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# The Virtual Correction to Bremsstrahlung in High-Energy $e^+e^-$ Annihilation: Comparison of Exact Results

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## Abstract

We have compared the virtual corrections to single hard bremsstrahlung as calculated by S. Jadach, M. Melles, B.F.L. Ward and S.A. Yost to several other expressions. The most recent of these comparisons is to the leptonic tensor calculated by J.H. Kühn and G. Rodrigo for radiative return. Agreement is found to within  $10^{-5}$  or better, as a fraction of the Born cross section, for most of the range of photon energies. The massless limits have been shown to agree analytically to NLL order.

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High precision studies of the Standard Model at proposed linear colliders will require per-mil level control of both the theoretical and experimental uncertainties in many critical processes to be measured. This will require computing higher order electroweak radiative corrections at least to order  $\mathcal{O}(\alpha^3 L^3)$  for the leading log effects and to the exact  $\mathcal{O}(\alpha^2)$ . One important contributions at this order is the virtual photon correction to the single hard bremsstrahlung in  $e^+e^-$  annihilations [1–3, 6, 7].

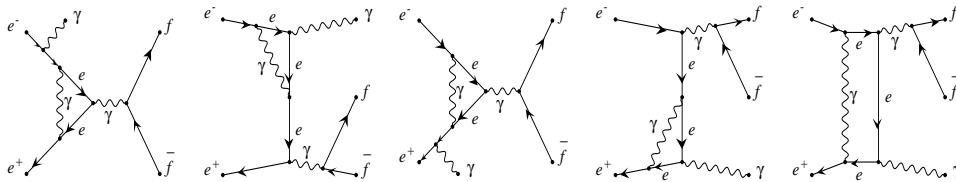


Figure 1: Feynman graphs for the virtual  $\mathcal{O}(\alpha)$  correction to the process  $e^+e^- \rightarrow f\bar{f} + \gamma$ .

In Refs. [3], some of us (S.J., B.F.L.W., S.A.Y.) have presented comparisons of the results in Refs. [1–3], and in general a very good agreement was found. In particular, all of these results can be shown to agree analytically to NLL order. However, some differences become apparent at the level of the NNLL (next-to-next-to leading log), depending on the different levels of “exactness” in the calculations. Specifically, the mass corrections are included in Ref. [3] in a fully differential way, whereas in Ref. [1] the mass corrections are included but the photon angular variable is integrated over and in Ref. [2] the results are fully differential but the mass corrections are incomplete. These comparisons therefore can not really test the NNLL, fully differential results in Ref. [3].

More recently, another fully differential expression for the virtual photon correction to single hard bremsstrahlung has appeared in Refs. [6, 7], where the result is obtained by an independent method via a leptonic tensor calculated to investigate radiative return, with particular emphasis on the 1-2 GeV cms energy regime. This permits a cross-check at the NNLL level of the corresponding results. Details have been presented in Refs. [4, 5]. An indirect comparison of the results in Refs. [3, 6, 7] was also reported in Ref. [8] via a comparison of the two Monte Carlo programs PHOKHARA [9] and  $\mathcal{K}\mathcal{K}$  MC [10], as these two Monte Carlos have the realizations of the results in Ref. [6, 7] (PHOKHARA) and Ref. [3] ( $\mathcal{K}\mathcal{K}$  MC). Agreement at the per-mil level was found on selected observables.

Relevant Feynman graphs for the process  $e^+e^- \rightarrow f\bar{f} + \gamma$  are illustrated in Fig. 1. In Ref. [3] the ISR matrix element is evaluated using the helicity spinor amplitude method of CALKUL [11], Xu *et al.* [12] and Kleiss and Stirling [13]. Computer algebra techniques [14] were used for reducing the loop integrals.

