Dimensional Reduction over Fuzzy Coset Spaces

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Abstract: We examine gauge theories on Minkowski space-time times fuzzy coset spaces. This means that the extra space dimensions instead of being a continuous coset space S/R are a corresponding finite matrix approximation. The gauge theory defined on this non-commutative setup is reduced to four dimensions and the rules of the corresponding dimensional reduction are established. We investigate in particular the case of the fuzzy sphere including the dimensional reduction of fermion fields.

Keywords: Non-Commutative Geometry, Field Theories in Higher Dimensions, Fuzzy Coset Spaces, Dimensional Reduction.

Contents

1. Introduction

The theoretical efforts to establish a deeper understanding of Nature has led to very interesting frameworks such as String theories and Non-commutative Geometry both of which aim to describe physics at the Planck scale. Looking for the origin of the idea that coordinates might not commute we might have to go back to the days of Heisenberg. In the recent years the birth of such speculations can be found in refs. [1, 2]. In the spirit of Non-commutative Geometry also particle models with non-commutative gauge theory were explored [3] (see also [4]), [5, 6]. On the other hand the present intensive research has been triggered by the natural realization of non-commutativity of space in the string theory context of D-branes in the presence of a constant background antisymmetric field [7]. After the work of Seiberg and Witten [8], where a map (SW map) between non-commutative and commutative gauge theories has been described, there has been a lot of activity also in the construction of non-commutative phenomenological lagrangians, for example various non-commutative standard model like lagrangians have been proposed [10, 11][1]. In particular in ref. [11], following the SW map methods developed in refs. [9], a non-commutative standard model with $SU(3) \times SU(2) \times U(1)$ gauge group has been presented. These non-commutative models represent interesting generalizations of the SM and hint at possible new physics. However they do not address the usual problem of the SM, the presence of a plethora of free parameters mostly related to the ad hoc introduction of the Higgs and

[1] These SM actions are mainly considered as effective actions because they are not renormalizable. The effective action interpretation is consistent with the SM in [11] being anomaly free [12]. Non-commutative phenomenology has been discussed in [13].

Yukawa sectors in the theory. At this stage it is worth recalling that various schemes, with the Coset Space Dimensional Reduction (CSDR) [14, 15, 16, 17] being pioneer, were suggesting that a unification of the gauge and Higgs sectors can be achieved in higher dimensions. Moreover the addition of fermions in the higher-dimensional gauge theory leads naturally after CSDR to Yukawa couplings in four dimensions. In the successes of the CSDR scheme certainly should be added the possibility to obtain chiral theories in four dimensions [18, 19, 20, 21] as well as softly broken supersymmetric or non-supersymmetric theories starting from a supersymmetric gauge theory defined in higher dimensions [22].

In this paper we combine and exploit ideas from CSDR and Non-commutative Geometry. We consider the dimensional reduction of gauge theories defined in high dimensions where the internal space is a fuzzy space (matrix manifold). In the CSDR one assumes that the form of space-time is $M^D = M^4 \times S/R$ with S/R a homogeneous space (obtained as the quotient of the Lie group S via the Lie subgroup R). Then a gauge theory with gauge group G defined on M^D can be dimensionally reduced to M^4 in an elegant way using the symmetries of S/R, in particular the resulting four dimensional gauge group is a subgroup of G. In the present work we will apply the method of CSDR in the case where the internal part of the space-time is a finite approximation of the homogeneous space S/R, i.e. a fuzzy coset (fuzzy cosets are studied in [24, 25, 26, 27]). In particular we study the fuzzy sphere case [23]. Fuzzy spaces are obtained by deforming the algebra of functions on their commutative parent spaces. The algebra of functions (from the fuzzy space to complex numbers) becomes finite dimensional and non-commutative, indeed it becomes a matrix algebra. Therefore, instead of considering the algebra of functions $Fun(M^D) \sim Fun(M^4) \times Fun(S/R)$ we consider the algebra $A = Fun(M^4) \times M_n$ where $Fun(M^4)$ is the usual commutative algebra of functions on Minkowski space M^4 and M_n is the finite dimensional non-commutative algebra of matrices that approximates the coset; on this finite dimensional algebra we still have the action of the symmetry group S. This very property will allow us to apply the CSDR scheme to fuzzy cosets. In the parent theory on $M^D = M^4 \times (S/R)_F$ the non-commutativity will lead us to consider the gauge groups $G = U(1)$ and more generally $G = U(P)$ [2]. Notice that there is no a priori relation between the gauge group $G = U(P)$ and the groups S and R.

In summary, gauge theories have been studied on non-commutative Minkowski space as well as on the product space commutative Minkowski times internal non-commutative space [3, 4, 5, 6], see also ref. [28] and ref. [29], where the internal space is the lattice of a finite group (and non-commutative geometry techniques allow to describe this lattice as a manifold [30]). CSDR is a unification scheme for obtaining realistic particle models, and the study of CSDR in the non-commutative context provides new particle models that might be phenomenologically relevant. One could study CSDR with the whole parent

[2] Alternatively one could also formulate the non-commutativity of $(S/R)_F$ in terms of a star product. Then, using SW map, it is possible as in [9] to consider arbitrary gauge groups G. This approach relies on a perturbative expansion in the non-commutativity parameter $\theta^{\hat{a}\hat{b}}(\hat{X}) = [\hat{X}^{\hat{a}}, \hat{X}^{\hat{b}}]$ and therefore, see e.g. (2.3), is particularly promising when the fuzzy manifold is described by $n \times n$ matrices in the limit $n \to \infty$.

space M^D being non-commutative or with just non-commutative Minkowski space or non-commutative internal space. We specialize here to this last situation and thus in the end obtain Lorentz covariant theories on commutative Minkowski space. We further specialize to fuzzy non-commutativity, i.e. to matrix kind non-commutativity. We thus consider non-commutative spaces like those studied in refs. [5, 6] and implementing the CSDR principle on these spaces we obtain new particle models.

The paper is organized as follows. We first recall the geometry of fuzzy coset spaces, with the leading example of the fuzzy sphere. In particular we study the Lie derivative on spinors. Non-commutative gauge fields and non-commutative gauge transformations are then also recalled. In Section 3 we briefly review the CSDR scheme in the commutative case and then implement the CSDR principle on fuzzy cosets, we thus obtain a set of contraints –namely the CSDR constraints– that the gauge and matter fields must satisfy. Next we first reinterpret an action on $M^4 \times (S/R)_F$ with $G = U(P)$ gauge group as an action on M^4 with $U(nP)$ gauge group. We then impose and solve the CSDR constraints and obtain the gauge group and the particle content of the reduced four-dimensional action. Discussions and conclusions are in Section 4.

2. Fuzzy sphere and fuzzy coset spaces geometry

In this section first we describe the fuzzy sphere and study spinor fields on the fuzzy sphere, then we briefly present more general fuzzy coset spaces. For the definition of the fuzzy sphere and the gauge theory over the fuzzy sphere we follow ref. [23] (see also ref. [31]). A fuzzy manifold is a discrete matrix approximation to a continuous manifold. The approximation is such that the discretized space preserves its continuum symmetries [27], a fact that will allow us to apply the CSDR. A method in order to discretize a manifold is to single out a (finite) subspace of the space of functions on the manifold. One would also like this subspace to be invariant under multiplication. As a simple example consider the Fourier analysis of a function on a circle,

$$f(\theta) = \sum_{n=-\infty}^{\infty} f_n e^{in\theta}. \tag{2.1}$$

A discretized version of the circle can be achieved replacing the algebra of functions on the circle with the space of functions that do not exceed a given frequency N. We then write

$$f_N(\theta) = \sum_{n=-N}^{N} f_n e^{in\theta}. \tag{2.2}$$

for a generic function, $f_N(\theta)$ being an approximation of $f(\theta)$. However the product of two such functions will in general extend to frequencies up to $2N$ and so the space of truncated functions does not close under multiplication, we cannot speak of an algebra of truncated functions. The same is true for the harmonic analysis on the sphere or any other coset S/R. The solution is in the definition of a non-commutative product (a matrix product) such that the space of truncated functions closes under this new product.

The algebra of functions on the ordinary sphere can be generated by the coordinates of $\mathbf{R}^3$ modulo the relation $\sum_{\hat{a}=1}^{3} x_{\hat{a}} x_{\hat{a}} = r^2$. The fuzzy sphere S_F^2 at fuzziness level N is the non-commutative manifold whose coordinate functions $\hat{X}_{\hat{a}} = \hat{X}^{\hat{a}}$ are $(N+1) \times (N+1)$ hermitian matrices proportional to the generators of the $(N+1)$-dimensional representation of $SU(2)$, $\hat{X}_{\hat{a}} = \kappa J^{\hat{a}}$. They satisfy the condition $\sum_{\hat{a}=1}^{3} \hat{X}_{\hat{a}} \hat{X}_{\hat{a}} = r^2$ and the following commutation relations

$$[\hat{X}_{\hat{a}}, \hat{X}_{\hat{b}}] = i\kappa C_{\hat{a}\hat{b}\hat{c}} \hat{X}_{\hat{c}}, \tag{2.3}$$

where $\kappa = \lambda_N r$ with $\lambda_N = 1/\sqrt{\frac{N}{2}(\frac{N}{2}+1)}$ [we use $J^2 = \frac{N}{2}(\frac{N}{2}+1)$ for the $N+1$ dimensional irrep. of $SU(2)$]. If we define

$$X_{\hat{a}} = \frac{1}{i\kappa r} \hat{X}_{\hat{a}} = \frac{1}{ir} J_{\hat{a}} \tag{2.4}$$

we have

$$[X_{\hat{a}}, X_{\hat{b}}] = C_{\hat{a}\hat{b}\hat{c}} X_{\hat{c}} \tag{2.5}$$

with $C_{\hat{a}\hat{b}\hat{c}} = \epsilon_{\hat{a}\hat{b}\hat{c}}/r$ and

$$\sum_{\hat{a}=1}^{3} X_{\hat{a}} X_{\hat{a}} = -\frac{\lambda_N^{-2}}{r^2}.$$

In order to describe the algebra of the fuzzy sphere S_F^2 we can equivalently use the $\hat{X}_{\hat{a}}$ or the $X_{\hat{a}}$ generators; in the following we will work in the latter basis.

