6D Beam-Beam Kick including Linear Coupling

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Abstract

The 6D beam-beam interaction as developed in 1992 by Hirata, Moshammer and Ruggiero [1, 2] has been extended to include linear coupled motion and an arbitrary crossing plane. Moreover, a synchro-beam mapping is given for solenoid fields which allows to investigate the beam-beam kick within a solenoid.

1 INTRODUCTION

The beam-beam interactions is studied in storage rings in order to incorporate the beam-beam kick in the computer programs MAD [3] and SIXTRACK [4], using the formalism developed by Hirata, Moshammer and Ruggiero (synchro-beam mapping; a Lorentz boost transforming the collision with an angle to a collision head-on). In this approach the strong bunch is split into several longitudinal slices where each slice is described by an electromagnetic potential of the form

$$U(x, y; \Sigma_{11}, \Sigma_{33}) = -\frac{r_e}{\gamma_0}\int_0^\infty \frac{\exp\left(-\frac{x^2}{2\Sigma_{11}+u} - \frac{y^2}{2\Sigma_{33}+u}\right)}{\sqrt{2\Sigma_{11}+u}\,\sqrt{2\Sigma_{33}+u}}du. \qquad (1)$$

Here r_e is the classical electron radius, γ_0 is the gamma of the test particle and Σ is a 6×6 matrix where

$$\Sigma_{ij} \equiv < X_i X_j > - < X_i >< X_j > \qquad (2)$$

where the lowercase x, y and the uppercase X, Y stand for the coordinates of the test particle and the strong bunch, respectively. In addition, a new technique of symplectic mapping in the six-dimensional phase space, called synchro-beam mapping (SBM), has been introduced by these authors in Ref. [1]. It allows to include the bunch length effect at the collision point and the energy variation caused by the electric field of the opposite bunch. This mapping is formulated only for head-on collision, but Hirata has shown that a crossing angle can be eliminated by a Lorentz-boost [2].

Eq. 1 is valid for the case of uncoupled motion. The aim of this report is to extend the formalism so as to include 6-dimensional linear coupling.

2 BEAM-BEAM KICK FOR COUPLED MOTION

2.1 The electromagnetic field due to a tilted bunch

The generalisation of the analysis in Refs. [1, 2] by including coupling and a tilted strong bunch (caused by coupling) can be achieved in a straight forward manner by describing the particle motion in the framework of the fully coupled 6-dimensional formalism and by replacing the electric potential of Eq. 1 of an untilted bunch by a new potential

$$\hat{U}(x, y; \hat{\Sigma}_{11}, \hat{\Sigma}_{33}; \Theta) \equiv U(\hat{x}, \hat{y}; \hat{\Sigma}_{11}, \hat{\Sigma}_{33}). \qquad (3)$$

The coupling has to be considered for the test particle as well as for the strong bunch.

Test particle

The potential (3) is obtained from (1) by introducing a rotated coordinate system of the test particle (for details see Appendix A of [8]):

$$\begin{aligned} \hat{x} &= \quad x\cos\Theta + y\sin\Theta; \\ \hat{y} &= -x\sin\Theta + y\cos\Theta, \end{aligned} \qquad (4)$$

where Θ denotes the twist angle of the strong bunch given by:

$$\begin{aligned} \sin 2\Theta &= -\frac{2\Sigma_{13}}{\sqrt{[\Sigma_{11}-\Sigma_{33}]^2+4\Sigma_{13}^2}}; \\ \cos 2\Theta &= \frac{\Sigma_{11}-\Sigma_{33}}{\sqrt{[\Sigma_{11}-\Sigma_{33}]^2+4\Sigma_{13}^2}}; \\ \Longrightarrow \tan 2\Theta &= -\frac{2\Sigma_{13}}{\Sigma_{11}-\Sigma_{33}} \end{aligned} \qquad (5)$$

