part of the arc of this circumference, which contains the point $t_1 = 1$ will be designated as C_1.The angle between the arcs C_0 and C_1 will be α, and that between the arcs C_1 and C_η will be β. The domain, bounded with the arcs C_0 and C_η will be marked as B_1.Similar constructions are to be performed in the plane t_2.The domain between the arcs C_0 and C_η in t_2 plane will be B_2.Let $t_2 = 1$, then applying the theorem about two constants for the case with B_1 domain, we will have

$$\left|G(t_1=1,t_2=1)\right| \leq \left[\max_{t_1\in C_0}\left|G(t_1,t_2=1)\right|\right]^{\omega_1}\left[\max_{t_1\in C_\eta}\left|G(t_1,t_2=1)\right|\right]^{1-\omega_1} \qquad /\ 43\ /$$

On the other hand, using the theorem about two constants for the function G in the domain B_2, when $t_1 \in C_0$ and $t_1 \in C_\eta$ we will find

$$\left|G(t_1\in C_0,t_2=1)\right| \leq \left[\max_{t_2\in C_0}\left|G(t_1\in C_0,t_2)\right|\right]^{\omega_2}\left[\max_{t_2\in C_\eta}\left|G(t_1\in C_0,t_2)\right|\right]^{1-\omega_2} \qquad /\ 44\ /$$

$$\left|G(t_1\in C_\eta,t_2=1)\right| \leq \left[\max_{t_2\in C_0}\left|G(t_1\in C_\eta,t_2)\right|\right]^{\omega_2}\left[\max_{t_2\in C_\eta}\left|G(t_1\in C_\eta,t_2)\right|\right]^{1-\omega_2} \qquad /\ 45\ /$$

where ω_1 and ω_2 are harmonic measures for the domains B_1 and B_2 equal to:

$$\omega_1 = \omega_2 = \frac{\beta}{\alpha+\beta} \qquad /\ 46\ /$$

Having taken the maximum for t_1 in the two latter inequalities, we will substitute the values, obtained into inequality /43/, then

$$|G(t_1=1,t_2=1)| \leq \left[\max_{t_1\in C_0}\max_{t_2\in C_0}|G(t_1,t_2)|\right]^{\omega_1\omega_2}\cdot\left[\max_{t_1\in C_0}\max_{t_2\in C_\eta}|G(t_1,t_2)|\right]^{\omega_1(1-\omega_2)}\times$$

$$\times\left[\max_{t_1\in C_\eta}\max_{t_2\in C_0}|G(t_1,t_2)|\right]^{(1-\omega_1)\omega_2}\cdot\left[\max_{t_1\in C_\eta}\max_{t_2\in C_\eta}|G(t_1,t_2)|\right]^{(1-\omega_1)(1-\omega_2)} \qquad /\ 47\ /$$

Simplicity we always designate $G_{m_1m_2}(Cos\theta_0,Cos\theta_0;t_1,t_2;s)=G(t_1,t_2)$. As the function $G(t_1,t_2)$ in the analyticity domain has the upper bound in the form of inequality / see 42 /

$$|G(t_1,t_2)| \leq M$$

then on the basis of /47/ we will obtain

$$\left|G_{m_1m_2}(Cos\theta_0,Cos\theta_0;1,1;s)\right| \leq \left[\max_{t_1\in C_0}\max_{t_2\in C_0}\left|G_{m_1m_2}(Cos\theta_0,Cos\theta_0;t_1,t_2;s)\right|\right]^{\omega_1\omega_2}\cdot M^{1-\omega_1\omega_2} \qquad /\ 48\ /$$

Using decomposition /32/ and unitarity condition /7/ we will find

$$\max_{t_1\in C_0}\max_{t_2\in C_0}\left|G_{m_1m_2}(Cos\theta_0,Cos\theta_0;t_1,t_2;s)\right| \leq \frac{[1+|m_1|(1-e^{-\zeta})]\cdot[1+|m_2|(1-e^{-\zeta})]}{(1-e^{-\zeta})^4} \qquad /\ 49\ /$$

