COULOMB SCATTERING WITHIN A SPHERICAL BEAM BUNCH IN HIGH CURRENT LINEAR ACCELERATORS *

A.V. Fedotov† and R.L. Gluckstern
Physics Department, University of Maryland, College Park, MD 20742

Abstract

Beam halo formation issues are important for the design of high current linear ion accelerators. Various mechanisms can potentially cause beam halo. Some recent studies have suggested that Coulomb collisions in the beam bunch can contribute significantly to beam bunch growth and halo development in linear accelerators. Despite the general belief that collisions are not important, it is clear that a rigorous treatment of this question is needed. In an effort to explore this issue in detail we have undertaken an analysis of the effects of Coulomb scattering between ions in a self-consistent spherical bunch.

1 INTRODUCTION

During the last several years, interest has grown in the design of high current linear ion accelerators for a variety of important applications. Since the beam bunch spends a very short time in a linac, compared with a circular machine or a storage ring, the general expectation is that collisions between individual ions will take place on too long a time scale to be important. However, there have been some recent numerical studies which suggested that small angle single Coulomb scattering may not be negligible for spheroidal bunches in a linac. Thus, the concern has arisen with regard to the possibility that Coulomb collisions between ions may contribute significantly to emittance growth and/or halo formation. It is clear that, because of the importance of this question for the design of high current linear ion accelerators, a more rigorous treatment of the effect of single Coulomb collisions is needed. In an effort to explore this issue in detail we have undertaken an analysis of the effects of Coulomb scattering between ions in a self-consistent [1] spherical bunch.

2 GENERAL CONSIDERATIONS

Our quantitative study of non-stationary distributions [2] shows that the parameters of halo formation are more or less the same as those caused by mismatches of an otherwise self-consistent phase space distribution.

We therefore start with our family of self-consistent, equipartitioned phase space distributions [1]:

$$f(\boldsymbol{r},\boldsymbol{v}) = \left\{ \begin{array}{ccc} N(H_0-H)^n & , & H<H_0 \\ 0 & , & H>H_0 \end{array} \right\}, \qquad (1)$$

where the Hamiltonian is

$$H(\boldsymbol{r},\boldsymbol{v}) = mv^2/2 + kr^2/2 + e\Phi_{\rm sc}(r), \qquad (2)$$

and use

$$G(r) \equiv H_0 - kr^2/2 - e\Phi_{\rm sc}(r), \qquad (3)$$

with k being the smoothed restoring force gradient. We choose the space charge potential $\Phi_{\rm sc}(r)$ which vanishes at $r=\infty$, H_0 is a constant, the external forces are linear, and $f(\boldsymbol{r},\boldsymbol{v})$ is normalized such that

$$\rho(\boldsymbol{r}) = Q\int d\boldsymbol{v} f(\boldsymbol{r},\boldsymbol{v})\ ,\ \int d\boldsymbol{r}\rho(\boldsymbol{r}) = Q, \qquad (4)$$

where Q is the total bunch charge.

The probability per unit time (in the coordinate system of the bunch) for a Coulomb collision between ions with velocity $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, is then

$$\begin{aligned}\frac{dP}{dt} &= \int d\boldsymbol{r}\int d\boldsymbol{v}_1 f(\boldsymbol{r},\boldsymbol{v}_1) \\ &\times \int d\boldsymbol{v}_2 f(\boldsymbol{r},\boldsymbol{v}_2)|\boldsymbol{v}_1-\boldsymbol{v}_2| d\Omega_s \frac{d\sigma}{d\Omega_s}, \qquad (5)\end{aligned}$$

where $d\sigma/d\Omega_s$ is the classical differential Rutherford cross section.