Strong bunch

For the strong beam we have the same transformation (4) for X and Y of the coordinates $\vec{X} \equiv (X, Y, Z; P_X, P_Y, P_Z)^T$. The particle motion can be represented as a superposition of eigenmodes as shown in [7]

$$\vec{X}(s) = \sum_{k=I,II,III} \sqrt{J_k}[\vec{v}_k(s)e^{-i\phi_k} + \vec{v}_k{}^* e^{i\phi_k}], \qquad (6)$$

whereby $\vec{v}_k(s)$ ($k = I, II, III$) describe the eigenmotion with the linear 6D transfer matrix from longitudinal position s_0 to s:

$$\vec{v}_k(s) = M(s, s_0)\vec{v}_k(s_0) \qquad (7)$$

with

$$M(s_0 + L, s_0)\vec{v}_k(s_0) = e^{-i2\pi Q_k}\vec{v}_k(s_0) \qquad (8)$$

(L is the circumference of the accelerator and Q_k the betatron tune for the k^{th} mode). The rotated $\hat{\Sigma}$ can be expressed

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by the elements of the unrotated Σ-matrix:

$$
\begin{aligned}
\hat{\Sigma}_{11} &\equiv <\hat{X}^2> \\
&= \frac{1}{2}\left\{[\Sigma_{11}+\Sigma_{33}] + \sqrt{[\Sigma_{11}-\Sigma_{33}]^2 + 4\Sigma_{13}^2}\right\} \\
\hat{\Sigma}_{33} &\equiv <\hat{Y}^2> \\
&= \frac{1}{2}\left\{[\Sigma_{11}+\Sigma_{33}] - \sqrt{[\Sigma_{11}-\Sigma_{33}]^2 + 4\Sigma_{13}^2}\right\}.
\end{aligned}
\tag{9}
$$

These elements are a function of the eigenvectors:

$$
\begin{aligned}
\Sigma_{11} \equiv < X^2 >= &\sum_{k=I,II,III} 2J_k v_{k1} v_{k1}^*; \\
\Sigma_{33} \equiv < Y^2 >= &\sum_{k=I,II,III} 2J_k v_{k3} v_{k3}^*; \\
\Sigma_{13} \equiv < XY >= &\sum_{k=I,II,III} J_k [v_{k1} v_{k3}^* + v_{k1}^* v_{k3}].
\end{aligned}
\tag{10}
$$

Note that

$$
E_1 = \sqrt{\hat{\Sigma}_{11}},\, E_2 = \sqrt{\hat{\Sigma}_{33}} \tag{11}
$$

are the principal axes of the elliptical cross section

$$
\frac{\hat{X}^2}{E_1} + \frac{\hat{X}^2}{E_2} = 1 \tag{12}
$$

in the $(\hat{X} - \hat{Y})$-plane.

Conversely to Ref. [2], the crossing angle 2ϕ can be chosen in an arbitrary crossing plane, defined by an angle α (see Fig. 1). We can write the components of the strong bunch in a Cartesian coordinate system (X, Y, Z; P_x, P_y, P_z) defined for the laboratory frame:

$$
\begin{aligned}
P_x &= P\sin 2\phi \cos\alpha; \\
P_y &= P\sin 2\phi \sin\alpha; \\
P_z &= -P\cos 2\phi,
\end{aligned}
\tag{13}
$$

with P the momentum of the bunch.

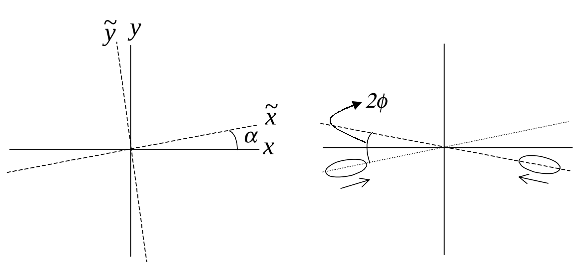

Figure 1: Coordinate system with a crossing angle 2ϕ and an arbitrary crossing plane defined by an angle α.

