Unified Theories from Fuzzy Extra Dimensions

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Abstract: We combine and exploit ideas from Coset Space Dimensional Reduction (CSDR) methods and Non-commutative Geometry. We consider the dimensional reduction of gauge theories defined in high dimensions where the compact directions are a fuzzy space (matrix manifold). In the CSDR one assumes that the form of space-time is $M^D = M^4 \times S/R$ with S/R a homogeneous space. Then a gauge theory with gauge group G defined on M^D can be dimensionally reduced to M^4 in an elegant way using the symmetries of S/R, in particular the resulting four dimensional gauge is a subgroup of G. In the present work we show that one can apply the CSDR ideas in the case where the compact part of the space-time is a finite approximation of the homogeneous space S/R, i.e. a fuzzy coset. In particular we study the fuzzy sphere case.

1   Introduction

Coset Space Dimensional Reduction (CSDR) [1, 2] is a unification scheme for obtaining realistic particle models from gauge theories on higher D-dimensional spaces M^D. It suggests that a unification of the gauge and Higgs sectors of the Standard Model can be achieved in higher than four dimensions. Moreover the addition of fermions in the higher-dimensional gauge theory leads naturally, after CSDR, to Yukawa couplings in four dimensions.

   We study CSDR in the non-commutative context and set the rules for constructing new particle models that might be phenomenologically relevant. One could study CSDR with the whole parent space M^D being non-commutative or with just non-commutative Minkowski space or non-commutative internal space. We specialize here to this last

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situation and therefore eventually we obtain Lorentz covariant theories on commutative Minkowski space. We further specialize to fuzzy non-commutativity, i.e. to matrix type non-commutativity. Thus, following [7], we consider non-commutative spaces like those studied in refs. [3, 4, 5] and implementing the CSDR principle on these spaces we obtain new particle models.

2 Fuzzy sphere

The fuzzy sphere [6, 5] is a matrix approximation of the usual sphere S^2. The algebra of functions on S^2 (for example spanned by the spherical harmonics) is truncated at a given frequency and thus becomes finite dimensional. The truncation has to be consistent with the associativity of the algebra and this can be nicely achieved relaxing the commutativity property of the algebra. The fuzzy sphere is the "space" described by this non-commutative algebra. The algebra itself is that of $N \times N$ matrices. More precisely, the algebra of functions on the ordinary sphere can be generated by the coordinates of $\mathbf{R}^3$ modulo the relation $\sum_{\hat{a}=1}^3 x_{\hat{a}} x_{\hat{a}} = r^2$. The fuzzy sphere S_F^2 at fuzziness level $N-1$ is the non-commutative manifold whose coordinate functions $iX_{\hat{a}}$ are $N \times N$ hermitian matrices proportional to the generators of the N-dimensional representation of $SU(2)$. They satisfy the condition $\sum_{\hat{a}=1}^3 X_{\hat{a}} X_{\hat{a}} = \alpha r^2$ and the commutation relations

$$[X_{\hat{a}}, X_{\hat{b}}] = C_{\hat{a}\hat{b}\hat{c}} X_{\hat{c}} \ , \tag{1}$$

where $C_{\hat{a}\hat{b}\hat{c}} = \varepsilon_{\hat{a}\hat{b}\hat{c}}/r$ while the proportionality factor α goes as N^2 for N large. Indeed it can be proven that for $N \to \infty$ one obtains the usual commutative sphere.

On the fuzzy sphere there is a natural $SU(2)$ covariant differential calculus. This calculus is three-dimensional and the derivations $e_{\hat{a}}$ along $X_{\hat{a}}$ of a function f are given by $e_{\hat{a}}(f) = [X_{\hat{a}}, f]$. Accordingly the action of the Lie derivatives on functions is given by

$$\mathcal{L}_{\hat{a}} f = [X_{\hat{a}}, f] \ ; \tag{2}$$

these Lie derivatives satisfy the Leibniz rule and the $SU(2)$ Lie algebra relation

$$[\mathcal{L}_{\hat{a}}, \mathcal{L}_{\hat{b}}] = C_{\hat{a}\hat{b}\hat{c}} \mathcal{L}_{\hat{c}}. \tag{3}$$