A function on the fuzzy sphere is a symmetric polynomial in the $X^{\hat{a}}$ coordinates. Since these coordinates are proportional to the $N+1$ dimensional irrep. of $SU(2)$ we have that any polynomial in the $X^{\hat{a}}$ can be rewritten as a symmetric polynomial of degree $\leq N$, and any $(N+1) \times (N+1)$ matrix can be expanded as a symmetric polynomial in the $X^{\hat{a}}$. Thus the space of functions on the fuzzy sphere S_F^2 at level N has dimension $(N+1)^2$. A convenient basis for this space is provided by the constant function $\mathbf{1}$ (the identity matrix) plus the non-commutative spherical harmonics up to level N

$$\hat{Y}_{lm} = r^{-l} \sum_{\hat{a}} f^{(lm)}_{\hat{a}_1,\hat{a}_2,\ldots,\hat{a}_l} X^{\hat{a}_1} \ldots X^{\hat{a}_l} \quad , \qquad l \leq N \tag{2.6}$$

with $f^{lm}_{\hat{a}_1,\hat{a}_2,\ldots,\hat{a}_l}$ the traceless and symmetric tensor of the ordinary spherical harmonics. Finally a generic function on the fuzzy sphere takes the form

$$f = \sum_{l=0}^{N} \sum_{m=-l}^{l} f_{lm} \hat{Y}_{lm} \ , \tag{2.7}$$

i.e. corresponds to an ordinary function on the commutative sphere with a cutoff on the angular momentum. Obviously this space of truncated functions is closed under the non-commutative $(N+1) \times (N+1)$ matrix product.

On the fuzzy sphere there is a natural $SU(2)$ covariant differential calculus. This calculus is three dimensional; the fact that the tangent space to the fuzzy sphere is three and not two dimensional is a typical aspect of non-commutative spaces. The three derivations $e_{\hat{a}}$ along $X_{\hat{a}}$ of a function f are given by

$$e_{\hat{a}}(f) = [X_{\hat{a}}, f] \, . \tag{2.8}$$

Accordingly the action of the Lie derivatives on functions is given by

$$\mathcal{L}_{\hat{a}} f = [X_{\hat{a}}, f] \ , \tag{2.9}$$

they satisfy the Leibniz rule and the $SU(2)$ Lie algebra relation

$$[\mathcal{L}_{\hat{a}}, \mathcal{L}_{\hat{b}}] = C_{\hat{a}\hat{b}\hat{c}} \mathcal{L}_{\hat{c}}. \tag{2.10}$$

In the $N \to \infty$ limit the derivations $e_{\hat{a}}$ become

$$e_{\hat{a}} = C_{\hat{a}\hat{b}\hat{c}} x^{\hat{b}} \partial^{\hat{c}} \tag{2.11}$$

and only in this commutative limit the tangent space becomes two dimensional. The exterior derivative is given by

$$df = [X_{\hat{a}}, f] \theta^{\hat{a}} \tag{2.12}$$

with $\theta^{\hat{a}}$ the one-forms dual to the vector fields $e_{\hat{a}}$, $< e_{\hat{a}}, \theta^{\hat{b}} >= \delta^{\hat{b}}_{\hat{a}}$. The space of one-forms is generated by the $\theta^{\hat{a}}$'s in the sense that for any one-form $\omega = \sum_i f_i(dh_i)\, t_i$ we can always write $\omega = \sum_{\hat{a}=1}^{3} \omega_{\hat{a}} \theta^{\hat{a}}$ with given functions $\omega_{\hat{a}}$ depending on the functions f_i, h_i and t_i. From $0 = \mathcal{L}_{\hat{a}} < e_{\hat{b}}, \theta^{\hat{c}} >=< \mathcal{L}_{\hat{a}} e_{\hat{b}}, \theta^{\hat{c}} > + < e_{\hat{b}}, \mathcal{L}_{\hat{a}} \theta^{\hat{c}} >$ and $\mathcal{L}_{\hat{a}}(e_{\hat{b}}) = C_{\hat{a}\hat{b}\hat{c}} e_{\hat{c}}$ [cf. (2.10)] we obtain the action of the Lie derivatives on one-forms,

$$\mathcal{L}_{\hat{a}}(\theta^{\hat{b}}) = C_{\hat{a}\hat{b}\hat{c}} \theta^{\hat{c}}. \tag{2.13}$$

It is then easy to check that the Lie derivative commutes with the exterior differential d, i.e. $SU(2)$ invariance of the exterior differential. On a general one-form $\omega = \omega_{\hat{a}} \theta^{\hat{a}}$ we have

$$\begin{aligned} \mathcal{L}_{\hat{b}} \omega = \mathcal{L}_{\hat{b}}(\omega_{\hat{a}} \theta^{\hat{a}}) &= (\mathcal{L}_{\hat{b}} \omega_{\hat{a}}) \theta^{\hat{a}} - \omega_{\hat{a}} C^{\hat{a}}_{\ \hat{b}\hat{c}} \theta^{\hat{c}} \\ &= \left[X_{\hat{b}}, \omega_{\hat{a}}\right] \theta^{\hat{a}} - \omega_{\hat{a}} C^{\hat{a}}_{\ \hat{b}\hat{c}} \theta^{\hat{c}} \end{aligned} \tag{2.14}$$

and therefore

$$(\mathcal{L}_{\hat{b}} \omega)_{\hat{a}} = \left[X_{\hat{b}}, \omega_{\hat{a}}\right] - \omega_{\hat{c}} C^{\hat{c}}_{\ \hat{b}\hat{a}} \ ; \tag{2.15}$$

this formula will be fundamental for formulating the CSDR principle on fuzzy cosets. Similarly, from $\mathcal{L}_{\hat{b}}(v) = \mathcal{L}_{\hat{b}}(v^{\hat{a}} e_{\hat{a}}) = [X_{\hat{b}}, v^{\hat{a}}] + \mathcal{L}_{\hat{b}}(e_{\hat{a}})$ we have

$$(\mathcal{L}_{\hat{b}} v)^{\hat{a}} = \left[X_{\hat{b}}, v^{\hat{a}}\right] - v_{\hat{c}} C_{\hat{c}\hat{b}\hat{a}} \ . \tag{2.16}$$

The differential geometry on the product space Minkowski times fuzzy sphere, $M^4 \times S^2_F$, is easily obtained from that on M^4 and on S^2_F. For example a one-form A defined on $M^4 \times S^2_F$ is written as

$$A = A_\mu dx^\mu + A_{\hat{a}} \theta^{\hat{a}} \tag{2.17}$$

with $A_\mu = A_\mu(x^\mu, X_{\hat{a}})$ and $A_{\hat{a}} = A_{\hat{a}}(x^\mu, X_{\hat{a}})$.

There are different approaches to the study of spinor fields on the fuzzy sphere [33, 34]. Here we follow ref. [2] (section 8.2)[3]. In the case of the product of Minkowski space and the

[3] For a discussion of chiral fermions and index theorems on matrix approximations of manifolds see ref. [35].

fuzzy sphere, $M^4 \times S^2_F$, we have seen that the geometry resembles in some aspects ordinary commutative geometry in seven dimensions. As $N \to \infty$ it returns to the ordinary six-dimensional geometry. Let g_{AB} be the Minkowski metric in seven dimensions and Γ^A the associated Dirac matrices which can be in the form

$$\Gamma^A = (\Gamma^\mu, \Gamma^{\hat{a}}) = (1 \otimes \gamma^\mu, \sigma^{\hat{a}} \otimes \gamma_5). \tag{2.18}$$

The space of spinors must be a left module with respect to the Clifford algebra. It is therefore a space of functions with values in a vector space $\mathcal{H}'$ of the form

$$\mathcal{H}' = \mathcal{H} \otimes C^2 \otimes C^4,$$

where $\mathcal{H}$ is an M_{N+1} module. The geometry resembles but is not really seven-dimensional, e.g. chirality can be defined and the fuzzy sphere admits chiral spinors. Therefore the space H' can be decomposed into two subspaces $\mathcal{H}'_\pm = \frac{1\pm\Gamma}{2}\mathcal{H}'$, where Γ is the chirality operator of the fuzzy sphere [2, 36]. The same holds for other fuzzy cosets such as $(SU(3)/U(1)\times U(1))_F$ [25].

In order to define the action of the Lie derivative $\mathcal{L}_{\hat{a}}$ on a spinor field Ψ, we write

$$\Psi = \zeta_\alpha \psi_\alpha, \tag{2.19}$$

where ψ_α are the components of Ψ in the ζ_α basis. Under a spinor rotation $\psi_\alpha \to S_{\alpha\beta}\psi_\beta$ the bilinear $\bar{\psi}\Gamma^{\hat{a}}\psi$ transforms as a vector $v^{\hat{a}} \to \Lambda_{\hat{a}\hat{b}}v^{\hat{b}}$. The Lie derivative on the basis ζ_α is given by

$$\mathcal{L}_{\hat{a}}\zeta_\alpha = \zeta_\beta \tau^{\hat{a}}_{\beta\alpha}, \tag{2.20}$$

where

$$\tau^{\hat{a}} = \frac{1}{2} C_{\hat{a}\hat{b}\hat{c}}\Gamma^{\hat{b}\hat{c}} \quad , \qquad \Gamma^{\hat{b}\hat{c}} = -\frac{1}{4}(\Gamma^{\hat{b}}\Gamma^{\hat{c}} - \Gamma^{\hat{c}}\Gamma^{\hat{b}}) \ . \tag{2.21}$$

Using that $\Gamma^{\hat{b}\hat{c}}$ are a rep. of the orthogonal algebra and then using the Jacoby identities for $C_{\hat{a}\hat{b}\hat{c}}$ one has $[\tau^{\hat{a}}, \tau^{\hat{b}}] = C_{\hat{a}\hat{b}\hat{c}}\tau^{\hat{c}}$ from which it follows that the Lie derivative on spinors gives a representation of the Lie algebra,

$$[\mathcal{L}_{\hat{a}}, \mathcal{L}_{\hat{b}}]\zeta_\alpha = C_{\hat{a}\hat{b}\hat{c}}\, \mathcal{L}_{\hat{c}}\, \zeta_\alpha \ . \tag{2.22}$$