2.2 *Lorentz boost*

The following relations of Ref. [2] remain valid:

$$
\begin{pmatrix} ct \\ x_C \\ z_C \\ y_C \end{pmatrix} = \underline{A} \begin{pmatrix} z(s) \\ x(s) \\ s \\ y(s) \end{pmatrix} \tag{14}
$$

where

$$
\underline{A} = \underline{A}^{-1} = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{15}
$$

and

$$
\begin{pmatrix} \mathcal{H}/c - P_0 \\ p_{xC} \\ p_{zC} - P_0 \\ p_{yC} \end{pmatrix} = P_0 \underline{B} \begin{pmatrix} p_z \\ p_x \\ h \\ p_y \end{pmatrix} \tag{16}
$$

with

$$
\underline{B} = \underline{B}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{17}
$$

and P_0 being the absolute value of the three-momentum of the test particle. They describe the connection between the Cartesian coordinate (x_C, y_C, z_C; p_{xC}, p_{yC}, p_{zC}; $\mathcal{H}$, t) with $\mathcal{H} = cP$ and the accelerator coordinate $\vec{x}$ =(x, y, z; p_x, p_y, p_z; h, s) of the test particle with the Hamiltonian

$$
h(p_x, p_y, p_z) = p_z + 1 - \sqrt{(p_z+1)^2 - p_x^2 - p_y^2}. \tag{18}
$$

In this case we had to apply the ultrarelativistic approximation $v_0 \approx c$.

The Lorentz boost

$$
\underline{L}_0 = \begin{pmatrix} 1/\cos\phi & -\sin\phi & -\tan\phi\sin\phi & 0 \\ -\tan\phi & 1 & \tan\phi & 0 \\ 0 & -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{19}
$$

used in Ref. [2] makes the collision head-on for $\alpha = 0$, so that the synchro-beam mapping can be applied.

We now include the arbitrary crossing angle by the following similarity transformation:

$$
\underline{L} = \underline{R}^{-1} \underline{L}_0 \underline{R} \tag{20}
$$

with

$$
\underline{R} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & 0 & \sin\alpha \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\alpha & 0 & \cos\alpha \end{pmatrix}. \tag{21}
$$

The coordinates in the transformed frame are:

$$
\begin{pmatrix} ct^* \\ x_C^* \\ z_C^* \\ y_C^* \end{pmatrix} = \underline{L} \begin{pmatrix} ct \\ x_C \\ z_C \\ y_C \end{pmatrix}; \tag{22}
$$

$$
\begin{pmatrix} \mathcal{H}^*/c \\ p_{xC}^* \\ p_{zC}^* \\ p_{yC}^* \end{pmatrix} = \underline{L} \begin{pmatrix} \mathcal{H}/c \\ p_{xC} \\ p_{zC} \\ p_{yC} \end{pmatrix}. \tag{23}
$$

Inserting Eq. 13 into Eq. 23, we get for the transformed momentum of the strong bunch ($\mathcal{H}/c = P$): $P_x^* = 0$;

$P_y^* = 0$ and the test particle $p_x = p_y = 0$, $\mathcal{H} = cP_0$ is transformed into $p_x^* = p_y^* = 0$ and $\mathcal{H}^* = cP_0^* = cP_0 \cos\phi$, i.e. the collision is indeed head-on.