As is clear from inequality /48/ one should find a harmonic measure and choose the magnitude $e^{-\zeta}$ in the proper way so as to obtain the bound. Directly from the construction it follows that

$$tg\,\alpha = \frac{2\,tg\frac{\chi}{2}\,th\frac{\zeta}{2}}{tg^2\frac{\chi}{2}-th^2\frac{\zeta}{2}} \qquad /\ 50\ /$$

$$tg\frac{\chi}{2} = \frac{r_0^+(s)+\eta\,Sin\,\theta_0-e^{-\zeta}}{r_0^+(s)+\eta\,Sin\,\theta_0+e^{-\zeta}} \qquad /\ 51\ /$$

If we choose

$$th\frac{\zeta}{2} = tg\frac{\chi}{2}\cdot\frac{1}{N+6+2(|m_1|+|m_2|)}\cdot\frac{1}{\ln\frac{s}{s_0}} \qquad /52/$$

then, taking into account the fact, that $\alpha+\beta$ according to the construction equals $\pi/2$, we will find, that the harmonic measure satiesfies the inequality

$$1-\omega = \frac{2\alpha}{\pi} < \frac{2}{\pi}tg\,\alpha < \frac{4}{\pi[N+6+2(|m_1|+|m_2|)]}\cdot\frac{1}{\ln\frac{s}{s_0}} \qquad /53/$$

Having solved equations /51/ and /52/ together we will find

$$1-e^{-\zeta} = \frac{[N+6+2(|m_1|+|m_2|)]^{-1}}{\ln\frac{s}{s_0}}\cdot\frac{\overset{+}{r_0}(s)-1+\eta\,Sin\,\theta_0}{r_0^+(s)+1+\eta\,Sin\,\theta_0} \qquad /54/$$

Substituting this expression into inequality /49/ we have

$$\max_{t_1\in C_0}\max_{t_2\in C_0}\left|G_{m_1m_2}(Cos\theta_0,Cos\theta_0;t_1,t_2;s)\right| \leq const\left\{[|m_1|+|m_2|+1]\frac{\overset{+}{r_0}(s)+1+\eta\,Sin\,\theta_0}{r_0^+(s)-1+\eta\,Sin\,\theta_0}\right\}^4 \ln^4\frac{s}{s_0}$$

On the basis of this inequality, as well as /49/ and /53/ we will obtain the required upper bound for the function $\Phi_{m_1m_2}$:

$$\left|\Phi_{m_1m_2}(Cos\theta_0,Cos\theta_0;s)\right| \leq const\,(|m_1|+|m_2|+1)^4\cdot\left[\frac{\overset{+}{r_0}(s)+1+\eta\,Sin\,\theta_0}{r_0^+(s)-1+\eta\,Sin\,\theta_0}\right]^4 \ln^4\frac{s}{s_0} \qquad /55/$$

§ 5. Upper bound for decrease of inclusive differential cross-section.

As it was said, the function $\Phi(\cos\theta, \omega_1; \cos\theta, \omega_2; s)$ is analytic in ω_1 and ω_2 in the product of the rings of form /16/ for any physical values of the variable $\cos\theta$. This analyticity is a consequence of the general principles of theory. Let us designate the ring domain in the plane ω_i as K_i.
Let $\tilde{\mathcal{O}}_1(\varphi_0, \lambda)$ and $\tilde{\mathcal{O}}_2(\varphi_0, \lambda)$ be some fixed neighbourhoods of physical points $\omega_1 = e^{i\varphi_0}$ and $\omega_2 = e^{i\varphi_0}$, $\varphi_0 \in [0, 2\pi]$. λ characterized the sizes of the neghbourhood. Generally speaking φ_0 may depend on θ_0. The union of the domains K_i and $\tilde{\mathcal{O}}_i$ will be dezignated as Δ_i.

