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A Comparison of Different Ways for the Evolution of α_s

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Abstract

Using different methods based on the renormalization group equation to evolve the strong coupling constant α_s from one energy to another gives different results. The difference between the maximum and the minimum of these results is an estimate for the theoretical uncertainty in the evolution of α_s.

Evolving $\alpha_s(m_\tau)$ to $\alpha_s(m_Z)$ the uncertainty in $\alpha_s(m_Z)$ due to different ways of crossing flavour thresholds and the uncertainty due to missing higher order coefficients were both found to be about 2.5%.

1 Introduction

The value of the strong coupling constant α_s depends on energy (running coupling constant). This behaviour is shown in Fig. 1. Its energy dependence is given by the Renormalization Group Equation (RGE):

$$q^2 \frac{\partial \alpha_s}{\partial q^2} = -\sum_{k=0}^{\infty} b_k \, \alpha_s^{k+2} \tag{1}$$

where the coefficients b_k depend of the number of active quark flavours $n_f(q)$. Theoretical uncertainties arise from the fact that the b_k are only known up to next-to-next-to-leading order (b_2) and that $n_f(q)$ is not defined unambigously. This work gives a quantitative estimate of these uncertainties.

2 Methods Used to Evolve α_s

The RGE is given in leading order as [1]

$$\begin{aligned} q^2 \frac{\partial \alpha_s}{\partial q^2} &= -b_0 \alpha_s^2 \\ b_0 &= \frac{33 - 2n_f}{12\pi} \end{aligned} \tag{2}$$

with n_f being the number of active flavours. For $n_f = const$ the equation can be solved analytically and one gets the solution

$$\alpha_s = \frac{1}{b_0 \ln \frac{q^2}{\Lambda^2}} \tag{3}$$

For the evolution of $\alpha_s(q)$ over a wider range of energies the next-to-leading order approximation of the RGE has to be used:

$$\begin{aligned} q^2 \frac{\partial \alpha_s}{\partial q^2} &= -b_0 \alpha_s^2 - b_1 \alpha_s^3 \\ b_0 &= \frac{33 - 2n_f}{12\pi} \\ b_1 &= \frac{153 - 19n_f}{24\pi^2} \end{aligned} \tag{4}$$

Including next-to-leading order terms this differential equation is solved by:

$$\alpha_s = \frac{1}{b_0 \ln \frac{q^2}{\Lambda^2}} \left(1 - \frac{b_1}{b_0^2} \frac{\ln \ln \frac{q^2}{\Lambda^2}}{\ln \frac{q^2}{\Lambda^2}} \right) \tag{5}$$

To the same order this solution can as well be written as

$$\alpha_s = \frac{1}{b_o \ln \frac{q^2}{\Lambda^2} + \frac{b_1}{b_0} \ln \ln \frac{q^2}{\Lambda^2}} \tag{6}$$

In order to use the solutions involving the parameter Λ this parameter has to be calculated from a given pair of q and $\alpha_s(q)$ first. In higher than leading order Λ is calculated numerically. In next-to-leading order it is possible to find an approximate formula for Λ [2]

$$\Lambda = q \exp\left(-\frac{1}{2\,b_0\,\alpha_s(q)} - \frac{b_1}{2\,b_0^2}\ln\left(\frac{\frac{b_1}{b_0}\,\alpha_s(q)}{\frac{b_1}{b_0}\,\alpha_s(q)+1}\right)\right) \tag{7}$$

The RGE is known to order α_s^4:

$$\begin{aligned} q^2\frac{\partial\alpha_s}{\partial q^2} &= -b_0\alpha_s^2 - b_1\alpha_s^3 - b_2\alpha_s^4 \\ b_0 &= \frac{33-2n_f}{12\pi} \\ b_1 &= \frac{153-19n_f}{24\pi^2} \\ b_2 &= \frac{2857-\frac{5033}{18}n_f+\frac{325}{54}n_f^2}{(4\pi)^3} \end{aligned} \tag{8}$$

An approximate solution of this equation is:

$$\alpha_s = \frac{1}{b_0\,\ln\frac{q^2}{\Lambda^2}}\left(1 - \frac{b_1}{b_0^2}\frac{\ln\ln\frac{q^2}{\Lambda^2}}{\ln\frac{q^2}{\Lambda^2}} + \left(\frac{b_1}{b_0^2}\frac{\ln\ln\frac{q^2}{\Lambda^2}}{\ln\frac{q^2}{\Lambda^2}}\right)^2 - \frac{b_1^2\left(\ln\ln\frac{q^2}{\Lambda^2}+1\right)-b_0\,b_2}{b_0^3\,\ln^2\frac{q^2}{\Lambda^2}}\right) \tag{9}$$

It is also possible to find a solution of the RGE which directly relates α_s at different energies without a parameter Λ. In next-to-leading order one gets: [3]

$$\begin{aligned} \omega &= 1 - b_0\,\alpha_s(q_0)\,\ln\frac{q_0^2}{q^2} \\ \alpha_s(q) &= \frac{\alpha_s(q_0)}{\omega}\left(1-\frac{b_1}{b_0}\frac{\alpha_s(q_0)}{\omega}\ln\omega\right) \end{aligned} \tag{10}$$

As already mentioned, one problem which causes theoretical uncertainties is the energy dependence of n_f. Here, three different ways to calculate the effective number of flavours shall be compared: [4]

1. $n_f(q)$ is a step function which is increased by one at the quark masses m_q.

2. $n_f(q)$ is a step function but it is increased by one at $2\,m_q$.

3. $n_f(q)$ is the "MOM-function"

$$\begin{aligned} n_f(q) &= \sum_{quarks} K\left(\frac{m_q^2}{q^2}\right) \\ K(x) &= 1 - 6x + \frac{12x^2}{\sqrt{1+4x}}\ln\left(\frac{\sqrt{1+4x}+1}{\sqrt{1+4x}-1}\right) \end{aligned} \tag{11}$$

These functions $n_f(q)$ are shown in Fig. 2. The quark masses used are:

m_u	5 MeV
m_d	10 MeV
m_s	200 MeV
m_c	1.5 GeV
m_b	5 GeV

To compare the ways of evolving α_s mentioned above, different measurements of α_s at certain energies [5] were evolved to $\alpha_s(m_Z)$. Firstly, the RGE was solved in leading (2), next-to-leading (4) and next-to-next-to-leading order (8) with the Runge-Kutta algorithm. These results will be referred to as the "numerical solution" in the following. Then the solutions (3,5,9) involving the parameter Λ were used. For $n_f(q)$ being the "MOM-function" (11) it is necessary to evolve in small steps of q and calculate n_f for each step. The second order solution (7) for Λ was also compared with the other methods. Finally, the direct solution (10) without Λ in next-to-leading order was compared with the second order numerical solution of the RGE.

The results for all methods are given for $n_f(q)$ based on the three above mentioned schemes.

The errors of $\alpha_s(m_Z)$ were calculated by evolving the values of $\alpha_s(q) + \sigma_{\alpha_s(q)}$ and $\alpha_s(q) - \sigma_{\alpha_s(q)}$ to the Z-mass. The error is the difference between the two results divided by two. Numerical errors are of the order 0.1%.