In the $N \to \infty$ limit the derivations $e_{\hat{a}}$ become $e_{\hat{a}} = C_{\hat{a}\hat{b}\hat{c}} x^{\hat{b}} \partial^{\hat{c}}$ and only in this commutative limit the tangent space becomes two dimensional. The exterior derivative is given by

$$df = [X_{\hat{a}}, f] \theta^{\hat{a}} \tag{4}$$

with $\theta^{\hat{a}}$ the one-forms dual to the vector fields $e_{\hat{a}}$, $< e_{\hat{a}}, \theta^{\hat{b}} >= \delta_{\hat{a}}^{\hat{b}}$. The space of one-forms is generated by the $\theta^{\hat{a}}$'s in the sense that for any one-form $\omega = \sum_i f_i(dh_i) t_i$ we can always write $\omega = \sum_{\hat{a}=1}^3 \omega_{\hat{a}} \theta^{\hat{a}}$ with given functions $\omega_{\hat{a}}$ depending on the functions f_i, h_i and t_i. The action of the Lie derivatives on one-forms is given by

$$\mathcal{L}_{\hat{a}}(\theta^{\hat{b}}) = C_{\hat{a}\hat{b}\hat{c}} \theta^{\hat{c}} \tag{5}$$

and it is easily seen to commute with the exterior differential d. On a general one-form $\omega = \omega_{\hat{a}} \theta^{\hat{a}}$ we have $\mathcal{L}_{\hat{b}} \omega = \mathcal{L}_{\hat{b}}(\omega_{\hat{a}} \theta^{\hat{a}}) = [X_{\hat{b}}, \omega_{\hat{a}}] \theta^{\hat{a}} - \omega_{\hat{a}} C^{\hat{a}}_{\hat{b}\hat{c}} \theta^{\hat{c}}$ and therefore

$$(\mathcal{L}_{\hat{b}} \omega)_{\hat{a}} = [X_{\hat{b}}, \omega_{\hat{a}}] - \omega_{\hat{c}} C^{\hat{c}}_{\hat{b}\hat{a}} \ ; \tag{6}$$

this formula will be fundamental for formulating the CSDR principle on fuzzy cosets.

The differential geometry on the product space Minkowski times fuzzy sphere, $M^4 \times S_F^2$, is easily obtained from that on M^4 and on S_F^2. For example a one-form A defined on $M^4 \times S_F^2$ is written as

$$A = A_\mu dx^\mu + A_{\hat{a}} \theta^{\hat{a}} \tag{7}$$

with $A_\mu = A_\mu(x^\mu, X_{\hat{a}})$ and $A_{\hat{a}} = A_{\hat{a}}(x^\mu, X_{\hat{a}})$.

One can also introduce spinors on the fuzzy sphere and study the Lie derivative on these spinors. Although here we have sketched the differential geometry on the fuzzy sphere, one can study other (higher dimensional) fuzzy spaces (e.g. fuzzy CP^M) and with similar techniques their differential geometry.

3   CSDR over fuzzy coset spaces

First we consider on $M^4 \times (S/R)_F$ a non-commutative gauge theory with gauge group $G = U(P)$ and examine its four dimensional interpretation. The action is

$$\mathcal{A}_{YM} = \frac{1}{4} \int d^4x \, Tr \, tr_G \, F_{MN} F^{MN}, \tag{8}$$

where Tr denotes integration over the fuzzy coset $(S/R)_F$ described by $N \times N$ matrices, and tr_G is the gauge group G trace. The higher-dimensional field strength F_{MN} decomposed in four-dimensional space-time and extra-dimensional components reads as follows $(F_{\mu\nu}, F_{\mu\hat{b}}, F_{\hat{a}\hat{b}})$; explicitly the various components of the field strength are given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu], \tag{9}$$

$$F_{\mu\hat{a}} = \partial_\mu A_{\hat{a}} - [X_{\hat{a}}, A_\mu] + [A_\mu, A_{\hat{a}}],$$

$$F_{\hat{a}\hat{b}} = [X_{\hat{a}}, A_{\hat{b}}] - [X_{\hat{b}}, A_{\hat{a}}] + [A_{\hat{a}}, A_{\hat{b}}] - C^{\hat{c}}_{\hat{a}\hat{b}} A_{\hat{c}}. \tag{10}$$

