

Longitudinal Space-Charge Geometric Factor for an Elliptical Beam

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Abstract

The longitudinal space-charge impedance for a particle beam, for which the azimuthal extent is much longer than the transverse, is typically expressed[3] in terms of a geometrical factor, g_0 , which depends on the transverse charge distribution and the geometry of the vacuum chamber. Expressions for g_0 for a round beam in a concentric round pipe are well known[3, 7]; and values may be given for the on-axis impedance or for the impedance ensemble-averaged over the beam cross-section. We have obtained analogous expressions for the on-axis and ensemble-average factors g_0 for the case of an elliptical charge distribution $\rho(m, x, y) = [1 - (x/a)^2 - (y/b)^2]^m$ inside a confocal elliptic metallic pipe for the indices $m = 0, 1, 2$. However, our results are not completely general; for example, the pipe cannot have the same eccentricity as the beam.

1 INTRODUCTION

1.1 Uniform elliptic beam in metallic enclosure

Given the 2D transverse electrical potential of a charge distribution with a metallic boundary condition, it is a straight-forward matter to write down a longitudinal geometric factor. Laslett [5, 6] gave the direct and image potentials for uniform elliptical beams enclosed by a variety of boundaries from flat plates to hyperbolic pole-faces; and these results could easily have been used to write g_0 factors circa 1977. [One result[4] does not satisfy $\nabla \wedge \mathbf{E} = \mathbf{0}$.]

1.2 Non-uniform elliptic beam in free space

As noted by Kellogg[1], and Houssais and Sacherer [2], for the case of free space in two-dimensions, the field in the interior of a disc with elliptical equi-density contours $\rho(m, x, y)$ where $m > 0$ is an integer, may be evaluated from definite integrals. For example the uniform ellipse with unit charge, $m = 0$, has fields:

$$E_x = \frac{x}{a(a+b)\pi} \quad \text{and} \quad E_y = \frac{y}{b(a+b)\pi} . \quad (1)$$

The fields in the case $m = 1$ have previously been given by Lapostolle [8, 9] *et al.* Similar problems but with a confocal elliptic conducting boundary are much more difficult and have to be treated by solving the Poisson equation.

1.3 Poisson's equation

It is natural to solve an electrostatics problem by finding a potential Φ which satisfies Poisson's equation $\nabla^2 \Phi = -\rho$

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and then forming $\mathbf{E} = -\nabla \Phi$. Solution of the Poisson equation is facilitated by adopting a coordinate system whose level surfaces coincide with the boundary of the charge distribution and or a metallic enclosing boundary.

1.4 Elliptic confocal coordinates

We consider an ellipse with major and minor semi-axes a and b , respectively; and with foci at $\pm c$ where $c^2 = a^2 - b^2$. We adopt elliptic coordinates u, v which are related to the cartesian coordinates x, y as follows:

$$x = c \cosh(u) \cos(v) \quad \text{and} \quad y = c \sinh(u) \sin(v) . \quad (2)$$

The level surfaces of constant u and constant v are families of ellipses and hyperbolae, respectively, with common foci. These level surfaces are sketched in figure 1. The ellipse

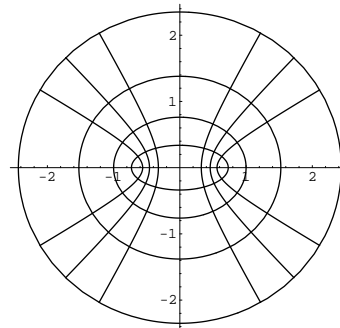


Figure 1: Level surfaces $u = 1/2, 1, 3/2, 2$ and $v = \pm\pi/6, \pm\pi/4, \pm\pi/3$, etc. for $u_b = 1$.

with major and minor semi-axes a, b is given by $\tanh u = b/a$ and we denote this value

$$u_b = \frac{1}{2} \log_e \left[\frac{a+b}{a-b} \right] = \ln \left[\frac{a+b}{c} \right] . \quad (3)$$

Similarly, the wall ellipse with semi-axes p, q is given by $u_w = \ln[(p+q)/c]$ or $\tanh u_w = q/p$ (where $p > q$).