Assumption B

Let at physical values $z_1 = z_2 = \cos\theta_0$ the function Φ be analytic in the product of the domains $\Delta_1 \times \Delta_2$. As usually we will consider the function $\Phi(\cos\theta_0, \omega_1; \cos\theta_0, \omega_2; s)$ to be polynomially bounded in s in the analyticity domain in ω_1, ω_2, i.e.

$$\left| \Phi(\cos\theta_0, \omega_1; \cos\theta_0, \omega_2; s) \right| \leq \left(\frac{s}{s_0} \right)^{N_0}, \quad s \gg s_0 . \qquad /56/$$

We will proceed to the estimation of the function $|\Phi(\cos\theta_0, e^{i\varphi_0}; \cos\theta_0, e^{i\varphi_0}; s)|$. In decomposition /10/ of the function Φ in powers ω_1 and ω_2 it is more convenient to separate summations in positive and negative values for m_i. For this purpose, we will use the Caushy theorem

$$\Phi(\cos\theta_0, \omega_1; \cos\theta_0, \omega_2; s) = \frac{1}{2\pi i} \left(\oint_{\Gamma_+} \frac{d\omega_1'}{\omega_1' - \omega_1} + \oint_{\Gamma_-} \frac{d\omega_1'}{\omega_1' - \omega_1} \right) \times \qquad /57/$$

$$\times \frac{1}{2\pi i} \left(\oint_{\Gamma_+} \frac{d\omega_2'}{\omega_2' - \omega_2} + \oint_{\Gamma_-} \frac{d\omega_2'}{\omega_2' - \omega_2} \right) \Phi(\cos\theta_0, \omega_1'; \cos\theta_0, \omega_2'; s)$$

From /57/ we have

$$\Phi(\cos\theta_0,\omega_1;\cos\theta_0,\omega_2;s) = \Phi^{(++)} + \Phi^{(+-)} + \Phi^{(-+)} + \Phi^{(--)} \qquad /\ 58\ /$$

where

$$\Phi^{(jk)} = \frac{1}{2\pi i}\oint_{r_j}\frac{d\omega_1'}{\omega_1'-\omega_1}\,\frac{1}{2\pi i}\oint_{r_k}\frac{d\omega_2'}{\omega_2'-\omega_2}\,\Phi(\cos\theta_0,\omega_1';\cos\theta_0,\omega_2';s), \qquad /\ 59\ /$$

$$j=(+,-),\ k=(+,-),\ r^- = \frac{1}{r_+}.$$

The sign + (-) corresponds to integration in the positive /negative/ direction.
As all the further reasonings are similar for any function $\Phi^{(jk)}$, we will treat just $\Phi^{(++)}$.
In the planes ω_1 and ω_2 and the neighbourhoods of $e^{i\varphi_0}$ we will carry out the constructions similar to those, made in the planes t_1 and t_2 in the neighbourhood of the points $t_1=1$, $t_2=1$.

If we designate the radius of the inner circle as $e^{-\tau}$, $\tau>0$, then having made use of decomposition /10/ and estimation /55/ for the function $\Phi_{m_1m_2}(\cos\theta_0,\cos\theta_0;s)$ we will find

$$\max_{\omega_1\in C_0}\max_{\omega_2\in C_0}\left|\Phi^{(++)}(\cos\theta_0,\omega_1;\cos\theta_0,\omega_2;s)\right| \leqslant const\left[\frac{r_0^+(s)+1+\eta\sin\theta_0}{r_0^+(s)-1+\eta\sin\theta_0}\right]^4\frac{\ln^4\frac{s}{s_0}}{(1-e^{-\tau})^6} \qquad /\ 60\ /$$

τ and χ are related to each other through the same relation, as the variables ζ and χ.
Taking

$$th\frac{\tau}{2} = \frac{1}{N_0}\,tg\frac{\chi}{2}\,\frac{1}{\ln\frac{s}{s_0}}$$

we will find

$$1-e^{-\tau} = \frac{N_0}{\ln\frac{s}{s_0}}\left[\frac{r^+ + \lambda\sin\varphi_0 - 1}{r^+ + \lambda\sin\varphi_0 + 1}\right] \qquad /\ 61\ /$$

