Relativistic Turbulence: A Long Way from Preheating to Equilibrium

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We study, both numerically and analytically, the development of equilibrium after preheating. We show that the process is characterized by the appearance of Kolmogorov spectra and the evolution towards thermal equilibrium follows self-similar dynamics. Simplified kinetic theory gives values for all characteristic exponents which are close to what is observed in lattice simulations. The resulting time for thermalization is long, and temperature at thermalization is low, $T \sim 100$ eV in the simple $\lambda\Phi^4$ inflationary model. Our results allow a straightforward generalization to realistic models.

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Introduction.—The dynamics of equilibration and thermalization of field theories is of interest for various reasons. In high-energy physics understanding of these processes is crucial for applications to heavy ion collisions and to reheating of the early Universe after inflation. Inflation solves the flatness and the horizon problems of the standard big bang cosmology and provides a calculable mechanism by which initial density perturbations were generated [1]. At the end of inflation the Universe was in a vacuumlike state. In the process of decay of this state and subsequent thermalization (reheating) the matter content of the Universe is created. It was realized recently that the initial stage of reheating, dubbed preheating [2], is a fast, explosive process. This initial stage by now is well understood [3–7]. Strong and fast amplification of fluctuation fields at low momenta may lead to various interesting physical effects, such as nonthermal phase transitions [8], peculiar baryogenesis [9], generation of high-frequency gravitational waves [10], etc.

Understanding of the subsequent stages of reheating and thermalization processes and calculation of the final equilibrium temperature is important for various applications, most notably baryogenesis and the problem of overabundant gravitino production in supergravity models [11]. The time, at which the expansion becomes radiation dominated during reheating, determines also the abundances of other relics, such as superheavy dark matter [12]. Thermalization of field theories was discussed already; see, e.g., Ref. [13]. However, at present the process of thermalization after preheating is still far away from being well understood. The problem is that at the preheating stage the occupation numbers are very large, of the order of the inverse coupling constant. In addition, in many models the zero mode of the inflaton field does not decay completely. Therefore, a simple kinetic approach is not applicable.

Fortunately, the description in terms of classical field theory is valid in this situation [3], and the process of preheating, as well as subsequent thermalization, can be studied on a lattice. In this Letter we adopt this approach. Our goal is to integrate the system on a lattice sufficiently accurately and sufficiently far in time to be able to see generic features, and possibly to the stage at which the kinetic description becomes a good approximation scheme. Lattice studies of thermalization, similar to ours, were done in Ref. [14]. Several generic rules of thermalization were formulated, such as the early equipartition of energy between coupled fields. However, the problem is very complicated and there are other unanswered important questions: What is the final thermalization temperature? At what stage does the kinetic description become valid? What is the functional form of particle distributions during the thermalization stage?

For our study we use a higher accuracy, improved version of the LATTICEEASY code [15]. We show that the distribution functions in the ultrarelativistic regime follow a *self-similar* evolution related to the turbulent transport of wave energy. This property enables us to estimate the physical reheating temperature, which turns out to be very low. The concept should be rather model independent since typical ranges of particle momenta at preheating and in thermal equilibrium are widely separated. However, in this Letter we restrict our numerical integration (but not the discussion) to the "minimal" inflationary model, the massless $\lambda\Phi^4$ theory.

The model.—With conformal coupling to gravity and after a rescaling of the field, $\varphi \equiv \Phi a$, where $a(t)$ is the cosmological scale factor, the equation of motion in comoving coordinates describes a φ^4 theory in Minkowski space-time,

$$\Box\varphi + \lambda\varphi^3 = 0. \tag{1}$$

At the end of inflation the field is homogeneous, $\varphi = \varphi_0(t)$. Later on fluctuations develop, but the homogeneous component of the field, which corresponds to the zero momentum in the Fourier decomposition, may be dynamically important and is referred to as the "zero mode." In such situations it is convenient to make a further rescaling of the field, $\phi \equiv \varphi/\varphi_0(t_0)$, and of the space-time coordinates, $x^\mu \to \sqrt{\lambda}\varphi_0(t_0)x^\mu$, which transforms the equation of motion (1) into dimensionless and parameter-free form,

$$\Box\phi + \phi^3 = 0. \tag{2}$$

Here t_0 corresponds to the initial moment of time (end of inflation), and in what follows we denote dimensionless time as τ. With this rescaling the initial condition for the zero-mode oscillations is $\phi_0(\tau_0) = 1$. All model dependence on the coupling constant λ and on the initial amplitude of the field oscillations is encoded now in the initial conditions for the small (vacuum) fluctuations of the field with nonzero momenta [3]. The physical normalization of the inflationary model corresponds to a dimensionful initial amplitude of $\varphi_0(t_0) \approx 0.3M_{\rm Pl}$ and a coupling constant $\lambda \sim 10^{-13}$ [1]. The reparametrization property of the system allows one to choose a larger value of λ for numerical simulations. We have used $\lambda = 10^{-8}$.

