[74]

ON SOME CONSEQUENCES OF ANALYTICITY AND UNITARITY

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Abstract

In this paper we present the results concerning the asymptotic behaviour of the cross sections of elastic and inelastic processes. Some of these results were obtained together with M.A.Mestvirishvili and Nguyen ngoc Thuan. For the sake of simplicity we assume all particles to be spinless.

1. Upper bounds for the cross-sections of binary processes

First we consider some process of elastic scattering

$$a + b \longrightarrow a + b \tag{1}$$

We assume the amplitude of this process $F(s, z)$ to be analytical in $z = \cos\theta$, θ being the scattering angle in the centre of mass system, inside the Mandelstam ellipse E_c with foci at ± 1 and with the major semi-axis $c \sim 1 + \frac{\gamma}{s}$, $\gamma > 0$, at $s \to \infty$. For a number of processes such analytic properties were proved on the basis of the fundamental postulates of the local quantum field theory [1]. Let us expand $F(s, z)$ in partial waves

$$F(s, z) = 8\pi \frac{\sqrt{s}}{p} \sum_{\ell=0}^{\infty} (2\ell+1)\, a_\ell(s)\, P_\ell(z) . \tag{1}$$

As is well known, from the analyticity in z and from the polynomial boundedness at $s \to \infty$ it follows that $a_\ell(s)$ decreases exponentially when ℓ increases [2]

$$|a_\ell(s)| \leqslant R(s) \left[c + \sqrt{c^2 - 1} \right]^{-\ell} , \tag{2}$$

where $R(s)$ is a polynomial in s. Let L be that value of ℓ at which the r.h.s. of the relation (2) equals 1, and L_0 be the smallest integer number still greater than L. Now we rewrite the sum in (1) as follows

$$F(s, z) = 8\pi \frac{\sqrt{s}}{p} \sum_{\ell=0}^{nL_0} (2\ell+1)\, a_\ell(s)\, P_\ell(z) + 8\pi \frac{\sqrt{s}}{p} \sum_{\ell=nL_0+1}^{\infty} (2\ell+1) . \tag{3}$$

$$a_\ell(s)\, P_\ell(z) .$$

By using the inequality (2) we can prove that a finite integer number n can always be chosen to make the second term in the r.h.s. of (3) decrease faster than any power of s at $s \to \infty$.

We remember that if we substitute for $a_\ell(s)$ in the first sum their upper unitarity bound

$$|a_\ell(s)| \leq 1 \tag{4}$$

then, in virtue, of the relation

$$nL_0 \sim \text{const} \sqrt{s} \ln s \tag{5}$$

we obtain the Froissart bounds

$$|F(s,1)| \leq \text{const}\; s \ln^2 s, \tag{6}$$

$$|F(s,\cos\theta)| \leq \text{const} \frac{s^{3/4} \ln^{3/2} s}{\sqrt{\sin\theta}}, \quad \theta \neq 0, \pi . \tag{7}$$

It is possible to improve these results using in the first sum of (3) the Schwartz inequality instead of the substitution $|a_\ell(s)| \to 1$. Then we have

$$|F(s,z)|^2 \sim \left| \sum_{\ell=0}^{nL_0} (2\ell+1)\, a_\ell(s)\, P_\ell(z) \right|^2 \leq \leq \sum_{\ell=0}^{nL_0} (2\ell+1) |P_\ell(z)|^2 \cdot \sum_{\ell=0}^{\infty} (2\ell+1) |a_\ell(s)|^2 . \tag{8}$$

We denote by $\frac{d\sigma_{el}}{d\cos\theta}$ and σ_{el} the differential and the total cross-sections of the process (I). We obtain, as a consequence of the relation (8), the inequalities [3]

$$\left.\frac{d\sigma_{el}}{d\cos\theta}\right|_{\theta=0} \leq \text{const}\; s\ln^2 s\;\sigma_{el} \;, \tag{9}$$

$$\left.\frac{d\sigma_{el}}{d\cos\theta}\right|_{\theta\neq 0,\pi} \leq \text{const}\;\frac{\sqrt{s}\,\ln s}{\sin\theta}\;\sigma_{el} \;. \tag{10}$$

