CENTROID THEORY OF TRANSVERSE ELECTRON-PROTON TWO-STREAM INSTABILITY IN A LONG PROTON BUNCH REVISITED *

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Abstract

We have identified a few technical errors in our recent paper on the electron-proton instability [1]. Although the overall qualitative results in the original paper are still correct, an error in Eq. (42) does cause some minor changes in the quantitative results including three figures and alternations in several equations. The most significant change is the required frequency spread for stability given in Eq. (68), which is now found to be twice that given in the original paper. This revision and related changes have been published in a recent Erratum [2]. This note summarizes the main results of the original paper and the highlight of the main changes made in the Erratum.

INTRODUCTION

In our recent paper on the study of the transverse electron-proton (e-p) instability in a long proton bunch [1], a few errors have been found. Although the overall qualitative results in the paper are still correct, an error in Eq. (42) does cause some minor changes in the quantitative results including three figures and alternations in several equations. Revisions and changes have been published in a recent Erratum [2].

The purpose of this note is two-fold. First, we review the main results in the original paper and summarize the major changes made in the Erratum. Second, to supplement the original paper and the Erratum, we include qualitative explanations and discussions of some of the results and the physical basis of the approach taken in the original paper for exploring the e-p instability.

In order to make it convenient for cross-checking, the equations here are numbered the same as they are in the original paper [1]. The summaries are arranged to be relatively self-contained. For readers not intending to go through the details in the original paper and the Erratum, this note can be read as a stand-alone paper.

REVIEW OF EARLIER RESULTS

In Ref. 1, we derived the equations of motion for the centroids of the proton bunch and the electron cloud by averaging the linearized equations of the single particle motion over the frequencies of the transverse motion. Damping effects are incorporated consistently as a result of the averaging process. In the model studied, we neglected the synchrotron oscillation of the protons, and the axial motion of the electrons, as well as the impedance of the surrounding structure. We showed that the damping exponent is linear in time for a Lorentzian distribution and quadratic in time for a Gaussian distribution.

The stability analysis of the e-p interaction was focused on the frequency spreads of a Lorentzian distribution. In the laboratory frame, the equations of centroid motion are

$$\left(\frac{\partial}{\partial t} - v\frac{\partial}{\partial z}\right)^2 Y_p + 2\Delta_p\left(\frac{\partial Y_p}{\partial t} - v\frac{\partial Y_p}{\partial z}\right) + (\omega_\beta^2 + \Delta_p^2)Y_p = G(z)Y_e \ , \qquad (18)$$

and

$$\frac{d^2Y_e}{dt^2} + 2\Delta_e\frac{dY_e}{dt} + [\Omega^2(z,t) + \Delta_e^2]Y_e = \Omega^2(z,t)Y_p \ , \qquad (19)$$

where $Y_q = Y_q(z,t)$ is the transverse centroid displacement at the axial position z and time t, the subscripts q stands for p (protons) or e (electrons), v is the axial speed of the protons, Δ_q characterizes the spread of transverse oscillation frequency of the particles, ω_β is the averaged angular betatron frequency of the protons, $\Omega(z) = (c/a)\sqrt{2r_e[\lambda_p(z) - \lambda_e(z)]} \approx (c/a)\sqrt{2r_e\lambda_p(z)}$ is the electron bounce frequency inside the proton bunch, c is the speed of light, a is the proton beam radius, r_q is the classical particle radius, $\lambda_q(z)$ is the line density of particles, $G(z) = 2r_pc^2\lambda_e(z)/(a^2\omega_\beta^2\gamma)$, $\gamma = (1 - v^2/c^2)^{-1/2}$, and the minus sign is used in the terms containing $\partial/\partial z$ because the proton bunch is assumed to propagate in the negative z direction.

Our interest is to investigate the general characteristics of the e-p instability from the asymptotic solution of Eqs. (18) and (19) instead of examining the evolution of any specific initial perturbation. A result "colored" by the initial conditions is not regarded to be a general or an intrinsic characteristics of the e-p instability. For example, the larger growth rate in the tail of the proton bunch due to a larger initial perturbation in the middle of the bunch that happens during the early development of the instability is not considered to be a general characteristic of the e-p instability. We therefore formulated an analytical approach to derive an approximation to the time-domain asymptotic solution of Eqs. (18) and (19) for a proton bunch of nonuniform line density propagating through a stationary electron background (the one pass interaction). This approximate solution of Eqs. (18) and (19), referred to as the "e-p mode",

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has a wavelength proportional to the electron bounce frequency. For small initial perturbations such as sinusoidal or noisy initial conditions, the "e-p mode" solution should be a good approximation to the asymptotic state which occurs when $t \gg \tau$ and $z' \gg v/f_e$, where τ is the instability e-folding time, z' is the distance from the head of the proton bunch, and f_e is the typical electron bounce frequency inside the proton bunch. In fact, this property has been observed in many numerical solutions of Eqs. (18) and (19).