3 HALO EXTENT

Using Eqs. (1) and (3) we find

$$\rho(r) = QN\int_0^{v_0} 4\pi v^2 dv[G(r)-mv^2/2]^n, \qquad (6)$$

where $v_0^2 = 2G(r)/m$. We then examine the kinematics of the Coulomb collision [3]. Clearly the ions which travel to the largest radius before turning back are the ones which have the largest possible velocity at a given r and where, after the collision, one is left at rest and the other travels radially with all the kinetic energy. Such a collision can only conserve energy and momentum when the colliding ion velocities are at 90° to one another. With the maximum velocity before the collision given by $mv^2/2 = G(r)$, the maximum energy after the collision is

$$\frac{mv_{\rm max}^2}{2} = 2G(r). \qquad (7)$$

The maximum radius R then satisfies

$$2G(r) + \frac{kr^2}{2} + e\Phi_{\rm sc}(r) = \frac{kR^2}{2} + e\Phi_{\rm sc}(R), \qquad (8)$$

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† fedotov@physics.umd.edu

where $r < a$, $R > a$. We then have

$$\frac{k(R^2 - a^2)}{2} - \frac{eQ}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{R}\right) = G(r). \quad (9)$$

We proceed further in this analysis for $n = -1/2$, where

$$G(r) = \frac{3k}{\kappa^2}\left[1 - \frac{i_0(\kappa r)}{i_0(\kappa a)}\right]. \quad (10)$$

Here

$$\kappa^2 = \frac{eQ}{4\pi\epsilon_0 \int_0^a r^2 dr G(r)} \quad (11)$$

and

$$i_0(w) = \frac{\sinh w}{w}. \quad (12)$$

From Eq. (9) it is clear that outermost particles are those which start near the bunch center. We then rewrite Eq. (9) as

$$\begin{aligned} u^3 \quad &+ \quad \left[-\frac{12}{\kappa^2 a^2} + \frac{6}{\kappa a}\frac{(1+c)}{s} - 3\right]u \\ &+ \quad 6(1 - \kappa a\frac{c}{s})/\kappa^2 a^2 + 2 = 0, \end{aligned} \quad (13)$$

where $u = R/a$, $s = \sinh(\kappa a)$, $c = \cosh(\kappa a)$. For large κa, Eq. (13) can be easily solved, and we obtain

$$\begin{aligned} \frac{R-a}{a} \quad &\simeq \quad \frac{\sqrt{3}-1}{\sqrt{\kappa^2 a^2 + 15/2}} \\ &\to \quad \frac{\sqrt{2}(\sqrt{3}-1)}{\sqrt{15}}\eta_{\mathrm{rms}} \cong 0.27\eta_{\mathrm{rms}}, \end{aligned} \quad (14)$$

where η_{rms} is an rms tune depression [3].

An interesting consequence of Eq. (14) is that the thickness of the shell populated by the scattered ions decreases as the beam becomes more space charge dominated. We remind the reader that Eq. (14) was obtained for the self-consistent equipartitioned distribution with $n = -1/2$. In his report [4] on numerical studies for distributions which were not self-consistent, Pichoff observed a similar dependence on tune depression.

At first glance this thin shell does not resemble our earlier description of a halo [1, 2]. However, the shell thickness given by Eq. (14) was obtained for an equipartitioned beam. We now use Pichoff's definition of the equipartitioning factor $\chi = v_z/v_x$. The maximum possible velocity after the collision can be rewritten as $mv_{\max}^2/2 = (1+\chi^2)G$, and we obtain for the shell thickness

$$\frac{R-a}{a} \simeq \frac{\sqrt{1+2\chi^2}-1}{\sqrt{15/2}}\eta_{\mathrm{rms}}. \quad (15)$$

Clearly, when the beam is non-equipartitioned or the beam with the stationary distribution is rms mismatched, the thickness of the shell can be significantly larger, depending on the equipartitioning factor. Such an increased shell thickness can then easily resemble the typical halo extent [1, 2]. Thus, in order to understand whether scattering can contribute to the halo formation we need to calculate the rate at which this shell becomes populated.

