Is it possible to recover information from the black-hole radiation?

M. Schiffer

CERN, CH-1211 Geneva 23, Switzerland

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In the framework of communication theory, we analyze the gedanken experiment in which beams of quanta bearing information are flashed toward a black hole. We show that stimulated emission at the horizon provides a correlation between incoming and outgoing radiations consisting of bosons. For fermions, the mechanism responsible for the correlation is the Fermi exclusion principle. Each one of these mechanisms is responsible for the partial transfer of the information originally coded in the incoming beam to the black-hole radiation. We show that this process is very efficient whenever stimulated emission overpowers spontaneous emission (bosons). Thus, black holes are not "ultimate waste baskets of information."

I. INTRODUCTION

In the wake of Hawking's seminal paper in which he proved that black holes radiate with a (distorted) black-body spectrum [1], a fundamental question touching the basis of quantum mechanics emerged.

The transmission of information by means of black-body radiation is thermodynamically forbidden. Therefore, it is widely believed that all information stored in a physical system is inexorably lost as it crosses the black-hole event horizon because, so it seems, it cannot be recovered from the black-hole radiation. Accordingly, as a black hole evaporates completely pure states could be converted into mixed ones (thermal radiation), threatening the very fundamentals of quantum mechanics that predict unitary evolution of quantum states.

There are many approaches to the problem. Hawking [1,2] advocates that all information regarding the black-hole past history is lost forever and that quantum mechanics has to be reformulated to accommodate this fact. Others believe that this information remains stored inside a black hole until the last moments of evaporation. Then either (i) the black hole stabilizes at some radius of the order of Planck's length and all the information in question is retained in its interior [3] or (ii) all this information is instantaneously liberated to the environment [4–6]. However, any of these scenarios requires a huge amount of information to be confined within a tiny region of space-time, something against our intuition and in conflict with the entropy bound formulated some years ago by Bekenstein [7,8]. Another group believes that the resolution of the paradox lies in the physics of super-strings: black holes are assumed to have some quantum W hair that, in principle, could be detected via Bohm-Aharonov experiments [9–11]. A more conservative stand was taken by 't Hooft [12,13] who suggested that the information in question leaks from a black hole by some yet unknown mechanism that correlates the outgoing and incoming radiations. In this direction, Bekenstein [14] very recently explored the fact that the coefficient of transmission through the potential barrier that surrounds a black hole is not unity (Hawking's radiation is not exactly

blackbody), to show that there is enough room from a thermodynamical point of view for the black hole to leak all the information it stored along its past history.

Our aim in this paper is by far less ambitious than the scope of the involved paradox. Here, we shall consider the gedanken experiment where information is coded in beams of quanta (very much as is done inside optical fibers used for telephonic communication), which are then flashed toward the black hole. Common wisdom asserts that as the beam crosses the horizon all the information it bore is lost forever. However, this neglects a fundamental aspect of black-hole radiance: the approach of the incoming beam at the horizon is followed by the stimulated emission of other quanta. In the framework of communication theory we shall prove that, thanks to this stimulated emission, information coded in an incoming beam consisting of bosons is partially transferred to the outgoing radiation and that this process is very efficient for all modes satisfying $\hbar\omega \ll T_{\rm BH}$, provided that the mean number of quanta in the incoming beam $\bar{n} \gg 1$, because stimulated emission then overpowers spontaneous emission. Under these conditions *most of the information originally coded in the ingoing beam can be recovered from the outgoing flux*. In the case of a beam consisting of fermions, the exclusion principle provides the mechanism responsible for a similar correlation.

II. THE ROLE OF STIMULATED EMISSION

An isolated black hole emits (spontaneously) bosons with a spectrum

$$\bar{n} = \frac{\Gamma}{e^x - 1}, \tag{1}$$

where $x = \hbar\omega/T_{\rm BH}$ and Γ is the coefficient of transmission through the potential barrier that surrounds the black hole [1], a function of black-hole and field-mode parameters which cannot be cast in a closed form. For this reason, it became a widespread practice to set $\Gamma = 1$ while discussing black-hole radiance, mainly because it is believed that the essence of the effect is captured by the

thermal factor of this expression alone.