3 Results

First, the different solutions of the RGE shall be compared with its numerical solutions. It is clear that the first order results are identical because the leading order RGE is solved exactly. The results of the next-to-leading order approximations are more interesting: One sees that the direct solution (10) without using Λ comes closest to the numerical solution of the RGE. A comparison of (5) and (6) reveals that these equations lead to quite similar results, with (6) being closer to the numerical solution when evolving from 1.78 GeV, (5) otherwise. The last method, solving for Λ to second order (7), shows some severe problems: The result for step functions in $n_f(q)$ seems to be systematically too high and when reducing the step length of q, as it is necessary for the calculation with the MOM-function, it even diverges. Consequently it should be used only when not crossing any flavour thresholds.

Another interesting aspect is a strong dependence of the numerical value for Λ on the order used (Fig. 3-5). It is evident that Λ calculated by the first order order solution is not a good approximation for higher order Λ's.

Next, theoretical uncertainties arising from using different ways to solve the RGE at different orders shall be examined. The second order calculation of Λ is excluded for the reasons mentioned above. Generally, the numerical solution with the flavour thresholds at m_q leads to the highest value of $\alpha_s(m_Z)$ and the solution with Λ using the MOM-function to get $n_f(q)$ gives the smallest value. An example for the difference between these two values is shown in Fig. 6. An illustration of this difference in different orders is shown in Fig. 7: α_s is expressed in units of the first order solution (3) with $n_f(q)$ according to the "MOM-scheme" (11). The relatively small uncertainty observed for the first order evolution is only due to the different treatment of flavour thresholds, whereas for the second and third order solutions the uncertainty also includes the effect of the various approximations discussed above.

In order to have an estimate for the theoretical error for $\alpha_s(m_Z)$ depending on the energy q where the evolution started, $\alpha_s(q)$ was calculated by evolving from $\alpha_s(m_Z)$=0.12 to $\alpha_s(q)$ using the solutions (3,5,9) with Λ and the "MOM-function" for $n_f(q)$. This result was evolved back to m_Z with the numerical solution of the RGE, taking the flavour thresholds at m_q. The difference between this value and 0.12 is shown in Fig. 8.

4 Conclusions

The first important conclusion is the fact that the numerical value for Λ depends strongly on the order used for the solution of the RGE. Consequently it is necessary to state which order one is using when giving a value for Λ.

Next, the theoretical error due to different assumptions for the energy dependence of the active number of flavours is found to be approximately 2.5% when evolving α_s from m_τ to m_Z.

Taking the difference of the second and third order numerical solution as an estimate for the theoretical uncertainty associated with the truncation of the RGE at next-to-next-to-leading order, this uncertainty is again found to be about 2.5% for $\alpha_s(m_Z)$ when evolving from m_τ to m_Z.

References

[1] Altarelli, Kleiss, Verzegnassi (Ed.), CERN 89-08 Vol. 1(1989) 376ff

[2] REV. OF PARTICLE PROPERTIES(1992) III.54

[3] ALEPH Collaboration, CERN-PPE/92-33(1992) 12

[4] Davier, Le Diberder, Martinez, ALEPH 91-111(1991)

[5] Bethke, Catani, CERN-TH.6484/92

TABLES

$$\alpha_s(1.78GeV) = 0.32 \pm 0.04 \qquad \Longrightarrow \qquad \alpha_s(91.20GeV)$$

NUMERICAL SOLUTION

order	1	2	3
steps at m_q	0.1244 ± 0.0061	0.1186 ± 0.0051	0.1154 ± 0.0044
steps at $2\,m_q$	0.1224 ± 0.0059	0.1163 ± 0.0049	0.1132 ± 0.0042
MOM-function	0.1217 ± 0.0058	0.1155 ± 0.0048	0.1124 ± 0.0041

SOLUTION WITH Λ

order	1	2	3
steps at m_q	0.1244 ± 0.0061	0.1171 ± 0.0046	0.1150 ± 0.0044
steps at $2\,m_q$	0.1224 ± 0.0059	0.1144 ± 0.0043	0.1126 ± 0.0042
MOM-function	0.1217 ± 0.0058	0.1136 ± 0.0042	0.1117 ± 0.0041

2^{nd} ORDER SOLUTIONS

	$1/(1+...)$	Λ in 2^{nd} order	no Λ
steps at m_q	0.1175 ± 0.0049	0.1211 ± 0.0055	0.1183 ± 0.0050
steps at $2\,m_q$	0.1151 ± 0.0046	0.1199 ± 0.0057	0.1160 ± 0.0048
MOM-function	0.1143 ± 0.0046	$*****$	0.1154 ± 0.0048

MAXIMA AND MINIMA (EXCEPT Λ SECOND ORDER)

order	1	2	3
minima:	0.1217 ± 0.0058	0.1136 ± 0.0042	0.1117 ± 0.0041
maxima:	0.1244 ± 0.0061	0.1186 ± 0.0051	0.1154 ± 0.0044

$$\alpha_s(5GeV) = 0.193 \pm 0.019 \quad \Longrightarrow \quad \alpha_s(91.20GeV)$$

NUMERICAL SOLUTION

order	1	2	3
steps at m_q	0.1146 ± 0.0067	0.1120 ± 0.0062	0.1108 ± 0.0059
steps at $2\,m_q$	0.1137 ± 0.0066	0.1108 ± 0.0061	0.1096 ± 0.0058
MOM-function	0.1131 ± 0.0065	0.1102 ± 0.0060	0.1089 ± 0.0057

SOLUTION WITH Λ

order	1	2	3
steps at m_q	0.1146 ± 0.0067	0.1118 ± 0.0061	0.1106 ± 0.0059
steps at $2\,m_q$	0.1137 ± 0.0066	0.1105 ± 0.0059	0.1093 ± 0.0057
MOM-function	0.1130 ± 0.0065	0.1098 ± 0.0058	0.1086 ± 0.0056

2^{nd} ORDER SOLUTIONS

	$1/(1+...)$	Λ in 2^{nd} order	no Λ
steps at m_q	0.1116 ± 0.0061	0.1131 ± 0.0064	0.1119 ± 0.0062
steps at $2\,m_q$	0.1104 ± 0.0060	0.1127 ± 0.0064	0.1108 ± 0.0061
MOM-function	0.1097 ± 0.0059	$*****$	0.1101 ± 0.0060

MAXIMA AND MINIMA (EXCEPT Λ SECOND ORDER)

order	1	2	3
minima:	0.1130 ± 0.0065	0.1097 ± 0.0059	0.1086 ± 0.0056
maxima:	0.1146 ± 0.0067	0.1120 ± 0.0062	0.1108 ± 0.0059

$$\alpha_s(7.1GeV) = 0.18 \pm 0.014 \quad \Longrightarrow \quad \alpha_s(91.20GeV)$$

NUMERICAL SOLUTION

order	1	2	3
steps at m_q	0.1153 ± 0.0058	0.1130 ± 0.0054	0.1120 ± 0.0052
steps at $2\,m_q$	0.1149 ± 0.0057	0.1125 ± 0.0053	0.1114 ± 0.0051
MOM-function	0.1143 ± 0.0056	0.1118 ± 0.0053	0.1107 ± 0.0050