We note that solutions given by Eqs. (14)-(15) can be used with good accuracy only for large values of κa ($\kappa a > 4$) corresponding to tune depressions $\eta < 0.6$, which covers our range of interest [5]. For $\eta \geq 0.6$ a better solution of Eq. (13) should be used. For completeness, we present below some values based on numerical solution of Eq. (13). For equipartitioned beam ($\chi = 1$), the shell extent becomes $R/a =$ 1.41, 1.37, 1.32, 1.27, 1.22 for $\eta =$ 1, 0.9, 0.8, 0.7, 0.6, respectively. Similar values for the shell thickness were obtained by Pichoff for the distribution functions used [4]. In fact, in the limit of zero space-charge, the shell thickness becomes independent of the distribution, and, from Eq. (8), is simply given by $R/a = \sqrt{1+\chi^2}$ (see also [4]).

4 COULOMB SCATTERING RATE

The heart of our calculation is the evaluation [3] of the 11 dimensional integral in Eq. (5) over the range of $\boldsymbol{r}$, $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, Ω_s which enables particles to leave the bunch. The detailed analyses are presented in [3]. Here we just present an order of magnitude estimate of Eq. (5), assuming a spherical bunch of radius a. We use $|\boldsymbol{v}_1| \sim |\boldsymbol{v}_2| \sim v$ and $\int d\boldsymbol{v} f(\boldsymbol{r}, \boldsymbol{v}) \sim 1/a^3$ and assume all integrals are finite to obtain the estimate

$$\frac{dP}{cdt} \sim \frac{r_p^2}{\epsilon_N^3}, \quad (16)$$

where the classical radius of the ion is given by

$$r_p = \frac{e^2}{4\pi\epsilon_0 mc^2}, \quad (17)$$

and where the (projected) normalized emittance of the bunch is

$$\epsilon_N \simeq av/c. \quad (18)$$

For a proton bunch with $r_p = 1.5 \times 10^{-18}$ [m] and $\epsilon_N \simeq 1 \times 10^{-6}$ [m rad], Eq. (16) yields

$$\frac{dP}{cdt} \sim 10^{-15}/\mathrm{km}, \quad (19)$$

clearly negligible for a linac.

The problem with the foregoing estimate is that it ignores the divergence of the Coulomb cross section at $\theta_s = 0$, as well as the possible divergence of the distribution $f(\boldsymbol{r}, \boldsymbol{v})$ near $mv^2/2 = G(\boldsymbol{r})$ for values of $n < 0$ in Eq. (1). We address these issues and present evaluation of the 11 dimensional integral given by Eq. (5) in [3].

After rigorous calculation [3] we finally find that the fraction of ions which leave the beam (and form a spherical layer around the bunch) per unit length is

$$\frac{dP}{cdt} \sim \left\{ \begin{array}{ccc} r_p^2/\epsilon_N^3 & , & n > 0 \\ (r_p^2/\epsilon_N^3)\ell n(\epsilon_N^2/r_p a) & , & n = 0 \\ (r_p^2/\epsilon_N^3)(\epsilon_N^2/r_p a)^n & , & -1 < n < 0 \end{array} \right\}, \quad (20)$$

where, for the distributions with $0 \geq n > -1$, we assume that the Coulomb force between ions is screened at the Debye length λ_D. Using $r_p = 1.5 \times 10^{-18}$ [m], $\epsilon_N \cong 10^{-6}$ [m rad] and $a \cong 10^{-2}$ [m], we then have

$$\frac{dP}{cdt} \sim \left\{ \begin{array}{ccc} 10^{-15}/\mathrm{km} & , & n > 0 \\ 10^{-14}/\mathrm{km} & , & n = 0 \\ 10^{-11}/\mathrm{km} & , & n = -.5 \\ 10^{-8}/\mathrm{km} & , & n = -.9 \end{array} \right\}. \qquad (21)$$

