IMPLICATIONS OF SPACETIME QUANTIZATION FOR THE BAHCALL-WAXMAN NEUTRINO BOUND

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ABSTRACT

There is growing interest in quantum-spacetime models in which small departures from Lorentz symmetry are governed by the Planck scale. In particular, several studies have considered the possibility that these small violations of Lorentz symmetry may affect various astrophysical observations, such as the evaluation of the GZK limit for cosmic rays, the interaction of TeV photons with the Far Infrared Background and the arrival time of photons with different energies from cosmological sources. We show that the same Planck-scale departures from Lorentz symmetry that lead to a modification of the GZK limit which would be consistent with the observations reported by AGASA, also have significant implications for the evaluation of the Bahcall-Waxman bound on the flux of high-energy neutrinos produced by photo-meson interactions in sources of size not much larger than the proton photo-meson mean free path.

1 Introduction

There is a growing interest recently [1-11] in the possibility that some novel quantum properties of spacetime may have important implications for the analysis of Lorentz transformations. Some approaches to the quantum-gravity problem attribute to the Planck scale E_p ($E_p \sim 10^{28} eV$) the status of an intrinsic characteristic of space-time structure. For example E_p can have a role in spacetime discretization or in the commutation relations between spacetime observables. It is very hard [9] (perhaps even impossible) to construct discretized versions or non-commutative versions of Minkowski space-time which enjoy ordinary Lorentz symmetry. Pedagogical illustrative examples of this observation have been discussed, *e.g.*, in Ref.[12] for the case of discretization and in Refs.[13, 14, 15] for the case of non-commutativity. The action of ordinary (classical) boosts on discretization length scales (or non-commutativity length scales) will naturally be such that different inertial observers would attribute different values to these lengths scales, just as one would expect from the mechanism of FitzGerald-Lorentz contraction.

Models based on an approximate Lorentz symmetry, with Planck-scale-dependent departures from exact Lorentz symmetry, have been recently considered in most quantum-gravity research lines, including models based on "spacetime foam" pictures [1, 2], "loop quantum gravity" models [3, 11], certain "string theory" scenarios [8], and "noncommutative geometry [9, 10].

Interest in tests of modifications of Lorentz symmetry has also increased recently as a result of the realization [4, 5, 16, 17] that these modifications of Lorentz symmetry provide one of the possible solutions of the so-called "cosmic-ray paradox". The spectrum of observed cosmic rays was expected to be affected by a cutoff at the scale $E_{\rm GZK} \sim 5{\cdot}10^{19}$ eV. Cosmic rays emitted with energy higher than $E_{\rm GZK}$ should interact with photons in the cosmic microwave background and lose energy by pion emission, so that their energy should have been reduced to the $E_{\rm GZK}$ level by the time they reach our Earth observatories. However, the AGASA observatory has reported [18] several observations of cosmic rays with energies exceeding the $E_{\rm GZK}$ limit [19] by nearly an order of magnitude. As other experiments do not see an excess of particles above the GZK limit, this experimental puzzle will only be established when confirmed by larger observatories, such as Auger [20]. Furthermore, numerous other solutions have been discussed in the literature. Still, it is noteworthy that Planck-scale modifications of Lorentz symmetry can raise [4, 5] the threshold energy for pion production in collisions between cosmic rays and microwave photons, and the increase is sufficient to explain away the puzzle associated with the mentioned ultra-high-energy cosmic-ray observations.

Bahcall-Waxman [21] have shown that the same particles which we observe as high-energy cosmic rays should also lead to neutrino production at the source. Using the observed cosmic ray fluxes they derive a bound (the Bahcall-Waxman bound) on the flux of high-energy neutrinos that can be revealed in astrophysics observatories. We show here that the same Lorentz-symmetry violations that can extend the cosmic-ray spectrum also affect the chain of particle-physics processes that arise in the neutrino production and hence in establishing the Bahcall-Waxman bound [21]. Thus, the departures from Lorentz symmetry that are capable of explaining the "cosmic-ray paradox" inevitably lead to modification of this limit.