SOLUTION WITH Λ

order	1	2	3
steps at m_q	0.1153 ± 0.0058	0.1129 ± 0.0053	0.1118 ± 0.0051
steps at $2\,m_q$	0.1149 ± 0.0057	0.1123 ± 0.0052	0.1112 ± 0.0050
MOM-function	0.1143 ± 0.0056	0.1115 ± 0.0052	0.1105 ± 0.0050

2^{nd} ORDER SOLUTIONS

	$1/(1+...)$	Λ in 2^{nd} order	no Λ
steps at m_q	0.1127 ± 0.0053	0.1142 ± 0.0055	0.1130 ± 0.0054
steps at $2\,m_q$	0.1121 ± 0.0053	0.1143 ± 0.0056	0.1124 ± 0.0053
MOM-function	0.1114 ± 0.0052	* * * * *	0.1118 ± 0.0053

MAXIMA AND MINIMA (EXCEPT Λ SECOND ORDER)

order	1	2	3
minima:	0.1143 ± 0.0056	0.1114 ± 0.0052	0.1105 ± 0.0050
maxima:	0.1153 ± 0.0058	0.1130 ± 0.0054	0.1120 ± 0.0052

$$\alpha_s(10GeV) = 0.167 \pm 0.015 \qquad \Longrightarrow \qquad \alpha_s(91.20GeV)$$

NUMERICAL SOLUTION

order	1	2	3
steps at m_q	0.1151 ± 0.0071	0.1132 ± 0.0068	0.1124 ± 0.0065
steps at $2\,m_q$	0.1151 ± 0.0071	0.1132 ± 0.0068	0.1124 ± 0.0065
MOM-function	0.1145 ± 0.0071	0.1124 ± 0.0067	0.1116 ± 0.0064

SOLUTION WITH Λ

order	1	2	3
steps at m_q	0.1151 ± 0.0071	0.1131 ± 0.0067	0.1122 ± 0.0065
steps at $2\,m_q$	0.1151 ± 0.0071	0.1131 ± 0.0066	0.1122 ± 0.0065
MOM-function	0.1145 ± 0.0071	0.1123 ± 0.0066	0.1114 ± 0.0064

2^{nd} ORDER SOLUTIONS

	$1/(1+...)$	Λ in 2^{nd} order	no Λ
steps at m_q	0.1129 ± 0.0067	0.1144 ± 0.0069	0.1132 ± 0.0067
steps at $2\,m_q$	0.1129 ± 0.0067	0.1144 ± 0.0069	0.1132 ± 0.0067
MOM-function	0.1121 ± 0.0066	* * * * *	0.1124 ± 0.0067

MAXIMA AND MINIMA (EXCEPT Λ SECOND ORDER)

order	1	2	3
minima:	0.1145 ± 0.0071	0.1121 ± 0.0066	0.1114 ± 0.0064
maxima:	0.1151 ± 0.0071	0.1132 ± 0.0068	0.1124 ± 0.0065

$$\alpha_s(34GeV) = 0.163 \pm 0.022 \quad \Longrightarrow \quad \alpha_s(91.20GeV)$$

NUMERICAL SOLUTION

order	1	2	3
steps at m_q	0.1363 ± 0.0154	0.1349 ± 0.0149	0.1343 ± 0.0146
steps at $2m_q$	0.1363 ± 0.0154	0.1349 ± 0.0149	0.1343 ± 0.0146
MOM-function	0.1362 ± 0.0154	0.1348 ± 0.0149	0.1342 ± 0.0146

SOLUTION WITH Λ

order	1	2	3
steps at m_q	0.1363 ± 0.0154	0.1348 ± 0.0148	0.1342 ± 0.0146
steps at $2m_q$	0.1363 ± 0.0154	0.1348 ± 0.0148	0.1342 ± 0.0146
MOM-function	0.1362 ± 0.0154	0.1347 ± 0.0148	0.1341 ± 0.0146

2^{nd} ORDER SOLUTIONS

	$1/(1+...)$	Λ in 2^{nd} order	no Λ
steps at m_q	0.1347 ± 0.0148	0.1367 ± 0.0154	0.1349 ± 0.0149
steps at $2m_q$	0.1347 ± 0.0148	0.1367 ± 0.0154	0.1349 ± 0.0149
MOM-function	0.1346 ± 0.0148	$*****$	0.1348 ± 0.0149

MAXIMA AND MINIMA (EXCEPT Λ SECOND ORDER)

order	1	2	3
minima:	0.1362 ± 0.0154	0.1346 ± 0.0148	0.1341 ± 0.0146
maxima:	0.1363 ± 0.0154	0.1349 ± 0.0149	0.1343 ± 0.0146

$$\alpha_s(35GeV) = 0.140 \pm 0.020 \quad \Longrightarrow \quad \alpha_s(91.20GeV)$$

NUMERICAL SOLUTION

order	1	2	3
steps at m_q	0.1203 ± 0.0148	0.1194 ± 0.0145	0.1191 ± 0.0143
steps at $2\,m_q$	0.1203 ± 0.0148	0.1194 ± 0.0145	0.1191 ± 0.0143
MOM-function	0.1202 ± 0.0148	0.1194 ± 0.0144	0.1190 ± 0.0142

SOLUTION WITH Λ

order	1	2	3
steps at m_q	0.1203 ± 0.0148	0.1194 ± 0.0144	0.1190 ± 0.0142
steps at $2\,m_q$	0.1203 ± 0.0148	0.1194 ± 0.0144	0.1190 ± 0.0142
MOM-function	0.1202 ± 0.0148	0.1193 ± 0.0144	0.1189 ± 0.0142

2^{nd} ORDER SOLUTIONS

	$1/(1+...)$	Λ in 2^{nd} order	no Λ
steps at m_q	0.1193 ± 0.0144	0.1208 ± 0.0148	0.1194 ± 0.0144
steps at $2\,m_q$	0.1193 ± 0.0144	0.1208 ± 0.0148	0.1194 ± 0.0144
MOM-function	0.1192 ± 0.0144	$*****$	0.1194 ± 0.0144

MAXIMA AND MINIMA (EXCEPT Λ SECOND ORDER)

order	1	2	3
minima:	0.1202 ± 0.0148	0.1192 ± 0.0144	0.1189 ± 0.0142
maxima:	0.1203 ± 0.0148	0.1194 ± 0.0145	0.1191 ± 0.0143

$$\alpha_s(20GeV) = 0.136 \pm 0.025 \quad \Longrightarrow \quad \alpha_s(91.20GeV)$$

NUMERICAL SOLUTION

order	1	2	3
steps at m_q	0.1086 ± 0.0160	0.1076 ± 0.0155	0.1072 ± 0.0152
steps at $2\,m_q$	0.1086 ± 0.0160	0.1076 ± 0.0155	0.1072 ± 0.0152
MOM-function	0.1084 ± 0.0159	0.1074 ± 0.0154	0.1070 ± 0.0152

