

Some remarks on $O(a)$ improved twisted mass QCD*

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Twisted mass QCD (tmQCD) has been introduced as a solution to the problem of unphysical fermion zero modes in lattice QCD with quarks of the Wilson type. We here argue that $O(a)$ improvement of the tmQCD action and simple quark bilinear operators can be more economical than in the standard framework. In particular, an improved and renormalized estimator of the pion decay constant in two-flavour QCD is available, given only the Sheikholeslami-Wohlert coefficient c_{sw} and an estimate of the critical mass m_c .

1. INTRODUCTION

Twisted mass QCD (tmQCD) [1,2] is a theoretically sound method to eliminate unphysical fermionic zero modes, which are at the origin of both conceptual and technical problems in lattice QCD with quarks of the Wilson type. In the continuum limit, tmQCD is equivalent to QCD with a standard mass term, provided the parameters and correlation functions are correctly matched [2]. This physical equivalence implies that the angle α , defined by the ratio of twisted to standard mass parameter,

$$\tan(\alpha) = \mu_R/m_R, \quad (1)$$

is unphysical. Furthermore, the matching of theories defined at different values of α induces a mapping between composite fields, which can be used to circumvent certain lattice renormalization problems. Examples for this are the definition of the order parameter of chiral symmetry, the pion decay constant, and some matrix elements of the effective weak hamiltonian [2,3].

Here we want to review some features of $O(a)$ improved tmQCD [4,5]. We restrict attention to the action and quark bilinear operators which appear in the PCAC and PCVC relations, i.e. the simplest Ward identities associated with chiral and flavour symmetries.

2. SET-UP

The lattice Dirac operator of two-flavour QCD with a chirally twisted mass term is given by

$$D_{\text{twist}} \stackrel{\text{def}}{=} D_W + m_0 + i\mu_q \gamma_5 \tau^3. \quad (2)$$

where D_W denotes the massless Wilson-Dirac operator and the Pauli matrix τ^3 acts in flavour space. Assuming that $O(a)$ improvement has been implemented in the massless theory [6], the $O(a)$ improved bare parameters of the action are given by

$$\tilde{g}_0^2 = g_0^2(1 + b_g am_q), \quad (3)$$

$$\tilde{m}_q = m_q + b_m am_q^2 + \tilde{b}_m a\mu_q^2, \quad (4)$$

$$\tilde{\mu}_q = \mu_q(1 + b_\mu am_q), \quad (5)$$

where $m_q = m_0 - m_c$ is the subtracted bare mass. In a mass-independent scheme the renormalized parameters then take the form

$$g_R^2 = \tilde{g}_0^2 Z_g(\tilde{g}_0^2, a\mu), \quad (6)$$

$$m_R = \tilde{m}_q Z_m(\tilde{g}_0^2, a\mu), \quad (7)$$

$$\mu_R = \tilde{\mu}_q Z_\mu(\tilde{g}_0^2, a\mu). \quad (8)$$

$O(a)$ improvement of the action thus introduces the improvement coefficients b_μ and \tilde{b}_m , in addition to the standard coefficients b_m and b_g . At this point we note that none of the coefficients proportional to m_q is needed if $m_q = O(a)$, as their contribution to physical quantities is then of $O(a^2)$ and thus negligible in the spirit of $O(a)$ improvement. Furthermore, eq. (7) then implies $m_R = O(a)$ so that the physical quark mass

*presented by S. Sint at the XIX International Symposium on Lattice Field Theory "Lattice 2001", August 19 - 24, 2001, Berlin, Germany

$(m_R^2 + \mu_R^2)^{1/2}$ is essentially defined by μ_R , and the angle α is close to $\pi/2$.