Under an infinitesimal G gauge transformation $\lambda = \lambda(x^\mu, X^{\hat{a}})$ we have

$$\delta A_{\hat{a}} = -[X_{\hat{a}}, \lambda] + [\lambda, A_{\hat{a}}] , \tag{11}$$

thus F_{MN} is covariant under local G gauge transformations: $F_{MN} \to F_{MN} + [\lambda, F_{MN}]$. This is an infinitesimal abelian $U(1)$ gauge transformation if λ is just an antihermitian function of the coordinates $x^\mu, X^{\hat{a}}$ while it is an infinitesimal nonabelian $U(P)$ gauge transformation if λ is valued in $\text{Lie}(U(P))$, the Lie algebra of hermitian $P \times P$ matrices. In the following we will always assume $\text{Lie}(U(P))$ elements to commute with the coordinates $X^{\hat{a}}$. In fuzzy/non-commutative gauge theory and in Fuzzy-CSDR a fundamental role is played by the covariant coordinate,

$$\varphi_{\hat{a}} \equiv X_{\hat{a}} + A_{\hat{a}} . \tag{12}$$

This field transforms indeed covariantly under a gauge transformation, $\delta(\varphi_{\hat{a}}) = [\lambda, \varphi_{\hat{a}}]$. In terms of φ the field strength in the non-commutative directions reads,

$$F_{\mu\hat{a}} = \partial_\mu \varphi_{\hat{a}} + [A_\mu, \varphi_{\hat{a}}] = D_\mu \varphi_{\hat{a}}, \tag{13}$$

$$F_{\hat{a}\hat{b}} = [\varphi_{\hat{a}}, \varphi_{\hat{b}}] - C^{\hat{c}}_{\hat{a}\hat{b}} \varphi_{\hat{c}} ; \tag{14}$$

and using these expressions the action reads

$$\mathcal{A}_{YM} = \int d^4x \, Tr \, tr_G \left(\frac{1}{4} F_{\mu\nu}^2 + \frac{1}{2} (D_\mu \varphi_{\hat{a}})^2 - V(\varphi) \right), \tag{15}$$

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where the potential term $V(\varphi)$ is the $F_{\hat{a}\hat{b}}$ kinetic term (recall $F_{\hat{a}\hat{b}}$ is antihermitian so that $V(\varphi)$ is hermitian and non-negative)

$$V(\varphi) = -\frac{1}{4}Tr\, tr_G \sum_{\hat{a}\hat{b}} F_{\hat{a}\hat{b}}F_{\hat{a}\hat{b}} = -\frac{1}{4}Tr\, tr_G \left([\varphi_{\hat{a}}, \varphi_{\hat{b}}][\varphi^{\hat{a}}, \varphi^{\hat{b}}] - 4C_{\hat{a}\hat{b}\hat{c}}\varphi^{\hat{a}}\varphi^{\hat{b}}\varphi^{\hat{c}} + 2r^{-2}\varphi^2\right).$$

This action is naturally interpreted as an action in four dimensions. The infinitesimal G gauge transformation with gauge parameter $\lambda(x^{\mu}, X^{\hat{a}})$ can indeed be interpreted just as an M^4 gauge transformation. We write

$$\lambda(x^{\mu}, X^{\hat{a}}) = \lambda^{\alpha}(x^{\mu}, X^{\hat{a}})\mathcal{T}^{\alpha} = \lambda^{h,\alpha}(x^{\mu})T^h\mathcal{T}^{\alpha} , \tag{16}$$

where $\mathcal{T}^{\alpha}$ are hermitian generators of $U(P)$, $\lambda^{\alpha}(x^{\mu}, X^{\hat{a}})$ are $N \times N$ antihermitian matrices and thus are expressible as $\lambda(x^{\mu})^{\alpha,h}T^h$ with T^h antihermitian generators of $U(N)$. Now the Lie algebra is the tensor product of $\text{Lie}(U(N))$ and $\text{Lie}(U(P))$, it is indeed $\text{Lie}(U(NP))$. Similarly we rewrite the gauge field A_{ν} as $A_{\nu}(x^{\mu}, X^{\hat{a}}) = A_{\nu}^{\alpha}(x^{\mu}, X^{\hat{a}})\mathcal{T}^{\alpha} = A_{\nu}^{h,\alpha}(x^{\mu})T^h\mathcal{T}^{\alpha}$ and interpret it as a $\text{Lie}(U(NP))$ valued gauge field on M^4. Similarly we treat $\varphi_{\hat{a}}$.

Up to now we have just performed a fuzzy Kaluza-Klein reduction. Indeed in the commutative case the expression (15) corresponds to rewriting the initial lagrangian on $M^4 \times S^2$ using spherical harmonics on S^2. Here the space of functions is finite dimensional and therefore the infinite tower of modes reduces to the finite sum given by Tr.