SOLUTION WITH Λ

order	1	2	3
steps at m_q	0.1086 ± 0.0160	0.1076 ± 0.0154	0.1071 ± 0.0152
steps at $2\,m_q$	0.1086 ± 0.0160	0.1076 ± 0.0154	0.1071 ± 0.0152
MOM-function	0.1084 ± 0.0159	0.1073 ± 0.0154	0.1069 ± 0.0151

2^{nd} ORDER SOLUTIONS

	$1/(1+...)$	Λ in 2^{nd} order	no Λ
steps at m_q	0.1075 ± 0.0154	0.1087 ± 0.0158	0.1076 ± 0.0155
steps at $2\,m_q$	0.1075 ± 0.0154	0.1087 ± 0.0158	0.1076 ± 0.0155
MOM-function	0.1072 ± 0.0153	$*****$	0.1074 ± 0.0154

MAXIMA AND MINIMA (EXCEPT Λ SECOND ORDER)

order	1	2	3
minima:	0.1084 ± 0.0159	0.1072 ± 0.0153	0.1069 ± 0.0151
maxima:	0.1086 ± 0.0160	0.1076 ± 0.0155	0.1072 ± 0.0152

$$\alpha_s(80.6GeV) = 0.123 \pm 0.027 \quad \Longrightarrow \quad \alpha_s(91.20GeV)$$

NUMERICAL SOLUTION

order	1	2	3
steps at m_q	0.1208 ± 0.0260	0.1207 ± 0.0260	0.1206 ± 0.0259
steps at $2\,m_q$	0.1208 ± 0.0260	0.1207 ± 0.0260	0.1206 ± 0.0259
MOM-function	0.1208 ± 0.0260	0.1206 ± 0.0260	0.1206 ± 0.0259

SOLUTION WITH Λ

order	1	2	3
steps at m_q	0.1208 ± 0.0260	0.1206 ± 0.0259	0.1206 ± 0.0259
steps at $2\,m_q$	0.1208 ± 0.0260	0.1206 ± 0.0259	0.1206 ± 0.0259
MOM-function	0.1208 ± 0.0260	0.1206 ± 0.0259	0.1206 ± 0.0259

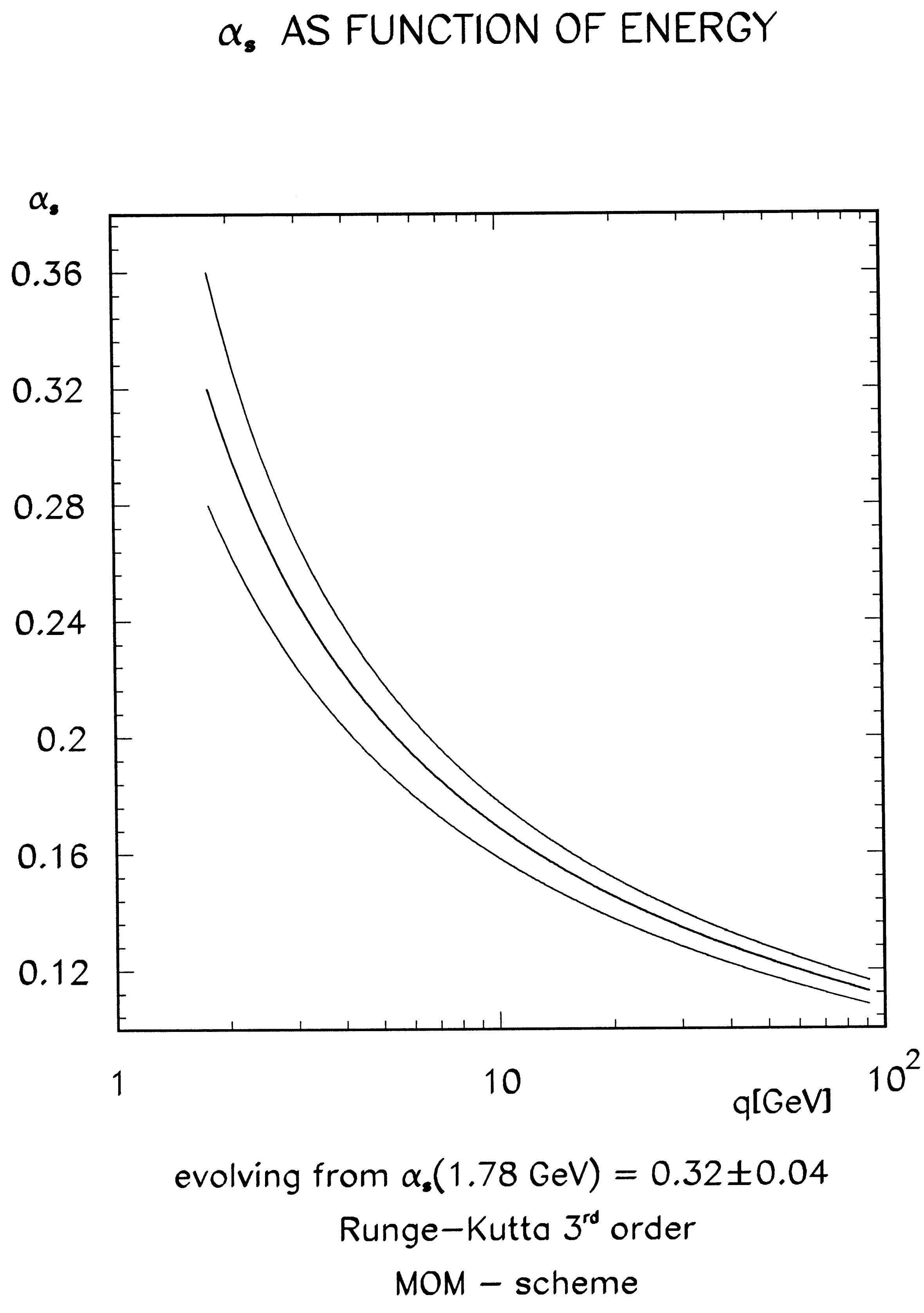
2^{nd} ORDER SOLUTIONS

	$1/(1+...)$	Λ in 2^{nd} order	no Λ
steps at m_q	0.1206 ± 0.0259	0.1220 ± 0.0266	0.1207 ± 0.0260
steps at $2\,m_q$	0.1206 ± 0.0259	0.1220 ± 0.0266	0.1207 ± 0.0260
MOM-function	0.1206 ± 0.0259	* * * * *	0.1206 ± 0.0260

MAXIMA AND MINIMA (EXCEPT Λ SECOND ORDER)

order	1	2	3
minima:	0.1208 ± 0.0260	0.1206 ± 0.0259	0.1206 ± 0.0259
maxima:	0.1208 ± 0.0260	0.1207 ± 0.0260	0.1206 ± 0.0259

FIGURES



α_s AS FUNCTION OF ENERGY
α_s
0.36
0.32
0.28
0.24
0.2
0.16
0.12
1
10
10^2
q[GeV]
evolving from α_s(1.78 GeV) = 0.32±0.04
Runge–Kutta 3rd order
MOM – scheme