Since a relatively large variety of quantum-gravity pictures is being considered in the analysis of the cosmic-ray paradox, in this first study we only intend to illustrate our point within a simple phenomenological model, leaving for future studies the task of more precise analysis of specific quantum-gravity models. For similar reasons we only focus on one of the chains of processes that are relevant for the Bahcall-Waxman bound: the case in which a proton at the source undergoes photo-pion interactions of the type $p + \gamma \to X + \pi^+$ before escaping the source, then giving rise to neutrino production through the decays $\pi^+ \to \mu^+ + \nu_\mu$ and $\mu^+ \to e^+ + \nu_e + \bar{\nu}_\mu$.

The phenomenological model on which we focus is the simplest one in the literature, which evolved primarily through the studies reported in Refs. [1, 4, 5]. This is a kinematic in which the Planck-scale E_p enters the energy/momentum dispersion relation

$$m^2 = E^2 - \vec{p}^2 + f(E, \vec{p}; E_p) \simeq E^2 - \vec{p}^2 + \eta \vec{p}^2 \left(\frac{E}{E_p}\right)^n \tag{1}$$

while the laws of energy-momentum conservation remain unaffected[1] by the Planck scale. η is a dimensionless coefficient which one expects to be roughly of order 1 (but cannot be reliably predicted at the present preliminary level of development of the relevant quantum-gravity models). The power n, which should also be treated as a phenomenological parameter, is a key element of this phenomelogical scenario, since it characterizes the first nonvanishing contribution in a (inverse-)Planck-scale power series of the quantum-gravity-induced correction $f(E, \vec{p}; E_p)$. It is usually expected that $n = 1$ and $n = 2$ are most likely.

In Section 2 we revisit the analysis of the emergence of "threshold anomalies" due to (1) in the study of particle production in collision processes. For positive η ($\eta \sim 1$) and $n \leq 2$, according to the Planck-scale effect (1) one expects [5] an increase in the threshold energy for pion production in collisions between cosmic rays and microwave photons, and the increase is sufficient to explain away the GZK puzzle raised by the AGASA observations. We also comment on another potentially observable threshold anomaly that concerns electron-positron pair production in photon-photon collisions. Throughout Section 2 we also emphasize the differences between the case of positive η and the case of negative η.

In Section 3 we show that (1) also affects significantly the at-the-source processes of the type $p + \gamma \rightarrow X + \pi^+$ that are relevant for the Bahcall-Waxman analysis. If $\eta \sim 1$ and $n \leq 2$ (*i.e.* for the same departures from Lorentz symmetry that would explain the cosmic-ray paradox) (1) leads to the prediction of a strongly reduced probability for the process $p + \gamma \rightarrow X + \pi^+$ to occur before the proton escapes the source. Correspondingly one expects sharply reduced neutrino production, and as a result the "quantum-gravity-modified Bahcall-Waxman bound" should be expected to be many orders of magnitude lower than the standard Bahcall-Waxman bound. The opposite effect is found for negative η ($\eta \sim -1$): in that case one would expect the standard Bahcall-Waxman bound to be violated, *i.e.* for negative η one could find a neutrino flux that exceeds the standard Bahcall-Waxman bound.

In Section 4 we show that also the decays $\pi^+ \rightarrow \mu^+ + \nu_\mu$ and $\mu^+ \rightarrow e^+ + \nu_e + \bar{\nu}_\mu$ are significantly affected by the Planck-scale effect (1). Again the effect goes in the direction of reducing neutrino production for positive η. However we also observe that the dominant quantum-gravity modification of the Bahcall-Waxman bound comes at the level of analysis of processes of the type $p + \gamma \rightarrow X + \pi^+$, where a several-order-of-magnitude modification would be expected, whereas the additional modification encountered at the level of the processes $\pi^+ \rightarrow \mu^+ + \nu_\mu$ and $\mu^+ \rightarrow e^+ + \nu_e + \bar{\nu}_\mu$ is not as significant. Section 5 is devoted to our closing remarks.

2 Previous results on Planck-scale-induced threshold anomalies and the sign of η

We begin by considering the implications of Eq. (1) for the analysis of processes of the type $1 + \gamma \rightarrow 2 + 3$. The key point for us is that, for a given energy E_1 of the particle that collides

[1]Our analysis based on (1) and standard energy-momentum conservation should be applicable (up to small numerical modifications) to a large class of quantum-gravity models which are being considered as possible solutions of the cosmic-ray paradox. One exception is the "doubly-special relativity" framework [9], in which one could adopt a dispersion relation of type (1) but it would then be necessary to introduce a corresponding modification of the laws of energy-momentum conservation (in order to avoid the emergence of a preferred class of inertial observers [9]). Our conclusions are not applicable to that scheme.

with the photon, there is of course a minimal energy ϵ_{min} of the photon (γ) in order for the process to be kinematically allowed, and therefore achieve production of the particles 2 and 3. One finds that if η is positive the value of ϵ_{min} predicted according to the Planck-scale effect (1) is higher than the corresponding value obtained using ordinary Lorentz symmetry.