On a generic spinor Ψ, applying the Leibniz rule we have

$$\mathcal{L}_{\hat{a}}\Psi = \zeta_\alpha[X_{\hat{a}}, \psi_\alpha] + \zeta_\beta \tau^{\hat{a}}_{\beta\gamma}\psi_\gamma \tag{2.23}$$

and of course $[\mathcal{L}_{\hat{a}}, \mathcal{L}_{\hat{b}}]\Psi = C_{\hat{a}\hat{b}\hat{c}}\, \mathcal{L}_{\hat{c}}\, \Psi$; we also write

$$\delta_{\hat{a}}\psi_\alpha = (\mathcal{L}_{\hat{a}}\Psi)_\alpha = [X_{\hat{a}}, \psi_\alpha] + \tau^{\hat{a}}_{\alpha\gamma}\psi_\gamma \ . \tag{2.24}$$

The action of the Lie derivative $\mathcal{L}_{\hat{a}}$ on the adjoint spinor is obtained considering the adjoint of the above expression, since $(X_{\hat{a}})^\dagger = -X_{\hat{a}}$, $(\tau^{\hat{a}})^\dagger = -\tau^{\hat{a}}$, $[\tau^{\hat{a}}, \Gamma_0] = 0$ we have

$$\delta_{\hat{a}}\bar{\psi}_\alpha = [X_{\hat{a}}, \bar{\psi}_\alpha] - \bar{\psi}_\gamma \tau^{\hat{a}}_{\gamma\alpha}. \tag{2.25}$$

One can then check that the variations (2.24) and (2.25) are consistent with $\psi^\dagger\Gamma^0\psi$ being a scalar. Finally we have compatibility among the Lie derivatives (2.24), (2.25) and (2.15):

$$\delta_{\hat{a}}(\bar{\psi}\Gamma^{\hat{d}}\psi) = (\delta_{\hat{a}}\bar{\psi})\Gamma^{\hat{d}}\psi + \bar{\psi}\Gamma^{\hat{d}}\delta_{\hat{a}}\psi = [X_{\hat{a}}, \bar{\psi}\Gamma^{\hat{d}}\psi] + \bar{\psi}[\Gamma^{\hat{d}}, \tau^{\hat{a}}]\psi = [X_{\hat{a}}, \bar{\psi}\Gamma^{\hat{d}}\psi] - C_{\hat{a}\hat{d}\hat{c}}\bar{\psi}\Gamma^{\hat{c}}\psi$$

(and $\delta_{\hat{a}}(\bar{\psi}\Gamma^\mu\psi) = [X_{\hat{a}}, \bar{\psi}\Gamma^\mu\psi]$). This immediately generalizes to higher tensors $\bar{\psi}\,\Gamma_{\hat{d}_1}\ldots\Gamma_{\hat{d}_i}\psi$.

The sphere S^2 is the complex projective space CP^1 . The generalization of the fuzzy sphere construction to CP^2 and its *spin*^{c} structure was given in ref. [24], whereas the generalization to $CP^{M-1} = SU(M)/U(M-1)$ and to Grassmannian cosets was given in ref. [27]. While a set of coordinates on the sphere is given by the $\mathbf{R}^3$ coordinates $x^{\hat{a}}$ modulo the relation $\sum_{\hat{a}} x^{\hat{a}}x^{\hat{a}} = r^2$, a set of coordinates on CP^{M-1} is given by $x^{\hat{a}}$, $\hat{a} = 1,\ldots M^2-1$ modulo the relations

$$x^{\hat{a}}x_{\hat{a}} = \frac{2(M-1)}{M}r^2 \;, \quad d_{\hat{a}\hat{b}}{}^{\hat{c}}x^{\hat{a}}x^{\hat{b}} = \frac{2(M-2)}{M}rx^{\hat{c}}\,, \tag{2.26}$$

where $d_{\hat{a}\hat{b}}{}^{\hat{c}}$ are the components of the symmetric invariant tensor of $SU(M)$. Then CP^{M-1} is approximated, at fuzziness level N, by $n\times n$ dimensional matrices $X_{\hat{a}}$, $\hat{a} = 1,\ldots,M^2-1$. These are proportional to the generators $J_{\hat{a}}$ of $SU(M)$ considered in the $n = \frac{(M-1+N)!}{(M-1)!N!}$ dimensional irrep., obtained from the N-fold symmetric tensor product of the fundamental M-dimensional representation of $SU(M)$. As in (2.4) we set $X_{\hat{a}} = \frac{1}{ir}J_{\hat{a}}$ so that

$$\sum_{\hat{a}=1}^{3} X_{\hat{a}}X_{\hat{a}} = -\frac{C_n}{r^2} \quad , \qquad [X_{\hat{a}}, X_{\hat{b}}] = C_{\hat{a}\hat{b}\hat{c}}X_{\hat{c}} \tag{2.27}$$

where C_n is the quadratic casimir of the given n-dimensional irrep., and $rC_{\hat{a}\hat{b}\hat{c}}$ are now the $SU(M)$ structure constants. More generally [25] we consider fuzzy coset spaces $(S/R)_F$ described by non-commuting coordinates $X_{\hat{a}}$ that are proportional to the generators of a given n-dimensional irrep. of the compact Lie group S and thus in particular satisfy the conditions (2.27) where now $rC_{\hat{a}\hat{b}\hat{c}}$ are the S structure constants (the extra constraints associated with the given n-dimensional irrep. determine the subgroup R of S in S/R). The differential calculus on these fuzzy spaces can be constructed as in the case for the fuzzy sphere. For example there are $dimS$ Lie derivatives, they are given by eq. (2.9) and satisfy the relation (2.10). On these fuzzy spaces we consider the space of spinors to be a left module with respect to the Clifford algebra given by (2.18), where now the $\sigma^{\hat{a}}$'s are replaced by the $\gamma^{\hat{a}}$'s, the gamma matrices on R^{dimS}; in particular all the formulae concerning Lie derivatives on spinors remain unchanged.

2.1 Non-commutative gauge fields and transformations

Gauge fields arise in non-commutative geometry and in particular on fuzzy spaces very naturally; they are linked to the notion of covariant coordinate [32]. Consider a field $\phi(X^{\hat{a}})$ on a fuzzy space described by the non-commuting coordinates $X^{\hat{a}}$. An infinitesimal gauge transformation $\delta\phi$ of the field ϕ with gauge transformation parameter $\lambda(X^{\hat{a}})$ is defined by

$$\delta\phi(X) = \lambda(X)\phi(X). \tag{2.28}$$

This is an infinitesimal abelian $U(1)$ gauge transformation if $\lambda(X)$ is just an antihermitian function of the coordinates $X^{\hat{a}}$, it is an infinitesimal nonabelian $U(P)$ gauge transformation if $\lambda(X)$ is valued in Lie($U(P)$), the Lie algebra of hermitian $P \times P$ matrices; in the following we will always assume Lie($U(P)$) elements to commute with the coordinates $X^{\hat{a}}$. The coordinates X are invariant under a gauge transformation

$$\delta X_{\hat{a}} = 0 \; ; \tag{2.29}$$

multiplication of a field on the left by a coordinate is then not a covariant operation in the non-commutative case. That is

$$\delta(X_{\hat{a}}\phi) = X_{\hat{a}}\lambda(X)\phi, \tag{2.30}$$

and in general the right hand side is not equal to $\lambda(X)X_a\phi$. Following the ideas of ordinary gauge theory one then introduces covariant coordinates $\varphi_{\hat{a}}$ such that

$$\delta(\varphi_{\hat{a}}\phi) = \lambda\varphi_{\hat{a}}\phi \; , \tag{2.31}$$

this happens if

$$\delta(\varphi_{\hat{a}}) = [\lambda, \varphi_{\hat{a}}] \; . \tag{2.32}$$

We also set

$$\varphi_{\hat{a}} \equiv X_{\hat{a}} + A_{\hat{a}} \tag{2.33}$$

and interpret $A_{\hat{a}}$ as the gauge potential of the non-commutative theory; then $\varphi_{\hat{a}}$ is the non-commutative analogue of a covariant derivative. The transformation properties of $A_{\hat{a}}$ support the interpretation of $A_{\hat{a}}$ as gauge field; they arise from requirement (2.32),

$$\delta A_{\hat{a}} = -[X_{\hat{a}}, \lambda] + [\lambda, A_{\hat{a}}] \; . \tag{2.34}$$

Correspondingly we can define a tensor $F_{\hat{a}\hat{b}}$, the analogue of the field strength, as

$$F_{\hat{a}\hat{b}} = [X_{\hat{a}}, A_{\hat{b}}] - [X_{\hat{b}}, A_{\hat{a}}] + [A_{\hat{a}}, A_{\hat{b}}] - C^{\hat{c}}{}_{\hat{a}\hat{b}}A_{\hat{c}} \tag{2.35}$$

$$= [\varphi_{\hat{a}}, \varphi_{\hat{b}}] - C^{\hat{c}}{}_{\hat{a}\hat{b}}\varphi_{\hat{c}}. \tag{2.36}$$

This tensor transforms covariantly

$$\delta F_{\hat{a}\hat{b}} = [\lambda, F_{\hat{a}\hat{b}}] \; . \tag{2.37}$$

Similarly, for a spinor ψ in the adjoint representation, the infinitesimal gauge transformation is given by

$$\delta\psi = [\lambda, \psi] \; , \tag{2.38}$$

while for a spinor in the fundamental the infinitesimal gauge transformation is given by

$$\delta\psi = \lambda\psi \; . \tag{2.39}$$

3. Coset Space Dimensional Reduction (CSDR)

First we briefly recall the CSDR scheme in the commutative case. It is indeed instructive to compare the commutative and the fuzzy case. The latter is described in the next subsection which is self contained.