This practice gives the misleading impression that black holes are inert to incoming radiation. But they are not. Suppose that a black hole is impinged by n quanta. Then, the mean number of quanta in the outgoing flux is composed of spontaneous emission [Eq. (1)] and a fraction $(1 - \Gamma)$ of the original beam. For incoming thermal radiation at temperature T [15,16],

$$\bar{n} = \frac{\Gamma}{e^x - 1} + \frac{1 - \Gamma}{e^y - 1}, \qquad (2)$$

where $y = \hbar\omega/T$. On the other hand, the conditional probability $p(m|n)$ that the black hole returns m quanta given that n are incident (in a given mode) is defined via the equation

$$p_o(m) = \sum_{n=0}^{\infty} p(m|n)p_i(n), \qquad (3)$$

where $p_i(n)$ and $p_o(m)$ are the probabilities that n quanta are incident and m emitted, respectively. This conditional probability can be extracted from the above two equations using maxentropy techniques [15,16]:

$$p(m|n) = \frac{(e^x - 1)e^{nx}\Gamma^{m+n}}{(e^x - 1 + \Gamma)^{n+m+1}}$$

$$\times \sum_{k=0}^{\min(m,n)} \frac{(m + n - k)!}{k!(n - k)!(m - k)!} X^k,$$

$$\qquad (4)$$

$$X = 2\frac{1 - \Gamma}{\Gamma^2}(\cosh x - 1) - 1.$$

Similar results were also produced by field theoretic calculations [17]. Although this conditional probability is a quite complicated distribution, Bekenstein [16] disentangled it into distributions corresponding to three different processes, namely, elastic scattering, spontaneous, and stimulated emissions:

$$p(m|n) = \sum_{k=0}^{\min(m,n)} p_{scat}(k|n)p_{spont+stim}(m - k|n). \qquad (5)$$

In this convolution, the first factor stands for the probability that $n - k$ quanta are absorbed while k are scattered:

$$p_{scat}(k|n) = \binom{n}{k}\Gamma_0^{n-k}(1 - \Gamma_0)^k, \qquad (6)$$

where

$$\Gamma_0 = \frac{\Gamma}{1 - e^{-x}} \qquad (7)$$

and $1 - \Gamma_0$ stand, respectively, for the absorption and (elastic) scattering probabilities of one quantum. The binomial factor takes care of the correct statistics for many bosons.

The second factor in the convolution

$$p_{stim+spon}(m|n) = \binom{m + n}{m}(1 - e^{-\gamma})^{n+1}e^{-\gamma m}, \qquad (8)$$

where γ is implicitly given by

$$\frac{\Gamma}{e^x - 1} = \frac{1}{e^\gamma - 1}, \qquad (9)$$

represents spontaneous and stimulated emission as shown by evaluation of the mean number of returned quanta for a fixed number k of incoming ones:

$$\bar{m}_{spont+stim} = \sum_m p_{spont+stim}(m|k)m = \frac{1}{e^\gamma - 1}(k + 1). \qquad (10)$$

Whenever stimulated emission takes place, we expect a black hole to behave very much like a laser, producing amplification of the incoming beam (signal) with negligible degradation of the information it originally bore.

In what follows, we shall analyze this question in the context of communication theory [18,19]. Assume that the actual state $|n\rangle$ the incoming field mode is in is not known a priori, only its occurrence probability $p_i(n)$. The amount of ignorance concerning the signal's actual state is Shannon's entropy [20]:

$$H_i = -k \sum_n p_i(n) \ln p_i(n). \qquad (11)$$

The constant k fixes the units of information: for $k = 1/\ln 2$ it is measured in bits, etc. Upon detection of the signal an observer picks up one from all possible states, gaining an amount of information equal to Shannon's entropy.