Fig. 1

EFFECTIVE NUMBER OF FLAVOURS DEPENDING ON ENERGY

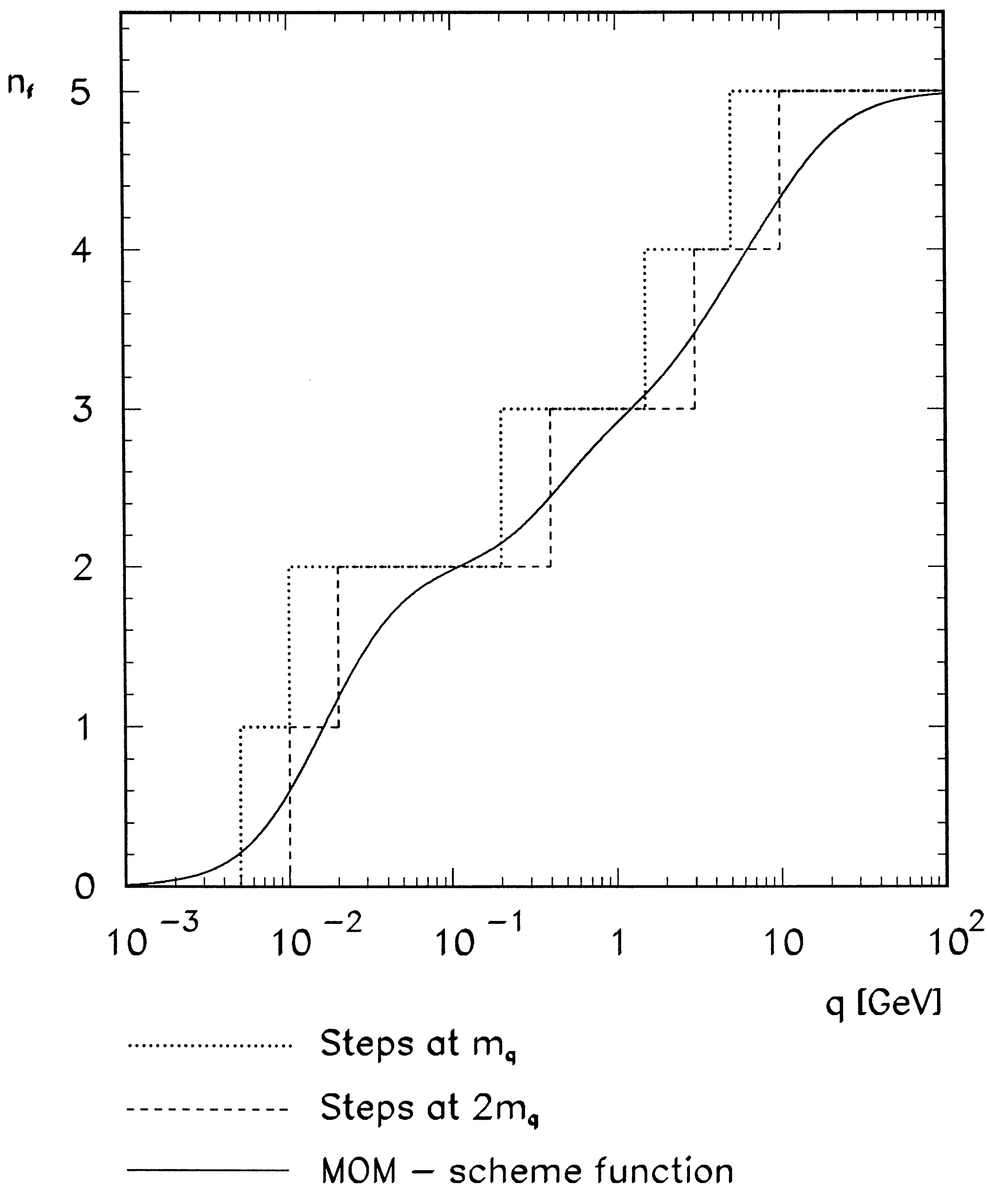
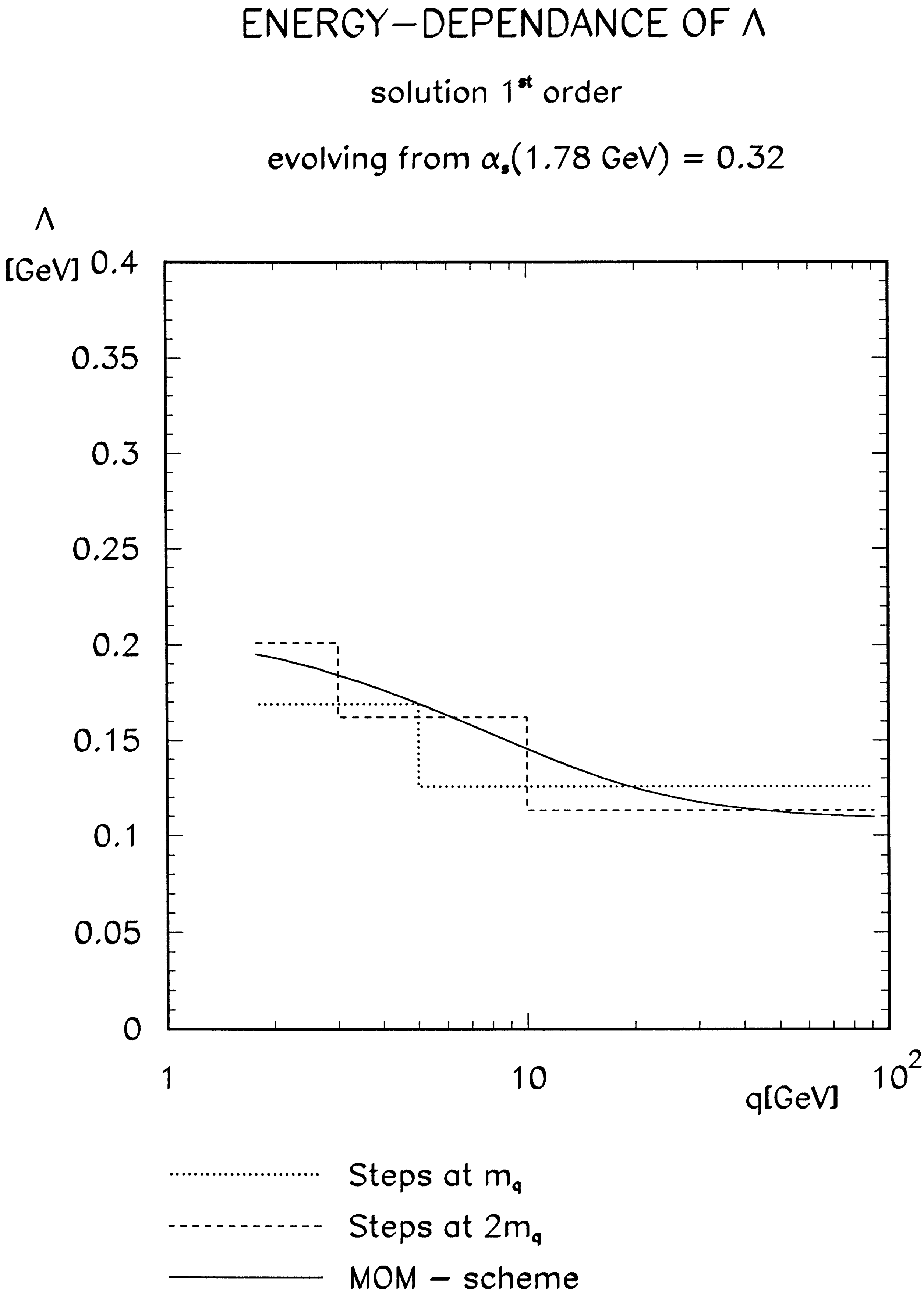
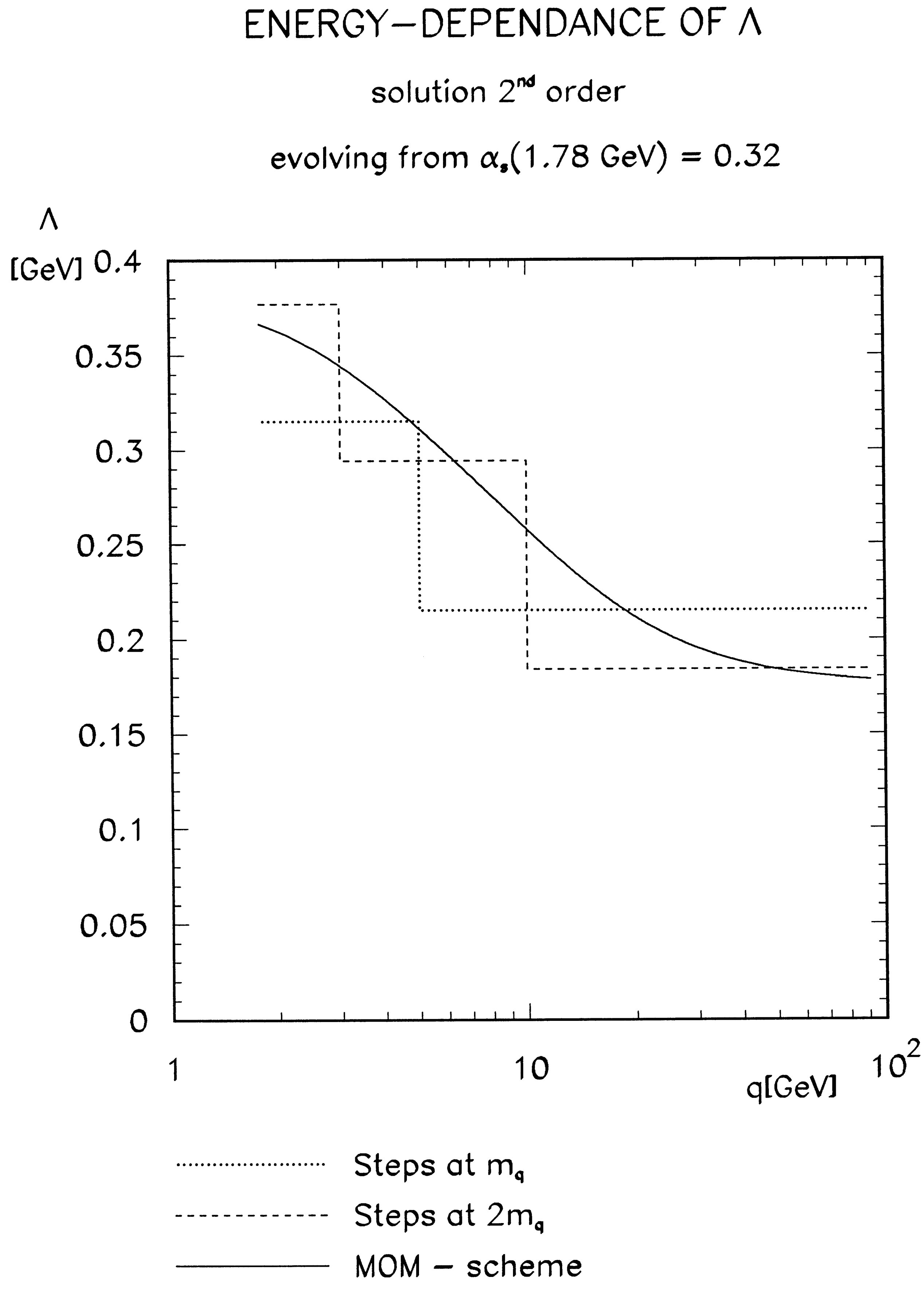