We consider now any binary process

$$a + b \to c + d \;, \tag{II}$$

and we decompose its amplitude $T(s,z)$ in partial waves

$$T(s,z) = 8\pi\frac{\sqrt{s}}{p}\sum_{\ell=0}^{\infty}(2\ell+1)\,b_\ell(s)\,P_\ell(z)\,. \tag{11}$$

This always can be done in the interval $-1 \leq z \leq 1$. The contribution from the partial amplitude $b_\ell(s)$ in the imaginary part $\operatorname{Im} a_\ell(s)$ equals $|b_\ell(s)|^2$

$$\operatorname{Im} a_\ell(s) = |a_\ell(s)|^2 + |b_\ell(s)|^2 + \cdots\cdots \tag{12}$$

Since all the terms in the r.h.s. of (12) are positive, we can write

$$|b_\ell(s)| < \sqrt{\operatorname{Im} a_\ell(s)} \leq \sqrt{|a_\ell(s)|}\,. \tag{13}$$

We conclude thus on the basis of relations (2) and (13) that also $b_\ell(s)$ decreases exponentially when ℓ increases and for the differential and total cross sections $\frac{d\sigma_{inel}}{d\cos\theta}$ and σ_{inel} we get the same inequalities as for the elastic processes

$$\left.\frac{d\sigma_{inel}}{d\cos\theta}\right|_{\theta=0} \leq \text{const}\; s\ln^2 s\;\sigma_{inel} \;, \tag{14}$$

$$\left.\frac{d\sigma_{inel}}{d\cos\theta}\right|_{\theta\neq 0,\pi} \leq const\ \frac{\sqrt{s}\ln s}{\sin\theta}\ \sigma_{inel}\ . \tag{15}$$

If the differential cross sections are considered at fixed t instead of $\cos\theta$ then we have

$$\left.\frac{d\sigma_{el}}{dt}\right|_{t=0} \leq const\ \ln^2 s\ \sigma_{el}\ , \quad \left.\frac{d\sigma_{el}}{dt}\right|_{t\neq 0} \leq const\ \frac{\ln s}{\sqrt{|t|}}\ \sigma_{el}\ , \tag{16}$$

$$\left.\frac{d\sigma_{inel}}{dt}\right|_{t=0} \leq const\ \ln^2 s\ \sigma_{inel}\ , \quad \left.\frac{d\sigma_{inel}}{dt}\right|_{t\neq 0} \leq const\ \frac{\ln s}{\sqrt{|t|}}\ \sigma_{inel}\ . \tag{17}$$

The relations (16), in particular, shows that the width of the diffraction peak for elastic and inelastic processes

$$\Delta_{el} = \frac{\sigma_{el}}{d\sigma_{el}/dt\,\big|_{t=0}}\ , \quad \Delta_{inel} = \frac{\sigma_{inel}}{d\sigma_{inel}/dt\,\big|_{t=0}} \tag{18}$$

can not decrease at $s\to\infty$ faster than $1/\ln^2 s$. This result for elastic processes was obtained earlier in a paper of Bessis who used a different method [4].

2. Upper bounds for the cross-sections of inelastic processes of multiple production

Now we are intended to show that the results obtained before are true also for the processes of multiple production of the type

$$a + b \longrightarrow c + \beta_j\ , \tag{III}$$

where B_j denotes a system of hadrons. We denote the angle between "a" and "c" particles momenta in the centre of mass system by θ. The amplitude of the process III $T^{jc}(s, \cos\theta, \ldots\ldots)$ will be presented in the form of a function of s, $z = \cos\theta$ and other independent variables, which always can be chosen in such a way that the integration domains in these variables are independent of the angle θ. The amplitude $T^{jc}(s, z, \ldots)$ can always be decomposed in the Legendre polynomials

$$T^{jc}(s, z, \ldots) = 4\pi \frac{\sqrt{s}}{p} \sum_{\ell=0}^{\infty} (2\ell+1)\, b_\ell^{jc}(s, \ldots)\, P_\ell(z). \qquad (19)$$