Following the general formalism, the discussions in the original paper [1] were concentrated on the growth rate of the "e-p mode" and the instability threshold inferred from it. We also derived a dispersion relation for the case of a uniform density electron background for the frequency-domain studies. We found that the threshold and the initial growth rate depend strongly on the initial conditions. For "non-e-p modes", the threshold and the initial growth rate can be quite different from those presented in the paper. It was also found that in the beam frame the electron oscillation frequency spread causes spatial damping but no temporal damping. Furthermore, the asymptotic amplitude ratio between the proton oscillation and the electron oscillation is independent of the frequency spread. For a Lorentzian distribution of transverse oscillation frequencies, the instability eventually damps for sufficiently long times $[t > (z'/v)+\omega_\beta \mathcal{J}(z')/(2\Delta_p^2)]$, where $\mathcal{J}(z')$ characterizes the coupling between the protons and the electrons, and is defined by

$$\mathcal{J}(z') \approx \frac{i}{(v\omega_\beta)^2 W} \int_0^{z'} \Omega^2(x) G(x) \Phi(x) \Psi(x) dx \;, \quad (44)$$

where $i = \sqrt{-1}$, W is the Wronskian of $\Phi(x)$ and $\Psi(x)$; $\Phi(x) = R(x)\mathrm{e}^{i\Theta(x)}$ and $\Psi(x) = R(x)\mathrm{e}^{-i\Theta(x)}$ are the two linear independent solutions of Eq. (19) under the conditions $\Delta_e = 0$ and $Y_p = 0$.

Numerical results were presented by considering proton line densities with uniform and parabolic profiles. For the parabolic proton line density, the "e-p modes" are parabolic cylinder functions. We found that Eq. (68) in the original paper overestimates the required frequency spread for stability.

SUMMARY OF REVISIONS

First, we add some comments to the procedure of obtaining the centroid equations [Eqs. (11) and (12)] and the applicability of Eq. (12) in the case of nonuniform proton line density. We note that although the centroid equations were derived from the solutions of the homogeneous parts of Eqs. (3) and (4) in the original paper [1], it is straightforward to show that these centroid equations can be deduced from the full solutions of Eqs. (3) and (4) by using the same procedure. We also note that only the uniform proton line density (λ_p) was considered in the process of deriving Eq. (12) given in the original paper. For nonuniform proton line density, the ω_e^2 term in Eq. (4) is replaced by a function proportional to λ_p, and $y_e = A(t)\exp[i\int\omega_e(t)dt] + A^*(t)\exp[-i\int\omega_e(t)dt]$. In this case, Eq. (12) is a good approximation for describing the motion of the electron centroid only when $\Delta_e \ll \omega_{eo}$ and $|A^{-1}dA/dt| \ll \omega_e$.

Next, we note that Eqs. (34), (36), (38) and (41) are homogeneous Volterra's integral equations of the second kind and zero would be the only solution for $\hat{Y}$ if the equal sign were used in these equations. However, we should keep in mind that these equations were derived from Eqs. (18) and (19) by neglecting the initial condition terms, otherwise one would obtain inhomogeneous integral equations (which are known to have a nonzero solution). As stated earlier, our interest is to study the general characteristics of the e-p instability from the asymptotic solution of Eqs. (18) and (19) instead of examining the evolution of any specific initial perturbation. Thus, we are actually looking for an approximate solution of an inhomogeneous Volterra's integral equation of the second kind at the stage where the instability has grown into oscillations with amplitude much larger than the small initial perturbation. The homogeneous integral equations occur in the process of excluding the direct contribution of initial conditions at large t/τ for easier access to the approximate functional form of the asymptotic solution. Hence, in Eqs. (34), (36) and (38), the equal sign (=) should be replaced by the approximate sign (≈) to indicate the omission of initial conditions.