Fig. 2



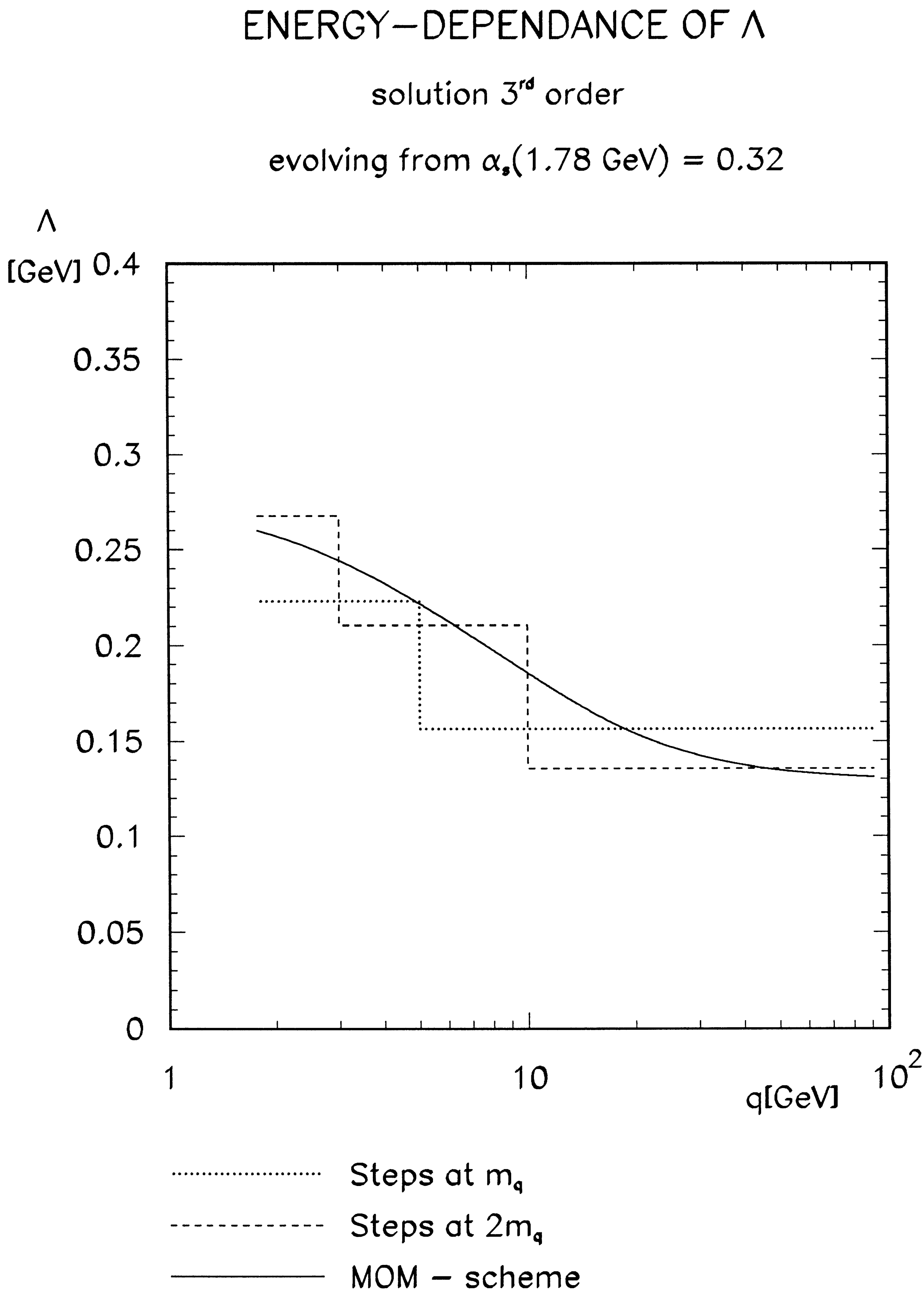
ENERGY–DEPENDANCE OF Λ
solution 1st order
evolving from α_s(1.78 GeV) = 0.32
Λ [GeV]
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
1
10
10^2
q[GeV]
Steps at m_q
Steps at $2m_q$
MOM – scheme