From the point of view of communication theory, a black hole acts as a source of noise, jamming the information borne by the original signal. In the presence of this noise, the outgoing radiation is associated with a larger entropy than is the incoming radiation, because this noise introduces a further measure of uncertainty in the signal. Nonetheless, this larger entropy does not correspond to a larger amount of useful information, i.e., the one that, in principle, could be recovered at the output. Thus, after reaching the horizon, H_i no longer represents the information borne by the signal because this has since been adulterated by noise. The procedure for dealing with this situation was outlined by Shannon [21] who noted that $H_{i|o}$, the conditional entropy of the input when the output is known:

$$H_{i|o} \equiv -\sum_{n,m} p(m|n)p_i(n) \ln\left[\frac{p(m|n)p_i(n)}{p_o(m)}\right] \qquad (12)$$

must represent the extra uncertainty introduced by the noise, which hinders reconstruction of the initial signal when the output is known. Thus, he interpreted

$$H_{useful} = H_i - H_{i|o} \qquad (13)$$

to be the useful information, meaning the one which we could, in principle, recover from the output signal, even

in the face of noise. Thus, H_{useful} represents the informational content of the outgoing radiation. We can also regard

$$H_{o|i} \equiv -\sum_{n,m} p(m|n) p_i(n) \ln p(m|n) \tag{14}$$

as the uncertainty in the output *for a given input*, as the effect of the noise. By means of Jannes identity, it is trivial to show that H_{useful} can be alternatively expressed in terms of $p_o(m)$:

$$H_{\text{useful}} = H_o - H_{o|i}. \tag{15}$$

Now, we wish to code information in the incoming beam in the most efficient way in order to optimize its transfer. In other words, we wish to maximize H_{useful} with respect to either $p_i(n)$ or $p_o(m)$. The variation of this quantity for a fixed mean number of quanta gives [22]

$$p_o(m) = e^{-[\alpha + \beta m + B(m)]}, \tag{16}$$

where the "chemical potential" $B(m)$ depends on the conditional probability through the equation

$$\sum_m B(m) p(m|n) = -\sum_m p(m|n) \ln p(m|n). \tag{17}$$

Inserting Eq. (16) into Eq. (15) we obtain the amount of information that can be transmitted in the presence of noise in the optimal regime:

$$I_{\text{max}} = \alpha + \beta \bar{m}. \tag{18}$$

In the above, α and β should be determined by normalization and mean number of quanta conditions. Namely,

$$\alpha = \ln \sum_m e^{-[\beta m + B(m)]} \tag{19}$$

and

$$\bar{m} = -\frac{\partial \alpha}{\partial \beta}. \tag{20}$$

Shannon proved the important result that the optimal regime for information transfer can always be achieved in practice by means of an efficient coding of the message, but that it can never be exceeded: any information we try to send in excess of H_{useful} will be washed out by noise [21].

We are now in a position to calculate the amount of information borne by the outgoing radiation in the optimal regime. Before doing so, it is worth recalling that the geometry becomes transparent ($\Gamma \to 1$) for very large frequency modes. In this limit, $X \to -1$ and the sum in Eq. (4) is unity. Thus, the outgoing radiation is purely thermal:

$$p(m|n) \to (1 - e^{-x}) e^{-mx}. \tag{21}$$

Notice that since the conditional probability is independent of n, $p_o(m) \to p(m|n)$. Inspecting Eq. (12) we see that in this limit $H_{i|o} \to H_i$ and consequently $H_{\text{useful}} \to 0$.

This result could be foreseen from thermodynamical considerations because the second law of thermodynamics forbids thermal radiation to convey any information.

In order to render these calculations feasible, we will have to resort to a simplification of the problem by setting $\Gamma_0 = 1$, which corresponds to omitting scattering processes. In other words, we assume that all incoming quanta are absorbed by the black hole. Accordingly, in so doing we shall be setting a lower bound on the amount of information that could be borne by the outgoing radiation, which we hope will be very close to its actual value. With this simplification, the relation between x and γ becomes very simple:

$$\gamma = \ln(e^x + 1). \tag{22}$$

Thus, our task now is to solve Eq. (17) for the distribution (8). This calculation is shown in the Appendix and the result is

$$B(m) = e^{\gamma m} (-1)^m \sum_{k=m}^{\infty} \binom{k+1}{m+1} \times \sum_{n=0}^{k} F(n+1)(-1)^n \binom{k}{n}, \tag{23}$$

where

$$F(n) \equiv -\sum_{m=0}^{\infty} \binom{m+n}{n} e^{-\gamma m} \ln p(m|n). \tag{24}$$