The most consequential revision is in Eq. (42) which should read [2]

$$\zeta(z',\omega) \approx \left(\frac{1}{\omega^2 - \omega_\beta^2}\right) \exp\left[\frac{i\omega_\beta^2 \mathcal{J}(z')}{\omega^2 - \omega_\beta^2}\right] . \quad (42)$$

In the solution for $\zeta(z',\omega)$ given here, the factor $1/(\omega^2 - \omega_\beta^2)$ before the exponential function is needed for the inverse Fourier transformation to yield the correct solution for $Y_p(z',t)$. This can be seen from the special case where in the absence of electrons, $\mathcal{J}(z') = 0$, and Eq. (42) in the original paper leads to an undefined solution instead of the betatron-oscillation solution in Eq. (45).

This change in Eq. (42) causes revisions in the solutions for Y_p and Y_e as well as many related equations [2]. The correct growing parts of Y_p and Y_e should read

$$Y_p(z',t) \approx C_p M_p(z')\mathrm{e}^{-\Delta_p t} \times \left[I_0^2(u) + \frac{\mathcal{J}^2(z') I_1^2(u)}{4u^2}\right]^{1/2} \cos T_p \;, \quad (55)$$

and

$$Y_e(z',t) \approx C_e M_e(z')\mathrm{e}^{-\Delta_p t} \times \left[\frac{I_1(u)}{u} - \frac{\mathcal{J}^2(z') I_2(u)}{8u^2}\right] \cos T_e \;, \quad (58)$$

where C_p and C_e are constants, $I_n(x)$ is the nth order modified Bessel function of the first kind. Here,

$$M_p(z') = \xi(z') R(z') \exp\Big[(\Delta_p - \Delta_e) z'/v\Big] \;, \quad (51)$$

$$M_e(z') = \mathcal{J}(z')R(z')\exp\Big[(\Delta_p - \Delta_e)z'/v\Big] \ , \qquad (59)$$

$$\xi(z') = 2r_p c^2 \lambda_e(z')/(a^2\gamma\omega_\beta^2) \ , \qquad (16)$$

$$u = \sqrt{2\omega_\beta \mathcal{J}(z')(t - z'/v)} \ , \qquad (47)$$

$$T_p = \sigma_p + \psi_p - \omega_\beta(t - z'/v) + \Theta(z') - \mathcal{J}(z')/4 \ , \quad (52)$$

$$T_e = \psi_e + \sigma_e - \omega_\beta(t - z'/v) + \Theta(z') - \mathcal{J}(z')/4 \ , \quad (60)$$

σ_p and σ_e are constants, $\psi_p = \cos^{-1}\{2uI_0(u)[4u^2I_0^2(u) + \mathcal{J}^2(z')I_1^2(u)]^{-1/2}\}$, and $\psi_e = 0$. The growth rate, $\Gamma_p(z', t)$, inferred from the Eq. (55) is

$$\Gamma_p(z', t) \approx - \Delta_p + \frac{\omega_\beta \mathcal{J}(z') I_1(u)\big[4u^2 I_0(u) + \mathcal{J}^2(z') I_2(u)\big]}{u\big[4u^2 I_0^2(u) + \mathcal{J}^2(z') I_1^2(u)\big]} \ . (65)$$

Applying the small argument expansions of Bessel functions to the corrected result in Eq. (65) leads to the following instability threshold or the maximal Δ_p needed for stability

$$\left(\frac{\Delta_p}{\omega_\beta}\right)_t \approx \mathrm{Max}\left\{\frac{\mathcal{J}(z')}{2}\left[\frac{1 + \mathcal{J}^2(z')/32}{1 + \mathcal{J}^2(z')/16}\right]\right\} , \qquad (68)$$

which is about twice as large as that given in the original paper [1]. As a result, the instability threshold presented in the numerical example should be near $(\Delta_p/\omega_\beta)_t \approx 2.9\%$ instead of 1.4% obtained previously [1].

The correction in Eq. (42) also causes some revisions in Fig. 3 - 4, as well as in Eqs. (A1), (A3), (A11)-(A13), (A16), and (A17) in Appendix A of the original paper [1]. The Erratum [2] also includes some corrections to some typographical errors in Eqs. (B4), (B6), (B8), and (B9).

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REFERENCES

[1] T. F. Wang, P. J. Channell, R. J. Macek and R. C. Davidson, Centroid Theory of Transverse Electron-Proton Two-Stream Instability in a Long Proton Bunch, Phys. Rev. ST Accel. Beams **6**, 014204 (2003).

[2] T. F. Wang, P. J. Channell, R. J. Macek and R. C. Davidson, Erratum: Centroid Theory of Transverse Electron-Proton Two-Stream Instability in a Long Proton Bunch [Phys. Rev. ST Accel. Beams **6**, 014204 (2003)], Phys. Rev. ST Accel. Beams **7**, 069901 (2004).