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CONSTRAINTS ON THE ASYMPTOTIC BEHAVIOUR OF TOTAL CROSS-SECTIONS

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ABSTRACT

We derive constraints on the asymptotic behaviour of total cross-sections which follow from dispersion relations and measured real parts of the forward scattering amplitudes. For πN and pp scattering, these constraints are calculated using recent results from FNAL and Serpukhov. The relation with other methods is discussed.

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1. INTRODUCTION

The analytic properties of scattering amplitudes have turned out to be a valuable tool in the analysis of experimental data, as well as for the test of models which describe only parts of the amplitudes or which are only valid in a restricted energy region.

Looking at the various attempts to make use of analyticity in the context of data analysis, there are mainly two kinds of problems. In the first case, data are given within a relatively large interval and analyticity is used either to get a local prediction (or to remove inconsistencies) of the amplitudes on the cut, or to define a continuation to nearby points in the analyticity domain. Since the unknown parts of the cuts are far away, they play a minor role in this kind of problem and do not affect very much the final result.

In the second case, analyticity is used in order to get an extrapolation of the amplitudes into regions where no data exist. It has been shown [1] that a meaningful answer to such a problem can be given only if additional constraints on the amplitudes can be established.

Several methods have been proposed in order to take into account the analytic properties. The most straightforward way consists of choosing a specific parametrization, which displays explicitly the correct analytic and crossing properties. The parameters are then adjusted by a best fit to the data [2]. However, one must be aware of the fact that the results are biased by the choice of the functions which, in general, do not account for the rather complicated behaviour of the amplitudes at low energies. More flexibility can be obtained by expanding the amplitudes in terms of functions which already have the correct analytical structure [3]. Problems encountered with this method were already discussed in Ref. 4).

Therefore, it seems to us preferable to use dispersion relations, since there is no need for a specific (model-inspired) parametrization and all possible low-energy effects are correctly taken into account.

In the present note we wish to show how much information on the asymptotic behaviour of amplitudes can be obtained by studying dispersion relations and the available experimental data. We shall confine ourselves to the case of forward scattering because real and imaginary parts of the amplitudes can be directly measured in this case. Moreover, they have in general small errors which turn out to be very essential in order to get non-trivial constraints on the amplitudes in the unknown region. The case of non-forward scattering could be treated in the same way, provided that sufficient observables are measured to perform an amplitude analysis without theoretical input. At present, this can be done only at a few energies leaving free an over-all phase which can be determined only by some interference experiment. The actual situation for non-forward scattering is therefore that analytic properties (together with plausible high-energy assumptions) are used in order to reduce the number of unknowns in an amplitude analysis, rather than to give constraints on the asymptotic behaviour.

Our method consists of deriving moments of the asymptotic total cross-section which represent constraints for any model with respect to the asymptotic behaviour. The definition of these moments and a discussion of some of their general properties are contained in Section 2. The constraints on πN and pp total cross-sections obtained by analyzing recent experimental data are presented in Section 3 and Section 4, respectively. Our final conclusions and criticisms of other approaches are given in Section 5.

2. DESCRIPTION OF THE METHOD

Our aim is to extract the information on the asymptotic behaviour of two-body scattering amplitudes contained in experimental data and in their analytic properties. The data consist of a discrete set of experimentally measured points -- affected with errors -- for the real and imaginary parts of the amplitudes within a certain energy interval. In general, the real and imaginary parts are not given at the same energy values.

In order to be able to study amplitudes which are even or odd under crossing symmetry, we assume to have the equivalent information also for the line reversed process.

The analytic properties are taken into account by considering fixed t dispersion relations *) for amplitudes with definite crossing symmetry:

$$F^{(\pm)} = \frac{1}{2} \left(F_{A\bar{B}} \pm F_{AB} \right) \qquad (2.1)$$

where F_{AB} denotes an invariant amplitude, free of kinematical singularities, for the process $A + B \rightarrow A + B$ and $F_{A\bar{B}}$ that of the line reversed process. For reasons which will be discussed below both amplitudes $F^{(+)}$ and $F^{(-)}$ are written in the once-subtracted form:

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*) In the following we write all formulae for the case of forward scattering. The generalization to $t \neq 0$ is straightforward.

$$\mathrm{Re}\, F^{(-)}_{(\nu)} = F^{(-)}_{B(\nu)} + F^{(-)}_{C(\nu)} + \frac{\nu}{\nu_0}\,\mathrm{Re}\, F^{(-)}_{(\nu_0)} + $$

$$+ \frac{2\nu}{\pi}(\nu^2-\nu_0^2)\int_{\nu_{th}}^{\infty} \frac{d\nu'}{\nu'^2-\nu^2}\,\frac{\mathrm{Im}\, F^{(-)}_{(\nu')}}{\nu'^2-\nu_0^2} \qquad (2.2a)$$

$$\mathrm{Re}\, F^{(+)}_{(\nu)} = F^{(+)}_{B(\nu)} + F^{(+)}_{C(\nu)} + \mathrm{Re}\, F^{(+)}_{(\nu_0)} + $$

$$+ \frac{2}{\pi}(\nu^2-\nu_0^2)\int_{\nu_{th}}^{\infty} \frac{d\nu'}{\nu'^2-\nu^2}\,\frac{\nu'\,\mathrm{Im}\, F^{(+)}_{(\nu')}}{\nu'^2-\nu_0^2} \qquad (2.2b)$$

where $\nu = s-u/4M^2 = \omega + t/4M^2$ is the crossing variable, and ω denotes the lab. energy of the incident particle. ν_{th} is the physical threshold for the process AB and ν_0 the subtraction point. $F_B^{(\pm)}$ stands for all possible Born terms and $F_C^{(\pm)}$ stands for contributions of possible unphysical cuts. It can easily be shown that these terms behave like

$$F_C^{(-)} \rightarrow \nu \qquad F_B^{(-)} \rightarrow \frac{1}{\nu}$$

$$F_C^{(+)} \rightarrow \mathrm{const.} \qquad F_B^{(+)} \rightarrow \mathrm{const.} \qquad (2.3)$$

as ν tends to infinity.

Cutting off the dispersion integrals in Eqs. (2.2) at $\bar{\nu}$, the highest energy for which the imaginary parts of the amplitudes are known *), we define the following functions

$$\Delta^{(-)}_{(\nu)} = \mathrm{Re}\, F^{(-)}_{(\nu)} - F^{(-)}_{B(\nu)} - \frac{2\nu}{\pi}(\nu^2-\nu_0^2)\int_{\nu_{th}}^{\bar{\nu}} \frac{d\nu'}{\nu'^2-\nu^2}\,\frac{\mathrm{Im}\, F^{(-)}_{(\nu')}}{\nu'^2-\nu_0^2} \qquad (2.4)$$

$$\Delta^{(+)}_{(\nu)} = \mathrm{Re}\, F^{(+)}_{(\nu)} - F^{(+)}_{B(\nu)} - \frac{2}{\pi}(\nu^2-\nu_0^2)\int_{\nu_{th}}^{\bar{\nu}} \frac{d\nu'}{\nu'^2-\nu^2}\,\frac{\nu'\,\mathrm{Im}\, F^{(+)}_{(\nu')}}{\nu'^2-\nu_0^2} \qquad (2.5)$$

*) If the situation is reversed, that means if information on the real parts tends to higher energies than that on the imaginary parts, Eqs. (2.2) should be replaced by inverse dispersion relations.