5 EFFECT OF MULTIPLE COLLISIONS

The shortcoming of the approach in Section 4 is that it does not take into account the effect of a large number of small scattering angle Coulomb collisions. For this purpose we also examine the evolution of the phase space distribution in time due to Coulomb collisions by starting with the Boltzmann equation for an otherwise equipartitioned beam bunch. For our self-consistent distribution, the Boltzmann equation, which accounts for the scattering of particles into and out of regions of velocity space, can be written as

$$\begin{aligned} \frac{\partial f(\boldsymbol{u}_1)}{\partial t} &= K n_D \int d\boldsymbol{u}_2 |\boldsymbol{u}_1 - \boldsymbol{u}_2| \\ \times \quad & \int d\Omega_s \left[\frac{d\sigma}{d\Omega_s}(\boldsymbol{u}_1', \boldsymbol{u}_2' \rightarrow \boldsymbol{u}_1, \boldsymbol{u}_2) f(\boldsymbol{u}_1') f(\boldsymbol{u}_2') \right. \\ - \quad & \left. \frac{d\sigma}{d\Omega_s}(\boldsymbol{u}_1, \boldsymbol{u}_2 \rightarrow \boldsymbol{u}_1', \boldsymbol{u}_2') f(\boldsymbol{u}_1) f(\boldsymbol{u}_2) \right]. \qquad (22) \end{aligned}$$

Here n_D is the ion particle density, and $d\sigma/d\Omega_s$ is the Coulomb scattering cross section for the initial and final states. Apart from constants, which are absorbed in K, $f(\boldsymbol{u})$ is $(1-u^2)^n$, where we renormalized all velocities as $v^2 = [2G(r)/m]u^2$.

We now calculate the rate of change of $\langle u_1^2 \rangle$ and $\langle u_1^4 \rangle$ and find [3] that $d\langle u_1^2 \rangle/dt = 0$ and that the lowest non-vanishing power occurs for $d\langle u_1^4 \rangle/dt$:

$$\frac{d\langle u_1^4 \rangle}{dt} = \frac{4\pi^2}{18} K n_D \left[\ell n \left(\frac{1}{\theta_{\min}} \right) - 1 \right]. \qquad (23)$$

It is possible to calculate the rate of change of the expectation value of $(1-u_1^2)^n$ for all values of n, with the same result as in Eq. (23), except for a numerical factor of order 1. In fact, the results obtained suggest the expected rounding of the $n = 0$ distribution near $u = 1$.

We therefore expect that multiple scattering simply leads to a generalization of our previous result in Section 4 with logarithmic behavior:

$$\frac{1}{c}\frac{dP}{dt} \sim \frac{r_p^2}{\epsilon_N^3} \ell n \left(\frac{1}{\theta_D} \right), \qquad (24)$$

where θ_D is minimum angle corresponding to Debye length impact parameter. If so, the rate of scattering due to multiple collisions is only slightly higher than the rate for single encounters for the distributions with $n > 0$. For the same parameters as those used in Section 4 the rate becomes clearly negligible with $dP/cdt \sim 10^{-14}$/km.

6 SUMMARY AND CONCLUSIONS

In Sections 3-4 we have calculated the effect of single Coulomb scattering of a self-consistent 6-D distribution for a spherical beam bunch. In this calculation we find:

- Single collisions are capable of populating a thin spherical shell around the beam bunch.
- When the beam is non-equipartitioned or the beam with the stationary distribution is rms mismatched, the thickness of the shell can be significantly larger, depending on the equipartitioning factor.
- For the relatively singular distribution with $n = -1/2$, a bunch with a normalized emittance $\epsilon_N \sim 10^{-6}$ [m rad] and a radius of 1 [cm] will populate the shell with a probability of 10^{-11} per kilometer of linac.
- For distributions with $n > 0$, this rate of population is further reduced by a factor 10^{-4}.

Our conclusion is that effect of single Coulomb collisions on halo development in high current ion linear accelerators is not important.

In Section 5 we related our analysis to diffusion caused by many small angle Coulomb collisions, with the conclusion that the effect of multiple Coulomb collisions in halo development in high current ion accelerators is also expected not to be important.

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