Proceedings of the 1999 Particle Accelerator Conference, New York, 1999

NEW LONGITUDINAL SPACE CHARGE ALGORITHM

Shane Koscielniak, TRIUMF, 4004 Wesbrook Mall, Vancouver, B.C., Canada

Abstract

We describe a new, improved longitudinal space-charge electric field calculation for particle beams with periodic modulation of the charge density. Whereas the usual method assumes long bunches with constant transverse cross-section, the new algorithm applies to long or short bunches with arbitrary binomial transverse distribution and incorporates dynamical corrections to the usual static approximation. The algorithm has been coded in Fortran and been made an option of the particle tracking program LONG1D. Based on tracking studies, it is concluded that these improvements to the physics model are most important for short bunches with high synchrotron frequency and high-order longitudinal multipole content.

1 INTRODUCTION

When simulating[6, 7] the longitudinal dynamics of a charged particle beam, it is customary to adopt a one-dimensional model that ignores the transverse coordinates and where each macro-particle is considered as a transverse slice or disc. The internal forces due to mutual Coulombic repulsion of like charges are called 'space-charge'. It is a common procedure to model longitudinal 'space charge' by forming the spatial derivative of the longitudinal charge density[4]. When this density is represented by a Fourier series, taking the derivative becomes particularly simple: each harmonic component is multiplied by its corresponding wave number (i.e. spatial frequency). This practice involves three assumptions:

- the beam bunches are long compared with the vacuum pipe cross-sectional radius
- the beam cross-section has constant charge density
- the field can be obtained (in the beam frame) from electrostatics.

1.1 Long bunches

The first assumption allows one to adopt a two-dimensional model for calculation of the transverse electric fields based on line charges (i.e. infinitely long filaments). This practice is inevitably dubious at the head and tail of the bunch. If one considers a moving point charge, with relativistic energy $\gamma m_0 c^2$, the longitudinal electric field is reduced by $1/\gamma^2$ and the field lines are 'compressed' into a transverse toroid that is coaxial with the motion. Hence it is clear that the assumption of line charges is best fulfilled by ultra-relativistic particles; but this is also the regime where longitudinal space charge is least important.

More realistic models that do not assume long bunches are available and have been used in computer simulations – but not widely so. The simplest such model is that of Morton[1] (see also Refs.[3, 5]) in which the geometric factor g_0 is made to roll off with increasing spatial frequency. A more elaborate model is that of Lebedev[2], and we shall follow a similar procedure below.

1.2 Beam cross section

The field distribution depends on the transverse charge density distribution. It is also influenced by the proximity of the vacuum pipe which is taken to be a perfectly conducting cylindrical wall concentric with the beam. Further, the longitudinal electric field varies over the beam cross section and must be ensemble-averaged (transversally) so as to obtain values that are representative. All of these effects taken together are usually rolled into a single geometric factor g_0. For example, for a uniform beam of radius a inside a pipe of radius b the on-axis geometric factor is $g_0 = 1 + 2\ln(b/a)$ and the ensemble-average geometric factor is $g_0 = 0.5 + 2\ln(b/a)$. These issues of transverse charge distribution and ensemble averaging have been addressed by Baartman[8], and will be pursued below.

1.3 Dynamics versus statics

Starting from the wave equation one may find an exact expression for the space-charge force in the frequency domain. Using this solution in a time-domain particle-tracking program leads to the following contradiction: the fields at each time step are calculated assuming the charge distribution is static and in equilibrium; however we also expect the beam distribution to be changing turn-by-turn in the synchrotron, or else there is little point performing a simulation. This contradiction is usually dismissed because "the effect is small"; the error incurred in the field estimate is of order the change in the beam distribution multiplied by the pipe radius and divided by the longitudinal distance moved in the time step. However, for machines with high synchrotron frequency and longitudinal distributions with significant high-order multipole content, changes in the distribution could be large. Moreover, at a fundamental philosophical level it is unsettling in a dynamical problem to use fields calculated for a statics problem.

