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# A NEW SUM RULE TO DETERMINE $|V_{ub}|/|V_{cb}|$

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**Abstract**

We present a new sum rule which does not contain any Landau-pole effect and allows a determination of $|V_{ub}|/|V_{cb}|$ with a theoretical error of $O\,(5\%)$ .

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# 1 Introduction

Semileptonic and radiative $B$-decays,

$$B \to X_u + l + \nu \qquad \text{and} \qquad B \to X_s + \gamma, \tag{1}$$

are affected by a variety of non-perturbative effects:

1. *Beauty mass entering all decay rates.* As well known, inclusive as well as semi-inclusive rates are proportional to the fifth power of the $b$-quark — not $B$-meson — mass, because the elementary process in (1) is

$$b \to \widehat{X}_q + (\text{non QCD partons})\,. \tag{2}$$

There is a non-perturbative splitting of the initial energy $\mathcal{E} = m_B$ coming from the $B$-mass into a fraction $m_b$ partecipating to the hard process and a remaining part $m_B - m_b$ acting as spectator. This problem is usually solved by taking ratios of distributions, in which the $m_b^5$ dependence cancels out. Furthermore, the HQET allows to consistently include the non-perturbative corrections to branching fractions which are suppressed by inverse powers of $m_b$.

2. *Fermi-motion.* Dynamics become more complicated going from inclusive rates to distributions in the threshold region

$$m^2 \ll Q^2, \tag{3}$$

where $m$ is the jet mass and $Q = 2E \gg \Lambda$ is the *hard scale* [1], with $E$ the jet energy in the $B$ rest-frame and $\Lambda$ the QCD scale. There are various regimes satisfying condition (3): $i$) the region

$$\Lambda Q \ll m^2 \ll Q^2, \tag{4}$$

is characterized by the occurrence of double logarithms $\alpha_S\,(Q)^n \log^{2n}(1-z)$, where $z \equiv 1 - m^2/Q^2$ is the fundamental variable in semi-inclusive heavy flavour decays. To have a consistent result, these large logarithms need to be resummed to all orders in perturbation theory. Region (4) is then described by resummed perturbation theory. Let us observe that, in the real world, the window (4) is quite small [2]; $ii$) if one pushes the jet mass to even smaller values, in the slice

$$m^2 \sim \Lambda\, Q, \tag{5}$$

resummation of infrared logarithms is no longer sufficient[1] since new non-perturbative phenomena come into play, the well-known Fermi motion effects [3]. The exchange of soft momenta between the beauty quark and the light valence quark in the $B$-meson cannot be neglected. In sec. 3 we show that it is possible to construct ratios of distributions in which the Fermi-motion effects completely cancel.

[1] Actually, the resummed distribution becomes singular in region (5) because of Landau pole effects.

3. *Final state hadronization.* If the jet mass is pushed to even smaller values, $m^2 \sim \Lambda^2$, the exclusive or resonance region is encountered. Additional non-perturbative effects come into play, related to the hadronization of the partons in the final state.

# 2 Fermi motion effects, universality and factorization

In the rest of this talk, we concentrate on the non-perturbative phenomena occurring in region (5). In agreement with classical intuition [3], Fermi motion effects have universality properties which allow one to write general factorization formulae. Actually, in QCD, this follows from the universal properties of soft and collinear radiation. The photon spectrum in the rare decay in (1) can be written as [1]:

$$\frac{d\Gamma_{rd}}{dx} = C_{rd}\left(\alpha_S\right)\ f\left(x;\alpha_S\right) + D_{rd}\left(x;\alpha_S\right), \tag{6}$$

where $x \equiv 2E_\gamma/m_B$. The coefficient function $C_{rd}\left(\alpha_S\right)$ and the remainder function $D_{rd}\left(x;\alpha_S\right)$ are process-dependent and short-distance dominated; they can therefore be safely computed in fixed-order perturbation theory:

$$C_{rd}\left(\alpha_S\right) \ = c_{rd}^0 + \frac{\alpha_S C_F}{\pi} c_{rd} + \cdots ; \qquad D_{rd}\left(x;\alpha_S\right) = \frac{\alpha_S C_F}{\pi} d_{rd}\left(x\right) + \cdots . \tag{7}$$