Fig. 3



ENERGY–DEPENDANCE OF Λ
solution 2nd order
evolving from α_s(1.78 GeV) = 0.32
Λ [GeV]
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
1
10
10^2
q[GeV]
Steps at m_q
Steps at 2m_q
MOM – scheme

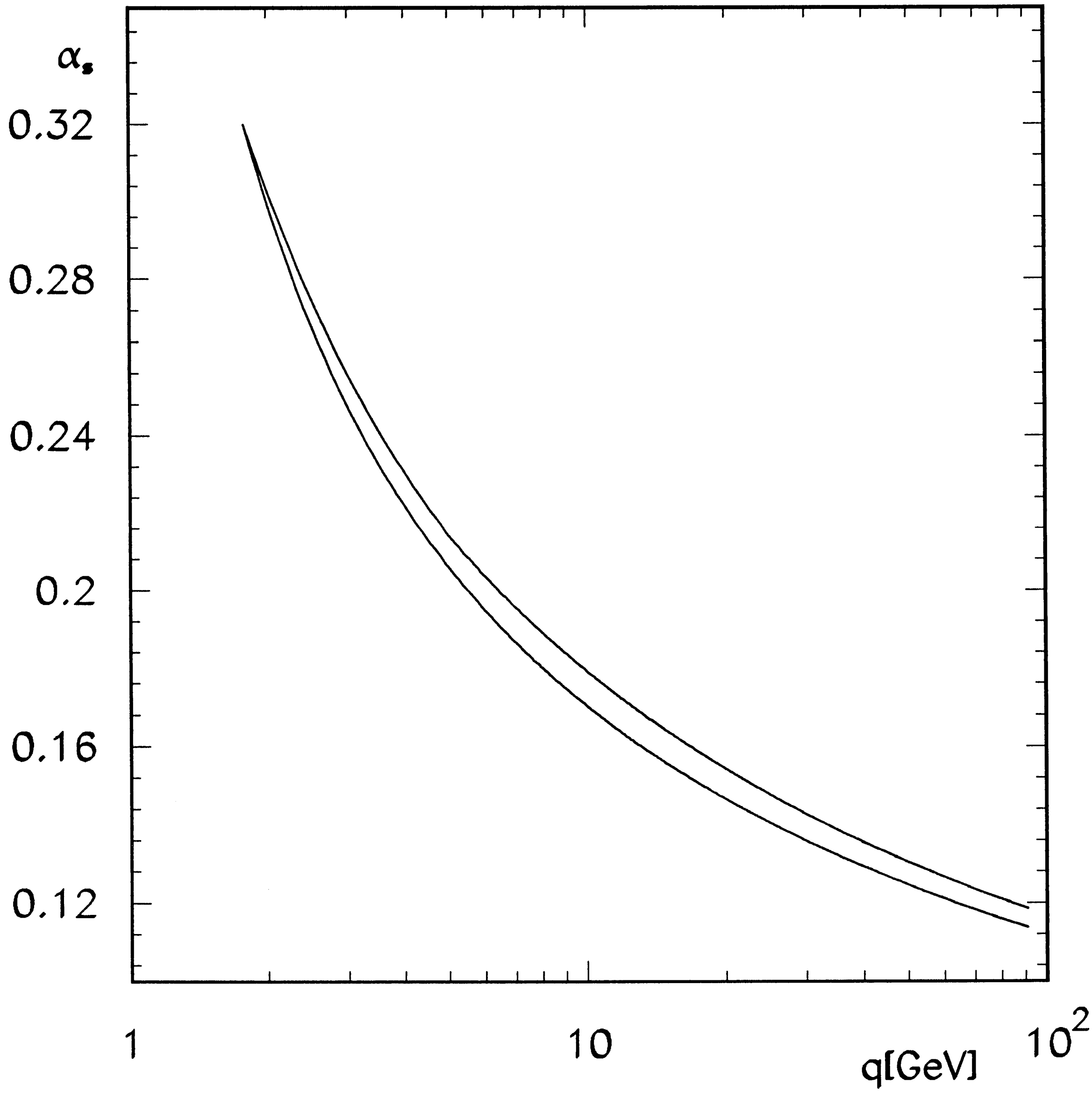
Fig. 4



ENERGY–DEPENDANCE OF Λ
solution 3rd order
evolving from $\alpha_s(1.78\ \text{GeV}) = 0.32$
Λ [GeV]
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
1
10
10^2
q[GeV]
Steps at m_q
Steps at $2m_q$
MOM – scheme