Numerical procedure and results.—We use a 3D cubic lattice with periodic boundary conditions. The finite-differences scheme that was used is second order in time and fourth order in space. The results displayed here are taken from a simulation with 256^3 lattice sites and a physical box size $L = 14\pi$. With this box size the infrared modes which belong to the resonance band are still well represented, while the ultraviolet lattice cutoff is sufficiently far away from the occupied modes; therefore, the particle spectra are not distorted even at late times. We have studied the dependence of our results on the lattice and the box size to avoid lattice artifacts. Various quantities are measured and monitored both in configuration space [zero mode, $\phi_0 \equiv \langle\phi\rangle$, and the variance, $\mathrm{var}(\phi) \equiv \langle\phi^2\rangle - \phi_0^2$] and in Fourier space. Using Fourier transformed fields we first define the wave amplitudes (which correspond to annihilation operators in the quantum problem),

$$a(\vec{k}) \equiv (\omega_k\phi_{\vec{k}} + i\dot{\phi}_{\vec{k}})/(2\pi)^{3/2}\sqrt{2\omega_k}, \tag{3}$$

where the effective frequency $\omega_k \equiv \sqrt{k^2 + m_{\rm eff}^2}$ is determined by the effective mass $m_{\rm eff}^2 = 3\lambda\langle\phi^2\rangle$. Making use of a_k, we calculate various correlators, $n(k) \equiv \langle a^\dagger a\rangle$, $\sigma(k) \equiv \langle aa\rangle$, $\langle a^\dagger a^\dagger aa\rangle$, etc. The first one, which corresponds to the particle occupation numbers, is of prime interest.

We begin the discussion of our numerical results with the evolution of the zero mode and the variance of the field, which are shown in Fig. 1. Initially we see an exponentially fast transfer of the zero-mode energy

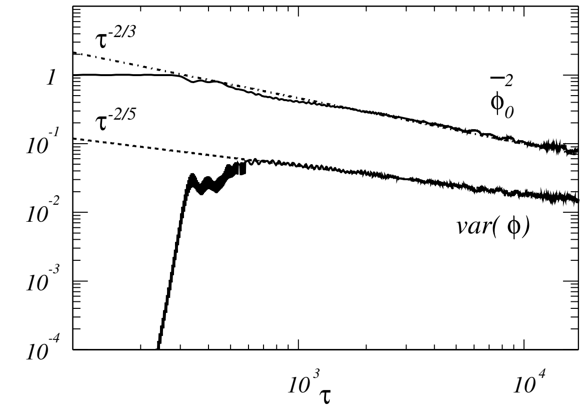

FIG. 1. Amplitude of the zero-mode oscillations, $\overline{\phi_0^2}$, and variance of the field fluctuations as functions of time τ.

into fluctuations during preheating (up to $\tau \sim 300$). It is followed by a long and slow relaxation phase. In this regime ($\tau > 1500$) the amplitude of the zero-mode oscillations decreases according to $\sim\tau^{-1/3}$; the variance of the field (averaged over high-frequency oscillations) drops according to $\sim\tau^{-2/5}$. This is consistent with previous results [3]. In addition we find that in this regime the zero mode is in a nontrivial dynamical equilibrium with the bath of highly occupied modes: when the zero mode is artificially removed, it is recreated on a short time scale (Bose condensation).

At early times the distribution functions of particles over momenta, see Fig. 2, have a peaky structure. The first peak which corresponds to the parametric resonance is initially at the theoretically predicted value of $k \sim 1.27$ [5]. Peaks at larger momenta cannot be created by the parametric resonance in this model [5]; they are due to rescattering of waves from the first resonance peak [3]. At late times ($\tau > 1500$) the spectra become smooth and at small k approach a power law, $n_k \sim k^{-s}$, where s fluctuates in the range of 1.5–1.7, depending on time and the range of k where it is fitted. This power law clearly differs