$D_{rd}$ starts at order $\alpha_S$ (the explicit expressions of these functions can be found in [1]). The function $f\left(x;\alpha_S\right)$ contains all the non-perturbative effects related to Fermi motion and is universal, i.e. process-independent[2]. Condition (5) corresponds to $x \sim 1-\Lambda/m_B \sim 0.9 \div 0.95$[3]. Integrating on both sides of eq. (6) over $x$, the total rate is correctly reproduced. A measure of the photon spectrum allows a direct determination of $f$. A similar factorization formula can also be written for the triple-differential distribution in the semi-leptonic decay (1) [1]:

$$\frac{d^3\Gamma_{sl}}{dx_e dw dz} = C_{sl}\left(w,x_e;\alpha_S\right)\ f\left(z;\alpha_S\right) + D_{sl}\left(w,x_e,z;\alpha_S\right), \tag{8}$$

where $x_e \equiv 2E_e/m_B$ and $w \equiv Q/m_B$. The coefficient function now depends also on $x_e$ and $w$ : the main point however is that $C_{sl}$ does not depend on the "Fermi motion variable" $z$. Equation (8), from which any other distribution is obtained by integration, then describes the effects of Fermi motion in any semileptonic decay spectrum. An interesting quantity is the hadron energy spectrum $d\Gamma_{sl}/dE$, whose behaviour for $E = m_B/2 + O\left(\Lambda\right)$ is non-perturbative and is controlled by

[2] The function $f$ does not exactly coincide with the shape function $f^{ET}$ defined in the low-energy effective theory, because it also contains some short-distance effects.

[3] In the region $x \ll 1-\Lambda/m_B$ the function $f$ is perturbative, as this corresponds to condition (4).

the cumulative distribution $F$ (the integral of $f$). A measure of $d\Gamma_{sl}/dE$ in this region would allow an independent determination of $f$ [1]. However, the most promising distribution to determine $f$ is that in the $z$-variable. Integrating over the energies $w$ and $x_e$, one obtains:

$$\frac{d\Gamma_{sl}}{dz} = C_{sl}\left(\alpha_S\right)\, f\left(z;\alpha_S\right) + D_{sl}\left(z;\alpha_S\right) \tag{9}$$

where

$$\frac{C_{sl}\left(\alpha_S\right)}{\Gamma^0_{sl}} = 1 + \frac{\alpha_S C_F}{\pi}\left(\frac{115}{144} - \frac{\pi^2}{2}\right) \tag{10}$$

and

$$\begin{aligned}\frac{d_{sl}\left(z\right)}{\Gamma^0_{sl}} \quad = \quad & \frac{8}{35}\,\frac{80+145t+44t^2+11t^3}{\left(1+t\right)^5\left(1-t\right)}\log\frac{1+t}{1-t} + \frac{4}{1-t^2}\log\frac{1-t^2}{4} + \\ & + \frac{21+t-49t^2-21t^3}{3\left(1+t\right)^5},\end{aligned} \tag{11}$$

with $t \equiv \sqrt{4\left(1-z\right)} = m/E$ and $\Gamma^0_{sl} = G_F^2\, m_b^5 |V_{ub}|^2/(192\pi^3)$. A measure of this distribution then would provide still another direct and independent determination of $f$[4]. The idea undelying the unifying representations (6) and (8) is that kinematics is different in the two processes, but dynamics is the same.