Substituting this expression into /60/ we will obtain

$$\max_{\omega_1\in C_0}\max_{\omega_2\in C_0}\left|\Phi^{(++)}(\cos\theta_0,\omega_1;\cos\theta_0,\omega_2;s)\right|\leq const\left[\frac{r_0^+(s)+1+\eta\sin\theta_0}{r_0^+(s)-1+\eta\sin\theta_0}\right]^4\cdot\left[\frac{r_+^+ 1+\lambda\sin\varphi_0}{r_-^+ 1+\lambda\sin\varphi_0}\right]^6 \ln^{10}\frac{s}{s_0} \qquad /62/$$

Repeating the previous reasonings and using the theorem about two constants we will find

$$\left|\Phi^{(++)}(\cos\theta_0,e^{i\varphi_0};\cos\theta_0,e^{i\varphi_0};s)\right|\leq const\left[\frac{r_0^+(s)+1+\eta\sin\theta_0}{r_0^+(s)-1+\eta\sin\theta_0}\right]^4\cdot\left[\frac{r_+^+ 1+\lambda\sin\varphi_0}{r_-^+ 1+\lambda\sin\varphi_0}\right]^6 \ln^{10}\frac{s}{s_0} \qquad /63/$$

This bound is valid for the functions $\Phi^{(+-)}, \Phi^{(-+)}, \Phi^{(--)}$ **and consequently for** $\Phi(\cos\theta_0, e^{i\varphi_0};\cos\theta_0, e^{i\varphi_0}; s)$ **too.**

Thus from /9/ and the bound, obtained for Φ **in the domain asymptotic in** s **the inclusive cross-section is bounded with inequality**

$$\left.\frac{d^2\sigma_{ab\to cd}}{d\cos\theta_0 d\varphi_0}\right|_{\substack{\theta=\theta_0\\ \varphi=\varphi_0}} \leq const\,\frac{\ln^{10}\frac{s}{s_0}}{s}\left[\frac{r_0^+(s)+1+\eta\sin\theta_0}{r_0^+(s)-1+\eta\sin\theta_0}\right]^4\cdot\left[\frac{r^+ + 1+\lambda\sin\varphi_0}{r^+ - 1+\lambda\sin\varphi_0}\right]^6 \qquad /64/$$

The assumptions A and B on the basis of which we have obtained this inequality are of a local character, therefore /64/ is valid for the points θ_0 **and** φ_0 **.**

If the assumptions A and B are true for any physical values of the angles θ **and** φ **in the range**

$$0<\theta<\pi\,,\quad 0<\varphi<\pi\,,\quad \pi<\varphi<2\pi \qquad /65/$$

then bound /64/ is true for all the physical values of θ **and** φ **.For the angles** θ **and** φ **of interval /65/ we have**

$$\frac{d^2\sigma_{ab\to cd}}{d\cos\theta d\varphi}\leq const\,\frac{\ln^{10}\frac{s}{s_0}}{s\,\sin^4\theta\,\sin^6\varphi} \qquad /66/$$

This bound may be improved to a certain extend.Really, when obtaining /49/ we made use of inequality $|d^{\ell}_{|m|}(\theta)|\leq 1$. **This inequality may be improved for the angles** $\theta\neq 0,\pi$ **.**

From the theorem of group properties we have

$$\left[d^{\ell}_{m}(\theta)\right]^{2} \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} d\varphi \left| P_{\ell}\left(\cos^{2}\theta + \sin^{2}\theta \cos\varphi\right)\right|$$

taking into account that

$$\left| P_{\ell}(\cos\Omega)\right| < \frac{1}{\sqrt{\ell \sin\Omega}}$$

after some calculations we will find

$$\left| d^{\ell}_{m}(\theta)\right| < \sqrt[4]{\frac{3}{(2\ell+1)\sin\theta}} \qquad / \; 67 \; /$$

In our choice $e^{-\zeta}$ /see 54 / this inequality allows to obtain the following inequality

$$\max_{t_1 \in C_0} \max_{t_2 \in C_0} \left| G_{m_1 m_2}(\cos\theta, \cos\theta; t_1, t_2; s)\right| \leq \frac{const}{\sqrt{\sin\theta}\,(1-e^{-\zeta})^{\frac{7}{2}}}$$

/ instead of /49/ /.