One way to dimensionally reduce a gauge theory on $M^4 \times S/R$ with gauge group G to a gauge theory on M^4, is to consider field configurations that are invariant under S/R transformations. Since the action of the group S on the coset space S/R is transitive (i.e., connects all points), we can equivalently require the fields in the theory to be invariant under the action of S on S/R. Infinitesimally, if we denote by $\zeta_{\hat{a}}$ the Killing vectors on S/R associated to the generators $T^{\hat{a}}$ of S, we require the fields to have zero Lie derivative along $\zeta_{\hat{a}}$. For scalar fields this is equivalent to requiring independence under the S/R coordinates. The CSDR scheme dimensionally reduces a gauge theory on $M^4 \times S/R$ with gauge group G to a gauge theory on M^4 imposing a milder constraint, namely the fields are required to be invariant under the S action up to a G gauge transformation [14, 15, 16]. Thus we have, respectively for scalar fields ϕ and the one-form gauge field A

$$\mathcal{L}_{\zeta_{\hat{a}}}\phi = \delta^{W_{\hat{a}}}\phi = W_{\hat{a}}\phi \quad , \tag{3.1}$$

$$\mathcal{L}_{\zeta_{\hat{a}}}A = \delta^{W_{\hat{a}}}A = -DW_{\hat{a}} \ , \tag{3.2}$$

where $\delta^{W_{\hat{a}}}$ is the infinitesimal gauge transformation relative to the gauge parameter $W_{\hat{a}}$ that depends on the coset coordinates (in our notations A and $W_{\hat{a}}$ are antihermitian and the covariant derivative reads $D = d + A$). The gauge parameters $W_{\hat{a}}$ obey a consistency condition which follows from the relation

$$[\mathcal{L}_{\xi_{\hat{a}}}, \mathcal{L}_{\xi_{\hat{b}}}] = \mathcal{L}_{[\xi_{\hat{a}}, \xi_{\hat{b}}]} \tag{3.3}$$

and transform under a gauge transformation $\phi \to g\phi$ as

$$W_{\hat{a}} \to gW_{\hat{a}}g^{-1} + (\mathcal{L}_{\xi_{\hat{a}}}g)g^{-1}. \tag{3.4}$$

Since two points of the coset are connected by an S-transformation which is equivalent to a gauge transformation, and since the Lagrangian is gauge invariant, we can study the above equations just at one point of the coset, let's say $y^a = 0$, where we denote by (x^μ, y^a) the coordinates of $M^4 \times S/R$, and we use $\hat{a}, a, i$ to denote S, S/R and R indices. In general, using (3.4), not all the $W_{\hat{a}}$ can be gauged transformed to zero at $y^a = 0$, however one can choose $W_a = 0$ denoting by W_i the remaining ones. Then the consistency condition which follows from eq. (3.3) implies that W_i are constant and equal to the generators of the embedding of R in G (thus in particular R must be embeddable in G; we write R_G for the image of R in G).

The detailed analysis of the constraints given in refs. [14, 15] provides us with the four-dimensional unconstrained fields as well as with the gauge invariance that remains in the theory after dimensional reduction. Here we give the results. The components $A_\mu(x, y)$ of the initial gauge field $A_M(x, y)$ become, after dimensional reduction, the four-dimensional gauge fields and furthermore they are independent of y. In addition one can find that they have to commute with the elements of the R_G subgroup of G. Thus the four-dimensional gauge group H is the centralizer of R in G, $H = C_G(R_G)$. Similarly, the $A_a(x, y)$ components of $A_M(x, y)$ denoted by $\phi_a(x, y)$ from now on, become scalars in four dimensions. These fields transform under R as a vector v, i.e.

$$\begin{aligned} S &\supset R \\ adjS &= adjR + v. \end{aligned} \tag{3.5}$$

Moreover $\phi_a(x, y)$ acts as an intertwining operator connecting induced representations of R acting on G and S/R. This implies, exploiting Schur's lemma, that the transformation properties of the fields $\phi_a(x, y)$ under H can be found if we express the adjoint representation of G in terms of $R_G \times H$:

$$\begin{aligned} G &\supset R_G \times H \\ adjG &= (adjR, 1) + (1, adjH) + \sum(r_i, h_i). \end{aligned} \tag{3.6}$$

Then if $v = \sum s_i$, where each s_i is an irreducible representation of R, there survives an h_i multiplet for every pair (r_i, s_i), where r_i and s_i are identical irreps. of R. If we start from a pure gauge theory on $M^4 \times S/R$, the four-dimensional potential (at $y^a = 0$) can be shown to be given by

$$V = \frac{1}{4} F_{ab} F^{ab} = \frac{1}{4} (C^{\hat{c}}{}_{ab} \phi_{\hat{c}} - [\phi_a, \phi_b])^2, \tag{3.7}$$

where we have defined $\phi_i \equiv W_i$. However, the fields ϕ_a are not independent because the conditions (3.2) at $y^a = 0$ constrain them. The solution of the constraints provides the physical dimensionally reduced fields in four dimensions; in terms of these physical fields the potential is still a quartic polynomial. Then, the minimum of this potential will determine the spontaneous symmetry breaking pattern.

Turning next to the fermion fields, similarly to scalars, they act as an intertwining operator connecting induced representations of R in G and in $SO(d)$, where d is the dimension of the tangent space of S/R. Proceeding along similar lines as in the case of scalars, and considering the more interesting case of even dimensions, we impose first the Weyl condition. Then to obtain the representation of H under which the four-dimensional fermions transform, we have to decompose the fermion representation ρ_F of the initial gauge group G under $R_G \times H$, i.e.

$$\rho_F = \sum(t_i, h_i), \tag{3.8}$$

and the spinor of $SO(d)$ under R

$$\sigma_d = \sum \sigma_j. \tag{3.9}$$

Then for each pair t_i and σ_i, where t_i and σ_i are identical irreps. there is an h_i multiplet of spinor fields in the four-dimensional theory. In order however to obtain chiral fermions in the effective theory we may have to impose further requirements [15, 19].

3.1 CSDR over fuzzy coset spaces

In the present case space-time has the form $M^4 \times (S/R)_F$, where $(S/R)_F$ is the approximation of S/R by finite $n \times n$ matrices. On $M^4 \times (S/R)_F$ we consider a non-commutative gauge theory with gauge group $G = U(P)$. We implement the CSDR scheme in the fuzzy case in three steps:

1) We state the CSDR principle on fuzzy cosets and reduce it to a set of contraints –the CSDR constraints (3.16), (3.18), (3.21), (3.24), (3.25)– that the gauge and matter fields must satisfy.

2) We reinterpret actions on $M^4 \times (S/R)_F$ with $G = U(P)$ gauge group as actions on M^4 with $U(nP)$ gauge group. More explicitely, we expand the fields on $M^4 \times (S/R)_F$ in Kaluza-Klein modes on $(S/R)_F$. Since the algebra of functions on $(S/R)_F$ is finite dimensional we obtain a finite tower of modes; since $(S/R)_F$ is described by $n \times n$ matrices a basis for this mode expansion is given by the generators of Lie($U(n)$). In this way we show that the different modes can be conveniently grouped togheter so that an initial Lie(G)-valued field on $M^4 \times (S/R)_F$ (with $G = U(P)$) is reinterpreted as a Lie($U(nP)$) valued field on M^4. Of course also the CSDR constraints can now be interpreted on M^4 instead of on $M^4 \times (S/R)_F$. This leads to their solution in step 3).

3) We solve the CSDR constraints and obtain the gauge group and the particle content of the reduced four-dimensional actions. This last step is first studied in the fuzzy sphere case, and then for more general fuzzy cosets.

3.1.1 CSDR principle

Since the Lie algebra of S acts on the fuzzy space $(S/R)_F$, we can state the CSDR principle in the same way as in the continuum case, i.e. the fields in the theory must be invariant under the infinitesimal S action up to an infinitesimal gauge transformation

$$\mathcal{L}_{\hat{b}}\phi = \delta^{W_{\hat{b}}}\phi = W_{\hat{b}}\phi \quad , \quad \mathcal{L}_{\hat{b}}A = \delta^{W_{\hat{b}}}A = -DW_{\hat{b}}, \tag{3.10}$$

where A is the one-form gauge potential $A = A_\mu dx^\mu + A_{\hat{a}}\theta^{\hat{a}}$, and $W_{\hat{b}}$ depends only on the coset coordinates $X^{\hat{a}}$ and (like A_μ, A_a) is antihermitian. We thus write $W_{\hat{b}} = W_{\hat{b}}^{\alpha}\mathcal{T}^{\alpha}$, $\alpha = 1, 2 \ldots P^2$, where $\mathcal{T}^i$ are hermitian generators of $U(P)$ and $(W_b^i)^\dagger = -W_b^i$, here † is hermitian coniugation on the $X^{\hat{a}}$'s. The principle gives for the space-time part A_μ

$$\mathcal{L}_{\hat{b}}A_\mu = [X_{\hat{a}}, A_\mu] = -[A_\mu, W_{\hat{b}}], \tag{3.11}$$

while for the internal part $A_{\hat{a}}$

$$[X_{\hat{b}}, A_{\hat{d}}] + A_{\hat{a}}C_{\hat{b}}{}^{\hat{a}}{}_{\hat{d}} = -[A_{\hat{d}}, W_{\hat{b}}] - \mathcal{L}_{\hat{d}}W_{\hat{b}}. \tag{3.12}$$

From the first of eqs. (3.10) we have $\mathcal{L}_{\hat{a}}\mathcal{L}_{\hat{b}}\phi = (\mathcal{L}_{\hat{a}}W_{\hat{b}})\phi + W_{\hat{b}}W_{\hat{a}}\phi$, then using the relation $[\mathcal{L}_a, \mathcal{L}_{\hat{b}}] = C_{\hat{a}\hat{b}}{}^{\hat{c}}\mathcal{L}_{\hat{c}}$ we obtain the consistency condition

$$[X_{\hat{a}}, W_{\hat{b}}] - [X_{\hat{b}}, W_{\hat{a}}] - [W_{\hat{a}}, W_{\hat{b}}] = C_{\hat{a}\hat{b}}{}^{\hat{c}}W_{\hat{c}}. \tag{3.13}$$

Under the gauge transformation $\phi \to \phi' = g\phi$ with $g \in G = U(P)$, we have $\mathcal{L}_{\hat{a}}\phi' = W'_{\hat{a}}\phi'$ and also $\mathcal{L}_{\hat{a}}\phi' = (\mathcal{L}_{\hat{a}}g)\phi + g(\mathcal{L}_{\hat{a}}\phi)$, and therefore

$$W_{\hat{a}} \to W'_{\hat{a}} = gW_{\hat{a}}g^{-1} + [X_{\hat{a}}, g]g^{-1} \ . \tag{3.14}$$

Now in order to solve the constraints (3.11), (3.12), (3.13) we cannot follow the strategy adopted in the commutative case where the constraints were studied just at one point of the coset (say $y^a = 0$). This is due to the intrinsic nonlocality of the constraints. On the other hand the specific properties of the fuzzy case (e.g. the fact that partial derivatives are realized via commutators, the concept of covariant derivative) allow to symplify and eventually solve the constraints. If we define

$$\omega_{\hat{a}} \equiv X_{\hat{a}} - W_{\hat{a}} \ , \tag{3.15}$$

we obtain the following form of the consistency condition (3.13)

$$[\omega_{\hat{a}}, \omega_{\hat{b}}] = C_{\hat{a}\hat{b}}{}^{\hat{c}}\omega_c, \tag{3.16}$$

where $\omega_{\hat{a}}$ transforms as

$$\omega_{\hat{a}} \to \omega'_{\hat{a}} = g\omega_{\hat{a}}g^{-1}. \tag{3.17}$$

Now eq. (3.11) reads

$$[\omega_{\hat{b}}, A_\mu] = 0. \tag{3.18}$$

Furthermore by considering the covariant coordinate,

$$\varphi_{\hat{d}} \equiv X_{\hat{d}} + A_{\hat{d}} \tag{3.19}$$

we have

$$\varphi \to \varphi' = g\varphi g^{-1} \tag{3.20}$$

and eq. (3.12) simplifies to

$$C_{\hat{b}\hat{d}\hat{e}}\varphi^{\hat{e}} = [\omega_{\hat{b}}, \varphi_{\hat{d}}]. \tag{3.21}$$

Therefore eqs. (3.16) (3.18) (3.21) are the constraints to be solved. Note that eqs. (3.20) and (3.21) have the symmetry

$$\varphi_{\hat{a}} \to \varphi_{\hat{a}} + \omega_{\hat{a}} \ , \tag{3.22}$$

suggesting that $\omega_{\hat{a}}$ is a ground state around which we calculate the fluctuations $\varphi_{\hat{a}}$, and indeed, as formula (3.32) for the potential shows, $\varphi_{\hat{a}} = \omega_{\hat{a}}$ minimize the potential; in fact the potential vanishes for this value of $\varphi_{\hat{a}}$.