Let $h^2 = c^2(\sinh^2 u + \sin^2 v)$. In the elliptic coordinates, Poisson's equation becomes:

$$\left[\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right] = -h^2(u, v) \rho(u, v) . \quad (4)$$

First consider the homogeneous equation when $\rho \equiv 0$. Let us introduce the dual-complex symbols $i^2 = -1$ and $j^2 = +1$ and identify the four directions $1, i, j, ij$. Possible solutions of $\nabla^2 \Phi = 0$ are then the components of

$$e^{ju} e^{iv} = (\cosh u + j \sinh u)(\cos v + i \sin v) . \quad (5)$$

Other possible solutions are $\Phi = 1, u, v$; and $\Phi = u$ is of particular interest because it is the far-field solution analogous to $\ln r$ in polar coordinates.

2 UNIFORM ELLIPTIC BEAM

We treat the case of constant density normalized to unit charge, $\rho = 1/\pi ab$. We follow, basically, the exposition of Symon [10] excepting that we shall give a correct expression for the field in the exterior region, as is essential to computing the geometric factor. The particular integral is

$$\text{P.I.} = -[\cosh(2u) + \cos(2v)]/[4\pi \sinh(2u_b)]. \quad (6)$$

We must now find the interior complementary function and the potential exterior to the beam Φ_{ext} ; both depend on whether it is free space or a metallic boundary.

2.1 Free space

The potential functions are

$$\Phi_{\text{int}} = \text{P.I.} + A_1 \cosh(2u) \cos(2v) \quad (7)$$

$$\Phi_{\text{ext}} = B_1 [\cosh(2u) - \sinh(2u)] \cos(2v) + B_2 \times u. \quad (8)$$

The azimuthal (i.e. v -dependence) of Φ_{ext} vanishes as $r \rightarrow \infty$ leaving only the logarithmic term $u \rightarrow \ln(r)$, as desired. The adjustable coefficients A_1, B_1, B_2 are found from the continuity of the field components E_u, E_v across the boundary $u = u_b$. The field is

$$\mathbf{E} = -\nabla\Phi = -\frac{1}{h} \left[\mathbf{e}_u \frac{\partial}{\partial u} + \mathbf{e}_v \frac{\partial}{\partial v} \right] \Phi. \quad (9)$$

Because the field component $E_v(u_b, v)$ tangential to the surface $u = u_b$ is not zero, so it follows that the boundary of the charge distribution is not an equipotential.

The result for the field in the interior of the disc is found to be identical with the components given above (1). The field exterior to the disc is given by

$$E_x = x \frac{[1 - \tanh u]}{(a-b)(a+b)\pi}, \quad E_y = y \frac{[\coth u - 1]}{(a-b)(a+b)\pi}. \quad (10)$$

2.2 Elliptic metallic boundary

We introduce a grounded, confocal elliptical conducting wall along the level surface $u = u_w > u_b$. The particular integral is unchanged, but we shall have to find a new complementary function and exterior potential to satisfy the condition $\mathbf{E} = \mathbf{0}$ at the wall. We take:

$$\Phi_{\text{int}} = \text{P.I.} + A_1 \cosh(2u) \cos(2v), \quad (11)$$

$$\Phi_{\text{ext}} = B_1 \cos(2v) \cosh(2u - 2u_w) + B_2 \times (u - u_w), \quad (12)$$

where we have arranged Φ_{ext} to be zero at the wall. Continuity of the field across the boundary of the charge distribution ($u = u_b$) determines the adjustable constants. For the interior region, cartesian components of the field are:

$$E_x = x \frac{[-b + a \tanh(2u_w)]}{a(a+b)(a-b)\pi}, \quad E_y = y \frac{[+a - b \tanh(2u_w)]}{b(a+b)(a-b)\pi}, \quad (13)$$

which clearly shows that the induced charges affect the field in the region enclosed by the conductor. Evaluation of the formulae is facilitated by noting that $\tanh(2u_w) = 2pq/(p^2 + q^2) \approx 1$. The confocal condition implies $c^2 = p^2 - q^2$ for the wall ellipse.