which can be evaluated by inserting the experimental values of the real parts and performing the principal value integration over the imaginary parts *). These functions $\Delta^{(\pm)}$ contain all information on $\mathrm{Im}\,F^{(\pm)}$ for $\nu > \bar{\nu}$ and possible low-energy effects following from analyticity and the experimental data. Under very general conditions for $\mathrm{Im}\,F^{(\pm)}$, $\Delta^{(\pm)}$ can be expanded in a power series:

$$\Delta^{(\pm)}_{(\nu)} = \left(\frac{\nu}{\bar{\nu}}\right)^{\frac{1\mp 1}{2}} \left[\sum_{n=1}^{\infty} \left(\frac{\nu}{\bar{\nu}}\right)^{2n} M_n^{(\pm)}(\bar{\nu})\right] + M_0^{(\pm)}$$

$$\text{for} \quad \nu_{th} \ll \nu < \bar{\nu} \tag{2.6}$$

The coefficients $M_n^{(\pm)}$ are given by moments over $\mathrm{Im}\,F^{(\pm)}$:

$$M_n^{(\pm)} = \frac{2}{\pi}\,\bar{\nu}^{\,2n+1} \int_{\bar{\nu}}^{\infty} \frac{d\nu'}{\nu'^2 - \nu_0^2}\, \frac{\mathrm{Im}\,F^{(\pm)}_{(\nu')}}{\nu'^{2n}} \left(\frac{\nu'}{\bar{\nu}}\right)^{\frac{1\pm 1}{2}} \qquad n \geq 1$$

$$\simeq \frac{2}{\pi}\,\bar{\nu}^{\,2n+1} \int_{\bar{\nu}}^{\infty} \frac{d\nu'}{\nu'^{2(n+1)}}\, \mathrm{Im}\,F^{(\pm)}_{(\nu')} \left(\frac{\nu'}{\bar{\nu}}\right)^{\frac{1\pm 1}{2}}$$

$$\text{if} \quad \nu_0 \ll \bar{\nu}$$

$$M_0^{(\pm)} = F_{C\,(\nu)}^{(\pm)} + \left(\frac{\nu}{\nu_0}\right)^{\frac{1\mp 1}{2}} \mathrm{Re}\,F^{(\pm)}_{(\nu_0)} \tag{2.7}$$

The question now is how many moments can be determined from the data. Since Δ is known only at a finite number of ν values and has experimental errors, it is clear that the number of moments which can be determined is limited. Obviously, this number is much less than the number of data points on $\Delta^{(\pm)}$. Furthermore, it will depend strongly on the magnitude of $(\nu/\bar{\nu})^2$; in other words, the nearer information on real parts tend to $\bar{\nu}$ the more coefficients M_n can be extracted.

To get an impression on the behaviour of the moments M_n, we consider the following bound:

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*) To calculate the principal value integrals numerically, we prefer to insert directly the data rather than to use a smooth interpolation. In order to avoid unreasonable local fluctuations, we replace the data in a small interval around the singularity by a second-order polynomial, which is determined by a best fit to the data points under the condition that the first derivatives at the edges are continuous. The main advantage of this method is that we obtain a well-defined error for the integral which is hard to get if data have been interpolated before. Moreover, there is no need for a careful selection of the input data since the method removes fluctuations which originate from different experiments.

$$\left| M_n^{(\pm)} \right| \leq \frac{2}{\pi} \bar{\nu}^{2n+1} \int_{\bar{\nu}}^{\infty} \frac{d\nu'}{\nu'^2 - \nu_0^2} \frac{\left| \operatorname{Im} F^{(+)}_{(\nu')} \right|}{\nu'^{2n}} \left(\frac{\nu'}{\nu} \right)^{\frac{1 \pm 1}{2}} \tag{2.8}$$

Replacing $\operatorname{Im} F^{\pm}$ by total cross-sections via the optical theorem

$$\operatorname{Im} F^{(\pm)} = \frac{k}{4\pi} \sigma^{(\pm)} \tag{2.9}$$

(k denotes the lab. momentum of the incident particle), we obtain the following upper limits for the moments:

$$\begin{aligned} \left| M_n^{(-)} \right| &\leq \frac{1}{2\pi^2} \frac{\sigma_F^{(+)}(\bar{\nu})}{2n+1} \\ & \qquad\qquad\qquad\qquad n \geq 1 \\ \left| M_n^{(+)} \right| &\leq \frac{1}{2\pi^2} \frac{\sigma_F^{(+)}(\bar{\nu})}{2n} \end{aligned} \tag{2.10}$$

with

$$\sigma_F^{(+)}(\nu) = \frac{C_{AB} + C_{A\bar{B}}}{2} \log^2 \nu \tag{2.11}$$

Taking C_{AB} and $C_{A\bar{B}}$ from Ref. 5), we find that in the case of πN scattering only the first two moments contribute more than 1% at $(\nu/\bar{\nu})^2 = 0.1$. We therefore expect that $\Delta^{(\pm)}$ are slowing varying functions of $(\nu/\bar{\nu})^2$, at least as long as $(\nu/\bar{\nu})^2$ is not near to 1.

Another important property of the moments holds in the case of an amplitude which is positive everywhere. Hence, all moments are positive and Δ must be a monotonously increasing function of $(\nu/\bar{\nu})$.

These general properties are very useful in order to decide whether a structure in Δ is an effect of local inconsistencies of the data, or whether it can be related to the M_n without violating generally accepted principles.

Finally, we wish to show what is expected in the case of conventional asymptotic behaviour of scattering amplitudes. We have therefore calculated the moments, assuming

i) a power law in ν (Regge behaviour)

$$\mathrm{Im}\, F^{(\pm)} = C^{(\pm)} \left(\frac{\nu}{\nu_1}\right)^{\alpha} \qquad \nu > \bar{\nu} \tag{2.12}$$

which gives

$$M_n^{(+)} = \frac{2}{\pi} C^{(+)} \left(\frac{\bar{\nu}}{\nu_1}\right)^{\alpha} \frac{1}{2n-\alpha} \qquad \alpha \neq 2n$$

$$M_n^{(-)} = \frac{2}{\pi} C^{(-)} \left(\frac{\bar{\nu}}{\nu_1}\right)^{\alpha} \frac{1}{2n+1-\alpha} \qquad \alpha \neq 2n+1 \tag{2.13}$$

ii) a power law in $\ell n\, \nu$ and ν

$$\mathrm{Im}\, F^{(\pm)} = C^{(\pm)} \left(\frac{\nu}{\nu_1}\right)^{\alpha} \ell n^{\beta} \left(\frac{\nu}{\nu_2}\right) \qquad \nu > \bar{\nu} \tag{2.14}$$

α real, $\beta = 1, 2$ or -1

Then we get

$$M_n^{(\pm)} = \frac{2}{\pi} C^{(\pm)} \left(\frac{\bar{\nu}}{\nu_1}\right)^{\alpha} f\left(2n + \tfrac{1}{2}(1 \pm 1) - \alpha, \beta\right)$$

$$\alpha \neq 2n + \tfrac{1}{2}(1 \pm 1) \tag{2.15}$$

with

$$f(m,2) = \ell n^2 \frac{\bar{\nu}}{\nu_2} + \frac{2\, \ell n \frac{\bar{\nu}}{\nu_2}}{m-1} + \frac{2}{(m-1)^2}$$

$$f(m,1) = \ell n \frac{\bar{\nu}}{\nu_1} + \frac{1}{m-1}$$

$$f(m,-1) = -\left(\frac{\nu_2}{\bar{\nu}}\right)^{-m+1} Ei\left[(-m+1)\, \ell n \frac{\bar{\nu}}{\nu_2}\right]$$

where $Ei(x)$ denotes the integral exponential function.