Using Eq. 20, the full Lorentz transformation is therefore a transformation from the accelerator coordinates to Cartesian coordinates, the Lorentz transformation and again a backwards transformation to the accelerator coordinates:

$$\vec{x}(0) \rightarrow \vec{x}^*(s^*) \tag{24}$$

leading to:

$$\begin{pmatrix} z^*(s^*) \\ x^*(s^*) \\ s^* \\ y^*(s^*) \end{pmatrix} = \underline{A}^{-1} \underline{L}\, \underline{A} \begin{pmatrix} z(0) \\ x(0) \\ 0 \\ y(0) \end{pmatrix} \tag{25}$$

and

$$\begin{pmatrix} p_z^*(s^*) \\ p_x^*(s^*) \\ h^* \\ p_y^*(s^*) \end{pmatrix} = \frac{P_0}{P_0^*} \underline{B}^{-1} \underline{L}\, \underline{B} \begin{pmatrix} p_z(0) \\ p_x(0) \\ h \\ p_y(0) \end{pmatrix}. \tag{26}$$

From Eq. 25 we have:

$$s^* = -x(0)\cos\alpha\sin\phi - y(0)\sin\alpha\sin\phi \tag{27}$$

so that in general $s = 0$ is not necessarily transformed to $s^* = 0$. Since we need a transformation from $\vec{x}(0)$ to $\vec{x}^*(0^*)$, an additional transformation

$$\vec{x}^*(s^*) \rightarrow \vec{x}^*(0^*) \tag{28}$$

has to be performed.

Following Ref. [2], the transformation (28) can be written as a first-order Taylor expansion:

$$\begin{aligned} w_i^*(0^*) &= w_i^*(s^*) - \frac{dw_i^*(0^*)}{ds^*} s^* \\ &= w_i^*(s^*) - h_i^* s^* \\ &= w_i^*(s^*) + h_i^* \sin\phi[x(0)\cos\alpha + y(0)\sin\alpha] \end{aligned} \tag{29}$$

with

$$w_i \equiv (x, y, z); \quad h_i^* = \frac{\partial}{\partial p_i^*} h^*(p_x^*, p_y^*, p_z^*; P_0^*). \tag{30}$$

Furthermore we obtain from (26) and the Hamiltonian (18):

$$\begin{gathered} h^*(p_x^*, p_y^*, p_z^*; P_0^*) = \\ \frac{1}{\cos^2\phi} h(p_x, p_y, p_z; P_0) = h(p_x^*, p_y^*, p_z^*; P_0^*). \end{gathered} \tag{31}$$

Combining the transformations (25, 26) and (29), we finally obtain the equations

$$\begin{aligned} x^* &= z\cos\alpha\tan\phi + x \\ &\quad + h_x^*[x\cos\alpha\sin\phi + y\sin\alpha\sin\phi] \\ y^* &= z\sin\alpha\tan\phi + y \\ &\quad + h_y^*[x\cos\alpha\sin\phi + y\sin\alpha\sin\phi] \\ z^* &= \frac{z}{\cos\phi} + h_z^*[x\cos\alpha\sin\phi + y\sin\alpha\sin\phi]; \\ p_x^* &= \frac{p_x}{\cos\phi} - h\cos\alpha\frac{\tan\phi}{\cos\phi}; \\ p_y^* &= \frac{p_y}{\cos\phi} - h\sin\alpha\frac{\tan\phi}{\cos\phi}; \\ p_z^* &= p_z - p_x\cos\alpha\tan\phi \\ &\quad - p_y\sin\alpha\tan\phi + h\tan^2\phi. \end{aligned} \tag{32}$$

The transformation $\mathcal{L}$ of Eq. 32 can be represented as the combination of a scale transformation

$$x, y, z; p_x, p_y, p_z \rightarrow \tilde{x}, \tilde{y}, \tilde{z}; \tilde{p}_x, \tilde{p}_y, \tilde{p}_z \tag{33}$$

with

$$\begin{gathered} \tilde{x} = x,\ \tilde{y} = y,\ \tilde{z} = z; \\ \tilde{p}_x = \frac{p_x}{\cos\phi},\ \tilde{p}_y = \frac{p_y}{\cos\phi},\ \tilde{p}_z = \frac{p_z}{\cos\phi} \end{gathered} \tag{34}$$

and a canonical transformation

$$\tilde{x}, \tilde{y}, \tilde{z}; \tilde{p}_x, \tilde{p}_y, \tilde{p}_z \rightarrow x^*, y^*, z^*; p_x^*, p_y^*, p_z^* \tag{35}$$