The  $\mathcal{O}(\alpha^2)$  virtual correction to single hard bremsstrahlung can be expressed in terms of a form factor multiplying the  $\mathcal{O}(\alpha)$  tree level matrix element [3]:

$$\mathcal{M}_1^{\text{ISR}(1)} = \frac{\alpha}{4\pi}(f_0 + f_1 I_1 + f_2 I_2)\mathcal{M}_1^{\text{ISR}(0)} \quad (1)$$

where  $\mathcal{M}_1^{\text{ISR}(0)}$  is the tree-level hard bremsstrahlung matrix element and  $\mathcal{M}_1^{\text{ISR}(1)}$  includes an additional virtual photon. The form factors  $f_i$  and spinor factors  $I_i$  can be found in Ref. [3] or Ref. [5]. (The latter form is equivalent to the earlier form, but is explicitly numerically stable in collinear limits.) In the Monte Carlo program, we actually calculate the YFS residuals [16]  $\bar{\beta}_1^{(2)}$  which are obtained by subtracting the soft limit  $4\pi B_{\text{YFS}}$  defined in Ref. [3] from  $f_0$  in Eq. 1, and using the matrix element to compute the cross-section.

The mass corrections are then added following the methods in Ref. [15], after checking that the exact expression for the mass corrections differs from the result obtained by the latter methods by terms which vanish as  $m_e^2/s \rightarrow 0$  where  $m_e$  is the electron mass and  $s$  is the squared cms energy. The effect is to add a correction

$$\begin{aligned} \langle f_0 \rangle_m &= \frac{2m_e^2}{s} \left( \frac{r_1}{r_2} + \frac{r_2}{r_1} \right) \frac{z}{(1-r_1)^2 + (1-r_2)^2} \\ &\times \left\{ \langle f_0 \rangle + (\ln z - 1)L - \frac{3}{2} \ln z + \frac{1}{2} \ln^2 z + 1 \right\} \end{aligned} \quad (2)$$

to the spin-averaged form factor  $\langle f \rangle_0$ , where  $L = \ln(s/m_e^2)$  is the ‘‘big logarithm’’ in the LL expansion,  $z = s'/s$  and  $r_i = 2p_i \cdot k/s$ , with incoming fermion momenta  $p_i$  and real photon momentum  $k$  defined as in Ref. [3].

To NLL order, an expression for the cross section which correctly reproduces all collinear limits can be obtained by setting  $f_1 = f_2 = 0$  and using the spin-averaged NLL form factor

$$\begin{aligned} \langle f_0 \rangle^{\text{NLL}} &= 2\{L - 1\} + \frac{r_1(1-r_1)}{1+(1-r_1)^2} + \frac{r_2(1-r_2)}{1+(1-r_2)^2} + 2\ln r_1 \ln(1-r_2) \\ &+ 2\ln r_2 \ln(1-r_1) - \ln^2(1-r_1) - \ln^2(1-r_2) + 3\ln(1-r_1) \\ &+ 3\ln(1-r_2) + 2\text{Sp}(r_1) + 2\text{Sp}(r_2) + \langle f_0 \rangle_m^{\text{NLL}} \end{aligned} \quad (3)$$

where  $\text{Sp}(x)$  is the Spence dilogarithm function and only terms surviving when  $r_1 \rightarrow 0$  or  $r_2 \rightarrow 0$  are needed in the mass correction. In the NLL limit, the  $\mathcal{O}(\alpha^2)$  YFS residual  $\bar{\beta}_1^{(2)}$  for real + virtual emission may be expressed in terms of the YFS residual  $\bar{\beta}_1^{(1)}$  for pure real emission using the result Eq. 3,

$$\bar{\beta}_1^{(2)} = \bar{\beta}_1^{(1)} \left( 1 + \frac{\alpha}{2\pi} \langle f_0 \rangle^{\text{NLL}} \right). \quad (4)$$

The new comparison uses the leptonic tensor of Ref. [7],

$$\begin{aligned} L_0^{\mu\nu} = & \frac{e^6}{s s'^2} \{ a_{00} s \eta^{\mu\nu} + a_{11} p_1^\mu p_1^\nu + a_{22} p_2^\mu p_2^\nu + a_{22} (p_1^\mu p_2^\nu + p_1^\nu p_2^\mu) \\ & + i\pi a_{-1} (p_1^\mu p_2^\nu + p_1^\nu p_2^\mu) \} \end{aligned} \quad (5)$$