The contribution from the process (III) to the imaginary part of the partial amplitude of the process (I) is equal to

$$\operatorname{Im} a_\ell(s)\Big|_{\mathrm{III}} = \eta_\ell^{jc}(s) = \frac{1}{2\pi} \frac{\sqrt{s}}{p} \int (2\pi)^4 \delta^4\left(p_a + p_b - p_c - \sum p_i\right) \qquad (20)$$

$$\frac{p_c^2\, dp_c}{2\varepsilon_c} \prod_i \frac{d^3 p_i}{(2\pi)^3\, 2\varepsilon_i} \left| b_\ell^{jc}(s, \ldots) \right|^2,$$

where p_a, p_b and p_c are the momenta of the corresponding particles, p_i are the momenta of the particles of the system B_j, and ε_c, ε_i are the time components of p_c and p_i. The differential cross section of the process (III) at a given θ, integrated over all other variables, reads

$$\frac{d\sigma_{inel}^{jc}}{d\cos\theta} = \frac{2\pi}{p^2} \sum_{\ell,\ell'} (2\ell+1)(2\ell'+1)\, P_\ell(\cos\theta)\, P_{\ell'}(\cos\theta)\, C_{\ell\ell'}^{jc}(s) \qquad (21)$$

where

$$C_{\ell\ell'}^{j^c}(s) = \frac{1}{2\pi}\frac{\sqrt{s}}{p}\int (2\pi)^4\delta^4(\,)\frac{p_c^2\,dp_c}{2E_c}\prod_i \frac{d^3p_i}{(2\pi)^3 2E_i}\, b_\ell^{j^c}(s,\dots)\, b_{\ell'}^{j^c}(s,\dots)^* \tag{22}$$

Due to the Schwartz inequality we can write

$$\left| C_{\ell\ell'}^{j^c}(s)\right|^2 \leq \eta_\ell^{j^c}(s)\,\eta_{\ell'}^{j^c}(s) \leq |a_\ell(s)||a_{\ell'}(s)| \,. \tag{23}$$

As a consequence of this inequality we have, on the basis of the relation (2), the exponential decrease of $C_{\ell\ell'}^{j^c}$ in ℓ and ℓ' . By means of the arguments used above it is easy to derive inequality of the type (14) and (15) for $\frac{d\sigma_{inel}^{j^c}}{d\cos\theta}$ (see [5])

$$\left.\frac{d\sigma_{inel}^{j^c}}{d\cos\theta}\right|_{\theta=0} \leq \text{const}\; s\,\ln^2 s\;\sigma_{inel}^{j^c} \,, \tag{24}$$

$$\left.\frac{d\sigma_{inel}^{j^c}}{d\cos\theta}\right|_{\theta\neq 0,\pi} \leq \text{const}\;\frac{\sqrt{s}\,\ln s}{\sin\theta}\cdot\sigma_{inel}^{j^c} \,. \tag{25}$$

Performing summation on the cross sections $\frac{d\sigma_{inel}^{j^c}}{d\cos\theta}$ over all the possible system B_j we obtain the expression for the total cross section for the production of a particles "c" at a given angle θ

$$\frac{d\sigma_{inel}^{c}}{d\cos\theta} = \sum_j \frac{d\sigma_{inel}^{j^c}}{d\cos\theta} \,.$$

Similar relations can be established also for these quantities

$$\left.\frac{d\sigma^{c}_{inel}}{d\cos\theta}\right|_{\theta=0} \leq \text{const}\; s\ln^{2}s\;\sigma^{c}_{inel}\,, \tag{26}$$

$$\left.\frac{d\sigma^{c}_{inel}}{d\cos\theta}\right|_{\theta\neq 0,\pi} \leq \text{const}\;\frac{\sqrt{s}\ln s}{\sin\theta}\;\sigma^{c}_{inel}\,. \tag{27}$$

Similarly to (18) we introduce the notion of the width of the diffraction peak for the inelastic processes

$$\Delta^{j^c}_{inel} = \frac{s\,\sigma^{j^c}_{inel}}{d\sigma^{j^c}_{inel}/d\cos\theta\big|_{\theta=0}}\,,\qquad \Delta^{c}_{inel} = \frac{s\,\sigma^{c}_{inel}}{d\sigma^{c}_{inel}/d\cos\theta\big|_{\theta=0}}\,. \tag{28}$$

Then, due to the relations (24) and (26), it follows that these widths can not decrease faster than $1/\ln^2 s$.