In the applications that are of interest here the particle that collides with the photon has a very high energy, $E_1 \simeq p_1 \gg m_1$, and its energy is also much larger than the energy of the photon with which it collides $E_1 \gg \epsilon$. This will allow some useful simplifications in the analysis.

Let us start by briefly summarizing the familiar derivation of ϵ_{min} in the ordinary Lorentz-invariant case. At the threshold (no momenta in the CM frame after the collision) energy conservation and momentum conservation become one dimensional:

$$E_1 + \epsilon = E_2 + E_3 \ , \tag{2}$$

$$p_1 - q = p_2 + p_3 \ , \tag{3}$$

where q is the photon's momentum. The ordinary Lorentz-invariant relations are

$$q = \epsilon \ , \quad E_i = \sqrt{p_i^2 + m_i^2} \simeq p_i + \frac{m_i^2}{2p_i} \ , \tag{4}$$

where we have assumed that, since E_1 is large as mentioned, E_2 and E_3 are also large ($E_{2,3} \simeq p_{2,3} \gg m_{2,3}$).

The threshold conditions are usually identified by transforming these laboratory-frame relations into center-of-mass-frame relations and imposing that the center-of-mass energy be equal to $m_2 + m_3$. However, in preparation for the discussion of deformations of Lorentz invariance it is useful to work fully in the context of the laboratory frame. There the threshold condition that characterizes ϵ_{min} can be identified with the requirement that the solutions for E_2 and E_3 as functions of ϵ (with a given value of E_1) that follow from Eqs. (2), (3) and (4) should be imaginary for $\epsilon < \epsilon_{min}$ and should be real for $\epsilon \geq \epsilon_{min}$. This straightforwardly leads to

$$\epsilon \geq \epsilon_{min} \simeq \frac{(m_2 + m_3)^2 - m_1^2}{4E_1} \ . \tag{5}$$

This standard Lorentz-invariant analysis is modified [4, 5] by the deformations codified in (1). The key point is that Eq. (4) is replaced by

$$\epsilon = q - \eta \frac{q^{n+1}}{2E_p^n} \ , \quad E_i \simeq p_i + \frac{m_i^2}{2p_i} - \eta \frac{p_i^{n+1}}{2E_p^n} \ . \tag{6}$$

Combining (2), (3) and (6) one obtains a modified kinematical requirement

$$\epsilon \geq \epsilon_{min} \simeq \frac{(m_2 + m_3)^2 - m_1^2}{4E_1} + \eta \frac{E_1^{n+1}}{4E_p^n} \left(1 - \frac{m_2^{n+1} + m_3^{n+1}}{(m_2 + m_3)^{n+1}} \right) \ . \tag{7}$$

where we have included only the leading corrections (terms suppressed by both the smallness of E_p^{-1} and the smallness of ϵ or m were neglected).

The Planck-scale "threshold anomaly" [5] described by Eq. (7) is relevant for the analysis of the GZK limit in cosmic-ray physics. In fact, the GZK limit essentially corresponds to the maximum energy allowed of a proton in order to travel in the CMBR without undergoing processes of the type $p + \gamma \rightarrow p + \pi$. For a proton of energy $E_1 \sim 5 \cdot 10^{19} eV$ the value of ϵ_{min} obtained from the undeformed equation (5) is such that CMBR photons can effectively act as targets for photopion production. But, for $\eta \sim 1$ and $n \leq 2$, the value of ϵ_{min} obtained from the Planck-scale deformed equation (7) places CMBR photons below threshold for photopion production by protons with energies as high as $E_1 \sim 10^{21} eV$, and would explain [4, 5] observations of cosmic rays above the GZK limit. For negative $\eta \sim -1$ one obtains the opposite result: photopion production should be even more efficient than in the standard case. Therefore negative η is disfavoured by the observations reported by various UHECR observations.