Calculating $F(n)$ entails a summation of the logarithm of a binomial [see Eq. (8)], a task that we were not able to accomplish analytically. Since the logarithm of a binomial, regarded as a function of its lower argument, is very close to a parabola (see Fig. 1) we adopted the approximation

$$\ln \binom{m+n}{n} \approx 4 \ln 2 \frac{mn}{m+n} \tag{25}$$

and carried out the summation detailed in the Appendix. The final result for $F(n)$ is

$$F(n) \approx (n+1) \frac{\gamma e^{-\gamma} - (1 - e^{-\gamma}) \ln(1 - e^{-\gamma})}{(1 - e^{-\gamma})^{n+2}} - n \frac{(4 \ln 2) e^{-\gamma}}{(1 - e^{-\gamma})^{n+1}}. \tag{26}$$
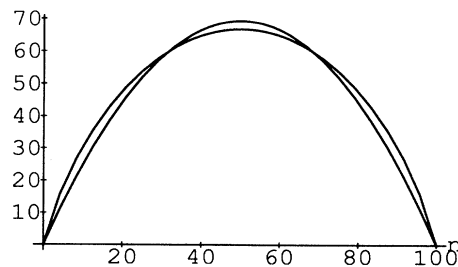
FIG. 1. Approximating $f(n) = \ln \binom{m}{n}$ by a parabola. Here, $m = 100$.

Inserting back this result into Eq. (23) we obtain, for $B(m)$,

$$B(m) \approx \mu m + \nu, \tag{27}$$

where

$$\mu = \gamma - (e^\gamma - 1)\ln(1 - e^{-\gamma}) - 4\ln 2(1 - e^{-\gamma})$$

and

$$\tag{28}$$

$$\nu = (4\ln 2)e^{-\gamma}.$$

Fixing α and β through Eqs. (19) and (20) in terms of $\bar{m}$ one obtains

$$I_{\text{max}}(x) \approx \ln(\bar{m} + 1) + \bar{m}\ln\left(\frac{\bar{m} + 1}{\bar{m}}\right) - \nu - \bar{m}\mu. \tag{29}$$

Recalling the definitions of μ and ν [Eq. (28)]; the relation between the mean numbers of incoming and outgoing quanta [Eq. (10)]; and the relation between γ and x [Eq. (22)], we plotted I_{max} against x for $\bar{n} = 1, 10, 20, 100$ (see Fig. 2). The horizontal line represents the information originally coded in the incoming wave. Notice that for $\bar{n}$ sufficiently large, I_{max} develops an unphysical negative tail, which must be a consequence of our (crude) strategy of approximating the logarithm of a binomial by a parabola. At any rate, there are features that are likely to be universal. First, for high-frequency modes $x \gg 1$, $I_{\text{max}} \to 0$, because in this limit thermal radiation overpowers stimulated emission ($\Gamma \to 1$). The other feature is that as the mean number of incoming quanta $\bar{n}$ grows, less and less information is degraded for all modes $x \ll 1$. This makes the case for the following picture: whenever stimulated emission overpowers thermal radiation, the amount of information that can be recovered from the black-hole radiation is close to that originally coded in the incoming one.

Now, what happens if the incoming beam contained fermions instead of bosons? At first thought we are led to say that the information carried by fermions is completely washed out by the black hole because the mechanism, responsible for correlating incoming and outgoing radiations, that worked for bosons, cannot take place (the exclusion principle forbids stimulated emission of fermions). Fortunately, this is not quite true.

III. THE ROLE OF THE EXCLUSION PRINCIPLE

Suppose that a black hole spontaneously radiating fermions with a (distorted) thermal distribution is hit by a fermion, which is then scattered. In order to conform to the exclusion principle, the black-hole response must be to suppress its own radiation of a similar fermion, leaving a definite imprint in the outgoing radiation. In other words, the exclusion principle provides the mechanism that correlates fermion incoming and outgoing radiations. As we shall see, this mechanism will be responsible for the partial transfer of the information stored in the beam to the outgoing radiation.