3. The exponential increase of the imaginary part of the amplitude of the elastic scattering at real $z > 1$.

We have in the foregoing sections established a number of relations assuming polynomial boundedness of the elastic scattering amplitude $F(s,z)$ for all finite z inside the ellipse E_c . Using the unitarity condition we can replace this assumption by a weaker one, the polynomial boundedness of $\operatorname{Im} F(s,z)$. If, at least, one of these inequalities is not fulfilled, it could mean that either $\operatorname{Im} F(s,z)$ is not analytical in z or it increases faster than any polynomial in this ellipse. We denote by $N(s)$ the maximum of the modulus of $\operatorname{Im} F(s,z)$ for any z inside E_c . It can easely be proved that $\operatorname{Im} F(s,z)$ reaches this maximum at $z=c$:

$$\operatorname{Im} F(s,c) = N(s) . \tag{29}$$

Again we assume $\operatorname{Im} F(s,z)$ to be analytical in E_c. Then, even in the case when only one of the established inequalities is broken this would mean that $N(s)$ at $s \to \infty$ increases faster then any polynomial. We show now that the study of the behaviour of the corresponding quantities (the cross-section or the width of the diffraction peak) enables us to guess the character of the increase of this function.

Indeed, by means of calculations used above we can, instead of the inequality (6), for example, derive relation

$$F(s,1) \leq \text{const}\; s \ln^2 N(s) \tag{30}$$

where $N(s)$ is determined by (29). This gives

$$N(s) \geqslant \exp\left[\text{const} \frac{|F(s,1)|}{s}\right]^{1/2} \sim \exp\left[\text{const} \frac{1}{s} \frac{d\sigma_{el}}{d\cos\theta}\Big|_{\theta=0}\right]^{1/4} \tag{31}$$

which is a generalization of the inequality

$$N(s) \geqslant \exp\left[\text{const}\; \sigma_{tot}\right]^{1/2}$$

obtained by Martin in a different way [2]. The following formulae can be derived analogously

$$N(s) \geqslant \exp\left[\text{const} \frac{\sin\theta}{\sqrt{s}} \frac{d\sigma_{el}}{d\cos\theta}\Big|_{\theta\neq 0,\pi}\right]^{1/3}, \tag{32}$$

$$N(s) \geqslant \exp\left[\text{const} \frac{1}{s\sigma_{el}} \frac{d\sigma_{el}}{d\cos\theta}\Big|_{\theta=0}\right]^{1/2}, \tag{33}$$

$$N(s) \geqslant \exp\left[\text{const} \, \frac{\sin\theta}{\sqrt{s}\,\sigma_{el}} \, \frac{d\sigma_{el}}{d\cos\theta}\Big|_{\theta\neq 0,\pi} \right], \tag{34}$$

$$N(s) \geqslant \exp\left[\text{const} \, \frac{1}{s} \, \frac{d\sigma_{inel}}{d\cos\theta}\Big|_{\theta=0} \right]^{1/4}, \tag{35}$$

$$N(s) \geqslant \exp\left[\text{const} \, \frac{\sin\theta}{\sqrt{s}} \, \frac{d\sigma_{inel}}{d\cos\theta}\Big|_{\theta\neq 0,\pi} \right]^{1/3}, \tag{36}$$

$$N(s) \geqslant \exp\left[\text{const} \, \frac{1}{s\,\sigma_{inel}} \, \frac{d\sigma_{inel}}{d\cos\theta}\Big|_{\theta=0} \right]^{1/2}, \tag{37}$$

$$N(s) \geqslant \exp\left[\text{const} \, \frac{\sin\theta}{\sqrt{s}\,\sigma_{inel}} \, \frac{d\sigma_{inel}}{d\cos\theta}\Big|_{\theta\neq 0,\pi} \right], \tag{38}$$

where σ_{inel} means either the cross section of a binary inelastic process, or the cross section of an inelastic process of multipole production.