There has also been some interest [4, 5, 6] in the implications of Eq. (7) for electron-positron pair production in collisions between astrophysical high-energy photons and the photons of the Far Infrared Background. Electron-positron pair production should start to be significant when the high-energy photon has energies of about 10 or 20 TeV. The Planck-scale correction in Eq. (7) would be significant, though not dominant, at those energies. Observations of TeV photons are becoming more abundant, but the field is still relatively young. Moreover, our knowledge of the Far Infrared Background is presently not as good as our knowledge of the CMBR. Therefore observations of TeV photons do not yet provide a significant insight on the Planck scale physics of interest here. Consistency with those observations only imposes a constraint [6] of the type $|\eta| < 100$, which (since the quantum-gravity intuition favours $|\eta| \sim 1$) is not yet significant from a quantum-gravity perspective.

A similar upper limit ($|\eta| < 100$) is obtained by considering the implications [1, 22] of the deformed dispersion relation for the arrival times of photons with different energies emitted (nearly-)simultaneously from cosmological sources.

In summary, the present situation justifies some interest for the case of positive η, particularly as a possible description of cosmic rays above the GZK limit. The case of negative η is disfavored by various UHECR observations. Additional phenomenological reasons to disfavour negative η have been found in analyses of photon stability (see, *e.g.*, Ref. [6]), which is instead not relevant for the positive η case. Moreover, the case of negative η appears to be also troublesome conceptually since it leads to superluminal velocities in a framework, such as the one adopted here, in which the new effects are simply motivated by the idea of a quantum-spacetime medium[2], and therefore do not naturally lead to the expectation of superluminal velocities. Still, as a contribution to this evolving understanding, we will consider the Bahcall-Waxman bound both for positive and negative η.

3 Planck-scale-induced threshold anomalies and the neutrino bound

A key observation for our analysis comes from the fact that the Planck-scale threshold anomaly described by Eq. (7) is significant for the Bahcall-Waxman bound for the same reasons that render it significant for the GZK limit in cosmic-ray physics. In fact, both the Bahcall-Waxman bound and the GZK limit involve the analysis of processes of the type $p + \gamma \to X + \pi$ in which a high-energy proton collides with a softer photon. In the case of the Bahcall-Waxman bound one finds that for a proton of energy $E_1 \sim 10^{19} eV$ which is emerging from a source (*e.g.* an AGN), according to the standard kinematical requirement (5) the photons in the environment that are eligible for production of charged pions π^+ are all the photons with energy $\epsilon \geq \epsilon_{min} \sim 0.01 eV$. But, for the Planck-scale scenario of (7) with $\eta \sim 1$ and $n = 1$ far fewer photons in the environment, viz. only photons with

[2]A deformed dispersion relation is generically expected in a special relativistic theory when a medium is present. The presence of the medium does not alter the principles of special relativity, and superluminal velocities should not be allowed. The situation is different in the context of the approach proposed in Ref. [9], in which the deformed dispersion relation is not motivated by the presence of a quantum-spacetime medium but rather by a role for the Planck scale in the relativity principles. In the framework of Ref. [9] superluminal velocities would not lead to paradoxical results.

energy $\epsilon \geq \epsilon_{min} \sim 10^9 eV$, are kinematically eligible for production of charged pions. In a typical source the abundance of photons with $\epsilon \geq 10^9 eV$ is much smaller, by several orders of magnitude, than the abundance of photons with $\epsilon \geq 0.01 eV$. Correspondingly the Planck-scale effect predicts a huge reduction in the probability that a charged pion be produced before the proton escapes the source, and in turn this leads (for $\eta > 0$ and $n = 1$) to a decrease in the expected high-energy neutrinos flux by many orders of magnitude below the level set by the Bahcall-Waxman bound.

The same qualitative picture applies to the case $\eta \sim 1$, $n = 2$, although the effect is somewhat less dramatic because of the large suppression of the effect that is due to the extra power of the Planck scale. In fact, for $\eta \sim 1$, $n = 2$ one finds that the photons in the environment that are energetically enough for the production of charged pions must have energy $\epsilon \geq \epsilon_{min} \sim 1 eV$.

Whereas for positive η the Planck-scale effect leads to a lower neutrino bound the reverse is true for negative η. In particular, for $\eta \sim -1$, $n \leq 2$ from (7) it follows that photons in the source with energies even below[3] $0.01 eV$ are viable targets for the production of charged pions by protons with energy $E_1 \sim 10^{19} eV$. Correspondingly, the Bahcall-Waxman bound would be weakened.