The mean number of fermions in the outgoing radiation is, as before, composed of the spontaneous emission and of a fraction $1 - \Gamma$ of the mean number of incoming fermions:

$$\bar{m} = \frac{\Gamma}{e^x + 1} + (1 - \Gamma)\bar{n}. \tag{30}$$

From this equation it is possible to read the conditional probabilities [16]:
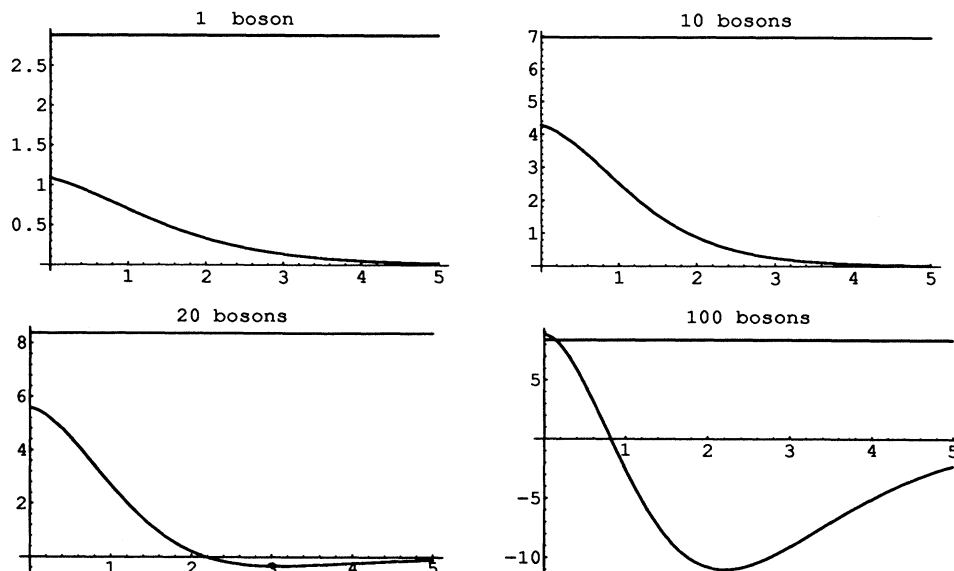
$$p(0|0) = 1 - \frac{\Gamma}{e^x + 1},$$

FIG. 2. $I_{\text{max}}(x)$ for $\bar{n} = 1, 10, 20$, and 100. Information is measured in bits.

$$p(1|0) = \frac{\Gamma}{e^x + 1},$$

$$p(0|1) = \frac{\Gamma}{e^{-x} + 1},$$

and

$$p(1|1) = \frac{\Gamma}{e^x + 1} + (1 - \Gamma). \tag{31}$$

In order to find the optimal information transmission regime we follow the same steps we took in the previous sections. First we have to solve Eq. (17) with the above distribution. For fermions there are only two B's:

$$B(0) = \frac{a\,p(1|1) - b\,p(1|0)}{1 - \Gamma} \tag{32}$$

and

$$B(1) = \frac{b\,p(0|0) - a\,p(0|1)}{1 - \Gamma}, \tag{33}$$

where

$$a = -p(0|0)\ln p(0|0) - p(1|0)\ln p(1|0) \tag{34}$$

and

$$b = -p(0|1)\ln p(0|1) - p(1|1)\ln p(1|1). \tag{35}$$

The optimal rate can be expressed in terms of these B's:

$$I_{\max} = - [(1 - \bar{m})\ln(1 - \bar{m}) + \bar{m}\ln\bar{m}] - B(0)$$
$$+ [B(0) - B(1)]\,\bar{m}. \tag{36}$$