4. <u>Lower bound for the cross sections of inelastic processes</u>

In this section we estiblish some lower bounds for the cross sections of inelastic processes of the type (III).

We write the differential cross section in the form

$$\frac{d\sigma^{j^c}_{inel}}{d\cos\theta} \doteq \frac{\pi^2}{p\sqrt{s}}\, G^{j^c}(s, \cos\theta) \tag{39}$$

where

$$G^{j^c}(s,\cos\theta) = \int \delta^4(p_a + p_b - p_c - \sum_i p_i)\, \frac{p_c^2\, dp_c}{2E_c} \prod_i \frac{d^3 p_i}{(2\pi)^3\, 2E_i} \left| T^{j^c}(s, \cos\theta \ldots)\right|^2. \tag{40}$$

We denote by $\operatorname{Im} F(s,\cos\theta)\big|_{j^c}$ the contribution to the imaginary part of the amplitude of the corresponding elastic process (1). Then we can write

$$\operatorname{Im} F(s,\cos\theta)\Big|_{j^c} = \pi \int d\Omega_{p_c}\, H^{j^c}(s, \cos\theta_1, \cos\theta_2), \tag{41}$$

where

$$H^{j^c}(s,\cos\theta_1,\cos\theta_2) = \int \delta^4(p_a + p_b - p_c - \sum p_i)\, \frac{p_c^2\, dp_c}{2E_c} \prod \frac{d^3 p_i}{(2\pi)^3\, 2E_i} \tag{42}$$
$$T^{j^c}(s, \cos\theta_1, \ldots)\, T^{j^c}(s, \cos\theta_2, \ldots)^{*}.$$

θ_1 and θ_2 denote here the angles between the momenta of initial and final particles "a" and that of the intermediate particles "c". The total cross section is

$$\sigma^{j^c}_{inel} = \int \frac{d\sigma^{j^c}_{inel}}{d\cos\theta}\, d\cos\theta = \frac{1}{2p\sqrt{s}} \operatorname{Im} F(s,1)\Big|_{j^c} \tag{43}$$

We remind that Martin has proved the analyticity in z of the imaginary part of the amplitude of the elastic scattering $F(s,z)$ inside the Mandelstam ellipse [1]. By the arguments similar to those in the preceeding section, it can be shown that $Im\, F(s,z)\big|_{jc}$ is also an analytical function in this domain. The study of the analytic properties of the Feynman diagrams shows, however, that the contribution from some simplest diagrams of the perturbation theory to the amplitude $F(s,z)$ satisfies a dispersion relation in z for the values of s in the physical domain of the s-channel. This was the motivation for the assumption that $F(s,z)$ is analytic in the complex z plane with the cuts and poles on the real axis. It is quite natural to assume that $Im\, F(s,z)\big|_{jc}$ for the values of s in the physical domain of the s — channel is also an analytic function in z in the complex plane with singularitites on the real axis. Let us assume that $Im\, F(s,z)\big|_{jc}$ is polynomially bounded, i.e. an $n>0$ exists, such that

$$\left| Im\, F(s,z)\big|_{jc} \right| \leq const\; s^{n}, \qquad s \to \infty$$

for all z from any finite domain. Then a certain relation between the behaviour of this function at z =1 and their behaviour at $s \to \infty$ in the interval $-1 < z < 1$ can be established. Let us assume that the total cross section σ_{inel}^{jc} does not decrease faster than some power of $1/s$ at $s \to \infty$:

$$\left| Im\, F(s,1)\big|_{jc} \right| > \frac{const}{s^{m}}, \qquad s \to \infty$$

where m is some positive constant (see relation (43)).Then the following lower bound can be written[6]

$$\mathrm{Im}\, F(s,\cos\theta)\Big|_{j^c} \geqslant \text{const}\; e^{-c(\theta)\sqrt{s}\ln s}, \quad s\to\infty, \qquad c(\theta)>0, \quad -1<\cos\theta<1. \tag{44}$$