We now consider the $O(a)$ improved bare composite fields [6,4]

$$(A_I)_\mu^a = A_\mu^a + c_A a \tilde{\partial}_\mu P^a + a \mu_q \tilde{b}_A \varepsilon^{3ab} V_\mu^b, \quad (9)$$

$$(V_I)_\mu^a = V_\mu^a + c_V a \tilde{\partial}_\nu T_{\mu\nu}^a + a \mu_q \tilde{b}_V \varepsilon^{3ab} A_\mu^b, \quad (10)$$

$$(P_I)^a = P^a, \quad (11)$$

with isospin index $a = 1, 2$. Renormalized improved operators are then multiplicatively related to the improved bare ones,

$$(A_R)_\mu^a = Z_A(\tilde{g}_0^2)(1 + b_A a m_q)(A_I)_\mu^a, \quad (12)$$

$$(V_R)_\mu^a = Z_V(\tilde{g}_0^2)(1 + b_V a m_q)(V_I)_\mu^a, \quad (13)$$

$$(P_R)^a = Z_P(\tilde{g}_0^2, a\mu)(1 + b_P a m_q)(P_I)^a. \quad (14)$$

While the improvement coefficients may be considered functions of the bare coupling g_0 , consistent $O(a)$ improvement implies that the Z -factors are functions of the improved bare coupling \tilde{g}_0^2 [6]. As non-perturbative determinations of b_g are not available (cf., however, [7]), this renders chiral extrapolations of $O(a)$ improved matrix elements difficult². Compared to the standard framework at $\mu_q = 0$ we note that in $O(a)$ improved tmQCD there are only two new coefficients (\tilde{b}_V and \tilde{b}_A) required to improve the above operators. Considering again $\alpha = \pi/2$ this means that the massive theory is improved with a single new coefficient \tilde{b}_m in the action (rather than two, b_g and b_m), while the operators are improved with \tilde{b}_A and \tilde{b}_V as compared to b_V, b_A, b_P . This comparison becomes even more favourable if one takes into account a generic

3. REDUNDANCY OF IMPROVEMENT COEFFICIENTS

To illustrate this point we consider the renormalized 2-point functions

$$G_A(x-y) = \langle (A_R)_0^1(x)(P_R)^1(y) \rangle, \quad (15)$$

$$G_V(x-y) = \langle (V_R)_0^2(x)(P_R)^1(y) \rangle, \quad (16)$$

which, for the proper choice of the improvement coefficients, are expected to converge to their con-

²This problem does not occur in the quenched approximation where $b_g = 0$.

tinuum limit with $O(a^2)$ corrections. If all improvement coefficients were necessary one would expect uncanceled $O(a)$ effects to arise if any of the improvement coefficients is modified by terms of $O(1)$. Denoting such shifts to $\tilde{b}_m, b_\mu, \tilde{b}_A, \tilde{b}_V$ by $\Delta\tilde{b}_m$ etc., we find that the induced $O(a)$ effect in G_A is of the form

$$\Delta G_A(x) \propto a \mu_R \left[\Delta\tilde{b}_m \mu_R \frac{\partial}{\partial m_R} G_A(x) + \Delta b_\mu c_1 m_R \frac{\partial}{\partial \mu_R} G_A(x) + \Delta\tilde{b}_A c_2 G_V(x) \right], \quad (17)$$

with some constants c_1, c_2 which can be easily worked out [4]. Now, due to the identity in the continuum limit,

$$\left(m_R \frac{\partial}{\partial \mu_R} - \mu_R \frac{\partial}{\partial m_R} \right) G_A(x) = -G_V(x), \quad (18)$$

one concludes that $\Delta\tilde{b}_m, \Delta b_\mu$ and $\Delta\tilde{b}_A$ need not vanish separately for $G_A(x)$ to remain $O(a)$ improved. We find this to be a generic feature of tmQCD, which can be traced back to the equivalence of correlation functions of tmQCD and QCD in the continuum limit. In fact, the l.h.s. of eq.(18) is nothing else but the derivative with respect to the unphysical parameter α .