4 Implications of the Planck-scale for particle decays and the neutrino bound

In the previous Section we have shown that the Planck-scale effects considered here would affect the production of charged pions before the ultra-high-energy cosmic-ray proton escapes the source. In this Section we analyze the implications of the same effects for the decay processes $\pi^+ \to \mu^+ + \nu_\mu$ and $\mu^+ \to e^+ + \nu_e + \bar{\nu}_\mu$ which are also relevant for the Bahcall-Waxman bound.

Since we are interested in both a two-body decay, $\pi^+ \to \mu^+ + \nu_\mu$, and a three-body decay, $\mu^+ \to e^+ + \nu_e + \bar{\nu}_\mu$ it is convenient for us to obtain a general result for N-body decays. This will also be a technical contribution to the study of the kinematics governed by (1). In fact, the implications of (1) for two-body decays have been previously analyzed [23], but for decays in three or more particles there are no previous results in the literature.

We start our analysis of the decay $A \to 1 + 2 + ... + N$ (A is the generic particle that decays into particles $1, 2, ...N$) with the energy-momentum conservation laws:

$$E_A = E_1 + E_2 + ... + E_N \tag{8}$$

$$\vec{p}_A = \vec{p}_1 + \vec{p}_2 + ... + \vec{p}_N \tag{9}$$

Denoting with θ_{ij} the angle between the linear momentum of particle i and that of particle j, and denoting with p the modulus of the 3-vector $\vec{p}$, we can use (9) to obtain ($i, j = 1, 2, ...N$)

$$p_A^2 = \sum_i p_i^2 + \sum_{i \neq j} p_i p_j \cos\theta_{ij} \tag{10}$$

and (8) to get

$$E_A^2 = \sum_i E_i^2 + \sum_{i \neq j} E_i E_j \ . \tag{11}$$

[3]Formally in this case Eq. (7) even admits photon targets with "negative energies". But, of course, considering the approximations we implemented, one can only robustly infer that photons with very low energies can lead to pion production.

From (10) and (11) it follows that

$$E_A^2 - p_A^2 = \sum_i (E_i^2 - p_i^2) + \sum_{i \neq j} (E_i E_j - p_i p_j \cos\theta_{ij}) \ . \tag{12}$$

Next we use the deformed dispersion relation (1), $E^2 - p^2 = m^2 - \eta E^n p^2 / E_p^n$, to obtain

$$m_A^2 - \eta E_A^2 p_A^2 / E_p^2 = \sum_i (m_i^2 - \eta E_i^n p_i^2 / E_p^n) + \sum_{i \neq j} (E_i E_j - p_i p_j \cos\theta_{ij}) \tag{13}$$

For simplicity let us consider separately the cases $n = 1$ and $n = 2$, starting with $n = 1$. It is convenient to rewrite the kinematical condition (13), for $n = 1$, in the following way

$$\sum_i m_i^2 - m_A^2 + \sum_{i \neq j} \left(p_i p_j + p_i \frac{m_j^2}{p_j} \right) + \frac{\eta}{E_p} \left(E_A^3 - \sum_i E_i^3 - \sum_{i \neq j} E_i E_j^2 \right) = \sum_{i \neq j} p_i p_j \cos\theta_{ij} \tag{14}$$

where we used again the deformed dispersion relation,

$$E = (p^2 + m^2 - \frac{\eta}{E_p} E p^2)^{\frac{1}{2}} \simeq p - \frac{\eta}{2E_p} p^2 + \frac{m^2}{2p} \ . \tag{15}$$

We are neglecting terms of order E_p^{-2} and higher, which are clearly subleading, and we are also neglecting terms of order $E_p^{-1} m^2$ which are negligible compared to terms of order $E_p^{-1} E^2$ since all particles involved in the processes of interest to us have very high momentum.