The unique piece missing is the transmission coefficient. We borrowed the low-frequency limit of Γ for a massless fermion with $l = s$ from Page's work on black-hole emission rates [23],

$$\Gamma \approx \left(\frac{x}{8\pi}\right)^2, \quad x < 1, \tag{37}$$

and plotted the graphic of $I_{\max}$ for $\bar{n} = 0.5$ and $0 \le x <\le 1$ in Fig. 3. Since a fermion is a binary system *par excel-*
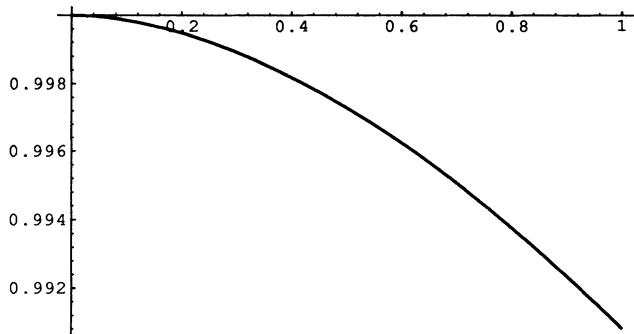

FIG. 3. $I_{\max}(x)$ for fermions $\bar{n} = 0.5$. Information is measured in bits.

lence, information is measured here in bits. Observe that at $x \approx 1$, the limit where Page's approximation breaks down, most of the information stored in the incoming beam is transferred to the outgoing radiation.

IV. CONCLUDING REMARKS

In this paper we showed that, contrary to common wisdom, black holes are not "ultimate waste baskets of information." When hit by radiation consisting of bosons, the black-hole response consists of stimulated emission of other quanta. For fermions the mechanism is entirely different. In order to conform to the exclusion principle, the black hole must suppress its own spontaneous emission of a fermion having the same quantum numbers as the one it scattered. Both mechanisms provide a correlation between outgoing and ingoing modes, which allows information originally stored in the incoming radiation to be partially transferred to the radiation emitted by the black hole.

Unfortunately, this mechanism is not efficient enough to resolve the black-hole information paradox because thermal radiation overpowers stimulated emission of bosons for the vast majority of modes. For fermions, whenever $x \gg 1$, the spontaneous emission is not affected because all incoming quanta are absorbed $(1 - \Gamma) \to 0$. If we wish to solve the "black-hole information paradox" from a conservative standpoint such as 't Hooft's, we have to search for new mechanisms that could account for a perfect correlation between incoming and outgoing radiation.

From a technical point of view, the crude approximation adopted for the logarithm of the binomial has to be overcome, as well as the negligence of scattering processes by the geometry. From a more fundamental standpoint, there also remains to be investigated the transmission of information by means of superradiant modes. Since in these modes outgoing and incoming radiations ought to be perfectly correlated, no information is expected to be degraded by the black hole there.

Recently two-dimensional dilatonic black holes became objects of intense research [24] because they allow backreaction effects to be taken into account. Let us recall that in two dimensions the coefficient $\Gamma \equiv 1$. According to our informational theoretic approach and the recent work by Bekenstein [14], two-dimensional, in contrast with four-dimensional black holes, do behave as perfect sinks of information. Thus, the physics of two- and four-dimensional black holes are entirely different, and it might well be possible that all technology learned in two dimensions might be useless for the real case of four-dimensional space-time.

ACKNOWLEDGMENTS

APPENDIX: THE INVERSE MATRIX OF A BINOMIAL

Inserting the distribution (8) into the equation for $B(m)$ [Eq. (17)] and canceling an overall factor $(1 - e^{-\gamma})^{n+1}$ yields

$$\sum_{m=0}^{\infty} B(m)e^{-\gamma m}\binom{m+n}{n} = F(n),\qquad (A1)$$

where

$$F(n) = -\sum_{m=0}^{\infty}\binom{m+n}{n}e^{-\gamma m}\ln p(m|n).\qquad (A2)$$

Solving Eq. (A1) for $B(m)$ is equivalent to the problem of finding the inverse matrix of the binomial (infinite matrix). To this end, let us multiply both sides of this equa-

tion by z^{-n}, where $|z| \geq 1$ is a fiducial complex number and sum over n. The result is

$$\sum_{m} B(m)e^{-\gamma m}\left(\frac{z}{z-1}\right)^{m+1} = \sum_{m}\frac{F(n)}{z^n},\qquad (A3)$$

where we used the fact that

$$\sum_{n}\binom{m+n}{n}z^{-n} = \left(\frac{z}{z-1}\right)^{m+1}.\qquad (A4)$$

Our next step is to expand the left-hand side of Eq. (A3) into powers of $\frac{1}{z-1}$:

$$\sum_{m,n\leq m+1} B(m)e^{-\gamma m}\binom{m+1}{n}\frac{1}{(z-1)^n} = \sum_{n}\frac{F(n)}{z^n}.\qquad (A5)$$