$O(a)$ improved tmQCD as introduced above may hence be regarded as a one-parameter family of $O(a)$ improved theories. Choosing \tilde{b}_m as the free parameter, we set

$$\tilde{b}_m = -\frac{1}{2}, \quad (19)$$

exactly. The other coefficients are then fixed, and in perturbation theory given by $[C_F = (N^2 - 1)/2N]$

$$b_\mu = -0.103(3) C_F g_0^2 + O(g_0^4), \quad (20)$$

$$\tilde{b}_A = 0.086(4) C_F g_0^2 + O(g_0^4), \quad (21)$$

$$\tilde{b}_V = 0.074(3) C_F g_0^2 + O(g_0^4). \quad (22)$$

Indeed, the choice (19) is partially motivated by the fact that the tree level values of these coefficients then vanish. Beyond perturbation theory, \tilde{b}_A can be determined through the PCAC relation [8], \tilde{b}_V can be obtained by imposing the physical parity symmetry in tmQCD at finite a , whereas the PCVC relation involves the combination $b_P + b_\mu$. We also note that tmQCD offers new ways to determine some of the standard

coefficients such as b_A , by matching appropriate correlation functions of tmQCD and standard QCD. This is not too surprising if one recalls that tmQCD and QCD are, in the continuum, related by a chiral symmetry transformation, and given the fact that most of the standard coefficients are determined by chiral Ward identities [9].

4. $O(a)$ IMPROVED F_π

In tmQCD the flavour symmetry is only softly broken by the twisted mass term. As a consequence, there exists a vector current \tilde{V}_μ^a , which satisfies the PCVC relation exactly,

$$\partial_\mu^* \tilde{V}_\mu^a = -2\mu_q \varepsilon^{3ab} P^b. \quad (23)$$

This vector current is protected against renormalization, which implies $Z_\mu = Z_P^{-1}$ in any scheme which respects the Ward identities. Recalling that the vector current in tmQCD at $m_R = 0$ is interpreted as the physical axial current, the pion decay constant F_π can be extracted from the asymptotic behaviour of the correlation function,

$$\mu_q \tilde{G}_P(x_0) = a^3 \sum_{\mathbf{x}} \mu_q \langle P^1(x) (P_R)^1(0) \rangle, \quad (24)$$

for large times x_0 , after division by m_π^2 and by the wave function renormalization of the interpolating field P^1 . Given the renormalization properties of both P^1 and μ_q we conclude that F_π is obtained with $O(a^2)$ errors only. First studies of F_π using this method in the quenched approximation have been presented in [10].

Generalizing this procedure to non-vanishing m_R , one considers the r.h.s. of the physical PCAC relation,

$$\sqrt{m_R^2 + \mu_R^2} \langle P_R^1(x) P_R^1(0) \rangle. \quad (25)$$

From this more general expression one infers that F_π remains $O(a)$ improved even if $m_R = O(a)$ rather than $O(a^2)$ [11].

5. CONCLUSIONS

We have argued that $O(a)$ improvement of tmQCD at $m_R = 0 \Leftrightarrow \alpha = \pi/2$ is more economical than in the standard theory. No additional free parameters arise in the action, so that

$m_R = 0$ can be satisfied up to $O(a^2)$, provided the coefficients c_{sw}, c_A are known³. Moreover, an $O(a)$ improved estimate of F_π can even be obtained with the knowledge of c_{sw} alone. This is to be confronted with the standard situation where also Z_A, b_g, c_A, b_A are required. An interesting aspect of $O(a)$ improved tmQCD is the absence of a quark mass dependent rescaling of g_0 , which allows for chiral extrapolations to be done at fixed g_0 , whilst maintaining $O(a)$ improvement.

This work is part of the ALPHA collaboration research programme. We thank Peter Weisz for an enjoyable collaboration and critical comments on a draft of this writeup. S. Sint is grateful to the Theory Division of DESY-Hamburg, where part of this work has been done.

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³The knowledge of c_A is required if one needs an $O(a)$ improved estimate of the critical mass from the PCAC relation.