On the basis of this bound and with reasoning similar to the previous ones, we will find

$$\max_{\omega_1 \in C_0} \max_{\omega_2 \in C_0} \left| \Phi^{(++)}(\cos\theta, \omega_1; \cos\theta, \omega_2; s)\right| \leq const \frac{1}{\sin^4\theta} \frac{\ln^{\frac{7}{2}}\frac{s}{s_0}}{\left[1 - e^{-\tau}\right]^{\frac{11}{2}}}$$

Application of this inequality allows to improve the bound for the inclusive differential cross-section

$$\frac{d^2\sigma_{ab\to cd}}{d\cos\theta\, d\varphi} \leq const \frac{\ln^{9}\frac{s}{s_0}}{s \sin^4\theta \,|\sin\varphi|^5 \sqrt{|\sin\varphi|}} \qquad / \; 68 \; /$$

For fixed values of θ and φ bounds /66/ and /68/ for inclusive cross-sections are somewhat better as compared with bounds, obtained in [4]. It should also be noted, that these bounds have been obtained at weaker assumptions on analyticity, that is in our opinion is very substantial. Now we will show, that these bounds cannot be improved in the sense of the power dependence on s.

§ 6. Is any further improvement of the bounds possible ?

In the present Section we will make an attempt to clarify the question, wether any further improvement of the bounds on the basis of analyticity and unitary boundedness is possible. From the first sight it seems to be obvious, that widening of the analyticity domain under A and B assumptions may lead to the improvement of the bounds. The maximum possible analyticity of the function $\Phi_{m_1 m_2}(z_1, z_2; s)$ in variables z_1 and z_2 under assumption A is the analyticity in every variable in the plane with the definite cuts along the real axis. Similarly under assumption B the maximum possible analyticity of the function Φ in variables ω_1 and ω_2 is the analyticity in every variable in the plane with definite cuts along the real axis. In this the variables z_1 and z_2 take physical values only : $z_1 = z_2 = \cos\theta$.

We will show, that the class of functions, satisfying the condition of the maximum analyticity in the above mentioned sense and unitary boundedness on the partial waves

$$\sum_{m=-\ell}^{\ell} \sum_{j} \int d\Gamma_j \left| T_\ell^m(s,\xi;j) \right|^2 \leq \frac{1}{4} \qquad /69/$$

always contains a function, decreasing with increase of s not more rapidly than

$$\lim_{s\to\infty} s \cdot \frac{d^2\sigma_{ab\to cd}}{d\cos\theta\, d\varphi} = const > 0. \qquad /70/$$

In that way the bounds of /66/ and /68/ obtained before may be improved by decreasing the power of logarythm of the system energy.

Let us now construct the function that satisfies the analyticity conditions and unitarity and decreases with the increase of S at θ and φ fixed, not more rapidly than /70/.

The coefficients $C^{\ell_1\ell_2}_{m_1m_2}(s)$ in decomposition/11/ will be chosen in the following way:

$$C^{\ell_1\ell_2}_{m_1m_2}(s)=\frac{1}{4}\left[\frac{i}{r_o^+(s)}\right]^{\ell_1}\left[\frac{-i}{r_o^+(s)}\right]^{\ell_2} d^{\ell_1}_{m_1 0}\left(\frac{\pi}{2}\right)d^{\ell_2}_{m_2 0}\left(\frac{\pi}{2}\right)(-i)^{m_1}\, i^{m_2} \qquad /71/$$

According to unitary bound /69/ diagonal elements of the matrix $C^{\ell_1\ell_2}_{m_1m_2}(s)$ should satisfy the condition:

$$\sum_{m=-\ell}^{\ell} C^{\ell\ell}_{mm}(s)\leq\frac{1}{4}$$

In our choice of the matrix we have

$$C^{\ell\ell}_{mm}(s)=\frac{1}{4}\left[\frac{1}{r_o^+(s)}\right]^{2\ell}\left[d^{\ell}_{m0}\left(\frac{\pi}{2}\right)\right]^2 \qquad /72/$$