In the following sections we report our results from an application of the method outlined above to πN and NN forward scattering. We also have tried to analyse the $K_L^0 p \to K_S^0 p$ regeneration amplitude at $t = 0$ by considering the extrapolated differential cross-sections for this process, the forward amplitude following from coherent regeneration experiments, and total cross-section data for $K^{\pm} n$. However, the errors of these data are so large that all moments are compatible with zero. This means that in this case we do not learn anything from the present data and analytic properties about the asymptotic behaviour of the regeneration amplitude.

3. πN FORWARD SCATTERING

3.1 The crossing odd amplitude

Our first example will be the calculation of the function $\Delta^{(-)}$ in πN scattering. The cut-off energy $\bar{\nu}$ in the integral of Eq. (2.4) is chosen to be 200 GeV, because up to this energy $\sigma^{(-)}$ is known from the $\pi^{\pm}p$ total cross-sections [6]. For convenience, we put the subtraction energy ν_0 equal to the pion mass m_π in order to have a simple relation between the subtraction constant and the scattering lengths. The real parts Re $F^{(-)}$ can be obtained in the most reliable way by an extrapolation *) of the πN charge exchange (CEX) differential cross-sections [7),8),9)] to the forward direction using isospin invariance

$$F^{(-)} = -\frac{F_{CEX}}{\sqrt{2}} \qquad (3.1)$$

and the optical theorem Eq. (2.9) for $\sigma^{(-)} = \frac{1}{2}(\sigma_{\pi^-p} - \sigma_{\pi^+p})$. Our results are shown in Fig. 1 where we have plotted $\tilde{\Delta}^{(-)}/\nu$ **)

$$\frac{\tilde{\Delta}^{(-)}}{\nu} \equiv \frac{\Delta^{(-)}}{\nu} - \frac{2}{\pi}\int_{\nu_{th}}^{\bar{\nu}} \frac{d\nu' \, \mathrm{Im}\, F^{(-)}(\nu')}{\nu'^2 - \nu_0^2} \qquad (3.2)$$

as a function of $(\nu/\bar{\nu})^2$. The errors of $\tilde{\Delta}^{(-)}$ originate mainly from the real parts; it turned out that the principal value integral gives only a minor contribution to the total errors. The figure shows clearly that the two recent high-energy experiments on CEX [9),10)] are inconsistent in the near forward direction. As we shall see, this discrepancy is only important, however, in the case where $\sigma^{(-)} \to 0$ asymptotically.

Owing to the restricted range in $(\nu/\bar{\nu})^2$, we are not able to determine more than the first moment M_0 which amounts to

$$M_0^{(-)} = (202 \pm 2) \text{ mb GeV} \qquad (3.3)$$

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*) In order to obtain the CEX differential cross-section at $t = 0$, we have used the same procedure as the authors of Ref. 10).

**) The integral in Eq. (3.2) is dominated by low-energy effects and can be calculated from total cross-section data with high accuracy. Since it dominates $\Delta^{(-)}/\nu$, we prefer to show $\tilde{\Delta}^{(-)}/\nu$ rather than $\Delta^{(-)}/\nu$ in order to demonstrate the influence of the high-energy data on the determination of the moments.

if data of Ref. 9) are used at high energies. If we take instead the results of Ref. 8), $M_0^{(-)}$ increases only by 1%.

Neglecting the extremely small contribution from the unphysical region which arises from the process $\pi^- p \to \gamma n$, $M_0^{(-)}/\bar{\nu}$ can be written in terms of s-wave scattering lengths and the pion-nucleon coupling constant

$$\frac{M_0^{(-)}}{\bar{\nu}} = \frac{1}{3} \; \frac{M + m_\pi}{M} (a_1 - a_3) \; - \; \frac{2 f^2 m_\pi}{m_\pi^2 - \nu_B^2} \tag{3.4}$$

a_1 and a_3 are the s-wave scattering lengths for isospin $I = \frac{1}{2}$ and $\frac{3}{2}$, respectively, $\nu_B = -m_\pi^2/2M$, and $f^2 = 0.08 \pm 0.002$ denotes the πNN coupling constant. From Eqs. (3.4) and (3.3) we obtain

$$a_1 - a_3 = (.29 \pm .015)\, m_\pi^{-1} \tag{3.5}$$

which has to be compared with a recent determination of this quantity from low-energy $\pi^\pm p$ angular distribution 11). These authors obtained $a_1 - a_3 = (0.262 \pm 0.004) m_\pi^{-1}$ which differs from our result by almost two standard deviations. We do not think that this discrepancy is due to Coulomb corrections of the $\pi^\pm p$ total cross-sections in the intermediate energy range as it was proposed in Ref. 12). It seems more likely to us that the authors of Ref. 11) underestimated their error. But there may also be correction terms to $a_1 - a_3$ and f^2 due to a breaking of charge independence at low energies. A careful analysis of the forthcoming measurements of πN CEX 13) in the region of the first resonance hopefully will solve this question.

Although the discrepancy in $a_1 - a_3$ amounts only to 10%, it has a tremendous effect for the prediction of Re $F^{(-)}$ at higher energies because it blows up linearly in energy. Inserting the values for $a_1 - a_3$ and f^2 from Ref. 11) into the subtracted dispersion relation, the authors of Ref. 14) obtain a zero of Re $F^{(-)}$ near 20 GeV which, however, is certainly excluded by the CEX data.

The preceding discussion referred to the more general case of an asymptotically non-vanishing $\sigma^{(-)}$. Here, as we have seen in Eq. (3.4), $M_0^{(-)}$ gave us only a relation between f^2 and $a_1 - a_3$ which must be fulfilled to produce the correct dispersive real parts. We have, however, no constraint on the asymptotic behaviour following from $M_0^{(-)}$.

Since most models on πN CEX predict $\sigma^{(-)} \to 0$ for $\nu \to \infty$, it is of interest to consider also the unsubtracted dispersion relation for $F^{(-)}$.

Then $M_0^{(-)}$ can be expressed by an integral over $\sigma^{(-)}$. Inserting this into the left-hand side of Eq. (3.4) we obtain the well-known Goldberger-Myazawa-Oehme (GMO) sum rule [15)]

$$\frac{1}{3}\left(1+\frac{m_\pi}{M}\right)(a_1-a_3) = \frac{2f^2 m_\pi}{m_\pi^2-\nu_B^2} + \frac{m_\pi}{2\pi^2}\int_0^\infty \frac{\sigma^{(-)}(k)\,dk}{\sqrt{k^2+m_\pi^2}} \qquad (3.6)$$

Subtracting the known part of the integral from $M_0^{(-)}/\bar{\nu}$ we obtain

$$\frac{m_\pi}{2\pi^2}\int_{\bar{k}}^\infty \frac{\sigma^{(-)}\,dk}{\sqrt{k^2+m_\pi^2}} = \begin{cases} 0.0031 \pm 0.0003 \text{ mb GeV} \\ \text{if data of Ref. 9)} \\ 0.0042 \pm 0.0003 \text{ mb GeV} \\ \text{if data of Ref. 8)} \end{cases} \qquad (3.7)$$

are used. Obviously, any model belonging to the class mentioned above, which describes present data on CEX [either those of Ref. 9) or those of Ref. 8)] and $\sigma^{(-)}$ and, which claims to have the correct analytic properties, can be checked by simply computing the integral of Eq. (3.7).

First of all we wish to discuss the consequences if the larger value corresponding to the data of Ref. 8) is the right one. Assuming a power behaviour for $\sigma^{(-)}$ we obtain from Eq. (2.13) $\alpha(0) = 0.55 \pm 0.05$ which agrees with the value from a best fit to the $\sigma^{(-)}$ data giving $\alpha(0) = 0.55 \pm 0.02$. This means that the single Regge pole model is consistent with the forward dispersion relation provided that the data of Ref. 8) are correct. This is not the case if we rely on the data of Ref. 9), since the corresponding value of Eq. (3.7) inserted into Eq. (2.13) gives $\alpha(0) = 0.41 \pm 0.08$ in net disagreement with the behaviour of $\sigma^{(-)}$ below $\bar{\nu} = 200$ GeV.