One proceeds in a similar way for the spinor fields. The CSDR principle relates the Lie derivative on a spinor ψ, that we consider in the adjoint representation of G, to a gauge transformation; recalling eqs. (2.21) and (2.24) we have

$$[X_{\hat{a}}, \psi] + \frac{1}{2}C_{\hat{a}\hat{b}\hat{c}}\Gamma^{\hat{b}\hat{c}}\psi = [W_{\hat{a}}, \psi], \tag{3.23}$$

where ψ denotes the column vector with entries ψ_α. Setting again $\omega_{\hat{a}} = X_{\hat{a}} - W_{\hat{a}}$ we obtain the constraint

$$-\frac{1}{2}C_{\hat{a}\hat{b}\hat{c}}\Gamma^{\hat{b}\hat{c}}\psi = [\omega_{\hat{a}}, \psi] \ . \tag{3.24}$$

We can also consider spinors that transform in the fundamental rep. of the gauge group G, we then have $[X_{\hat{a}}, \psi] + \frac{1}{2}C_{\hat{a}\hat{b}\hat{c}}\Gamma^{\hat{b}\hat{c}}\psi = W_{\hat{a}}\psi$, and setting again $\omega_{\hat{a}} = X_{\hat{a}} - W_{\hat{a}}$ we obtain

$$-\frac{1}{2}C_{\hat{a}\hat{b}\hat{c}}\Gamma^{\hat{b}\hat{c}}\psi = \omega_{\hat{a}}\psi - \psi X_{\hat{a}} \ . \tag{3.25}$$

3.1.2 Actions and Kaluza-Klein modes

Let us consider a pure YM action on $M^4 \times (S/R)_F$ and examine how it is reinterpreted in four dimensions. The action is

$$\mathcal{A}_{YM} = \frac{1}{4}\int d^4x \, Tr \, tr_G \, F_{MN}F^{MN}, \tag{3.26}$$

where Tr is the usual trace over $n \times n$ matrices and is actually the integral over the fuzzy coset $(S/R)_F$ [4], while tr_G is the gauge group G trace. The higher-dimensional field strength F_{MN} decomposed in four-dimensional space-time and extra-dimensional components reads as follows

$$(F_{\mu\nu}, F_{\mu\hat{b}}, F_{\hat{a}\hat{b}}) \ ; \tag{3.27}$$

explicitly the various components of the field strength are given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu], \tag{3.28}$$

$$\begin{aligned} F_{\mu\hat{a}} &= \partial_\mu A_{\hat{a}} - [X_{\hat{a}}, A_\mu] + [A_\mu, A_{\hat{a}}] \\ &= \partial_\mu \varphi_{\hat{a}} + [A_\mu, \varphi_{\hat{a}}] = D_\mu \varphi_{\hat{a}}, \end{aligned} \tag{3.29}$$

$$F_{\hat{a}\hat{b}} = [\varphi_{\hat{a}}, \varphi_{\hat{b}}] - C^{\hat{c}}{}_{\hat{a}\hat{b}}\varphi_{\hat{c}} \ ; \tag{3.30}$$

they are covariant under local G transformations: $F_{MN} \to gF_{MN}g^{-1}$, with $g = g(x^\mu, X^{\hat{a}})$.

In terms of the decomposition (3.27) the action reads

$$\mathcal{A}_{YM} = \int d^4x \, Tr \, tr_G \left(\frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2}(D_\mu\varphi_{\hat{a}})^2\right) - V(\varphi) \ , \tag{3.31}$$

where the potential term $V(\varphi)$ is the $F_{\hat{a}\hat{b}}$ kinetic term (recall $F_{\hat{a}\hat{b}}$ is antihermitian so that $V(\varphi)$ is hermitian and non-negative)

$$\begin{aligned} V(\varphi) &= -\frac{1}{4}Tr \, tr_G \sum_{\hat{a}\hat{b}} F_{\hat{a}\hat{b}}F_{\hat{a}\hat{b}} \\ &= -\frac{1}{4}Tr \, tr_G \sum_{\hat{a}\hat{b}} \left([\varphi_{\hat{a}}, \varphi_{\hat{b}}] - C^{\hat{c}}{}_{\hat{a}\hat{b}}\varphi_{\hat{c}}\right)\left([\varphi_{\hat{a}}, \varphi_{\hat{b}}] - C^{\hat{c}}{}_{\hat{a}\hat{b}}\varphi_{\hat{c}}\right) \end{aligned} \tag{3.32}$$

[4] Tr is a good integral because it has the cyclic property $Tr(f_1 \ldots f_{p-1}f_p) = Tr(f_pf_1 \ldots f_{p-1})$. It is also invariant under the action of the group S, that we recall to be infinitesimally given by $\mathcal{L}_{\hat{a}}f = [X_{\hat{a}}, f]$.

For sake of clarity we here recall that: Tr is the trace over the $n \times n$ matrices that describe the fuzzy coset $(S/R)_F$, tr_G is the trace over $G = U(P)$ matrices in the fundamental representation, φ is the covariant coordinate [cf. (3.19)] where $X^{\hat{a}}$ is normalized as in (2.27), $rC^{\hat{a}}_{\ \hat{b}\hat{c}}$ are the S structure constants.

The action (3.31) is naturally interpreted as an action in four dimensions. The infinitesimal G gauge transformation with gauge parameter $\lambda(x^\mu, X^{\hat{a}})$ can indeed be interpreted just as an M^4 gauge transformation. We write

$$\lambda(x^\mu, X^{\hat{a}}) = \lambda^\alpha(x^\mu, X^{\hat{a}})\mathcal{T}^\alpha = \lambda^{h,\alpha}(x^\mu)T^h\mathcal{T}^\alpha \ , \tag{3.33}$$

where $\mathcal{T}^\alpha$ are hermitian generators of $U(P)$, $\lambda^\alpha(x^\mu, X^{\hat{a}})$ are $n \times n$ antihermitian matrices and thus are expressible as $\lambda(x^\mu)^{\alpha,h}T^h$, where T^h are antihermitian generators of $U(n)$. The fields $\lambda(x^\mu)^{\alpha,h}$, with $h = 1, \ldots n^2$, are the Kaluza-Klein modes of $\lambda(x^\mu, X^{\hat{a}})^\alpha$. We now consider on equal footing the indices h and α and interpret the fields on the r.h.s. of (3.33) as one field valued in the tensor product Lie algebra $\mathrm{Lie}(U(n)) \otimes \mathrm{Lie}(U(P))$. This Lie algebra is indeed $\mathrm{Lie}(U(nP))$ [5]. Similarly we rewrite the gauge field A_ν as

$$A_\nu(x^\mu, X^{\hat{a}}) = A^\alpha_\nu(x^\mu, X^{\hat{a}})\mathcal{T}^\alpha = A^{h,\alpha}_\nu(x^\mu)T^h\mathcal{T}^\alpha \ , \tag{3.34}$$

and interpret it as a $\mathrm{Lie}(U(nP))$ valued gauge field on M^4, and similarly for $\varphi_{\hat{a}}$. Finally $Tr\, tr_G$ is the trace over $U(nP)$ matrices in the fundamental representation.

The above analysis applies also to more general actions, and to the field $\omega_{\hat{a}}$ and therefore to the CSDR constraints (3.16), (3.18), (3.21), (3.24), (3.25) that can now be reinterpreted as constraints on M^4 instead of on $M^4 \times (S/R)_F$. The action (3.31) and the minima of the potential (3.32), in the case $P = 1$, have been studied, without CSDR constraints, in refs. [5, 6]. It is imposing the CSDR constraints that we reduce the number of independent gauge and matter fields in the action (3.31), and that we therefore obtain new and richer particle models. We now solve these constraints. We first consider the fuzzy sphere case and then extend the results to more general fuzzy cosets.

3.1.3 CSDR constraints for the fuzzy sphere

We consider $(S/R)_F = S^2_F$, i.e. the fuzzy sphere, that we consider at fuzziness level N ($(N+1) \times (N+1)$ matrices). We first study the basic example where the gauge group G is just $U(1)$. There we begin by considering a specific solution –determined by a specific embedding of $SU(2)$ into $U(N+1)$– of constraint (3.16); we then solve also (3.18), (3.21) and (3.24), the latter concerns fermions in the adjoint of $G = U(1)$. We further write down the fermion action on $M^4 \times S^2_F$, reinterpret it as an action on M^4 (as we did for the pure YM action (3.31)), and then rewrite the complete YM plus fermion action in terms of the fields that satisfy the CSDR constraints. We then describe in full generality

[5] Proof: The $(nP)^2$ generators $T^h\mathcal{T}^\alpha$ are $nP \times nP$ antihermitian matrices. We just have to show that they are linearly independent. This is easy since it is equivalent to prove the linear independence of the $(nP)^2$ matrices $e_{ij}\varepsilon_{\rho\sigma}$ where $i = 1, \ldots n$, $\rho = 1, \ldots P$ and e_{ij} is the $n \times n$ matrix having 1 in the position (i, j) and zero elswere, and similarly for the $P \times P$ matrix $\varepsilon_{\rho\sigma}$.

how to solve the CSDR constraints (3.16), (3.18), (3.21) and (3.24). Finally we study the case of fermions that transform in the fundamental of the gauge group $G = U(1)$. The generalization of the above outlined analysis to the case of the gauge group $G = U(P)$ then follows.

