CALCULATION OF FINITE-LENGTH, HOLLOW-BEAM EQUILIBRIA

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Abstract

Finite-length equilibria occur in a number of intense-beam and plasma applications. Penning traps permit the study of intra-beam collective effects, as the additional freedom gained from having an internal conductor permits greater control over the plasma profile, so that monotonic, but not constant, plasma profiles can be obtained. On the basis that the thermal velocity of background neutrals and the drift velocity of the electrons are much lower than the thermal velocity of the electrons, and the rotation frequency is small compared to the gyrofrequency, the equilibrium equation can be reduced to a self-consistent Poisson equation where the source depends on the potential. We solve for these equilibria using a Gauss-Seidel relaxation method. Our results show the shape of the equilibria for various electrode configurations.

1 INTRODUCTION

Penning traps with nonneutral plasma under the influence of a magnetic field have been studied for a variety of experiments including plasma physics[1],[2], and Coulomb crystals[3]. Recently, these traps have been used for experimental tests of the CPT theorem, which predicts that various quantities such as masses, gyromagnetic ratios, and charge-to-mass ratios are equal for particles and antiparticles. The comparison of charge-to-mass ratios for the antiproton and proton in the trap is much more accurate than earlier comparisons made with other techniques[4].

In principle, plasma can be confined perfectly in an ideal trap with cylindrical symmetry. O'Neil and coworkers[5],[6],[7] derived and solved the equilibrium equations for a nonneutral plasma without a center conductor. They assume that the plasma is in thermodynamic equilibrium. A particular thermal equilibrium can be obtained from specific values of total number, total angular momentum, and total energy. Similarly, a nonneutral plasma beam in a solenoid at magnetic field has identical dynamics.

However, in practice, the particles cannot be confined indefinitely. Collisions with background neutrals and anomalous transport[8](which is independent of pressure) cause the plasma in the trap to expand radially[9]. Since the geometry and dynamics are similar to those of a Penning trap, these kinds of effects can be determined in the nonneutral beam. Thus penning traps permit study of intra-beam collective effects.

The equilibria of nonneutral plasma can be described by the self-consistent Poisson equation. Without a center conductor, the plasma evolves toward thermal equilibria in which the physical parameter, such as maximum number density, angular velocity, and temperature, are determined by the total number, the total angular momentum, and the total energy. However, for the trap with a center conductor, the bias potential is an additional parameter. This provides a mechanism to control the plasma. To study these systems, we have developed a method to find the solution numerically.

In this paper, we investigate the equilibria in two different cases. In the following section, we briefly explain how to get the thermal equilibria in both isothermal and adiabatic cases and show the results in one dimensional adiabatic process in which the entropy is constant, when the plasma is cylindrically symmetric, and it is long compared with the radius of the outer shell. In the next section, we develop a method to calculate two dimensional equilibria.

2 THERMAL EQUILIBRIUM

A modified Penning trap with a center conductor that is electrically biased allows control of equilibrium by changing the central potential. As shown in Fig. 1, an outer conducting cylinder is divided axially into three sections with a center conductor. Compared to the central section, the two remaining end sections are at more negative potential to confine negative charged particles axially. A uniform axial magnetic field with the electric field between two shells provides radial confinement.

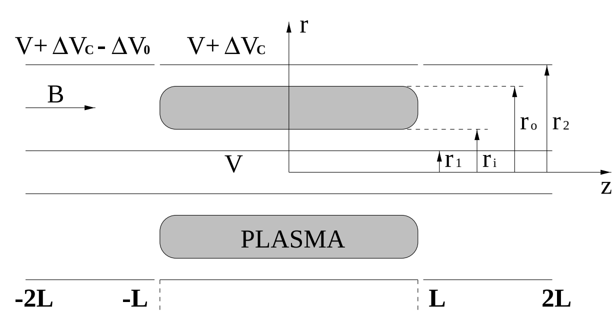

Figure 1: The side view of Modified Penning Trap. Here, $B = 350G$.

In order to understand the equilibrium state in the trap, we need to take an appropriate Hamiltonian. The plasma approximation (weak correlation) determines the equilibrium. The plasma approximation, that the plasma can be treated as a continuous fluid, requires that the number of particles in a Debye sphere be large, ($\bar{n}\lambda_D^3 \gg 1$). In this case the total Hamiltonian is the sum of the Hamiltonians for each particles, with the potential given by Poisson equa-

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tion where the charge density is the fluid density.

With such an approximation, the one particle Hamiltonian is

$$H = \frac{1}{2m}(p_r^2 + p_z^2) + \frac{1}{2mr^2}[p_\theta + \frac{e}{c}A_\theta(\vec{r})r]^2 - e\phi(\vec{r}) \quad (1)$$

where $A_\theta(\vec{r}) = \frac{Br}{2}$.