Next we reduce the number of gauge fields and scalars in the action (15) by applying the CSDR scheme. Since Lie $SU(2)$ acts on the fuzzy sphere $(SU(2)/U(1))_F$, and more in general the group S acts on the fuzzy coset $(S/R)_F$, we can state the CSDR principle in the same way as in the continuum case, i.e. the fields in the theory must be invariant under the infinitesimal $SU(2)$, respectively S, action up to an infinitesimal gauge transformation

$$\mathcal{L}_{\hat{b}}\varphi = \delta^{W_{\hat{b}}}\varphi = W_{\hat{b}}\varphi, \qquad \mathcal{L}_{\hat{b}}A = \delta^{W_{\hat{b}}}A = -DW_{\hat{b}}, \tag{17}$$

where A is the one-form gauge potential $A = A_{\mu}dx^{\mu} + A_{\hat{a}}\theta^{\hat{a}}$, and $W_{\hat{b}}$ depends only on the coset coordinates $X^{\hat{a}}$ and (like A_{μ}, A_a) is antihermitian. We thus write $W_{\hat{b}} = W_{\hat{b}}^{\alpha}\mathcal{T}^{\alpha}$, $\alpha = 1, 2 \dots P^2$, where $\mathcal{T}^i$ are hermitian generators of $U(P)$ and $(W_{\hat{b}}^i)^{\dagger} = -W_{\hat{b}}^i$, here † is hermitian conjugation on the $X^{\hat{a}}$'s. Now in order to solve the constraints (17) we cannot follow the strategy adopted in the commutative case, where the constraints were studied just at one point of the coset (say $y^a = 0$). This is due to the intrinsic nonlocality of the constraints. On the other hand the specific properties of the fuzzy case (e.g. the fact that partial derivatives are realized via commutators, the concept of covariant coordinate) allow to simplify and eventually solve the constraints. Indeed in terms of the covariant coordinate $\varphi_{\hat{d}} = X_{\hat{d}} + A_{\hat{d}}$ and of

$$\omega_{\hat{a}} \equiv X_{\hat{a}} - W_{\hat{a}} , \tag{18}$$

the CSDR constraints assume a particularly simple form, namely

$$[\omega_{\hat{b}}, A_{\mu}] = 0, \tag{19}$$

$$C_{\hat{b}\hat{d}\hat{e}}\varphi^{\hat{e}} = [\omega_{\hat{b}}, \varphi_{\hat{d}}]. \tag{20}$$

In addition we have a consistency condition following from the relation $[\mathcal{L}_a, \mathcal{L}_{\hat{b}}] = C_{\hat{a}\hat{b}}{}^{\hat{c}}\mathcal{L}_{\hat{c}}$:

$$[\omega_{\hat{a}}, \omega_{\hat{b}}] = C_{\hat{a}\hat{b}}{}^{\hat{c}}\omega_c, \tag{21}$$

where $\omega_{\hat{a}}$ transforms as $\omega_{\hat{a}} \rightarrow \omega_{\hat{a}}' = g\omega_{\hat{a}}g^{-1}$. One proceeds in a similar way for the spinor fields [7].

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4 Solving the CSDR constraints for the fuzzy sphere

We consider $(S/R)_F = S_F^2$, i.e. the fuzzy sphere, and to be definite at fuzziness level $N-1$ ($N \times N$ matrices). We study first the basic example where the gauge group $G = U(1)$. In this case the $\omega_{\hat{a}} = \omega_{\hat{a}}(X^{\hat{b}})$ appearing in the consistency condition (21) are $N \times N$ antihermitian matrices and therefore can be interpreted as elements of $\mathrm{Lie}(U(N))$. On the other hand the $\omega_{\hat{a}}$ satisfy the commutation relations (21) of $\mathrm{Lie}(SU(2))$. Therefore in order to satisfy the consistency condition (21) we have to embed $\mathrm{Lie}(SU(2))$ in $\mathrm{Lie}(U(N))$. Let T^h with $h = 1, \ldots, (N)^2$ be the generators of $\mathrm{Lie}(U(N))$ in the fundamental representation, we can always use the convention $h = (\hat{a}, u)$ with $\hat{a} = 1, 2, 3$ and $u = 4, 5, \ldots, N^2$ where the $T^{\hat{a}}$ satisfy the $SU(2)$ Lie algebra,

$$[T^{\hat{a}}, T^{\hat{b}}] = C^{\hat{a}\hat{b}}_{\phantom{\hat{a}\hat{b}}\hat{c}} T^{\hat{c}} . \tag{22}$$

Then we define an embedding by identifying

$$\omega_{\hat{a}} = T_{\hat{a}}. \tag{23}$$

The constraint (19), $[\omega_{\hat{b}}, A_\mu] = 0$, then implies that the four-dimensional gauge group K is the centralizer of the image of $SU(2)$ in $U(N)$, i.e.