We show now that the inequality (44) will be broken at $\pi-2\theta_0<\theta<2\theta_0$ if the following inequality holds for all θ in the interval $\pi-\theta_0\leqslant\theta\leqslant\theta_0$ for some θ_0 satisfying $\frac{\pi}{4}>\theta_0>0$:

$$\left[G^{j^c}(s,\cos\theta)\right]^{1/2} < \text{const}\; e^{-b(\theta)\sqrt{s}\ln s}, \quad s\to\infty, \qquad b(\theta)>0. \tag{45}$$

Hence we conclude that there exists a certain interval of angles in which we have

$$\frac{d\sigma^{j^c}_{inel}}{d\cos\theta} \geqslant \text{const}\; e^{-2b(\theta)\sqrt{s}\ln s}, \quad s\to\infty \tag{46}$$

To this end we consider the function $H^{j^c}(s,\cos\theta_1,\cos\theta_2)$. Comparing the expression (42) of this function with the expression (40) of $G^{j^c}(s,\cos\theta)$ and using the Schwartz inequality we get

$$\left|H^{j^c}(s,\cos\theta_1,\cos\theta_2)\right| \leqslant \left[G^{j^c}(s,\cos\theta_1)\, G^{j^c}(s,\cos\theta_2)\right]^{1/2}.$$

Thus

$$\left|\, \mathrm{Im}\, F(s,\cos\theta)\Big|_{j^c}\right| \leqslant \pi\int d\Omega_{p_c}\left[G^{j^c}(s,\cos\theta_1)\, G^{j^c}(s,\cos\theta_2)\right]^{1/2}.$$

Now, assuming (45) it is easy to derive on the basis of the last relation an inequality reciprocal to (44). We have thus proved the lower bound (46).

We have introduced the notion of the total cross section $\frac{d\sigma^{c}_{inel}}{d\cos\theta}$ with the production of a particle "c" at a given angle θ. It is possible to write a similar inequality also for this quantity

$$\frac{d\sigma^{c}_{inel}}{d\cos\theta} \geq \text{const}\, e^{-2b(\theta)\sqrt{s}\ln s}, \quad s \to \infty, \quad b(\theta) > 0, \tag{47}$$

for the values of θ in a certain interval.

We point out that our assumption that the cross sections σ^{jc}_{inel} and σ^{c}_{inel} do not decrease faster than a certain power of 1/s can be also proved experimentally. In the framework of the Regge theory such a behaviour of the cross sections for zero angles is valid for the majority of processes.

5. Asymptotic equality of the cross sections of crossing processes of multipole production

We consider now the inelastic crossing processes

$$a + b \to a_1 + a_2 + \cdots + a_n + b', \tag{IY$_1$}$$

$$\tilde{a} + b' \to \tilde{a}_1 + \tilde{a}_2 + \cdots + \tilde{a}_n + b. \tag{IV$_2$}$$

We denote the momenta of particles "b" and "b$'$" in (IV_1) and those of particles "b$'$" and "b" in (IY_2) by p and p$'$, the momenta of particles "a" and "a_i" (or those of their antiparticles by q and q_i . We put $k_i = \sum_{j=i}^{n} q_j$.
We can chose

$$s = -(p+q)^2, \quad t = -(p-p')^2, \tag{48}$$
$$W_i^2 = -k_i^2, \quad t_i = -(q-k_{i+1})^2, \quad \eta_i = e^{2\xi_i} = \frac{q_i(p+p')}{k_{i+1}(p+p')}$$

as suitable independent invariant variables.
Let $T^{1,2}(s, t, W_i^2, t_i, \xi_i)$ be the amplitudes of the processes under consideration. These amplitudes for fixed values of the variables t, t_i, W_i^2, η_i, generally speaking, do not satisfy the dispersion relation in s . At $s \to \infty$, however, they tend to certain asymptotic amplitudes $T_\infty^{1,2}(s, t, W_i^2, t_i, \xi_i)$ analytic in s in the complex plane with cuts on the real axis [3]. The crossing-symmetry relation holds for these asymptotic amplitudes

$$T_\infty^1(u, t, W_i^2, t_i, \eta_i) = T_\infty^2(s, t, W_i^2, t_i, \eta_i)^* \tag{49}$$

where $s + u + t = m_a^2 + m_b^2 + m_{b'}^2 + W_1^2$.