Next, we multiply both sides by $(z-1)^k, k = 0, 1, \ldots$, and perform a contour integration for any closed path C lying outside the region $|z| < 1$:

$$\sum_{m} B(m)e^{-\gamma m}\binom{m+1}{n}\oint_C\frac{dz}{(z-1)^{n-k}} = \sum_{n,p\leq n} F(n)\binom{k}{p}(-1)^{k-p}\oint_C\frac{dz}{z^{n-p}};\qquad (A6)$$

on the right-hand side (RHS) of this equation we have expanded the factor $(1-z)^k$ into powers of z. The contour integrals of the left- and right-hand sides are, respectively, $2\pi i\delta_{n,k+1}$ and $2\pi i\delta_{n,p+1}$. Thus,

$$\sum_{m} B(m)e^{-\gamma m}\binom{m+1}{k+1} = \sum_{n}\binom{k}{n-1}(-1)^{k-n+1}F(n).\qquad (A7)$$

A further trick is needed to isolate $B(m)$. Multiply both sides of our last result by y^{k+1}, where $|y| \leq 1$ is a new fiducial quantity, and sum over k ($0 \leq k \leq m$) to obtain

$$\sum_{n} B(n)e^{-\gamma(n+1)}\left[(1+y)^{n+1} - 1\right] = \sum_{n}\binom{k}{n-1}y^k(-1)^{k-n-1}F(n);\qquad (A8)$$

then apply the operator $\frac{1}{(m+1)!}\left(\frac{d}{dx}\right)^{(m+1)}$ on both sides of this equation:

$$\sum_{n} B(n)e^{-\gamma n+1}\binom{n}{m}(1+y)^{n-m} = \sum_{k,n}\binom{k}{n-1}\binom{k}{m}(-1)^{k-n-1}y^{(k-m)}F(n).\qquad (A9)$$

As our last step, we take the limit $y \to -1$ and obtain

$$B(m)e^{-\gamma m} = \sum_{n}\binom{k}{n-1}\binom{k}{m}(-1)^{(m-n-1)}F(n)\qquad (A10)$$

which, after substituting n by $n+1$, yields exactly Eq. (23) displayed in Sec. II.

Our next task is to evaluate

$$F(n) = -\sum_{m=0}^{\infty} e^{-\gamma m}\binom{m+n}{n}\ln p(m|n).\qquad (A11)$$

This calculation entails a summation of the logarithm of a binomial, a task that we were not able to accomplish. Thus, we resorted to the approximation

$$\ln\binom{m+n}{n} \approx 4\ln 2\frac{mn}{m+n}.\qquad (A12)$$

Under this approximation,

$$F(n) \approx -\sum_m \binom{m+n}{n} e^{-\gamma m} \left[-\gamma m + (n+1) \ln(1 - e^{-\gamma}) + 4 \ln 2 \frac{mn}{m+n} \right] . \tag{A13}$$

After some trivial manipulation, this can be rewritten as

$$F(n) \approx - \left[\gamma \frac{d}{d\gamma} + (n+1) \ln(1 - e^{-\gamma}) \right] \sum_m \binom{m+n}{n} e^{-\gamma m}$$

$$+ (4 \ln 2)n \left[n e^{\gamma n} \int_\gamma^\infty d\beta \sum_m \binom{m+n}{n} e^{-\beta(m+n)} - \sum_m \binom{m+n}{n} e^{-\gamma m} \right] . \tag{A14}$$

Performing all the sums, the derivative, and the integration, one obtains

$$F(n) \approx (n+1) \frac{\gamma e^{-\gamma} - (1 - e^{-\gamma}) \ln(1 - e^{-\gamma})}{(1 - e^{-\gamma})^{n+2}} - n \frac{4 \ln 2 e^{-\gamma}}{(1 - e^{-\gamma})^{n+1}} . \tag{A15}$$

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