Using the fact that $\cos\theta_{ij} \leq 1$ for every θ_{ij}, it follows from (14) that for the decay to be kinematically allowed a necessary condition is

$$\sum_i m_i^2 - m_A^2 + \sum_{i \neq j} \left(p_i p_j + p_i \frac{m_j^2}{p_j} \right) + \frac{\eta}{E_p} \left(E_A^3 - \sum_i E_i^3 - \sum_{i \neq j} E_i E_j^2 \right) \leq \sum_{i \neq j} p_i p_j \tag{16}$$

or equivalently

$$\sum_i m_i^2 - m_A^2 + \sum_{i \neq j} p_i \frac{m_j^2}{p_j} + \frac{\eta}{E_p} \left(E_A^3 - \sum_i E_i^3 - \sum_{i \neq j} E_i E_j^2 \right) \leq 0 \tag{17}$$

In the analysis of particle-decay processes relations of the type (17) impose constraints on the available phase space. For positive η the quantum-gravity effect clearly goes in the direction of reducing the available phase space; in fact, it is easily seen that

$$\left(E_A^3 - \sum_i E_i^3 - \sum_{i \neq j} E_i E_j^2 \right) = \left((\sum_i E_i)^3 - \sum_i E_i^3 - \sum_{i \neq j} E_i E_j^2 \right) > 0 \ . \tag{18}$$

The correction is completely negligible as long as $m_A^2 \gg E_A^3 / E_p$, but for $m_A^2 \ll E_A^3 / E_p$ there is clearly a portion of phase space in which, for positive η, condition (17) is not satisfied. (Think for example of the case $E_1 \sim E_2 \sim ... \sim E_N \sim E_A / N$.) Starting at $E_A \geq (m_A^2 E_p)^{1/3}$ the phase space available for the decay of particle A is gradually reduced as E_A increases. The difference between the standard Lorentz-symmetry prediction for the lifetime and the quantum-gravity-corrected prediction becomes more and more significant as the energy of the decaying particle is increased, and goes in the direction of rendering the particle more stable, i.e., rendering the decay more unlikely. For the relevant decays of pions and muons we expect that the quantum-gravity effect starts being important at pion/muon energies of order $(m_\pi^2 E_p)^{1/3} \sim (m_\mu^2 E_p)^{1/3} \sim 10^{15} eV$.

For positive η it is inevitable that at some energy a significant suppression of the decay probability kicks in, while for negative η it is easy to see that there is no effect on the size of the phase space available for the decays. The change in the sign of η turns as usual into a change of sign of the effect, which would go in the direction of extending the phase space available for the decay, but the relevant portion of parameter space (some neighborhood of $E_1 \sim E_2 \sim ... \sim E_N \sim E_A/N$) is already allowed even without the Planck scale effect, so for negative-η effect is not significant.

Completely analogous considerations apply to the case $n = 2$. From (13), for $n = 2$, one obtains

$$\sum_i m_i^2 - m_A^2 + \sum_{i \neq j} \left(p_i p_j + p_i \frac{m_j^2}{p_j} \right) + \frac{\eta}{E_p^2} \left(E_A^4 - \sum_i E_i^4 - \sum_{i \neq j} E_i E_j^3 \right) = \sum_{i \neq j} p_i p_j \cos\theta_{ij} \ , \quad (19)$$

and then, just following the same line of analysis we already adopted for the case $n = 1$, for $n = 2$ one finds that the decay is only allowed if

$$\sum_i m_i^2 - m_A^2 + \sum_{i \neq j} p_i \frac{m_j^2}{p_j} + \frac{\eta}{E_p^2} \left(E_A^4 - \sum_i E_i^4 - \sum_{i \neq j} E_i E_j^3 \right) \leq 0 \ . \quad (20)$$

Since

$$\left(E_A^4 - \sum_i E_i^4 - \sum_{i \neq j} E_i E_j^3 \right) = \left((\sum_i E_i)^4 - \sum_i E_i^4 - \sum_{i \neq j} E_i E_j^3 \right) > 0 \ , \quad (21)$$

we find again that for positive η the quantum-gravity correction inevitably goes in the direction of reducing the available phase space and therefore rendering the decay more unlikely. In this $n = 2$ case we expect that the quantum-gravity suppression of the decay probability starts being important at pion/muon energies of order $(m_\pi E_p)^{1/2} \sim (m_\mu E_p)^{1/2} \sim 10^{18} eV$.

Of course also for $n = 2$ one finds that the case of negative η does not have significant implications, for exactly the same reasons discussed above in considering the $n = 1$ case.