From this relation, we can prove [7] that for non-oscillating amplitudes $T^{1,2}$ at $s \to \infty$ we have the asymptotic equality of the differential cross sections of the processes (IY_1) and (IY_2)

for the finite values of the variables t, W_i^2, t_i, η_i.

Let us consider the processes (IY_1) and (IY_2) at fixed values of t and W_1^2. The physical domains of the variables W_i^2, $i \geq 2$ and t_i, η_i, $i \geq 1$ depend on s,t and W_1^2. However, these domains remain finite at $s \to \infty$. It means that at very high energies the domains of integration over the variables W_i^2, $i \geq 2$ and t_i, η_i, $i \geq 1$ are practically independent of s and the values of these variables are finite Both sides of the asymptotic equality for the differential cross sections of the processes (IY_1) and (IY_2) can therefore be integrated over all possible values of W_i^2, $i \geq 2$ and t_i, η_i, $i \geq 1$. As a result we have

$$\frac{\partial^2 \sigma_1}{\partial t \partial W_1^2} \sim \frac{\partial^2 \sigma_2}{\partial t \partial W_1^2}, \tag{50}$$

where σ_1 and σ_2 are the cross sections of the processes (IY_1) and (IY_2). For fixed values of W_1^2 the number of possible systems $a_1 + \cdots + a_n$ is finite. We can therefore perform a summation in both sides of the relation (50) over all possible channels. As a result we obtain

$$\frac{\partial^2 \sigma(a + b \to \cdots\cdots + b')}{\partial t \partial W_1^2} \sim \frac{\partial^2 \sigma(\tilde{a} + b' \to \cdots\cdots + b)}{\partial t \partial W_1^2}, \tag{51}$$

$$s \to \infty .$$

The dots in the brackets denote the particle systems with the effective mass W_1 . We note that for the determination of the cross-sections appearing in the relation (41) it is sufficient to measure the momentum of one of the particles (particle "b" or particle "b'") since this allows the definition of t and W_1^2 (the mixing mass method). Because of the C-invariance the cross section of the process (IY_2) is equal to that of the process

$$a + \tilde{b}' \rightarrow a_1 + a_2 + \cdots + a_n + \tilde{b} \qquad (IY_3)$$

So we have besides the relation (51) another equation

$$\frac{\partial^2 \sigma(a + b \rightarrow \cdots\cdots + b')}{\partial t\, \partial W_1^2} \sim \frac{\partial^2 \sigma(a + \tilde{b}' \rightarrow \cdots\cdots + \tilde{b})}{\partial t\, \partial W_1^2} . \qquad (52)$$

Concluding this section, we present some concrete relations (see also ref. [8])

$$\frac{\partial^2 \sigma(\pi^+ + p \rightarrow \cdots\cdots + p)}{\partial t\, \partial W_1^2} \sim \frac{\partial^2 \sigma(\pi^- + \bar{p} \rightarrow \cdots\cdots + \bar{p})}{\partial t\, \partial W_1^2}$$

$$\frac{\partial^2\sigma(\pi^+ + p \to \pi^+ + \cdots\cdots)}{\partial t\,\partial w_1^2} \sim \frac{\partial^2\sigma(\pi^- + p \to \pi^- + \cdots\cdots)}{\partial t\,\partial w_1^2},$$

$$\frac{\partial^2\sigma(\pi^+ + p \to K^+ + \cdots\cdots)}{\partial t\,\partial w_1^2} \sim \frac{\partial^2\sigma(K^- + p \to \pi^- + \cdots\cdots)}{\partial t\,\partial w_1^2},$$

$$\frac{\partial^2\sigma(p + p \to p + \cdots\cdots)}{\partial t\,\partial w_1^2} \sim \frac{\partial^2\sigma(\tilde{p} + p \to \tilde{p} + \cdots\cdots)}{\partial t\,\partial w_1^2},$$

$$\frac{\partial^2\sigma(p + p \to \cdots\cdots + p)}{\partial t\,\partial w_1^2} \sim \frac{\partial^2\sigma(\tilde{p} + p \to \cdots\cdots + p)}{\partial t\,\partial w_1^2}.$$

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