The $G = U(1)$ case. In this case the $\omega_{\hat{a}} = \omega_{\hat{a}}(X^{\hat{b}})$ that appear in the consistency condition (3.16), $[\omega_{\hat{a}}, \omega_{\hat{b}}] = C_{\hat{a}\hat{b}}{}^{\hat{c}}\omega_{\hat{c}}$, are $(N+1) \times (N+1)$ antihermitian matrices, i.e. we can interpret them as elements of Lie$(U(N+1))$. On the other hand $r\omega_{\hat{a}}$ satisfy the commutation relations (3.16) of Lie$(SU(2))$ (in fact $rC^{a}_{\ bc}$ are the $SU(2)$ structure constants). Therefore in order to satisfy the consistency condition (3.16) we have to embed Lie$(SU(2))$ in Lie$(U(N+1))$. Let T^h with $h = 1, \ldots, (N+1)^2$ be the generators of Lie$(U(N+1))$ in the fundamental representation and with normalization $Tr(T^hT^k) = -\frac{1}{2}\delta^{hk}$. We can always use the convention $h = (\hat{a}, u)$ with $\hat{a} = 1, 2, 3$ and $u = 4, 5, \ldots, (N+1)^2$ where the $T^{\hat{a}}$ satisfy the $SU(2)$ Lie algebra,

$$[T^{\hat{a}}, T^{\hat{b}}] = rC^{\hat{a}\hat{b}}{}_{\hat{c}}T^{\hat{c}} \ . \tag{3.35}$$

Then we define an embedding by identifying

$$r\omega_{\hat{a}} = T_{\hat{a}}. \tag{3.36}$$

Constraint (3.18), $[\omega_{\hat{b}}, A_\mu] = 0$, then implies that the four-dimensional gauge group K is the centralizer of the image of $SU(2)$ in $U(N+1)$, i.e.

$$K = C_{U(N+1)}(SU((2))) = SU(N-1) \times U(1) \times U(1) \ ,$$

here the last $U(1)$ is the $U(1)$ of $U(N+1) \simeq SU(N+1) \times U(1)$. The functions $A_\mu(x, X)$ are arbitrary functions of x but the X dependence is such that $A_\mu(x, X)$ is Lie(K) valued instead of Lie$(U(N+1))$, i.e. eventually we have a four-dimensional gauge potential $A_\mu(x)$ with values in Lie(K). Concerning constraint (3.21), $[\omega_{\hat{b}}, \varphi_{\hat{a}}] = C_{\hat{b}\hat{a}}{}^{\hat{e}}\varphi_{\hat{e}}$, we note that it is satisfied by choosing

$$\varphi_{\hat{a}} = \varphi(x) r\omega_{\hat{a}} \ , \tag{3.37}$$

i.e. the unconstrained degrees of freedom correspond to the scalar field $\varphi(x)$ that is a singlet under the four-dimensional gauge group K. This solution is unique since, given the embedding (3.36), the adjoint of $SU(2)$ is contained just once in the adjoint of $U(N+1)$ (see for example [37]).

The physical spinor fields are obtained by solving the constraint (3.24), $-\frac{1}{2}C_{\hat{a}\hat{b}\hat{c}}\Gamma^{\hat{b}\hat{c}}\psi = [\omega_{\hat{a}}, \psi]$. In the l.h.s. of this formula we can say that we have an embedding of Lie$(SU(2))$ in the spin representation of Lie$(SO(3))$. This embedding is given by the matrices $\tau^{\hat{a}} = \frac{1}{2}C_{\hat{a}\hat{b}\hat{c}}\Gamma^{\hat{b}\hat{c}}$; since Lie$(SU(2))$ = Lie$(SO(3))$ this embedding is rather trivial and indeed $\tau^{\hat{a}} = \frac{-i}{2r}\sigma^{\hat{a}}$. Thus the constraint (3.24) states that the spinor $\psi = \psi^h T^h = \binom{\psi_1}{\psi_2}$, where $T^h \in$ Lie$(U(N+1))$ and $\psi_{1(2)} = \psi^h_{1(2)}T^h$ are four-dimensional spinors, relate (intertwine) the fundamental rep. of $SU(2)$ to the representations of $SU(2)$ induced by the embedding

(3.36) of $SU(2)$ in $U(N+1)$, i.e. of $SU(2)$ in $SU(N+1)$. In formulae

$$\begin{aligned} SU(N+1) &\supset SU(2)\times SU(N-1)\times U(1) \\ (N+1)^2-1 &= (1,1)_0\oplus(3,1)_0\oplus(1,(N-1)^2)_0 \\ &\quad \oplus(2,(N-1))_{-(N+1)}\oplus(2,\overline{(N-1)})_{N+1}. \end{aligned} \tag{3.38}$$

Then we deduce that the fermions that satisfy constraint (3.24) transform as $(N-1)_{-(N+1),0}$ and $\overline{(N-1)}_{N+1,0}$ under $K=SU(N-1)\times U(1)\times U(1)$. In the case of the fuzzy sphere the embedding Lie$(SU(2))\subset$ Lie$(SO(3))$ is somehow trivial. If we had chosen instead the fuzzy $(SU(3)/U(1)\times U(1))_F$, then Lie$(SU(3))$ should be embedded in Lie$(SO(8))$.

In order to write the action for fermions we have to consider the Dirac operator $\mathcal{D}$ on $M^4\times S^2_F$. This operator can be constructed following the derivation presented in ref. [33] for the Dirac operator on the fuzzy sphere, see also ref. [36]. For fermions in the adjoint we obtain

$$\mathcal{D}\psi=i\Gamma^\mu(\partial_\mu+A_\mu)\psi+i\sigma^{\hat a}[X_{\hat a}+A_{\hat a},\psi]-\frac{1}{r}\psi\ , \tag{3.39}$$

where Γ^μ is defined in (2.18), and with slight abuse of notation we have written $\sigma^{\hat a}$ instead of $\sigma^{\hat a}\otimes 1$. Using eq. (3.19) the fermion action,

$$\mathcal{A}_F=\int d^4x\, Tr\,\bar\psi\mathcal{D}\psi \tag{3.40}$$

becomes

$$\mathcal{A}_F=\int d^4x\ \ Tr\,\bar\psi\left(i\Gamma^\mu(\partial_\mu+A_\mu)-\frac{1}{r}\right)\psi\ +i\,Tr\,\bar\psi\sigma^{\hat a}[\varphi_{\hat a},\psi]\ , \tag{3.41}$$

where we recognize the fermion masses $1/r$ and the Yukawa interactions.

Using eqs. (3.37), (3.24) the YM action (3.31) plus the fermion action reads

$$\begin{aligned} \mathcal{A}_{YM}+\mathcal{A}_F &= \int d^4x\ \ \frac{1}{4}Tr(F_{\mu\nu}F^{\mu\nu})-\frac{3}{4}D_\mu\varphi D^\mu\varphi-\frac{3}{8}(\varphi^2-r^{-1}\varphi)^2 \\ &\quad +\int d^4x\ \ Tr\,\bar\psi\left(i\Gamma^\mu(\partial_\mu+A_\mu)-\frac{1}{r}\right)\psi\ -\ \frac{3}{2}Tr\,\bar\psi\varphi\psi\ \ . \end{aligned} \tag{3.42}$$

The choice (3.36) defines one of the possible embeddings of Lie$(SU(2))$ in Lie$(U(N+1))$ [Lie$(SU(2))$ is embedded in Lie$(U(N+1))$ as a regular subalgebra], while on the other extreme we can embed Lie$(SU(2))$ in Lie$(U(N+1))$ using the irreducible $N+1$ dimensional rep. of $SU(2)$, i.e. we identify $\omega_{\hat a}=X_{\hat a}$. Constraint (3.18) in this case implies that the four-dimensional gauge group is $U(1)$ so that $A_\mu(x)$ is $U(1)$ valued. Constraint (3.21) leads again to the scalar singlet $\varphi(x)$.

In general, we start with a $U(1)$ gauge theory on $M^4\times S^2_F$. We solve the CSDR constraint (3.16) by embedding $SU(2)$ in $U(N+1)$. There are $p(N+1)$ embeddings where

$p(n)$ is the number of ways one can partition the integer n into a set of non-increasing positive integers [23] (for example the solution $\omega^{\hat{a}} = 0$ corresponds to the partition $(1, 1, \ldots 1)$, and the embedding using the n irrep. of $SU(2)$ corresponds to the partition (n)). Then constraint (3.18) gives the surviving four-dimensional gauge group. Constraint (3.21) gives the surviving four-dimensional scalars and eq. (3.37) is always a solution but in general not the only one. Setting $\varphi_{\hat{a}} = \omega_{\hat{a}}$ we always minimize the potential. This minimum is given by the chosen embedding of $SU(2)$ in $U(N+1)$. Constraint (3.24) gives the surviving four dimensional spinors.