2.3 Geometrical factor

To make Φ continuous at $u = u_b$ we add a constant to Φ_{int} . Because $\Phi_{\text{ext}}(u_w) = 0$ it follows that the line integral $\int \mathbf{E} \cdot d\mathbf{l}$ from any point u, v within the charged disc to any point on the wall is simply $\Phi_{\text{int}}(u, v)$. From this it follows that the space-charge geometrical factor is directly proportional to $\Phi_{\text{int}}(u, v)$.

On-axis g_0 Because of peculiar behaviour of the elliptic coordinate system as $u \rightarrow 0$, the beam centre is located at ($u = 0, v = \pm\pi/2$). Thus the on-axis impedance is

$$g_0 = 4\pi\Phi_{\text{int}}(0, \pi/2) = 2(u_w - u_b) + \tanh(2u_w). \quad (14)$$

$$\text{Simplifying: } g_0 = 2 \ln \left[\frac{p+q}{a+b} \right] + \frac{2pq}{(p^2+q^2)}. \quad (15)$$

The logarithmic term contains essentially the quotient of the average radii of each of the two ellipses; and the algebraic term in p, q is approximately 1. Hence this result is very similar to the ‘‘classical’’ formula $g_0 = 1 + 2 \ln(p/a)$ for a uniform circular beam and pipe.

Ensemble average g_0 The on-axis space-charge impedance gives the ‘‘worst case scenario’’. A more representative value is given by forming the ensemble average over the beam cross-section.

$$\langle g_0 \rangle = 4\pi \int_0^{u_b} \int_{-\pi}^{+\pi} \Phi_{\text{int}} \rho h^2 du dv. \quad (16)$$

For the case of constant density, the ensemble average is

$$\langle g_0 \rangle = 2(u_w - u_b) + \frac{1}{2} \tanh(2u_w) = 2 \ln \left[\frac{p+q}{a+b} \right] + \frac{pq}{(p^2+q^2)}. \quad (17)$$

This expression is very similar to the ensemble geometric factor $\langle g_0 \rangle = 2 \ln(p/a) + (1/2)$ for a round beam and pipe.

3 BEAM WITH PARABOLIC DENSITY

We now consider the slightly more complicated case that the charge distribution is $\rho(1, x, y)$. The particular integral:

$$\begin{aligned} \text{P.I.} = & A_1 [\cosh 2u + \cos 2v] + A_2 [\cosh 4u + \cos 4v] \\ & + A_3 [\cosh 4u \cos 2v + \cos 4v \cosh 2u]. \end{aligned} \quad (18)$$

We substitute into the Poisson equation and compare coefficients of $\cos(4v) \cosh(2u)$, $\cosh 4u$, and $\cosh 2u$ to obtain three equations for the adjustable constants.

3.1 Free space

We take the potential functions:

$$\begin{aligned} \Phi_{\text{int}} = & \text{P.I.} + A_1 \cosh(2u) \cos(2v) + A_2 \cosh(4u) \cos(4v) \\ \Phi_{\text{ext}} = & B_1 \cos(2v) [\cosh(2u) - \sinh(2u)] \\ & + B_2 \cos(4v) [\cosh(4u) - \sinh(4u)] + B_3 u. \end{aligned} \quad (19)$$

Continuity of the field components E_u, E_v at the boundary of the charge distribution ($u = u_b$) determines the adjustable constants A_1 through B_3 .

3.2 Elliptic metallic boundary

We introduce a grounded, confocal elliptical conducting wall along the level surface $u = u_w > u_b$. We take:

$$\begin{aligned}\Phi_{\text{int}} &= \text{P.I.} + A_1 \cosh(2u) \cos(2v) + A_2 \cosh(4u) \cos(4v) \\ \Phi_{\text{ext}} &= B_1 \cos(2v) \cosh[2(u - u_w)] \\ &+ B_2 \cos(4v) \cosh[4(u - u_w)] + B_3 \times (u - u_w).\end{aligned}\quad (20)$$

where we have arranged Φ_{ext} to be zero at the wall. Continuity of the field across the boundary determines the adjustable constants. To make Φ continuous at $u = u_b$ we add a constant to Φ_{int} .