As in accordance with the group properties

$$\sum_{m=-\ell}^{\ell} d^{\ell}_{m0}\left(\frac{\pi}{2}\right)d^{\ell}_{m0}\left(\frac{\pi}{2}\right)=d^{\ell}_{00}(0)=1$$

then we will obtain

$$\sum_{m=-\ell}^{\ell} C^{\ell\ell}_{mm}(s)=\frac{1}{4}\left[\frac{1}{r_o^+(s)}\right]^{2\ell}\leq\frac{1}{4}$$

As $r_o^+(s)>1$ then unitary bound is fulfilled. If into /11/ we substitute the values of the coefficients equal to /71/ and take into account decomposition /10/ we will find

$$\Phi(z_1,\omega_1;z_2,\omega_2;s)=\Phi_1(z_1,\omega_1;s)\,\Phi(z_2,\omega_2;s) \qquad /73/$$

where

$$\Phi_1(z_1,\omega_1;s)=\left[1+(r_o^-(s))^2\right]\left[1-r_o^-(s)\left(\omega_1-\frac{1}{\omega_1}\right)\sqrt{1-z_1^2}-(r_o^-(s))^2\right]^{-\frac{3}{2}} \qquad /74/$$

$$\Phi_2(z_2,\omega_2;s)=\left[1+(r_o^-(s))^2\right]\left[1+r_o^-(s)\left(\omega_2-\frac{1}{\omega_2}\right)\sqrt{1-z_2^2}-(r_o^-(s))^2\right]^{-\frac{3}{2}} \qquad /75/$$

In finding Φ_i we use relation

$$P_\ell\left(\sin\theta\cos\left(\varphi-\frac{\pi}{2}\right)\right)=\sum_{m=-\ell}^{\ell}(-i)^m e^{im\varphi} d^{\ell}_{m}(\theta)\,d^{\ell}_{m}\left(\frac{\pi}{2}\right) \qquad /76/$$

as well as decomposition for the generating function for the Legendre polynomials.

To investigate the analytic properties of the function Φ it is enough to consider only one of the factors of /73/, e.g. Φ_1. Let $z_1=\cos\theta_1$, $0\leq\theta_1\leq\pi$, then the function Φ_1 is analytic in the whole plane ω_1 with the cuts from $-r_o^-(s)$ upto 0 and from $r_o^+(s)$ upto infinity. Taking $\omega_1=e^{i\varphi_1}$, $0\leq\varphi_1\leq 2\pi$ one can easily get convinced that Φ_1 is analytic in the variable z_1 in the whole plane z_1 with the cuts along the real axis from $-\infty$ upto -1 and from 1 upto infinity. Similarly the function Φ_2 is analytic in the whole plane ω_2 at physical values of $z_2 = \cos\theta_2$ excluding the cuts along the real axis from 0 upto $r_o^-(s)$ and from $-\infty$ upto $-r_o^+(s)$. At physical values $\omega_2 = e^{i\varphi_2}$ the function Φ_2 is analytic in the variable z_2 in the same domain as the function Φ_1 in the variable z_1. Easy to see that the function Φ is polynomially bounded in s in the analyticity domain.

Thus the constructed function Φ satisfies the requirements of the maximum analyticity.

Now we will consider the magnitude corresponding to the inclusive cross-section

$$\frac{d^2\sigma}{d\cos\theta\, d\varphi} = \frac{1}{4|\vec{p}_a|\sqrt{s}}\, \frac{r_o^+(s)\, x_o^2(s)}{\left[x_o^2(s) - 1 + \sin^2\theta \sin^2\varphi\right]^{\frac{3}{2}}} \qquad /77/$$

at θ and φ fixed and not equal to $0, \pi, 2\pi$ in the domain asymptotic in s we will have

$$\frac{d^2\sigma}{d\cos\theta\, d\varphi} \simeq \frac{1}{2s \sin^3\theta\, |\sin^3\varphi|} \qquad /78/$$