Having only one constraint on $\sigma^{(-)}$ at our disposal, we did not look for more sophisticated models for the CEX amplitude because it is clear that Eq. (3.7) can be satisfied if a sufficient number of parameters are involved. As an example, we note that the model of Ref. 16) derived from b universality and a peripheral geometry for the flip amplitude has the needed tendency of a faster decreasing $\sigma^{(-)}$. We checked that this model is, in fact, in agreement with our constraint.

Finally, we wish to point out that the discrepancy could be due to the presence of a $J = 1$ pole singularity which contributes only to the real part because of crossing symmetry. This possibility would be of particular interest if the power behaviour of the present known imaginary parts persists also at higher energies. The existence of such an odd $J = 1$ pole term simply reflects in an additional constant A in the GMO sum rule. Assuming that the power behaviour of $\sigma^{(-)}$ can be continued to infinite energies with the same parameters, we obtain the following estimate for the pole residue at $t = 0$:

$$A \approx -.007 \text{ mb} \tag{3.8}$$

The numerical value of A depends, of course, very sensitively on what has been assumed for $\sigma^{(-)}$. Since it is small and negative, we expect a zero in the real part around 4000 GeV so that we still are far away from the asymptotic regime, where we would have

$$\frac{d\sigma_{CEX}}{dt} \longrightarrow 2\pi A^2 \quad , \quad \frac{Re F^{(-)}}{Im F^{(-)}} \longrightarrow \infty \tag{3.9}$$

The following table lists the prediction for the forward differential cross-section at some energies where results will become available in the near future.

k [GeV/c]	$[(d\sigma/dt)_{CEX}$ at $t = 0]\mu b/GeV/c^2$
150	14.7
200	10.6
300	7.3

We note that the presence of such a linear term was already discussed by several authors[2),17),18)]. Since they used older data for the real and imaginary parts of the CEX amplitude or considered only information on $\pi^{\pm}p$ amplitudes, their results differ from ours by orders of magnitude. The physical origin of such a singularity is still far from being clear. Only a similar analysis at $t \neq 0$, which, however, will be hard to perform, could clarify the situation. Therefore, we prefer not to speculate, but we do not think that the effect can be attributed to a fixed pole as was proposed by the authors of Ref. 18). Because of t-channel unitarity, such a pole must be accompanied by a shielding cut which, in general, also affects the imaginary part, at least asymptotically. This means that the integral in Eq. (3.7) is no longer convergent; in other words, we are back again at the subtracted case where $M_0^{(-)}$ is simply the subtraction constant and gives no constraint for the asymptotic behaviour of the amplitude.

3.2 <u>The crossing even amplitude</u>

Our next example will be the calculation of the function $\Delta^{(+)}$ in πN scattering. As in the previous case the cut-off energy $\bar{\nu}$ is taken at 200 GeV, the highest energy where total cross-sections for $\pi^{\pm}p$ scattering are available.

The real parts $\mathrm{Re}\,F^{(+)}$ can be taken in this case only from an analysis of Coulomb interference measurements of $\pi^{\pm}p$ scattering because the extrapolation of $d\sigma/dt$ to $t=0$ does not give a reliable result. This is due to the fact that the ratio $\rho^{(+)} = \mathrm{Re}\,F^{(+)}/\mathrm{Im}\,F^{(+)}$ at higher energies is one order of magnitude less than in the case of CEX scattering.

Since the process corresponding to $F^{(+)}$ is not experimentally feasible, we need the real parts of π^+p <u>and</u> π^-p scattering at the same energy in order to form $\mathrm{Re}\,F^{(+)}$. At energies where data exist only for one process, we used isospin invariance

$$F^{(+)} = F_{\mp} \pm \frac{1}{\sqrt{2}} F_{CEX} \qquad (3.10)$$

and took $\mathrm{Re}\,F_{CEX}$ from an interpolation of the CEX data (cf. subsection 3.1).

The result of our evaluation of Eq. (2.5) is shown in Fig. 2a. The consistency between the experimental real parts and the dispersion relation is even better than in the previous case. However, since the errors of the real parts in the energy range between 30 and 60 GeV are rather large, we are not able to obtain more than two moments which amount to

$$\begin{aligned} M_0^{(+)} &= -.23 \pm .04 \quad \text{mb GeV} \\ M_1^{(+)} &= 230 \pm 20 \quad \text{mb GeV} \end{aligned} \qquad (3.11)$$

An attempt to extract a third moment did not improve substantially χ^2 and was therefore considered as insignificant.

In our calculation the subtraction of Eq. (2.26) was made at the crossing symmetry point $\nu_0 = 0$, relating $M_0^{(+)}$ to the value of $F^{(+)}$ at $\nu = 0$ and the πNN coupling constant:

$$M_0^{(+)} = F^{(+)}_{(\nu=0)} - \frac{g^2_{\pi NN}}{4\pi M} \tag{3.12}$$

In order to compare our result with s-wave scattering lengths, we consider

$$\left(1+\frac{m_\pi}{M}\right)(a_1+2a_3) = \frac{m_\pi^2}{2\pi^2}\int_0^\infty \frac{dk\ \sigma^{(+)}}{k^2+m_\pi^2} - \frac{g^2_{\pi NN}\ m_\pi^4}{4\pi M(4M^2 m_\pi^2 - m_\pi^4)} + M_0^{(+)} \tag{3.13}$$

which follows from Eq. (2.26) by putting $\nu_0 = m_\pi$. With the value of Eq. (3.11) we get

$$a_1 + 2a_3 = .017 \pm .015 \quad m_\pi^{-1} \tag{3.14}$$

We note that the errors of $a_1 + 2a_3$ is only due to the uncertainty of $M_0^{(+)}$, since the integral occurring in Eq. (3.13) is rapidly converging and has an error of about 2%. The term related to the coupling constant also turns out to give no essential contribution to the error of $a_1 + 2a_3$.

The positive value for the s-wave scattering lengths is in slight disagreement with the results of other authors [cf. Ref. 19)] who based their analysis mainly on low-energy phase shifts. We do not think that this discrepancy is serious because the errors quoted there are only statistical and do not account for inconsistencies of the low-energy data.

The next moment $M_1^{(+)}$ represents a constraint on the total cross-section $\sigma^{(+)}$ above 200 GeV. Any model which describes the present data has to fulfil this additional condition, otherwise it would be incompatible with the dispersion relation.

In order to demonstrate the restrictions on $\sigma^{(+)}$ above 200 GeV in specific cases, we assumed that the asymptotic behaviour of $\sigma^{(+)}$ is given by either

$$\sigma^{(+)}(k) = \sigma^{(+)}_{(k_0)} + \hat{\sigma}\ \ln^2(k/k_0) \tag{3.15}$$

with

$$k_0 = 150 \quad GeV/c$$

or

$$\sigma^{(+)}_{(k)} = \sigma^{(+)}_{\infty} - \frac{\sigma_1}{\ln k/k_0} \tag{3.16}$$

with

$$k_0 = 1 \text{ GeV}/c$$

Although they have a quite different asymptotic limit, both parametrizations can be adjusted to fit present data in a certain energy range [cf. Ref. 20)]. By requiring continuity at $\nu = \bar{\nu}$ and using our value for $M_1^{(+)}$ we can compute the two remaining parameters which amount to

$$\sigma^{(+)}(k_0) = (23.7 \pm .1)\,\text{mb} \quad , \quad \hat{\sigma} = (.37 \pm .8)\,\text{mb}$$

and

$$\sigma_\infty^{(+)} = (23. \pm 1.3)\,\text{mb} \, , \quad \sigma_1 = (-5 \pm 8)\,\text{mb}$$

The possible regions for $\sigma^{(+)}$ above $\nu = \bar{\nu}$ which follow from either behaviour are shown in Fig. 2b *).