The coefficients  $a_{ij}$  may be found in Ref. [7] The squared matrix element for real photon emission can be obtained by contracting this with a final-state tensor

$$H^{\mu\nu} = e^2 (p_3^\mu p_4^\nu + p_3^\nu p_4^\mu - p_3 \cdot p_4 \eta^{\mu\nu}) \quad (6)$$

to obtain

$$|\mathcal{M}^{\text{ISR}}|^2 = z L_0^{\mu\nu} H_{\mu\nu}. \quad (7)$$

The coefficients  $a_{ij}$  can be decomposed as  $a_{ij} = a_{ij}^{(0)} + \frac{\alpha}{\pi} a_{ij}^{(1)}$  in terms of a real-photon emission term and a virtual correction of order  $\alpha$ . The YFS residual  $\bar{\beta}_1^{(2)}$  may be obtained by calculating the cross section using Eq. 7 with  $a_{ij}^{(1)}$  replaced by  $c_{ij} = a_{ij}^{(1)} - a_{ij}^{\text{IR}}$ , where the IR-divergent term is

$$a_{ij}^{\text{IR}} = a_{ij}^{(0)} \left[ 2(L-1) \ln v_{\text{min}} + \frac{1}{2}L - 1 + \frac{\pi^2}{3} \right] \quad (8)$$

with cutoff  $v_{\text{min}}$  on the fraction of the beam energy radiated into the real photon,  $v = 2E_\gamma/\sqrt{s}$ . Again, it is possible to find a relatively compact expression for the NLL contribution to the YFS residual in the massless limit. The result is

$$\bar{\beta}_1^{(2)} = \bar{\beta}_1^{(1)} \left( 1 + \frac{\alpha}{2\pi} C_1 \right) + C_2 \quad (9)$$

with coefficient functions

$$\begin{aligned} C_1 &= \frac{1}{2a_{00}^{(0)}} \left( \frac{c_{11}}{z} + z c_{22} - 2c_{12} \right), \\ C_2 &= \frac{c_{11}}{4z} + \frac{z c_{22}}{4} - \frac{c_{12}}{2} - c_{00}, \end{aligned} \quad (10)$$

where the IR-finite coefficients  $c_{ij}$  are to be evaluated in the collinear limits  $r_1 \rightarrow 0$  or  $r_2 \rightarrow 0$ . We have verified that in these limits,  $C_1 = \langle f_0 \rangle^{\text{NLL}}$  and  $C_2 = 0$ , so that the results agree analytically in the massless NLL limit.

The results are illustrated in Fig. 2 for the case  $f\bar{f} = \mu^-\mu^+$ . We show the complete  $\bar{\beta}_1^{(2)}$  distribution for our exact result JMWY as presented in Ref. [3], both with and without mass corrections, the result IN of Igarashi and Nakazawa *et al.* [2], the result BVNB of Berends *et al.* [1], and the new comparison with exact results KR of Kühn and Rodrigo in Ref. [7] (with mass corrections) and Ref. [6] (without mass corrections). The results were obtained using the YFS3ff MC generator (the EEX3 option of the  $\mathcal{K}\mathcal{K}$  MC in Ref. [10]) with  $10^8$  events. The NLL contribution calculated in Ref. [3] has been subtracted in each case to permit a clear NNLL comparison. We see that there is a very good general agreement between all of these results.

Specifically, of the results agree to within  $0.4 \times 10^{-5}$  for cuts below 0.75. For cuts between 0.75 and .95, the results agree to within  $0.5 \times 10^{-5}$ , if the result of Ref. [1] is not included, and within  $1.1 \times 10^{-5}$  if that result is included. For the last data point, at  $v = 0.975$ , the result of Ref. [1] is approximately  $1 \times 10^{-4}$  greater than the others and is off-scale in Fig. 2, while the remaining results agree to  $3 \times 10^{-5}$ . The difference between the KR and JMWY result attributable to differences in the mass corrections never exceeds  $10^{-5}$  over the entire range of  $v_{\text{max}}$ .