On-axis geometric factor The geometric factor is

$$g_0 = 2(u_w - u_b) + \frac{(9 \sinh 6u_w - 7 \sinh 2u_w)}{6(\cosh 2u_w + \cosh 6u_w)}.\quad (21)$$

This and subsequent expressions for g_0 contain the same logarithmic term $(u_w - u_b) = \ln[(p+q)/(a+b)]$ as in (15); this term represents the “far field” which is independent of the charge distribution so long as it is elliptic. However, the “near field” (in the exterior region) does depend on the details of the charge distribution and feeds into the hyperbolic functions term.

Ensemble average geometric factor After performing the integral (16) the result is

$$\langle g_0 \rangle = 2(u_w - u_b) + \frac{(33 \sinh 6u_w - 31 \sinh 2u_w)}{[72 \cosh 2u_w \cosh 4u_w]}.\quad (22)$$

4 BEAM WITH QUARTIC DENSITY

We now consider the more complicated case that the charge distribution is $\rho(2, x, y)$. Because we are only interested here in the geometric factor, we shall only consider the case of a metallic enclosure and not the free-space case. The particular integral is:

$$\begin{aligned}&= A_1(\cosh 2u + \cos 2v) + A_4(\cosh 4u \cos 2v + \cos 4v \cosh 2u) \\ &+ A_2(\cosh 4u + \cos 4v) + A_5(\cosh 6u \cos 2v + \cos 6v \cosh 2u) \\ &+ A_3(\cosh 6u + \cos 6v) + A_6(\cosh 6u \cos 4v + \cos 6v \cosh 4u)\end{aligned}$$

After substitution into the Poisson equation, comparison of the coefficients of $\cos 6v \cosh 4u$, $\cos 6v \cosh 2u$, $\cos 4v \cosh 2u$, $\cosh 6u$, $\cosh 4u$, $\cosh 2u$ gives simple equations for $A_6, A_5, A_4, A_3, A_2, A_1$, respectively.

4.1 Elliptic metallic boundary

We now find the interior complementary function and the exterior potential function.

$$\Phi_{\text{int}} = \text{P.I.} + A_1 \cosh(2u) \cos(2v) + A_2 \cosh(4u) \cos(4v) + A_3 \cosh(6u) \cos(6v)\quad (23)$$

$$\Phi_{\text{ext}} = B_1 \cos(2v) \sinh[2(u - u_w)] + B_4(u - u_w) + B_2 \cos(4v) \sinh[4(u - u_w)] + B_3 \cos(6v) \sinh[6(u - u_w)]\quad (24)$$

The adjustable coefficients are determined by the continuity of \mathbf{E} across the charge boundary at $u = u_b$. For the purpose of evaluating g_0 , we make the potential continuous across the boundary u_b by adding a constant to Φ_{int} .

On-axis geometric factor

$$\begin{aligned}g_0 &= 2(u_w - u_b) \\ &+ \frac{(127 \sinh(2u_w) - 90 \sinh(6u_w) + 55 \sinh(10u_w))}{60[-1 + 2 \cosh(4u_w)] \cosh(2u_w) \cosh(4u_w)}\end{aligned}\quad (25)$$

Ensemble average geometric factor

$$\begin{aligned}\langle g_0 \rangle &= 2(u_w - u_b) \\ &+ \frac{[986 \sinh(2u_w) - 675 \sinh(6u_w) + 365 \sinh(10u_w)]}{600[-1 + 2 \cosh(4u_w)] \cosh(2u_w) \cosh(4u_w)}\end{aligned}\quad (26)$$

5 CONCLUSION

We have found the electrical potential for an elliptical charge distribution $\rho \propto [1 - (x/a)^2 - (y/b)^2]^m$ within a confocal elliptical metallic boundary for the cases $m = 0, 1, 2$ and have obtained the corresponding on-axis and ensemble-average geometric factors g_0 . We have noted the influence of induced charges on the grounded conductor upon the fields within, by comparison of the potentials with the free-space case of no boundary. This paper is a much abridged version of a TRIUMF design note[11] which contains all the mathematical details and more discussion of 2D electrostatics problems. Formulae were generated with the aid of MATHEMATICA[12].

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