Obviously the bounds on $\sigma^{(+)}$ become stronger if we determine the parameters not only by continuity at $\nu = \bar{\nu}$ and by adjustment to $M_1^{(+)}$, but if we require additionally that the assumed parametrization is already valid in a certain energy range below $\nu = \bar{\nu}$.

From Fig. 2a 21)-24) we observe also that part of the data of Ref. 21) show a systematic deviation from the fitted curve. This different behaviour is obviously the reason for the somewhat strange result of Ref. 25), which claimed that a rise of $\sigma^{(+)}$ above 60 GeV can be excluded. It seems to us that the method of averaged dispersion relations which has been applied by these authors in order to get a bound on the asymptotic behaviour of $\sigma^{(+)}$, can produce misleading results because it is unable to detect local inconsistencies or systematic deviations of the data.

. THE CROSSING EVEN pp DIFFRACTION AMPLITUDE

As a third example we have calculated $\Delta^{(+)}$ from Eq. (2.5) for the case of pp forward scattering **). The input for the dispersion integral was calculated from $\sigma_{tot}(pp)$ up to $\bar{\nu} = 2200$ GeV and from $\sigma^{(-)} = \frac{1}{2}(\sigma_{p\bar{p}} - \sigma_{pp})$. Above 200 GeV, $\sigma^{(-)}$ was assumed to follow a power law

$$\sigma^{(-)} = C\,(k/k_1)^{\alpha - 1} \tag{4.1}$$

*) It is clear that the lower branch of the curve corresponding to Eq. (3.15) is not of physical interest since positivity does not allow for a negative $\hat{\sigma}$.

**) The definition of the amplitude is the same as in Ref. 27).

the parameters $C = 26.0 \pm 0.5$ mb, $\alpha = 0.42 \pm 0.02$, $k_1 =$ 1 GeV/c being determined by a best fit to the data below 200 GeV [cf. Ref. 27)].

As in the previous case the only source of information on real parts which is at present at our disposal are the Coulomb interference measurements. Since these experiments were only performed for pp scattering, we assumed that the phase of $F^{(-)}$ is given by the one Regge pole model. We then have for $\mathrm{Re}\, F^{(+)}$

$$\mathrm{Re}\, F^{(+)} = \frac{k}{4\pi} \rho_{pp} \sigma_{pp} + \frac{k}{4\pi} \mathrm{tg}\left(\frac{\pi\alpha}{2}\right) C \left(\frac{k}{k_1}\right)^{\alpha - 1} \tag{4.2}$$

where ρ_{pp} denotes the measured ratio of real to imaginary part of pp scattering which we have taken from Ref. 26).

Figure 3 shows our result for $\Delta^{(+)}$ according to Eq. (2.5) and the assumptions on $F^{(-)}$ stated above. It is clear that the present result would be changed considerably if $F^{(-)}$ has a "non-Regge term" similar to what was found in subsection 3.1. The magnitude of the errors of $\Delta^{(+)}$ and the range of $(\nu/\bar{\nu})^2$ where $\Delta^{(+)}$ is known allow us to determine two moments

$$\begin{aligned} M_0^{(+)} &= 3.6 \pm .28 \quad \text{mb GeV} \\ M_1^{(+)} &= 4900 \pm 350 \quad \text{mb GeV} \end{aligned} \tag{4.3}$$

$M_0^{(+)}$ contains the combined effect of the unphysical $\bar{p}p$ cut, the pion Born terms, the subtraction constant and the contribution of the threshold region, which behave like a constant at energies where we have evaluated $\Delta^{(+)}$. It is therefore not directly related to the scattering lengths, but it may be used in sum rules for meson-nucleon coupling constants which follow from a saturation of the unphysical cut by resonances.

The moment $M_1^{(+)}$ gives a constraint on the asymptotic behaviour of $\sigma^{(+)}$. With the neglect of $\sigma^{(-)}$, which is justified if $\sigma^{(-)}$ behaves according to Eq. (4.1) also at energies above 2200 GeV, we have calculated the bounds on σ_{pp} in the same way as we did in the previous section for πN scattering. The parameters corresponding to Eq. (3.15) and Eq. (3.16) have larger errors due to the fact that the cross-section at $\nu = \bar{\nu}$ is less precise than in the πN case. Figure 3b shows the allowed regions for σ_{pp} if the asymptotic law is given by either Eq. (3.15) or Eq. (3.16). We note again that the lower branch of the curve belonging to Eq. (3.15) is not interesting because of the positivity of σ_{pp}.

It is clear that the forthcoming measurements of ρ_{pp} at the highest attainable ISR energies will allow us to determine additional moments. Since we then have more constraints at our disposal, it will be possible to check more sophisticated asymptotic models where more than two free parameters are involved.

5. FINAL REMARKS

We have shown what are the consequences for the asymptotic behaviour of total cross-sections following from present data and analytic properties. In all three specific cases considered in this note, we obtained only one integral constraint for the asymptotic cross-section due to the fact that real parts are not measured at such high energies as the imaginary parts and that they have in general larger errors.

In the case of the isospin even amplitudes, our results show consistency with the increasing behaviour of total cross-sections as is indicated by present experimental results on σ_{tot}, an asymptotic constant behaviour, however, cannot be excluded. For the isospin odd πN forward amplitude,, we found net disagreement with the one-Regge pole model, if we rely on more recent, but still preliminary data, on CEX scattering. Our result imposes a valuable constraint on models which have the property of an asymptotically vanishing $\sigma^{(-)}$; we checked, in particular, that a model derived from b universality reproduces the moment obtained in our analysis. Furthermore, it is pointed out that if the present behaviour of $\sigma^{(-)}$ persists also at higher energies, a simple interpretation of the CEX forward amplitude can be given by assuming the presence of a purely real term which grows linearly with energy. For this case we calculated the forward differential cross-sections at some energies where experimental results will become available in the near future.

Finally, we would like to stress that our analysis gives completely model-independent results because we are using dispersion relations where the information contained in the data is extracted without any further assumption. In our opinion this is the most suitable way to study the consequences of analyticity on any model. One only has to evaluate a few simple integrals and compare them to the moments given by our analysis.

It is obvious that the analytic parametrization method [2] which has to concentrate on a specific behaviour of the amplitudes from the very beginning, tests only compatibility of the data with the chosen specific form. In this sense it is just a model which is adjusted to the data rather than a test of analyticity. The same arguments hold also for the recently proposed operator method [28] which replaces the non-local dispersion integral by a differential operator. Apart from the fact that it provides by no means a local relation *), it is also based on a specific parametrization. It is important to note that the admissible functions must belong to a very restricted class of functions, otherwise the operator method does not give a meaningful result as has been shown recently by the authors of Ref. 29). The agreement of the results of Ref. 28) with the experimentally measured real parts proves only that these authors have selected a favourable interpolation of the total cross-sect on data; there are certainly other parametrizations of σ_{tot} with equal or even better χ^2 which give completely different real parts if the differential operator is applied, whereas the dispersion relation reproduces essentially the same result for every parametrization.