5 Closing remarks

We found that Planck-scale effects can have important implications for the neutrino-producing chain of processes $p + \gamma \to X + \pi^+$, $\pi^+ \to \mu^+ + \nu_\mu$, $\mu^+ \to e^+ + \nu_e + \bar{\nu}_\mu$, which are relevant for the Bahcall-Waxman bound. We focused on a single simple example of Planck-scale kinematics. Since this analysis led to encouraging conclusions (significant implications for the Bahcall-Waxman bound) it should provide motivation for more detailed studies in more structured quantum-gravity models. Because of the simple kinematical origin of our argument it is reasonable to expect that these more detailed studies will confirm that, for positive η, the quantum-gravity effect leads to a neutrino flux that is many orders of magnitude below the level allowed by the Bahcall-Waxman bound. We have shown here that this is due primarily to a strong suppression of the production of high-energy charged pions by protons at the source. If future observations give us a neutrino flux which is close to the level allowed by the Bahcall-Waxman bound, the type of quantum-gravity physics considered here would be excluded (for positive η). On the other hand, a low neutrino flux will be harder to interpret as, a priori, it is not clear that the Bahcall-Waxman bound should be saturated.

The suppression present for positive η already found full support at the first step in the chain of processes, in the collisions $p+\gamma \rightarrow X+\pi^+$. We felt however that it was appropriate to consider also the processes further down in the chain, in the decays of $\pi^+ \rightarrow \mu^+ + \nu_\mu$, $\mu^+ \rightarrow e^+ + \nu_e + \bar{\nu}_\mu$. In fact, it was conceivable that perhaps at that level the production of neutrinos might receive a compensating boost form the quantum-gravity effect which, for positive η, suppresses the likelihood of the process $p+\gamma \rightarrow X+\pi^+$. This turned out not to be the case: for positive η one actually expects a further suppression of neutrino production, since the quantum-gravity effect renders ultrahigh-energy pions and muons more stable. This part of our analysis also provided a technical contribution to the study of the kinematics governed by (1), since previously the implications of (1) were only known for two-body decays, while here we obtained a generalization to N-body decays for arbitrary N.

As discussed in Section 2, the case of negative η is disfavoured conceptually and starts to be strongly constrained by preliminary observations in astrophysics. We found that it would have striking consequences for the Bahcall-Waxman bound: in the case of negative η the modification of the Bahcall-Waxman bound would amount to violating (raising) the Bahcall-Waxman bound by several orders of magnitude.

Our analysis contributes to ongoing work aimed at establishing a web of consequences of the type of Planck-scale kinematics considered here. It is not hard to find several different solutions to a single anomaly in ultrahigh-energy astrophysics, *e.g.* the cosmic-ray paradox, if confirmed by other observatories. However for the type of Planck-scale kinematics considered here, there are several correlated predictions and these together can be used to favor or rule out the scenario. In particular, evidence supporting both a cosmic-ray paradox and an unexpectedly low ultrahigh-energy neutrino flux would fit naturally within the Planck-scale-kinematics scenario (with positive η). More precisely, evidence supporting both a cosmic-ray paradox and an unexpectedly low ultrahigh-energy neutrino flux would favor solutions of the cosmic-ray paradox based on violations of Lorentz symmetry with respect to other proposed solutions of the cosmic-ray paradox. In fact, the correlation we have exposed here between a cosmic-ray paradox and a lowered Bahcall-Waxman bound is a characteristic models in which the kinematics of the processes $p+\gamma \rightarrow X+\pi$ is modified by a violation of Lorentz symmetry. In fact, the processes $p+\gamma \rightarrow X+\pi^{\pm,0}$ dominate the GZK threshold and the Bahcall-Waxman limit. An increase in the GZK limit and a lowered Bahcall-Waxman bound are found whenever the violation of Lorentz symmetry causes an increased energy-threshold condition for the processes $p+\gamma \rightarrow X+\pi$. Therefore it can distinguish between models with and without this Lorentz-violation effect, but it cannot establish whether the origin of the Lorentz violation is connected with quantum gravity. In particular, it is interesting to consider the suggestion of Coleman and Glashow [16] concerning a specific Lorentz-violation solution of the GZK paradox, which is not motivated by quantum gravity. Coleman and Glashow [16] consider a scheme in which different particles have a different "maximum attainable speed" (essentially a different "speed-of-light constant" for different particles). This can be cast into our formalism with a particle-dependent η and with $n=0$. It follows from our analysis that any model that resolves the UHECR GZK paradox using the Coleman-Glashow scheme will also lead to a stronger Bahcall-Waxman neutrino bound.

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