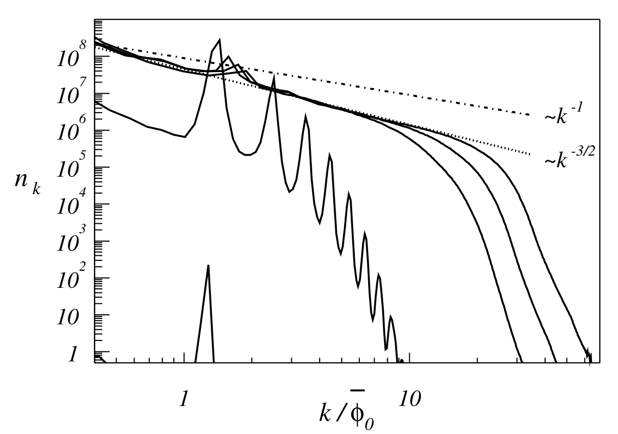

FIG. 2. Occupation numbers as function of $k\overline{\phi_0}^{-1}$ at $\tau = 100$, 400, 2500, 5000, and 10 000.

from the classical thermal equilibrium, $n_k \sim \omega_k^{-1}$. It is followed by the exponential cutoff, whose position monotonously shifts with time towards higher k. Pumping of energy from the zero mode stays effective at all times (note a small bump in the particle distributions in Fig. 2 at $k \sim 1$). It corresponds to the annihilation of four condensed particles into two quanta. Rescattering of two particles into two particles is also effective. One of the two particles can belong to the zero-mode condensate in either the initial or the final two particle state. We also can see in Fig. 2 that in the power-law region n_k is a function of $k/\overline{\phi}_0$ only, where $\overline{\phi}_0(\tau)$ represents the amplitude of the zero mode at time τ. This effect can be related to the above described dynamical equilibrium between zero mode and the bath of particles. [Note, in this respect, that the dimensionless equation (2) may be obtained at late times using the slowly varying current amplitude of the zero mode, such that $\phi(\tau) = 1$.]

The picture presented in Fig. 2 at late times resembles stationary Kolmogorov turbulence. This resemblance arises after rescaling of momenta by the current amplitude of the zero mode. However, in the present model the turbulence cannot be stationary because the amplitude of the zero mode (i.e., the strength of the source of turbulence) decreases. Further examination of Fig. 2 suggests that the evolution of particle spectra may be self-similar. We have tried therefore the following anzatz:

$$n(k, \tau) = \tau^{-q} n_0(k\tau^{-p}). \tag{4}$$

Spectra rescaled at several moments of time by the relation inverse to Eq. (4) are shown in Fig. 3. We have found that the evolution is indeed self-similar with $q \approx 3.5p$ and $p \approx 1/5$.

Discussion.—Here we discuss the question whether a simple kinetic theory gives predictions for turbulence and self-similarity exponents in agreement with the lattice calculations. Our lattice study of higher order correlators,

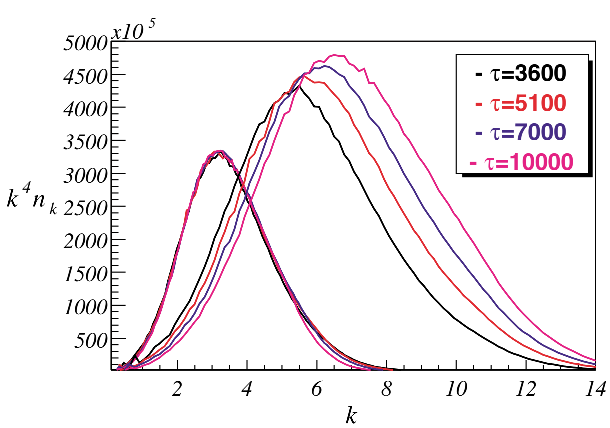

FIG. 3 (color online). On the right-hand side we plot the wave energy per decade found in lattice integration. On the left-hand side are the same graphs transformed according to the relation inverse to Eq. (4).

such as $\langle a^\dagger a^\dagger aa\rangle$, shows that the field distribution is very close to Gaussian; see also [13,14]. This facilitates the use of the kinetic approach. On the other hand, we have found that the magnitude of $\sigma(k) = \langle aa\rangle$ is of the order of a few percent compared to $n(k)$, and it is even larger in the region of resonant momenta. This means that the strict kinetic approach should include $\sigma(k)$. Nevertheless, we neglect related effects and write a kinetic equation in a simple form $\dot{n}_k = I_k$, where the collision integral for an m-particle interaction is given by