Fig. 5

α_s^{min} and α_s^{max}



evolving from $\alpha_s(1.78\ GeV) = 0.32$

2^{nd} order

Fig. 6

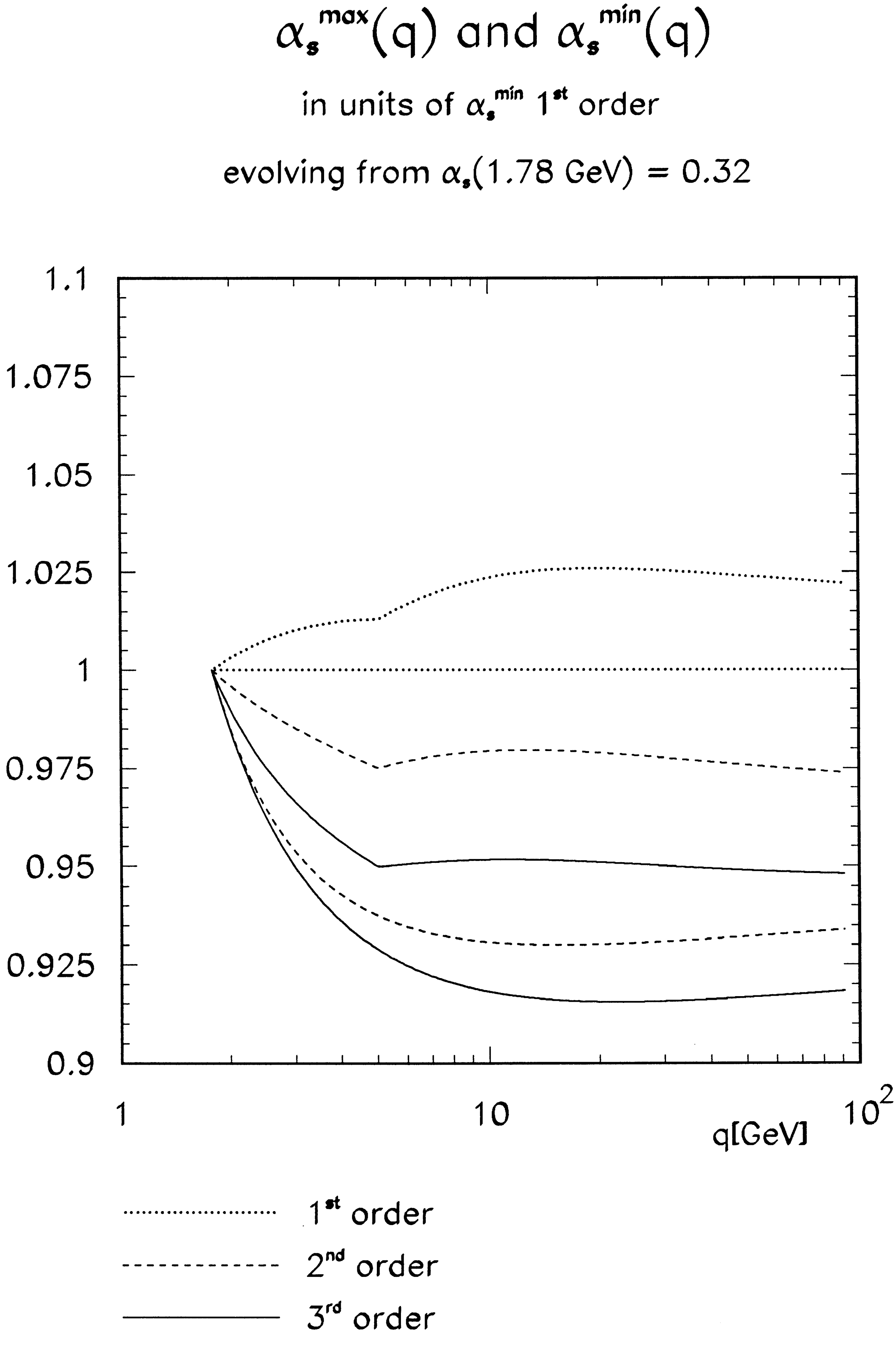

$\alpha_s^{max}(q)$ and $\alpha_s^{min}(q)$
in units of α_s^{min} 1st order
evolving from $\alpha_s(1.78\ \mathrm{GeV}) = 0.32$
1.1
1.075
1.05
1.025
1
0.975
0.95
0.925
0.9
1
10
10^2
q[GeV]
1st order
2nd order
3rd order

Fig. 7

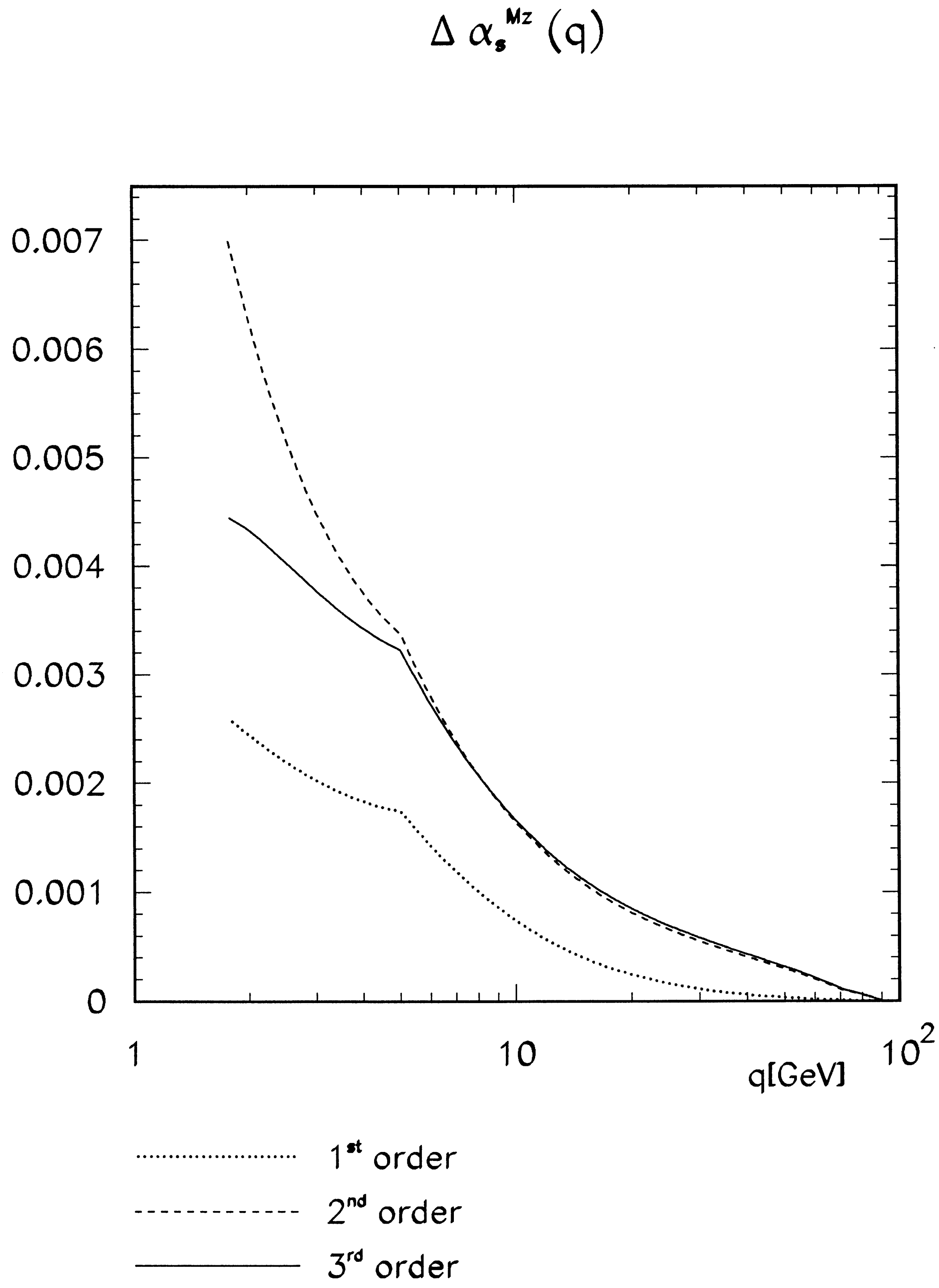

$\Delta\, \alpha_s^{Mz}\,(q)$
0.007
0.006
0.005
0.004
0.003
0.002
0.001
0
1
10
10^2
q[GeV]
1^{st} order
2^{nd} order
3^{rd} order

Fig. 8