Another method which is, however, closely related to ours, consists in studying averaged dispersion relations [25),30)]. The intention is to avoid principal value integrals which is achieved by integrating the dispersion relation over a certain energy interval $\omega_1 \leq \omega \leq \omega_2$. This means that the usual Cauchy kernel is replaced by a new kernel $K(\omega,\omega_1,\omega_2)$ which has only two logarithmic singularities at ω_1 and ω_2. It is certainly true that these singularities are easier to handle than a pole singularity if the input data have large fluctuations, but this is of no practical importance. In all cases considered in this note, total cross-section data have small errors and show only little fluctuations; therefore principal value integrals can be computed in a reliable way. On the other hand, since this method considers $<\Delta>_{\omega_1}^{\omega_2}$ and not $\Delta(\omega)$, inconsistencies of the data cannot be detected but may produce a misleading result. Moreover, it always gives only one constraint even in the ideal case where real and imaginary parts are known up to the same energy.

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*) According to the authors of Ref. 28), a few derivatives of σ_{tot} are sufficient to reproduce the real part. This can, of course, only be true if the local interpolation of the σ_{tot} data, continued to the low-energy region, forms an average of the physical amplitude. To make sure one has to study a FESR which is a non-local relation.

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FIGURE CAPTIONS

Fig. 1 : The quantity $\tilde{\Delta}^{(-)}/\nu$ as defined by Eq. (3.2) as a function of $(\nu/\bar{\nu})^2$. The real parts are taken from the forward values of Refs. 7), 8) and 9).

Fig. 2a : The function $\Delta^{(+)}$ for the isospin even πN amplitude as calculated from Coulomb interference data from Refs. 21), 22), 23) and 24). The solid line corresponds to our best fit.

Fig. 2b : Allowed regions for the total cross-section $\sigma^{(+)} = \frac{1}{2}(\sigma_{\pi^+p} + \sigma_{\pi^-p})$. The dashed line indicates the bounds if a logarithmic increase [Eq. (3.15)] is assumed above 200 GeV. The solid line belongs to a behaviour with a constant limit at infinity [Eq. (3.16)]. Data below 200 GeV are taken from Ref. 6).

Fig. 3a : $\Delta^{(+)}$ for the pp crossing even amplitude. The real parts are calculated according to Eq. (4.2) using the data on ρ_{pp} which are listed in Ref. 26). The solid line corresponds to a fit with M_o and M_1 given in Eq. (4.3).

Fig. 3b : Bounds on σ_{pp}^{tot}. The notation is the same as in Fig. 2b.

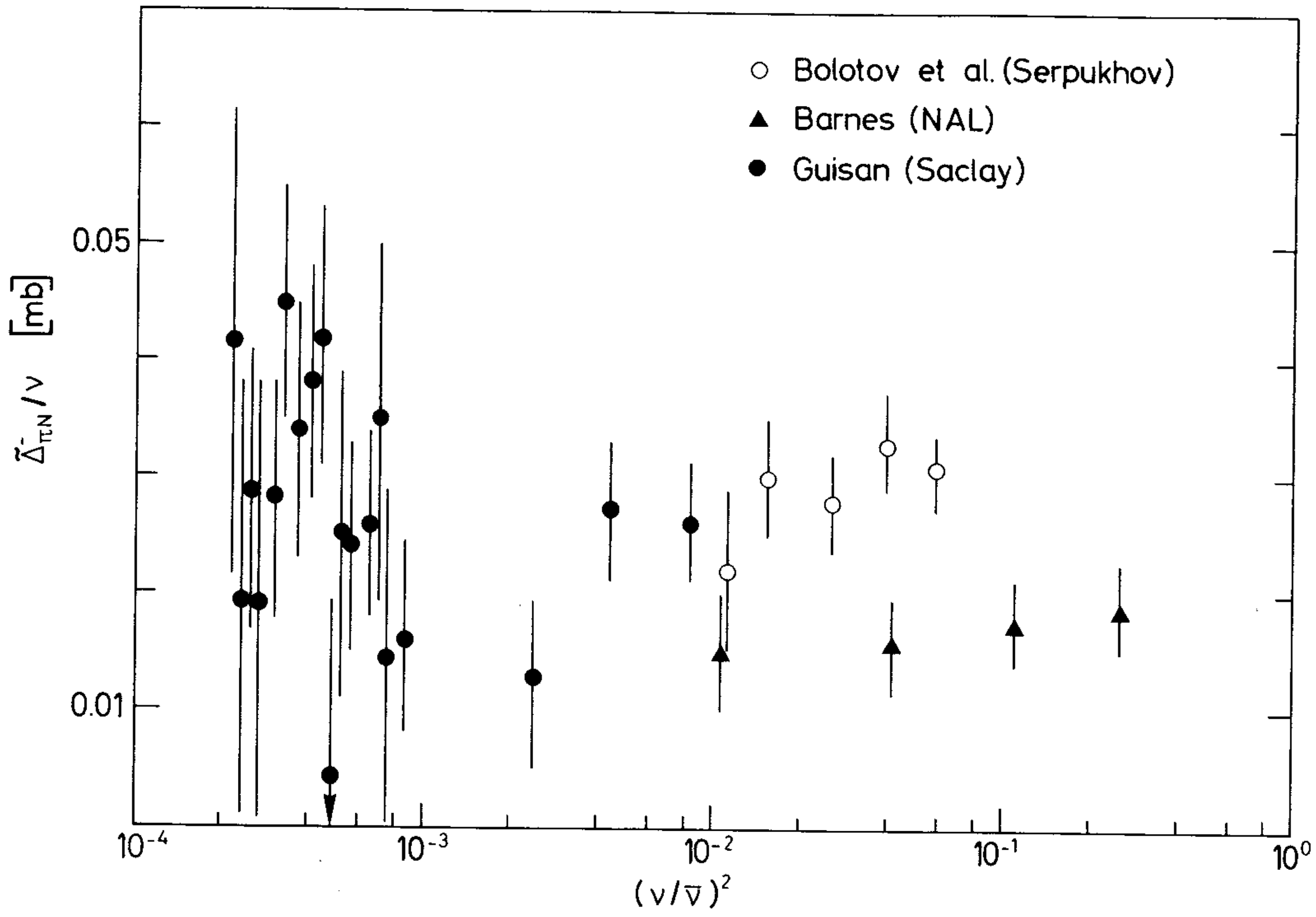

Bolotov et al. (Serpukhov)
Barnes (NAL)
Guisan (Saclay)
0.05
0.01
$\tilde{\Delta}^{-}_{\pi N}/\nu$ [mb]
10^{-4}
10^{-3}
10^{-2}
10^{-1}
10^{0}
$(\nu/\bar{\nu})^2$