2 NEW SPACE-CHARGE ALGORITHM

2.1 Field calculation

Let the electric field strength vector be $\mathbf{E}$, the charge density per unit volume be ρ. Suppose that the beam accelerates slowly. Let $s = ct$ where c is the speed of light and t

is the time coordinate. Let $\mathbf{e}$ be a unit vector and $\|$ denote the longitudinal axis. The electric field obeys the equation:

$$\left[\nabla^2 - \partial^2/\partial s^2\right]\mathbf{E} = (1/\epsilon_0)\left[\mathbf{e}_\| \beta_\| \partial/\partial s + \nabla\right]\rho\,. \quad (1)$$

Let r and z be the radial and longitudinal coordinates respectively, and $i = \sqrt{-1}$. Under the assumption that the charge distribution is longitudinally periodic and cylindrically symmetric, we may expand charge and field in Fourier series that contain a radial dependence for the coefficients:

$$\rho = \sigma(r)\sum_k \lambda^k(s) e^{ik(z-\beta s)\nu}, \quad \mathbf{E} = \sum_k \mathbf{E}^k(r,s) e^{ik(z-\beta s)\nu}\,. \quad (2)$$

Here k is the integer wave number and $\nu = h/R_s$ is the ratio of harmonic number h to the synchronous orbit radius R_s. Under the assumption that modulation frequencies are much smaller than the carrier frequencies, $\omega_k = k \times (\nu\beta_\| c)$, we may approximate the temporal field derivative as

$$\frac{\partial^2}{\partial s^2}\mathbf{E} \approx \sum_k -\left[(k\beta\nu)^2\mathbf{E}^k + 2i(k\beta\nu)\frac{\partial}{\partial s}\mathbf{E}^k\right] e^{ik(z-\beta s)\nu}\,. \quad (3)$$

We substitute expressions (2, 3) into the wave equation to obtain a relation for each Fourier component

$$\begin{aligned}\left[\nabla_\perp^2 - (k\nu)^2/\gamma^2\right] E_z^k + 2i(k\nu\beta)(E_z^k)' \\ = \frac{\sigma(r)}{\epsilon_0}\left[i(k\nu)\lambda^k/\gamma^2 + \beta_\|(\lambda^k)'\right]\,, \end{aligned} \quad (4)$$

where the superfix prime ($'$) denotes partial derivative w.r.t. s and $\nabla_\perp^2$ is the transverse part of the Laplacian operator.

Let us suppose the transverse charge distribution of radius a is given by the (unity-normalized) binomial form

$$\sigma(r) = [1-(r/a)^2]^\mu 2(\mu+1)/a^2\,. \quad (5)$$

Because there is no transverse multipole content, we expand each of the field coefficients in terms of the zeroth order Bessel function basis:

$$E_z^k(r,s) = \sum_j a_{kj}(s) J_0(\alpha_j r)\,. \quad (6)$$

To fulfil the boundary condition of a conducting wall at $r = b$ we take $J_0(\alpha_j b) = 0$ are consecutive zeros of the Bessel function. To find the time-dependent coefficients a_{kj} we make use of the orthogonality relation[11] between Bessel functions, leading to

$$\begin{aligned}(b^2/2)J_1^2(\alpha_j b)\left\{[-\alpha_j^2 - (k\nu/\gamma)^2]a_{kj} + 2i(k\nu\beta)a'_{kj}\right\} \\ = (1/\epsilon_0)[ik\nu\lambda^k + \beta_\|(\lambda^k)']B(\mu,\alpha_j a) \end{aligned} \quad (7)$$

where the function

$$B(\mu,x) \equiv (\mu+1)!2^{(\mu+1)}J_{\mu+1}(x)/x^{(\mu+1)}\,. \quad (8)$$

The longitudinal electric field is given by

$$E_z(r,z,s) = \sum_k e^{ik(z-\beta s)\nu}\sum_j a_{kj}(s)J_0(\alpha_j r)\,. \quad (9)$$

To find the effective longitudinal field for a one-dimensional particle simulation, we must ensemble average over the transverse distribution; and this leads to

$$\langle E_z\rangle(z,s) = \sum_k e^{ik(z-\beta s)\nu}\sum_j a_{kj}(s)B(\mu,\alpha_j a)\,. \quad (10)$$