Finally let us consider spinors that transform in the fundamental of the gauge group G. Then in the fermion action (3.40) we have the covariant Dirac operator $\mathcal{D}\psi = i\Gamma^\mu(\partial_\mu + A_\mu)\psi + i\sigma^{\hat{a}}[X_{\hat{a}}, \psi] + i\sigma^{\hat{a}} A_{\hat{a}}\psi - \frac{1}{r}\psi$, and instead of constraint (3.24) we have to use constraint (3.25). We thus obtain

$$\mathcal{A}_F = \int d^4x \;\; Tr\,\bar{\psi}\left(i\Gamma^\mu(\partial_\mu + A_\mu) - (\frac{5}{2r} + i\sigma^{\hat{a}}\omega_{\hat{a}})\right)\psi + i\,Tr\,\bar{\psi}\sigma^{\hat{a}}\varphi_{\hat{a}}\psi \;\; . \tag{3.43}$$

In the following we study constraint (3.25) in the example where constraint (3.16) is solved by considering the embedding

$$SU(N+1) \supset SU(2) \times U(1) \;, \tag{3.44}$$

obtained by identifying $\omega_{\hat{a}}$ with the generators of $SU(2)$ in the N dimensional irrep.. This embedding induces the embedding and the branching rule[6]

$$\begin{aligned} U(N+1) \;\simeq\; SU(N+1) \times U(1) \supset SU(2) \times U(1) \times U(1) \;, \\ (N+1)_{\sqrt{N}} = N_{1,\sqrt{N}} \oplus 1_{-N,\sqrt{N}} \quad . \end{aligned} \tag{3.45}$$

It follows that $SU(2)$ acts on ψ via the representation $(N \oplus 1) \times \overline{(N+1)}$ given by $\delta\psi = \omega_{\hat{a}}\psi - \psi X_{\hat{a}}$, cf. (3.25). We have

$$\begin{aligned} (N_{1,\sqrt{N}} \oplus 1_{-N,\sqrt{N}}) \times \overline{(N+1)} &= (N_{1,\sqrt{N}} \oplus 1_{-N,\sqrt{N}}) \times (N+1) \\ &= 2N_{1,\sqrt{N}} \oplus (2N-2)_{1,\sqrt{N}} \oplus (2N-4)_{1,\sqrt{N}} \ldots \oplus 2_{1,\sqrt{N}} \oplus (N+1)_{-N,\sqrt{N}} \;, \end{aligned} \tag{3.46}$$

where the indices denote the eigenvalues of the $U(1)$ generators appearing in the r.h.s. of (3.45). We can now solve constraint (3.25) that states that the spinor ψ intertwines the fundamental rep. of $SU(2)$ appearing in the l.h.s. of (3.25) with the rep. of $SU(2)$ appearing in the r.h.s. of (3.25). Since this latter in the present example contains the 2 of $SU(2)$ just once, we conclude that there exists one surviving four-dimensional spinor; this spinor has charges $(1, \sqrt{N})$ with respect to the four-dimensional gauge group $K = U(1) \times U(1)$. In general for fermions in the fundamental we consider the product of the $\overline{(N+1)}$ of $SU(2)$ times the representations of $SU(2)$ on ψ induced by the embedding of

[6]The generator $\lambda = diag(1, 1, \ldots 1, -N)$ of the first $U(1)$ appearing in the r.h.s. of (3.45) is normalized so that $Tr(\lambda^2) = N(N+1)$. This implies the normalization $\lambda' = \sqrt{N} diag(1, 1, \ldots 1, 1)$ for the generator of the second $U(1)$ appearing in the r.h.s. of (3.45), i.e. the $U(1)$ coming from $U(N+1) \simeq SU(N+1) \times U(1)$.

$SU(2)$ in $U(N+1)$ [in the above example the embedding defined by (3.45)]. There are as many four-dimensional spinors as many times the fundamental of $SU(2)$ appears in this product.

The $G = U(P)$ case. In this case $\omega_{\hat{a}} = \omega_{\hat{a}}(X^{\hat{b}}) = \omega_{\hat{a}}^{h,\alpha} T^h \mathcal{T}^\alpha$ is an $(N+1)P \times (N+1)P$ hermitian matrix and in order to solve the constraint (3.16) we have to embed Lie($SU(2)$) in Lie($U((N+1)P)$). All the results of the $G = U(1)$ case holds also here, we just have to replace $N+1$ with $(N+1)P$. This is true for the fermion sector too, provided that in the higher dimensional theory the fermions are considered in the adjoint of $U(P)$ (in the action (3.40) we then need to replace Tr with $Tr\, tr_{U(P)}$ i.e. $tr_{U((N+1)P)}$). We can also consider fermions in the fundamental of $U(P)$. Then an infinitesimal gauge transformation reads $\delta\psi = \lambda\psi$ and the four-dimensional spinors $\psi_{1(2)}$, where $\psi = \left(\begin{smallmatrix}\psi_1\\ \psi_2\end{smallmatrix}\right)$, transforms according to the fundamental of $U(P)$ and the $n \times \overline{n}$ of $U(n)$, i.e. they transform according to the fundamental of $U(nP)$ and the antifundamental of $U(n)$ (where $n = N+1$). In this case, in order to solve constraint (3.25) and find the surviving four-dimensional spinors, we have to consider the product of the $SU(2)$ representation $\overline{(N+1)} = (N+1)$ times the representations of $SU(2)$ on ψ induced by the embedding of $SU(2)$ in $U((N+1)P)$. The $SU(2)$ representation $\overline{(N+1)}$ arises from the $SU(2)$ action $\delta\psi = -\psi X_{\hat{a}}$ observing that $X_{\hat{a}}$ is an $SU(2)$ generator in the irrep. $N+1$. There are as many four-dimensional spinors as many times the fundamental of $SU(2)$ appears in this product of representations.

3.1.4 CSDR constraints for fuzzy cosets

Consider a fuzzy coset $(S/R)_F$ (e.g. fuzzy CP^M) described by $n \times n$ matrices, and let the higher dimensional theory have gauge group $U(P)$. Then we see that constraint (3.16) implies that we have to embed S in $U(nP)$. Constraint (3.18) then implies that the four dimensional gauge group K is the centralizer of the image $S_{U(nP)}$ of S in $U(nP)$, $K = C_{U(nP)}(S_{U(nP)})$.

Concerning fermions in the adjoint, in order to solve constraint (3.24) we consider the embedding

$$S \subset SO(dimS) \ ,$$

which is given by $\tau_{\hat{a}} = \frac{1}{2} C_{\hat{a}\hat{b}\hat{c}} \Gamma^{\hat{b}\hat{c}}$ that satisfies $[\tau^{\hat{a}}, \tau^{\hat{b}}] = C_{\hat{a}\hat{b}\hat{c}} \tau^{\hat{c}}$. Therefore ψ is an intertwining operator between induced representations of S in $U(nP)$ and in $SO(dimS)$. To find the surviving fermions, as in the commutative case [15], we decompose the adjoint rep. of $U(nP)$ under $S_{U(nP)} \times K$,

$$\begin{aligned} U(nP) &\supset S_{U(nP)} \times K \\ adjU(nP) &= \sum_i (s_i, k_i) \ . \end{aligned} \tag{3.47}$$

We also decompose the spinor rep. σ of $SO(dimS)$ under S

$$\begin{aligned} SO(dimS) &\supset S \\ \sigma &= \sum_e \sigma_e \ . \end{aligned} \tag{3.48}$$

Then, when we have two identical irreps. $s_i = \sigma_e$, there is a k_i multiplet of fermions surviving in four dimensions, i.e. four-dimensional spinors $\psi(x)$ belonging to the k_i representation of K.

Concerning fermions in the fundamental of the gauge group $U(P)$, we recall that they can be interpreted as transforming according to the fundamental of $U(nP)$ and the antifundamental $\overline{n}$ of $U(n)$. Moreover the coordinates $X_{\hat{a}}$ are generators of S in the irrep. n, so that the S action $\delta\psi = -\psi X_{\hat{a}}$ is given by the irrep. $\overline{n}$. In order to solve constraint (3.25) we therefore decompose the fundamental of $U(nP)$ under $S_{U(nP)} \times K$,

$$nP = \sum_i (t_i, h_i) \ , \tag{3.49}$$

and then consider the product representation

$$\sum_i (t_i \times \overline{n}, h_i) = \sum_\ell (u_\ell, h_\ell) \ , \tag{3.50}$$

where now u_ℓ are irreps. of S. When we have two identical irreps. $u_\ell = \sigma_e$, there is an h_ℓ multiplet of fermions surviving in four dimensions, i.e. four-dimensional spinors $\psi(x)$ belonging to the h_ℓ representation of K.

4. Discussion and Conclusions

Non-commutative Geometry has been regarded as a promising framework for obtaining finite quantum field theories and for regularizing quantum field theories. In general quantization of field theories on non-commutative spaces has turned out to be much more difficult and with less attractive ultraviolet features than expected [38, 39], see however ref. [40], and ref. [41], where pure Yang-Mills theory on the fuzzy sphere is quantized. Recall also that non-commutativity is not the only suggested tool for constructing finite field theories. Indeed four-dimensional finite gauge theories have been constructed in ordinary space-time and not only those which are $\mathcal{N} = 4$ and $\mathcal{N} = 2$ supersymmetric, and most probably phenomenologically uninteresting, but also chiral $\mathcal{N} = 1$ gauge theories [42] which already have been successful in predicting the top quark mass and have rich phenomenology that could be tested in future colliders [42, 43]. In the present work we have not adressed the finiteness of non-commutative quantum field theories, rather we have used non-commutativity to produce, via Fuzzy-CSDR, new particle models from particle models on $M^4 \times (S/R)_F$.

The Fuzzy-CSDR has different features from the ordinary CSDR leading therefore to new four-dimensional particle models. In this paper we have established the rules for the construction of these models; it may well be that Fuzzy-CSDR provides more realistic four-dimensional theories. Having in mind the construction of realistic models one can also combine the fuzzy and the ordinary CSDR scheme, for example considering $M^4 \times S'/R' \times (S/R)_F$.

A major difference between fuzzy and ordinary SCDR is that in the fuzzy case one always embeds S in the gauge group G instead of embedding just R in G. This is due to the fact that the differential calculus used in the Fuzzy-CSDR is based on $dimS$ derivations instead of the restricted $dimS - dimR$ used in the ordinary one. As a result the four-dimensional gauge group $H = C_G(R)$ appearing in the ordinary CSDR after the geometrical breaking and before the spontaneous symmetry breaking due to the four-dimensional Higgs fields does not appear in the Fuzzy-CSDR. In Fuzzy-CSDR the spontaneous symmetry breaking mechanism takes already place by solving the Fuzzy-CSDR constraints. The four dimensional potential has the typical "maxican hat" shape, but it appears already spontaneously broken. Therefore in four dimensions appears only the physical Higgs field that survives after a spontaneous symmetry breaking. Correspondingly in the Yukawa sector of the theory we have the results of the spontaneous symmetry breaking, i.e. massive fermions and Yukawa interactions among fermions and the physical Higgs field. Having massive fermions in the final theory is a generic feature of CSDR when S is embedded in G (see last ref. in [20]). We see that if one would like to describe the spontaneous symmetry breaking of the SM in the present framework, then one would be naturally led to large extra dimensions.

A fundamental difference between the ordinary CSDR and its fuzzy version is the fact that a non-abelian gauge group G is not really required in high dimensions. Indeed the presence of a $U(1)$ in the higher-dimensional theory is enough to obtain non-abelian gauge theories in four dimensions. We plan to elaborate further on this point, as well as on the possibility to construct realistic theories.