FIG. 1

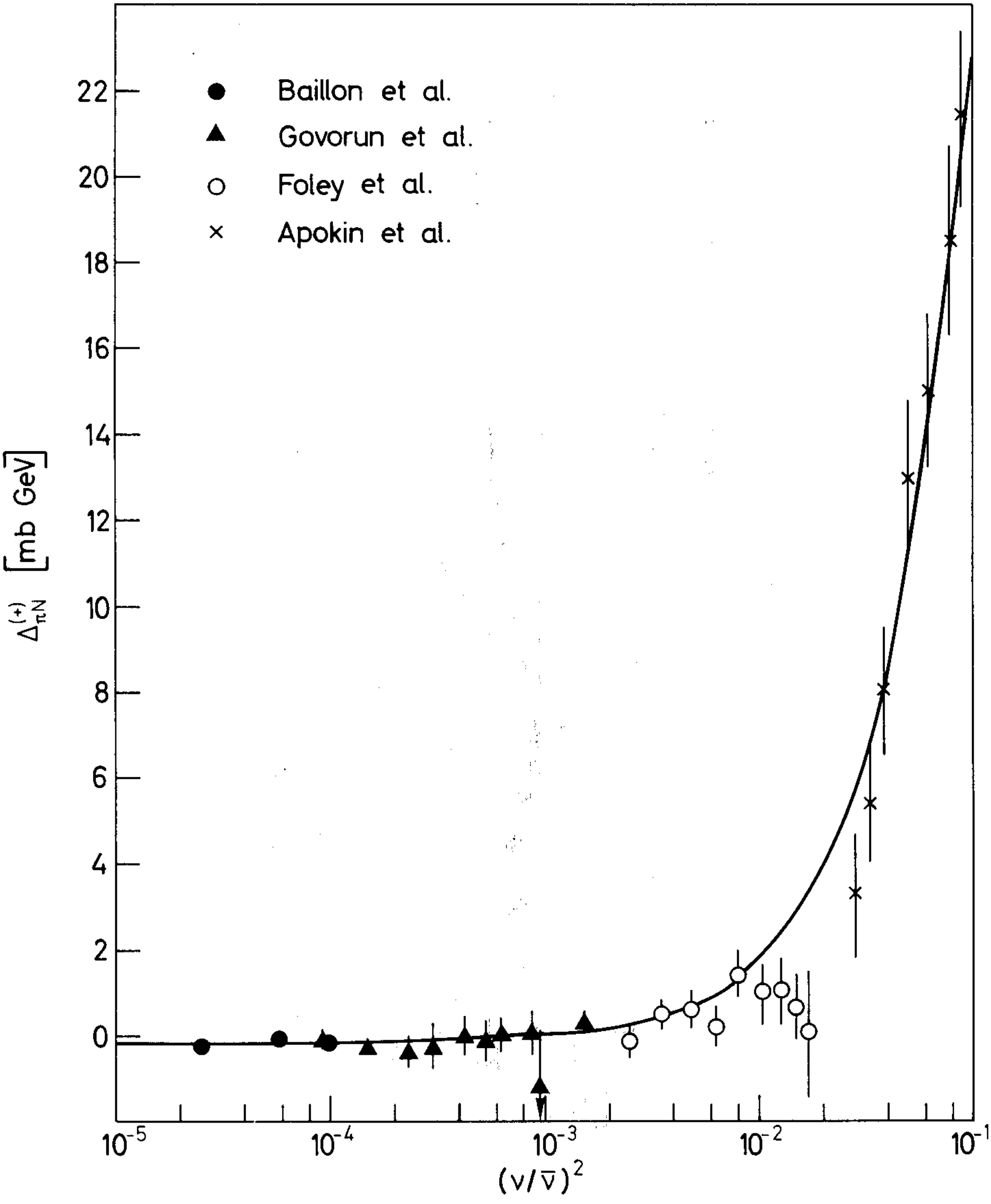

Baillon et al.
Govorun et al.
Foley et al.
Apokin et al.
$\Delta_{\pi N}^{(+)}$ [mb GeV]
$(\nu/\bar{\nu})^2$
0
2
4
6
8
10
12
14
16
18
20
22
10^{-5}
10^{-4}
10^{-3}
10^{-2}
10^{-1}

FIG. 2a

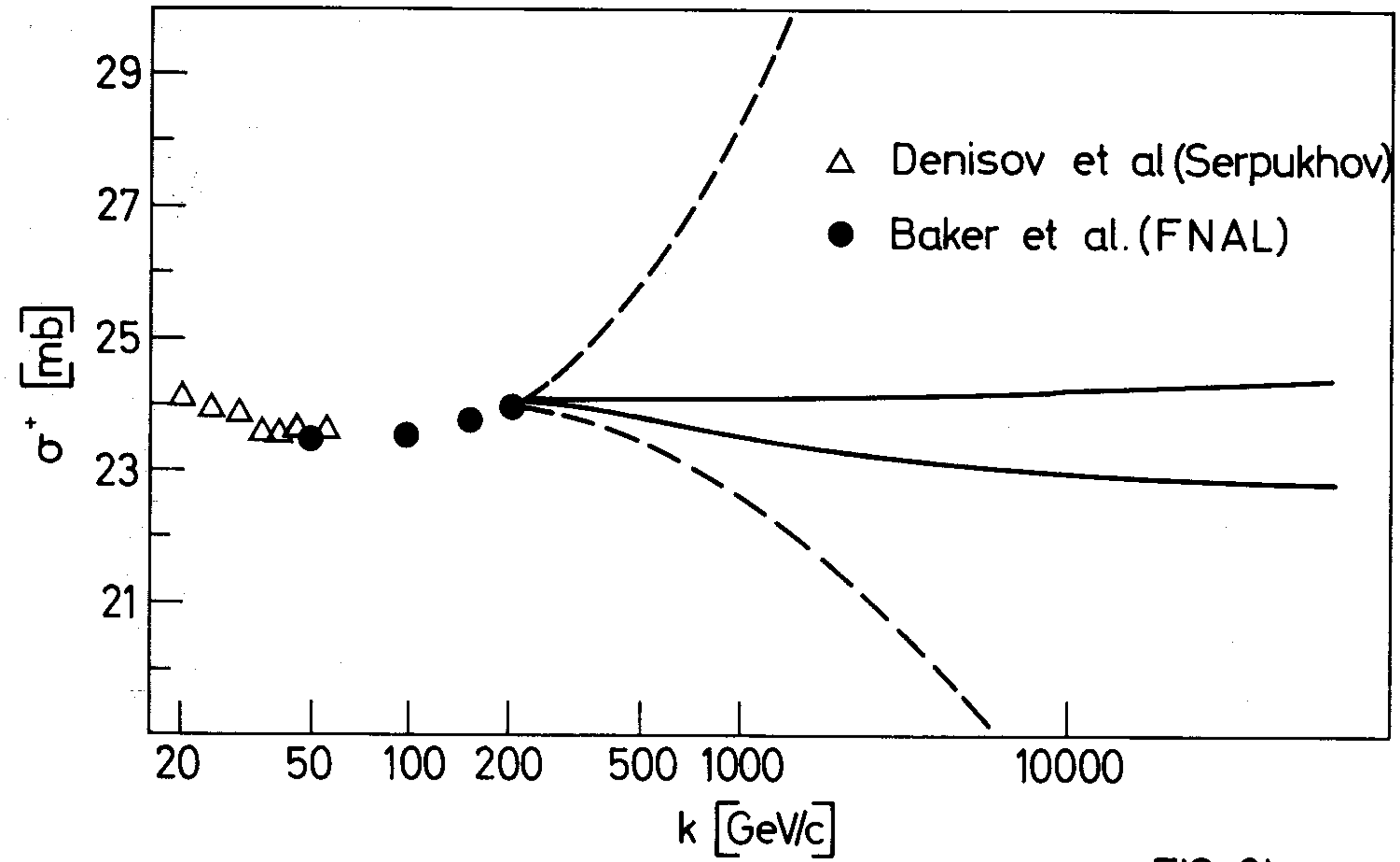

29
27
25
23
21
σ^{+} [mb]
20
50
100
200
500
1000
10000
k [GeV/c]
△ Denisov et al (Serpukhov)
● Baker et al. (FNAL)