From here it is clear that the asymptotic bound for inclusive cross-section cannot be improved in the sense of power dependence of s. Moreover from our initial conditions it is impossible to obtain a decreasing bound for the magnitude

$$\frac{d\sigma_{ab\to cd}}{d\cos\theta} = \int_0^{2\pi} d\varphi\, \frac{d^2\sigma_{ab\to cd}}{d\cos\theta\, d\varphi} \qquad /79/$$

Indeed integrating /77/ over φ we will get

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\, r_o^+(s)\, x_o^2(s)}{2|\vec{p}_a|\sqrt{s}\,[x_o^2(s) - \cos^2\theta]^{\frac{3}{2}}}\, F\left(\frac{1}{2}, \frac{3}{2}; 1; \frac{\sin^2\theta}{x_o^2(s) - \cos^2\theta}\right) \qquad /80/$$

Let θ be fixed and not equal to 0 and π then at $s \to \infty$ we will have

$$\frac{d\sigma}{d\cos\theta} \simeq \frac{\pi F(-\frac{1}{2}, \frac{1}{2}; 1; 1)}{2 m_\pi^2 \sin\theta} \qquad /81/$$

As

$$\frac{d\sigma_{ab\to c}}{d\cos\theta} \geqslant \frac{d\sigma_{ab\to cd}}{d\cos\theta} \qquad /82/$$

then from our initial conditions it is impossible to obtain the bound for the differential cross-section $d\sigma_{ab\to c}/d\cos\theta$ that will decrease with the increase of energy at large angles.

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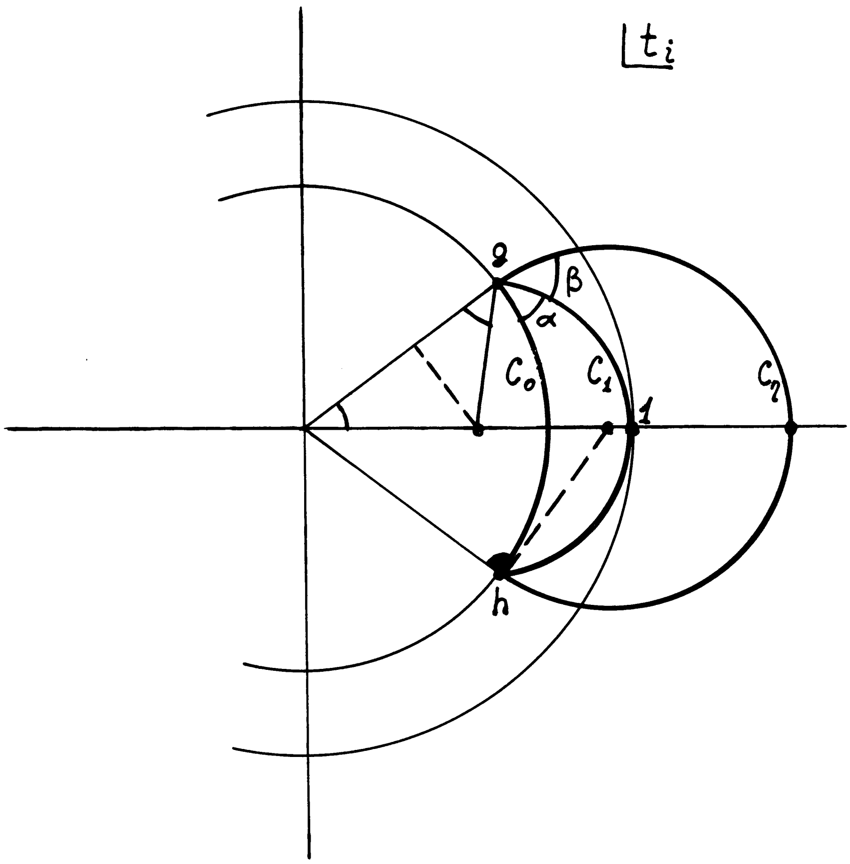

Fig. 1.