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References

[1] A. Connes, "Non-commutative Geometry", Academic Press 1994.

[2] J. Madore, "An Introduction to Non-commutative Differential Geometry and its Physical Applications", London Mathematical Society Lecture Note Series 257, Cambridge University Press 1999.

[3] A. Connes and J. Lott, Nucl. Phys. Proc. Suppl. **18B**, 29 (1991).

[4] C. P. Martin, J. M. Gracia-Bondia and J. C. Varilly, Phys. Rept. **294**, 363 (1998), hep-th/9605001.

[5] M. Dubois-Violette, J. Madore and R. Kerner, Phys. Lett. B **217**, 485 (1989); Class. Quant. Grav. **6**, 1709 (1989); M. Dubois-Violette, R. Kerner and J. Madore, J. Math. Phys. **31**, 323 (1990).

[6] J. Madore, Phys. Lett. B **305**, 84 (1993); Wroclaw 1992 Proceedings, Spinors, twistors, Clifford algebras and quantum deformations, 285-298, hep-ph/9209226.

[7] A. Connes, M. R. Douglas and A. Schwarz, JHEP **9802**, 003 (1998), hep-th/9711162.

[8] N. Seiberg and E. Witten, JHEP **9909**, 032 (1999), hep-th/9908142.

[9] B. Jurco, S. Schraml, P. Schupp and J. Wess, Eur. Phys. J. C **17**, 521 (2000), hep-th/0006246; B. Jurco, P. Schupp and J. Wess, Nucl. Phys. B **604**, 148 (2001), hep-th/0102129; B. Jurco, L. Moller, S. Schraml, P. Schupp and J. Wess, Eur. Phys. J. C **21**, 383 (2001), hep-th/0104153. G. Barnich, F. Brandt and M. Grigoriev, JHEP **0208**, 023 (2002), hep-th/0206003.

[10] M. Chaichian, P. Presnajder, M. M. Sheikh-Jabbari and A. Tureanu, "Noncommutative standard model: Model building", hep-th/0107055.

[11] X. Calmet, B. Jurco, P. Schupp, J. Wess and M. Wohlgenannt, Eur. Phys. J. C **23**, 363 (2002), hep-ph/0111115; P. Aschieri, B. Jurco, P. Schupp and J. Wess, Nucl. Phys. B **651**, 45 (2003), hep-th/0205214.

[12] F. Brandt, C. P. Martin and F. R. Ruiz, "Anomaly freedom in Seiberg-Witten noncommutative gauge theories," JHEP **0307** (2003) 068, hep-th/0307292.

[13] C. E. Carlson, C. D. Carone and R. F. Lebed, Phys. Lett. B **518**, 201 (2001), hep-ph/0107291; W. Behr, N. G. Deshpande, G. Duplancic, P. Schupp, J. Trampetic and J. Wess, Eur. Phys. J. C **29**, 441 (2003), hep-ph/0202121; I. Hinchliffe and N. Kersting, "Review of the phenomenology of noncommutative geometry", hep-ph/0205040; P. Schupp, J. Trampetic, J. Wess and G. Raffelt, "The photon neutrino interaction in non-commutative gauge field theory and astrophysical bounds", hep-ph/0212292.

[14] P. Forgacs and N. S. Manton, Commun. Math. Phys. **72**, 15 (1980).

[15] D. Kapetanakis and G. Zoupanos, Phys. Rept. **219**, 1 (1992).

[16] Y. A. Kubyshin, I. P. Volobuev, J. M. Mourao and G. Rudolph, "Dimensional Reduction Of Gauge Theories, Spontaneous Compactification And Model Building," *Lecture notes in Physics*, **Vol. 349**, Springer Verlag, Heidelberg 1989.

[17] F. A. Bais, K. J. Barnes, P. Forgacs and G. Zoupanos, Proc. *"HEP1985"*, Bari, Italy, p.60; P. Forgacs, D. Lust and G. Zoupanos, Proc. *"2nd Hellenic School on EPP*, Corfu, Greece, Sep 1-20, 1985, p.86; G. Zoupanos, K. Farakos, D. Kapetanakis, G. Koutsoumbas,Proc. *"23 Rencontre de Moriond" Electroweak interactions and unified theories*, Les Arcs, France, 1988, p.559; K. Farakos. D. Kapetanakis, G. Koutsoumbas, G. Zoupanos, Proc. *"ICHEP 1988"*, Munich, Germany, p.1137; G. Zoupanos, Proc. *"Int. Warsaw Meeting on EPP"* 27-31 May, 1991, Kazimierz, Poland p.446.

[18] G. Chapline and N. S. Manton, Nucl. Phys. **B 184**, 391 (1981); F. A. Bais, K. J. Barnes, P. Forgacs and G. Zoupanos, Nucl. Phys. **B 263**, 557 (1986); K. Farakos, G. Koutsoumbas, M. Surridge and G. Zoupanos, Nucl. Phys. **B 291**, 128 (1987); ibid., Phys. Lett. **B 191**, 135 (1987); Y. A. Kubyshin, J. M. Mourao, I. P. Volobujev, Int. J. Mod. Phys. **A 4**, 151 (1989).

[19] N. S. Manton, Nucl. Phys. **B 193**, 502 (1981); G. Chapline and R. Slansky, Nucl. Phys. B **209**, 461 (1982).

[20] C. Wetterich, Nucl. Phys. **B 222**, 20 (1983); L. Palla, Z. Phys. **C 24**, 195 (1984); K. Pilch and A. N. Schellekens, J. Math. Phys. **25**, 3455 (1984); P. Forgacs, Z. Horvath and L. Palla, Z. Phys. **C 30**, 261 (1986); K. J. Barnes, P. Forgacs, M. Surridge and G. Zoupanos, Z. Phys. **C 33**, 427 (1987).

[21] K. Farakos, D. Kapetanakis, G. Koutsoumbas, G. Zoupanos, Phys. Lett. **B 211**, 322 (1988); B. E. Hanlon and G. C. Joshi, Phys. Lett. **B 298**, 312 (1993).

[22] P. Manousselis and G. Zoupanos, JHEP **0203**, 002 (2002), hep-ph/0111125; ibid., Phys. Lett. B **518**, 171 (2001), hep-ph/0106033; ibid., Phys. Lett. B **504**, 122 (2001), hep-ph/0010141.

[23] J. Madore, Class. Quant. Grav. **9**, 69 (1992).

[24] H. Grosse and A. Strohmaier, geometry and the regularization problem of 4D quantum field theory," Lett. Math. Phys. **48**, 163 (1999), hep-th/9902138.

[25] S. P. Trivedi and S. Vaidya, JHEP **0009**, 041 (2000), hep-th/0007011.

[26] G. Alexanian, A. P. Balachandran, G. Immirzi and B. Ydri, J. Geom. Phys. **42**, 28 (2002), hep-th/0103023.

[27] A. P. Balachandran, B. P. Dolan, J. H. Lee, X. Martin and D. O'Connor, J. Geom. Phys. **43**, 184 (2002), hep-th/0107099; B. P. Dolan and O. Jahn, Int. J. Mod. Phys. A **18**, 1935 (2003), hep-th/0111020.

[28] L. Castellani, Class. Quant. Grav. **17**, 3377 (2000) hep-th/0005210.

[29] P. Aschieri, L. Castellani and A. P. Isaev, Int. J. Mod. Phys. A18, 3555 (2003), hep-th/0201223.

[30] A. Dimakis and F. Muller-Hoissen, J. Math. Phys. **35**, 6703 (1994).

[31] J. Madore, Annals Phys. **219**, 187 (1992); U. Carow-Watamura and S. Watamura, Commun. Math. Phys. **212**, 395 (2000), hep-th/9801195; S. Iso, Y. Kimura, K. Tanaka and K. Wakatsuki, Nucl. Phys. B **604**, 121 (2001), hep-th/0101102; P. Valtancoli, Int. J. Mod. Phys. A **18**, 967 (2003), hep-th/0206075.

[32] J. Madore, S. Schraml, P. Schupp and J. Wess, Eur. Phys. J. C **16**, 161 (2000), hep-th/0001203.

[33] H. Grosse, C. Klimcik and P. Presnajder, Commun. Math. Phys. **178**, 507 (1996), hep-th/9510083.

[34] U. Carow-Watamura and S. Watamura, Commun. Math. Phys. **183**, 365 (1997), hep-th/9605003.

[35] B. P. Dolan and C. Nash, JHEP **0207**, 057 (2002), hep-th/0207007.

[36] B. Ydri, "Fuzzy physics," hep-th/0110006.

[37] J. Fuchs and C. Schweigert, "Symmetries, Lie Algebras And Representations: A Graduate Course For Physicists," Cambridge University Press, 1997.

[38] T. Filk, Phys. Lett. B **376**, 53 (1996); J. C. Varilly and J. M. Gracia-Bondia, Int. J. Mod. Phys. A **14**, 1305 (1999), hep-th/9804001; M. Chaichian, A. Demichev and P. Presnajder, Nucl. Phys. B **567**, 360 (2000), hep-th/9812180.

[39] S. Minwalla, M. Van Raamsdonk and N. Seiberg, JHEP **0002**, 020 (2000), hep-th/9912072.

[40] H. Grosse and R. Wulkenhaar, "Renormalisation of phi**4 theory on noncommutative R**2 in the matrix base," hep-th/0307017.

[41] H. Steinacker, "Quantized gauge theory on the fuzzy sphere as random matrix model," hep-th/0307075.

[42] D. Kapetanakis, M. Mondragon and G. Zoupanos, Z. Phys. C **60**, 181 (1993), hep-ph/9210218; M. Mondragon and G. Zoupanos, Nucl. Phys. Proc. Suppl. **37C**, 98 (1995);

[43] J. Kubo, M. Mondragon and G. Zoupanos, Nucl. Phys. B **424**, 291 (1994); J. Kubo, M. Mondragon and G. Zoupanos, Phys. Lett. B **389**, 523 (1996), hep-ph/9609218; T. Kobayashi, J. Kubo, M. Mondragon and G. Zoupanos, Nucl. Phys. B **511**, 45 (1998), hep-ph/9707425; J. Kubo, M. Mondragon and G. Zoupanos, Acta Phys. Polon. B **27**, 3911 (1997), hep-ph/9703289. K. S. Babu, T. Enkhbat and I. Gogoladze, Phys. Lett. B **555**, 238 (2003), hep-ph/0204246.