FIG. 2b

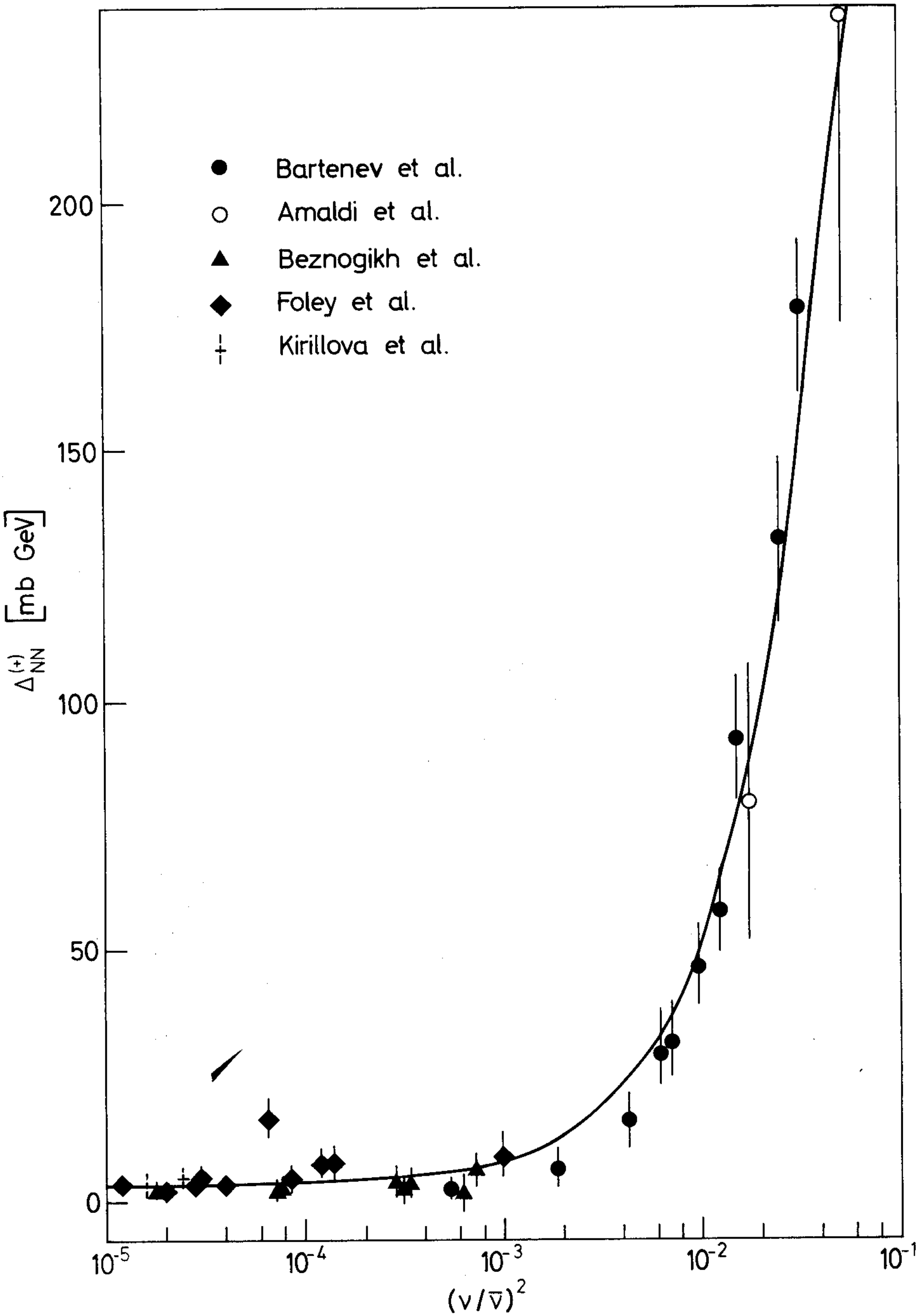
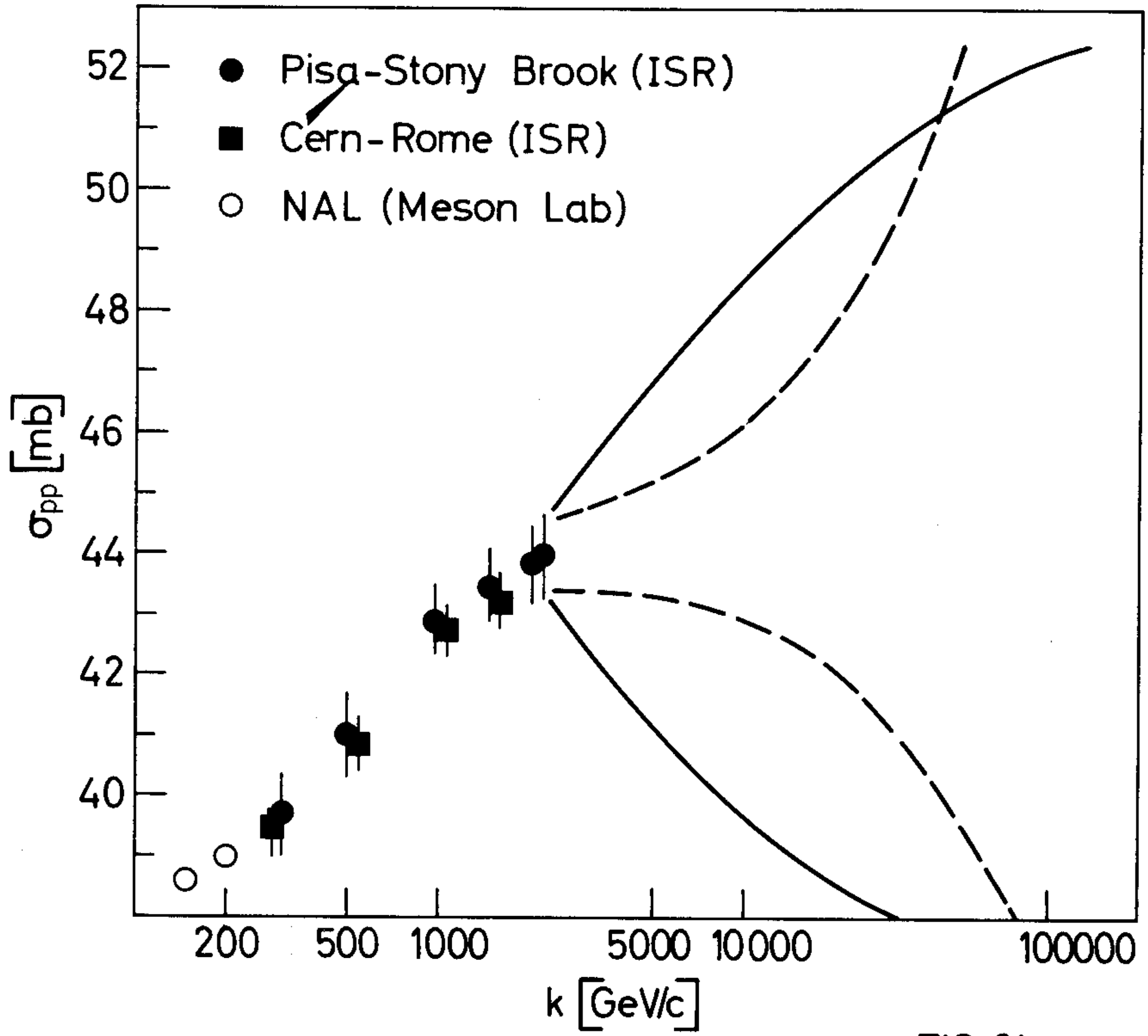

Bartenev et al.
Amaldi et al.
Beznogikh et al.
Foley et al.
Kirillova et al.
200
150
100
50
0
$\Delta_{NN}^{(+)}$ [mb GeV]
10^{-5}
10^{-4}
10^{-3}
10^{-2}
10^{-1}
$(\nu/\bar{\nu})^2$

FIG. 3a



Pisa-Stony Brook (ISR)
Cern-Rome (ISR)
NAL (Meson Lab)
52
50
48
46
44
42
40
σ_{pp} [mb]
200
500
1000
5000
10000
100000
k [GeV/c]

FIG. 3b