If the charge density does not change then we may find an explicit expression for the complex coefficients a_{kj}; and the static field is given by $\langle E_z\rangle(z,s) =$

$$\frac{1}{\epsilon_0}\sum_k e^{ik(z-\beta s)\nu}\sum_j \frac{-ik\nu\lambda^k B^2(\mu,\alpha_j a)}{(b^2/2)J_1^2(\alpha_j b)[(\alpha_j\gamma)^2 + (k\nu)^2]}\,. \quad (11)$$

For the case $k\nu \ll \gamma\alpha_j$ and $\mu = 0$ this expression leads to field values identical with the simple theory involving $g_0 = (1/2) + 2\ln(b/a)$.

2.2 *Discretization*

If the line charge coefficients λ^k are time dependent then Eqn. (7) must be solved numerically using a suitable scheme that discretizes the time steps and replaces the derivatives by finite difference representations. In the computer program LONG1D[9] we have chosen a scheme that is consistent with the leap-frog algorithm for integrating particle motion under space charge and also does not require any more evaluations of λ^k than does a naive scheme assuming static fields. Essentially, the field coefficients are propagated from old to new values by using the replacements:

$$2a_{kj} \Rightarrow a_{kj}^{new} + a_{kj}^{old} \quad (12)$$
$$\Delta s a'_{kj} \Rightarrow a_{kj}^{new} - a_{kj}^{old} \quad (13)$$
$$2\lambda^k \Rightarrow \lambda^k_{new} + \lambda^k_{old} \quad (14)$$
$$\Delta s(\lambda^k)' \Rightarrow \lambda^k_{new} - \lambda^k_{old} \quad (15)$$

in Equation (7) with $\Delta s = c\Delta t$. In such a scheme one needs a "start-up procedure" and the simplest is to assume that $\lambda^k_{new} = \lambda^k_{old}$ before the first time step.

3 EXAMPLES AND TESTING

As an initial test of these formulae, we took a bunch in a machine with parameters similar to the original PS Booster at 50 MeV. We took line density $\lambda(\phi) = [1-(\phi/L)^2]^3$ with $\phi = \nu z$ and $L = 1$ radian and transverse parameters $\mu = 0$, $a = 6$ cm in a pipe of radius $b = 10$ cm.

3.1 *Short bunches*

Under the assumption of statics, the field distribution was calculated according to equations (7) and (10) for harmonic numbers $h = 5$ and $h = 500$ and bunch lengths 10 m and 10 cm, respectively. The result is sketched in Figure 1.

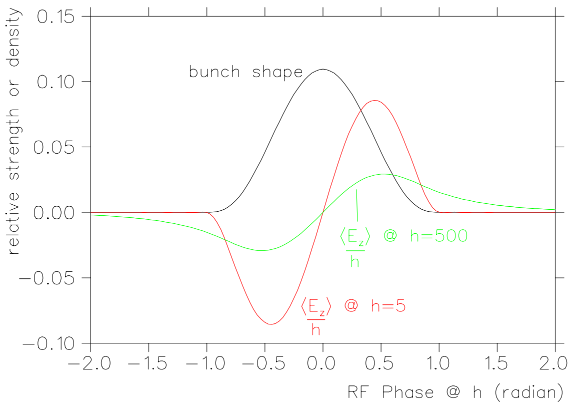

Figure 1: Comparison of $\langle E_z \rangle$ for long ($h = 5$) and short ($h = 500$) bunch lengths.

3.2 *Statics versus dynamics*

In order to see the dynamical correction to the static field approximation, the bunch was displaced 0.1 radian during a time step of one turn (so as to imitate part of a dipole oscillation) and the field calculated with and without the correction. The relative fractional error incurred by ignoring dynamical effects is presented as a mountain range plot in Figure 2. It is clear that the errors are worst at the head and tail of the bunch, and this is confirmed in the transverse ensemble-average, Figure 3, with peak errors of about 10% in narrow regions. However, these are the regions with fewest particles and so there is probably little impact on the beam dynamics of dipole oscillations.