These results are consistent with a total precision tag of  $1.5 \times 10^{-5}$  for our  $\mathcal{O}(\alpha^2)$  correction  $\bar{\beta}_1^{(2)}$  for an energy cut below  $v = 0.95$ . The NLL effect alone is adequate to within  $1.5 \times 10^{-5}$  for cuts below 0.95. The NLL effect has already been implemented in the  $\mathcal{K}\mathcal{K}$  MC in Ref. [10] and the attendant version of KK MC will be available in the near future [17].

These comparisons show that we now have a firm handle on the precision tag for an important part of the complete  $\mathcal{O}(\alpha^2)$  corrections to the  $f\bar{f}$  production process needed for precision studies of such processes in the final LEP2 data analysis, in the radiative return at  $\Phi$  and B-Factories, and in the future TESLA/ILC physics.

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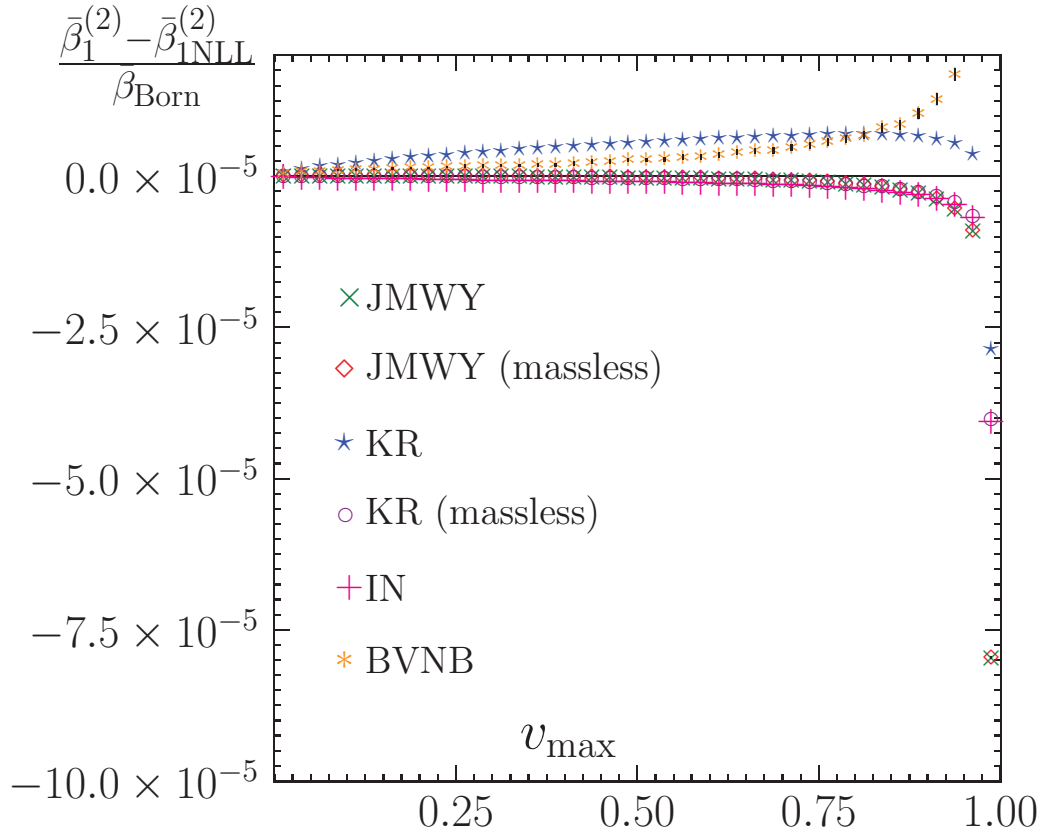


Figure 2: NNLL contribution  $\bar{\beta}_1^{(2)} - \bar{\beta}_{1\text{NLL}}^{(2)}$  for  $10^8$  events generated by the YFS3ff Monte Carlo as a function of the cut  $v_{\text{max}}$  on the fraction of the beam energy radiated to the photon. The results is in units of the non-radiative Born cross section for  $e^+e^- \rightarrow \mu^+\mu^-$ .