$$I_k = \int d\Omega_k U_k F[n]. \tag{5}$$

In d spatial dimensions the integration measure $d\Omega_k$ is given by $m - 1$ integrations over d-dimensional Fourier space. We include in $d\Omega_k$ the energy-momentum conservation δ functions. But we do not include there the relativistic $1/\omega(k_i)$ "on-shell" factors, which instead appear in the "matrix element" of the corresponding process, U_k. This will make the discussion of relativistic and nonrelativistic cases uniform. The function $F[n]$ is a sum of products of the type $n_{k_j}^{-1}\prod_{i=1}^m n_{k_i}$, where $j \in \{1, \ldots, m\}$ with appropriate signs and permutations of indices for incoming and outgoing particles. All dynamical aspects of turbulence follow from the scaling properties of the system [16]. Let ω_k, n_k, and U_k have defined weights under a ξ rescaling of Fourier space,

$$\begin{aligned} \omega(\xi k_i) &= \xi^\alpha \omega(k_i), \\ U(\xi k_1, \ldots, \xi k_m) &= \xi^\beta U(k_1, \ldots, k_m), \\ n(\xi k_i) &= \xi^\gamma n(k_i). \end{aligned} \tag{6}$$

The weight of the full collision integral under this reparametrization is

$$I_{\xi k} = \xi^{d(m-2)-\alpha+\beta+(m-1)\gamma} I_k. \tag{7}$$

It follows that the stationary turbulence with constant energy flux over momentum space is characterized by a power-law distribution function, $n_k \sim k^{-s}$, where $s = d + \beta/(m-1)$. The scaling properties also give the exponents of the self-similar distribution, Eq. (4). Assuming energy conservation in particles and using $\xi = \tau^{-p}$ we find $q = 4p$ and $p = 1/[(m-1)\alpha - \beta]$. For stationary turbulence we find that p should be $(m-1)$ times larger.

For a massless $\lambda\phi^4$ theory in three spatial dimensions and four-particle interaction we have $m = 4$, $\beta = -4\alpha$, and $\alpha = 1$. In this case $s = 5/3$ and $p = 1/7$. For three-particle interaction (the fourth particle belongs to the condensate in this case and the matrix element contains an additional factor of $\overline{\phi}_0^2$) we find $s = 3/2$ and a smaller value for p compared to the previous case. We cannot distinguish between 5/3 and 3/2 for s in our numerical integrations; s rather fluctuates between these two numbers, while 1/7 for p gives a fit to the data not as good as displayed in Fig. 3, where $p = 1/5$. However, during the integration time the energy in particles is not conserving. Namely, starting from the time at which the solution becomes self-similar, $\tau \sim 3000$, to the end of our integration, the energy influx from the zero mode to particles is about 20%. Correcting for this energy influx we find $q \approx 3.5p$ and $p \approx 1/6$. This should be considered as satisfactory agreement given the simplifications which were made.

Equilibration time and temperature.—At late times the influence of the zero mode should become negligible, but we still may expect the self-similar character of the evolution. Solution Eq. (4) with $p = 1/7$ should be valid in this case. This allows us to find the time needed to reach equilibrium. Indeed, the classical evolution will continue until the occupation numbers in the region of the peak in Fig. 3 will become of order one. At this time the classical approach becomes invalid and the distribution function should relax to the thermal Bose-Einstein one within quantum frameworks. Values of momenta where this happens are $k_{\max} \sim \lambda^{1/4}\varphi_0(\tau_0)$. On the other hand, the initial distribution is centered around $k_0 \sim \lambda^{1/2}\varphi_0(\tau_0)$ and moves to ultraviolet according to Eq. (4) as $\propto k_0\tau^p$. It follows that the time to reach equilibrium is $\tau \sim \lambda^{-7/4} \sim 10^{23}$, where in the second equality we assumed the normalization to the inflationary model. For the reheating temperature we find, rotating back from the conformal reference frame, $T_R \sim k_{\max}/a(\tau) \sim \lambda^2\varphi_0(t_0) \sim 10^{-26}M_{\rm Pl} \sim 100$ eV, where for the conformal scale factor we have used $a(\tau) = \tau$.

Conclusions.—Reheating after preheating appears to be a rather slow process. Although the "effective temperature" measured at low momentum modes during preheating may be high, in the model we have considered the resulting true temperature is parametrically the same as what could have been obtained in "naive" perturbation theory. Namely, equating the rate of scattering in thermal equilibrium to the Hubble expansion rate one obtains $T \sim \lambda^2 M_{\rm Pl}$ in this model. We anticipate this result should be applicable to more realistic models of inflation. Note that realistic models involve many fields and interactions and largest relevant coupling constants will determine the true temperature.

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