$$K = C_{U(N)}(SU((2))) = SU(N-2) \times U(1) \times U(1) ,$$

where the last $U(1)$ is the $U(1)$ of $U(N) \simeq SU(N) \times U(1)$. The functions $A_\mu(x, X)$ are arbitrary functions of x but the X dependence is such that $A_\mu(x, X)$ is $\mathrm{Lie}(K)$ valued instead of $\mathrm{Lie}(U(N))$, i.e. eventually we have a four-dimensional gauge potential $A_\mu(x)$ with values in $\mathrm{Lie}(K)$. Concerning the constraint (20), it is satisfied by choosing

$$\varphi_{\hat{a}} = r\varphi(x)\omega_{\hat{a}} , \tag{24}$$

i.e. the unconstrained degrees of freedom correspond to the scalar field $\varphi(x)$ which is a singlet under the four-dimensional gauge group K.

The choice (23) defines one of the possible embedding of $\mathrm{Lie}(SU(2))$ in $\mathrm{Lie}(U(N))$. For example we could also embed $\mathrm{Lie}(SU(2))$ in $\mathrm{Lie}(U(N))$ using the irreducible N dimensional rep. of $SU(2)$, i.e. we could identify $\omega_{\hat{a}} = X_{\hat{a}}$. The constraint (19) in this case implies that the four-dimensional gauge group is $U(1)$ so that $A_\mu(x)$ is $U(1)$ valued. The constraint (20) leads again to the scalar singlet $\varphi(x)$.

In general, we start with a $U(1)$ gauge theory on $M^4 \times S_F^2$. We solve the CSDR constraint (21) by embedding $SU(2)$ in $U(N)$. There exist p_N embeddings, where p_N is the number of ways one can partition the integer N into a set of non-increasing positive integers [6]. Then the constraint (19) gives the surviving four-dimensional gauge group. The constraint (20) gives the surviving four-dimensional scalars and eq. (24) is always a solution but in general not the only one. By setting $\varphi_{\hat{a}} = \omega_{\hat{a}}$ we obtain always a minimum of the potential. This minimum is given by the chosen embedding of $SU(2)$ in $U(N)$.

In the $G = U(P)$ case, $\omega_{\hat{a}} = \omega_{\hat{a}}(X^{\hat{b}}) = \omega_{\hat{a}}^{h,\alpha} T^h \mathcal{T}^\alpha$ is an $NP \times NP$ hermitian matrix and in order to solve the constraint (21) we have to embed $\mathrm{Lie}(SU(2))$ in $\mathrm{Lie}(U(NP))$. All the results of the $G = U(1)$ case holds also here, we just have to replace N with NP.

One proceeds in a similar way for more general fuzzy coset $(S/R)_F$ (e.g. fuzzy $CP^M = SU(M+1)/U(M)$) described by $N \times N$ matrices. The results are again similar, in particular one starts with a gauge group $G = U(P)$ on $M^4 \times (S/R)_F$, and then the CSDR constraints imply that the four-dimensional gauge group K is the centralizer of the image $S_{U(NP)}$ of S in $U(NP)$, $K = C_{U(NP)}(S_{U(NP)})$.

5   Discussion and Conclusions

The Fuzzy-CSDR has different features from the ordinary CSDR leading therefore to new four-dimensional particle models. Here we have stated the rules for the construction of such models; it may well be that Fuzzy-CSDR provides more realistic four-dimensional theories. Having in mind the construction of realistic models one can also combine the fuzzy and the ordinary CSDR scheme, for example considering $M^4 \times S'/R' \times (S/R)_F$.

A major difference between fuzzy and ordinary CSDR is that in Fuzzy-CSDR the spontaneous symmetry breaking mechanism takes already place by solving the Fuzzy-CSDR contraints. The four dimensional Higgs potential has the typical "mexican hat" shape, but it appears already spontaneously broken. Therefore in four dimensions appears only the physical Higgs field that survives after a spontaneous symmetry breaking. Correspondingly in the Yukawa sector of the theory [7] we obtain the results of the spontaneous symmetry breaking, i.e. massive fermions and Yukawa interactions among fermions and the physical Higgs field. We see that if one would like to describe the spontaneous symmetry breaking of the SM in the present framework, then one would be naturally led to large extra dimensions.

A fundamental difference between the ordinary CSDR and its fuzzy version is the fact that a non-abelian gauge group G is not really required in high dimensions. Indeed the presence of a $U(1)$ in the higher-dimensional theory is enough to obtain non-abelian gauge theories in four dimensions.

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