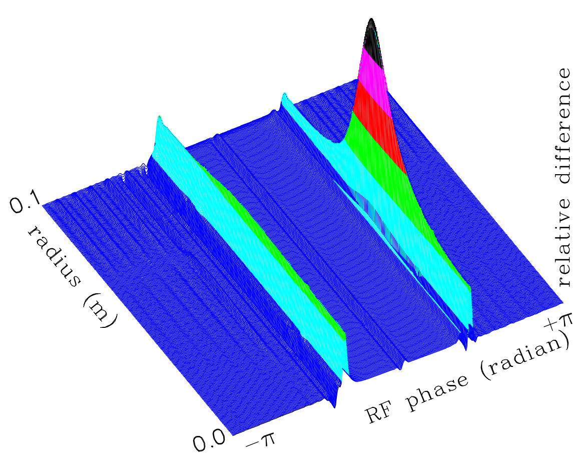

Figure 2: Comparison of relative fractional difference of $E_z(r, z)$, computed with and without dynamical corrections to the static approximation, versus cylindrical coordinates r, z. Vertical scale runs from -0.02 to $+0.24$.

3.3 *Particle tracking*

As a final test, the new space-charge algorithm was installed in the computer program LONG1D[9]. The evolution of a proton bunch in the TRIUMF KAON Accumulator ring mis-matched so as to give both dipole and quadrupole oscillations was tracked for 0.5 ms (about 4000 space-charge time steps) with and without the dynamical correction. The 450 MeV A-ring has $R_s = 34$ m and $h = 45$. Tracking of 5×10^4 macro-particles showed ensemble characteristics for the two cases to differ by only 1-to-2 parts in 10^4.

4 CONCLUSION

We have described a new, improved longitudinal space-charge electric field calculation[10] for particle beams with periodic modulation of the charge density. The algorithm has the following features:

- correct for short or long bunches
- general binomial transverse density distribution
- transverse ensemble averaging
- field is solution of electrodynamics problem with time varying longitudinal charge density.

The algorithm has been coded in Fortran and been made an option of the particle tracking program LONG1D. In early trials, with accumulator and booster-type ring parameters, the dynamical corrections seem to be of little importance for dipole and quadrupole mode oscillations.

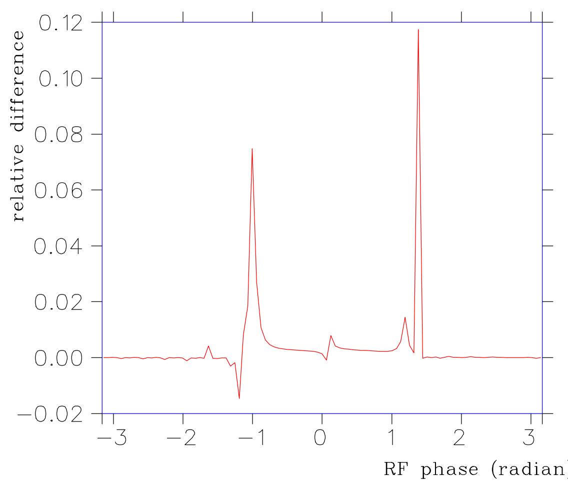

Figure 3: Comparison of relative fractional difference of $\langle E_z \rangle(z)$ computed with and without dynamical corrections.

5 REFERENCES

[1] P. Morton: Rev Sci Inst, **36**, p.429, 1965.

[2] A.N. Lebedev & E.A. Zhilkov: N.I.M. **45** pp.238-244, 1966.

[3] W. Hardt: Proc. HEACC-IX, Stanford, 1974, p.434.

[4] A, Hofmann: CERN 77-13, pp.141-145.

[5] J. Laclare: CERN 85-19, pp. 381-384.

[6] S. Koscielniak: Proc. EPAC88, Rome, 1988, pp 743-745.

[7] J.Wei et al: Particle Accelerators, **23**, 1989.

[8] R. Baartman: TRI-DN-92-K206.

[9] S. Koscielniak: TRI-DN-97-12.

[10] S. Koscielniak: TRI-DN-98-13.

[11] M. Gradshteyn & I. Rhyzik: *Tables of integrals, series and products*, Academic Press, 1964.