In a thermodynamic description[7] with conservation of total number, total angular momentum, and total energy, the distribution(a canonical ensemble) has the form

$$\rho(\vec{r}, \vec{p}) = Z^{-1}\, exp[-\frac{H - \omega p_\theta}{T}] \quad (2)$$

where Z can be determined from the total number. Integrating both sides over $\vec{p}$ gives the number density

$$n(\vec{r}) = \bar{n}\, exp\left[\frac{1}{T}\{e\phi(\vec{r}) - \frac{m}{2}\omega(\Omega - \omega)r^2\}\right] \quad (3)$$

where ω is the constant angular velocity in the trap and $\bar{n}$ is maximum number density.

Therefore, the self-consistent Poisson equation

$$\nabla^2\phi(\vec{r}) = 4\pi e n(\vec{r}) \quad (4)$$

which gives equilibrium states, will be reduced to a dimensionless equation

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial\psi}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{\partial^2\psi}{\partial\zeta^2} = e^{\psi} - (1 + \gamma) \quad (5)$$

in terms of new dimensionless variables

$$\gamma \equiv \frac{m\omega(\Omega - \omega)}{2\pi\bar{n}e^2} - 1,\ \ \psi \equiv \frac{e}{T}\phi - \frac{1+\gamma}{4}\rho^2,$$
$$\lambda_D^2 \equiv \frac{T}{4\pi\bar{n}e^2},\ \ \rho \equiv \frac{r}{\lambda_D},\ \ \zeta \equiv \frac{z}{\lambda_D}. \quad (6)$$

Now we will briefly discuss adiabatic variation of equilibria. As we mentioned, each isothermal equilibrium in one dimension can be found by solving the self-consistent Poisson equation with conservation of total number, angular momentum, and total energy when the longitudinal length of plasma is sufficiently long compared to the radius of the outer shell on which the potential is constant. However, for the trap with a center conductor, a small and slow change of bias potential permits some electrical work between two shells, which means that total energy in the system is no longer a conserved quantity. But a slow change of the potential guarantees no entropy change in this system. Therefore, we can get the equilibria by changing the bias potential slowly.

From the definition of entropy, we can easily redefine the entropy as

$$S = -2\pi\int_{r_1}^{r_2} dr\ r\, n(r)\, ln[n(r)\, T^{-3/2}] \quad (7)$$

where r_1 and r_2 is the radii of inner and outer shell[5]. With conservation of total number, angular momentum, and entropy, a particular bias gives other physical quantities, such as maximum number density, angular velocity, charges on the shells, and temperature.

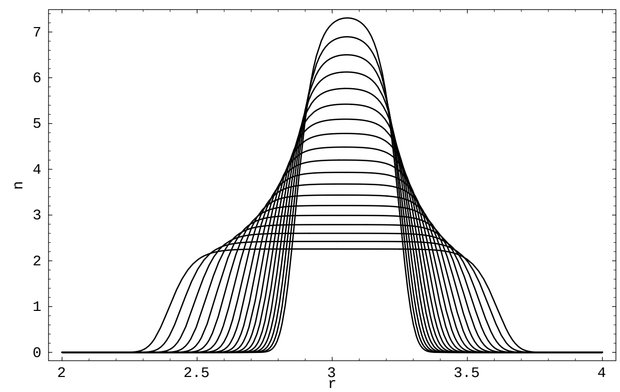

Figure 2: The density profiles $10^{-6}n(r)$'s vs $r(cm)$ in adiabatic process. Here, $r_1 = 0.32$, $r_2 = 5.10$, $N = 5 \times 10^7 cm^{-1}$, $\frac{P_\theta}{m\Omega} = -2.35 \times 10^8 cm$, and $S = -1.07 \times 10^8 cm^{-1}$.

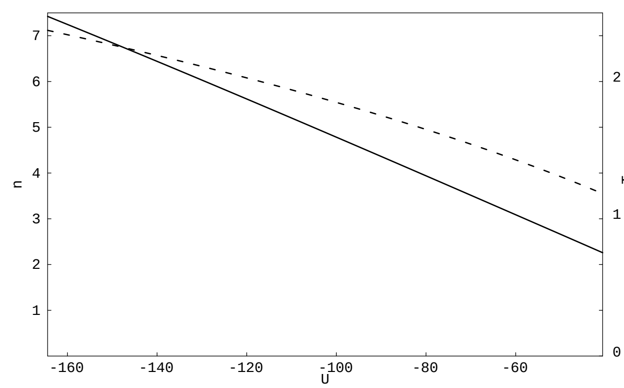

Figure 3: The maximum number density $10^{-6}n$ ($Solid\ line$) and The temperature $10^{-2}T(K)$ ($Dashed\ line$) vs potential difference $U(eV)$.

From Fig. 2 and Fig. 3, we can see how the profile can be changed as the potential between two shells is changed. The figures show that smaller potential difference gives wider annular profile. They also show that the temperature decreases as the difference goes down, which means that the trap can be used to cool an electron plasma.

3 FINITE-LENGTH EQUILIBRIUM

For the case of a finite length column, the number density and the potential are independent of θ and those are determined by three parameters, γ, ΔV_C, and ΔV_0(see Fig. 1).