# 3 A new sum rule to determine $|V_{ub}|/|V_{cb}|$

From the above discussion, it is clear that ratios of distributions can be constructed which do not depend on $f$ and are therefore short-distance quantities. Consider indeed:

$$R \equiv \left.\frac{d\Gamma_{rd}/dx - D_{rd}\left(x;\alpha_S\right)}{d\Gamma_{sl}/dz - D_{sl}\left(z;\alpha_S\right)}\right|_{z=x\geq 0.75} = \frac{C_{rd}\left(\alpha_S\right)}{C_{sl}\left(\alpha_S\right)}. \tag{12}$$

Non-perturbative effects completely cancel and there is consequently no Landau-pole singularity in $R$ for any value of $z$. The sum rule represented by eq. (12) is, as far as we know, new. Different sum rules for the determination of $|V_{ub}|/|V_{cb}|$ — involving the jet-mass distribution $d\Gamma_{sl}/dm$ or the electron spectrum $d\Gamma_{sl}/dx_e$ instead of $d\Gamma_{sl}/dz$ — have also been proposed in [4] and [5]. Other sum rules can be obtained by considering the hadron energy distribution. Considering for illustrative purposes the operator $O_7$ only, for which

$$\frac{C_{rd}}{\Gamma^0_{rd}} = 1 - \frac{\alpha_S C_F}{\pi}\left(\frac{\pi^2}{3} + \frac{13}{4}\right),\quad \frac{d_{rd}\left(x\right)}{\Gamma^0_{rd}} = \frac{1}{4}\left[7 + x - 2x^2 - 2\left(1+x\right)\log\left(1-x\right)\right], \tag{13}$$

[4]A measure of the distribution $d\Gamma_{sl}/dz$ is not available at present and a problem for the data analysis may be the large $b \to c l\nu$ background. It is however possible to impose the usual kinematical cut $m < m_D$ in $d\Gamma_{sl}/dz$ without changing the infrared structure [2].

with $\Gamma^0_{rd} = \alpha_{em} G_F^2 \, m_b^5 |V_{tb} V_{ts}^*|^2/(32\pi^4)$, the ratio $R$ reads[5]:

$$R \simeq \frac{6\alpha_{em}}{\pi} |C_7\,(m_b)\,|^2 \left[1 + \frac{\alpha_S C_F}{\pi}\left(\frac{\pi^2}{6} - \frac{583}{144}\right)\right] \frac{|V_{cb}|^2}{|V_{ub}|^2} \simeq 1.05 \cdot 10^{-3} \frac{|V_{cb}|^2}{|V_{ub}|^2}, \tag{14}$$

where we have taken $V_{tb} \simeq 1, V_{ts} \simeq V_{cb}$, $C_7\,(m_b) \simeq -0.31$ and $\alpha_S = \alpha_S\,(m_b) \simeq 0.21$[6]. A comparison of a more complete expression for $R$ with the data should allow an accurate determination of $|V_{ub}|/|V_{cb}|$ [2]. The "irreducible" theoretical error comes from the higher-twists: $\delta |V_{ub}|^2/|V_{cb}|^2 \sim \Lambda/m_B \sim 10\%$, so that $\delta |V_{ub}|/|V_{cb}| \sim O\,(5\%)$.

# 4 Conclusions

Fermi motion effects in semileptonic and rare $B$ decays can be factorized by means of a universal non-perturbative function $f\,(z)$. The latter can be determined by measuring the photon spectrum in the rare decay (1) in the end-point region. In the semileptonic case, any distribution can in principle be used for the experimental determination of $f$. Two distributions are particularly interesting: the hadron energy spectrum in the region $E = m_B/2 + O\,(\Lambda)$ and the $z$-distribution for $z \sim 1 - \Lambda/m_B$, as they are locally proportional[6] to $f$ in the non-perturbative region. A sum rule was presented which allows a model-independent extraction of $|V_{ub}|/|V_{cb}|$ with a theoretical error of $O\,(5\%)$.

# References


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[6] See for example: K. Chetyrkin, M. Misiak and M. Munz, Phys. Lett. B 400, 206 (1997); Erratum B 425, 414 (1998).


[5] A more complete analysis is in progress [2].
[6] By this we mean that no convolution of $f$ with some weight function is involved.