In this case, the dimensionless Poisson equation can be reduced to

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial\psi}{\partial\rho} + \frac{\partial^2\psi}{\partial\zeta^2} = e^{\psi} - (1+\gamma) \tag{8}$$

where the potential dependency of source term in the right-hand side makes it more difficult to solve the equation.

We solve this equation numerically by using a nonlinear variant of the Gauss-Seidel iteration procedure for elliptical equations. The value, $\psi_{i,j}$, at a grid point depends on the values of the nearest grid points, $\psi_{i+1,j}$, $\psi_{i-1,j}$, $\psi_{i,j+1}$, $\psi_{i,j-1}$, and $\psi_{i,j}$ itself because the source term depends on ψ. Therefore, the relation can be of a form

$$\psi_{i,j} = f(\psi_{i,j}, \psi_{i+1,j}, \psi_{i-1,j}, \psi_{i,j+1}, \psi_{i,j-1}). \tag{9}$$

Finally the equation can be reduced to

$$\begin{aligned} \psi_{i,j}^{(n+1)} &= \frac{1}{2(1+\frac{\Delta\rho^2}{\Delta\zeta^2})}[-\Delta\rho^2(e^{\psi_{i,j}^{(n)}} - 1 - \gamma) \\ &+ (1+\frac{\Delta\rho}{2\rho_i})\psi_{i+1,j}^{(n)} + (1-\frac{\Delta\rho}{2\rho_i})\psi_{i-1,j}^{(n)} \\ &+ \frac{\Delta\rho^2}{\Delta\zeta^2}(\psi_{i,j+1}^{(n)} + \psi_{i,j-1}^{(n)})]. \end{aligned} \tag{10}$$

With the convergence condition, $|\psi_{i,j}^{(n+1)} - \psi_{i,j}^{(n)}| < \delta$ for sufficiently small $\delta\,(=10^{-4})$, the equilibrium is obtained.

Since the length is sufficiently large compared to radius of outer shell, we may suppose that the cross section($z = constant$) near the center should coincide with the one dimensional solution that is independent of θ and z, and that the profile varies little with z in the central region far from the end sections of outer shell. Our numerical solution does bear this out. Fig. 4 shows the two dimensional density distribution.

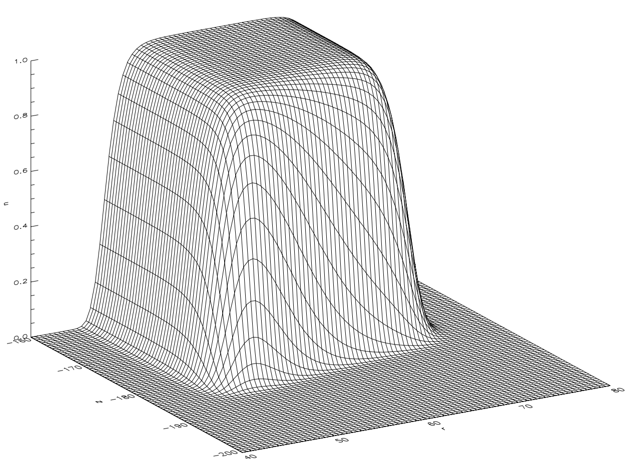

Figure 4: The 2D density profile $n(r,z)/\bar{n}$ near the $z = -L$ in isothermal process. Here, $L = 200\lambda_D$, $r_1 = 40\lambda_D$, $r_2 = 80\lambda_D$, $\gamma = 1.0\times10^{-5}$, $V = 3.70\,V$, $\Delta V_C = 12.30\,V$, $\Delta V_0 = 12.30\,V$, $T = 118.33\,K$, and $\lambda_D = 5.0\times10^{-2}\,cm$.

4 CONCLUSION

Modified Penning traps allows us to confine and cool electron plasma both radially and axially with two potential differences. By changing the radial potential difference slowly, all physical parameters including temperature can be controlled. In the adiabatic process, the plasma is wider and colder as the potential difference is smaller.

5 REFERENCES

[1] J. S. deGrassie and J. H. Malmberg, Phys. Fluids **23**, 63(1980).

[2] G. Rosenthal, G. Dimonte, and A. Y. Wong, Phys. Fluids **30**, 3257(1987).

[3] W. M. Itano, *et al.*, Science **279**, 686(1998).

[4] G. Gabrielse, *et al.*, Phys. Rev. Lett. **74**, 3544(1995).

[5] T. M. O'Neil and C. F. Driscoll, Phys. Fluids **21**, 266(1979).

[6] S. A. Prasad and T. M. O'Neil, Phys. Fluids**21**, 920(1978).

[7] T. M. O'Neil and Daniel H. E. Dubin, Phys. Plasma **5**, 2163(1998).

[8] D. L. Eggleston, Phys. Plasma **4**, 1196(1997).

[9] C. F. Driscoll, K. S. Fine, and J. H. Malmberg, Phys. Fluids **29**, 2015(1986).