resulting from the generating function

$$\begin{gathered} F_2(\tilde{x}, \tilde{y}, \tilde{z}; p_x^*, p_y^*, p_z^*) = \tilde{x}p_x^* + \tilde{y}p_y^* + \frac{\tilde{z}}{\cos\phi}p_z^* \\ + \tilde{z}\tan\phi[p_x^*\cos\alpha + p_y^*\sin\alpha] \\ + \sin\phi[\tilde{x}\cos\alpha + \tilde{y}\sin\alpha]h^*(p_x^*, p_y^*, p_z^*). \end{gathered} \tag{36}$$

Thus $\mathcal{L}$ is only quasi symplectic; the Jacobian of this transformation is $1/\cos^3\phi$. This defect of the symplecticity is restored in the backwards transformation $\mathcal{L}^{-1}$ after having applied the beam-beam force.

2.3 *Beam-beam force*

We approximate the strong bunch by a number of slices. Each slice is represented by its $Z^*(0^*)$ coordinate, which shall be denoted by $Z^\dagger$. Taking into account only terms linear with respect to dynamical variables in $\mathcal{L}$, the first and second momenta of the particle distribution at the locations of the slices are given by:

$$
\begin{aligned}
X^\dagger &= Z^\dagger \cos\alpha \sin\phi; \\
Y^\dagger &= Z^\dagger \sin\alpha \sin\phi; \\
P_X^\dagger &= 0; \\
P_Y^\dagger &= 0; \\
P_Z^\dagger &= 0; \\
\Sigma_{11}^\dagger &= \Sigma_{11}; \\
\Sigma_{22}^\dagger &= \frac{1}{\cos^2\phi}\Sigma_{22}; \\
\Sigma_{33}^\dagger &= \Sigma_{33}; \\
\Sigma_{44}^\dagger &= \frac{1}{\cos^2\phi}\Sigma_{44}; \\
\Sigma_{12}^\dagger &= \frac{1}{\cos\phi}\Sigma_{12}; \\
\Sigma_{34}^\dagger &= \frac{1}{\cos\phi}\Sigma_{34}; \\
\Sigma_{13}^\dagger &= \Sigma_{13}.
\end{aligned} \tag{37}
$$

Putting Eq. 37 into Eqs. 5 and 9 one obtains:

$$
\begin{aligned}
\Theta^\dagger &= \Theta; \\
\hat{\Sigma}_{11}^\dagger &= \hat{\Sigma}_{11}; \\
\hat{\Sigma}_{33}^\dagger &= \hat{\Sigma}_{33}; \\
\hat{\Sigma}_{13}^\dagger &= \hat{\Sigma}_{13},
\end{aligned} \tag{38}
$$

i.e. the cross section of the strong bunch remains unchanged.

In order to calculate the beam-beam kick, we need to transform $\hat{\Sigma}_{11}^\dagger$ and $\hat{\Sigma}_{33}^\dagger$ as well as $\Theta^\dagger$ from the interaction point (IP) to the collision point (CP). The distance between the two points is given by

$$
S = S(z^*, Z^\dagger) = \frac{z^* - Z^\dagger}{2} \tag{39}
$$

Using Eqs. 5, 9 and 37 we obtain:

$$
\begin{aligned}
\hat{\Sigma}_{11}^\dagger(S) &= \frac{1}{2}\Big\{[\Sigma_{11}^\dagger(S) + \Sigma_{33}^\dagger(S)] \\
&\quad + \sqrt{[\Sigma_{11}^\dagger(S) - \Sigma_{33}^\dagger(S)]^2 + 4\Sigma_{13}^\dagger(S)^2}\Big\}; \\
\hat{\Sigma}_{33}^\dagger(S) &= \frac{1}{2}\Big\{[\Sigma_{11}^\dagger(S) + \Sigma_{33}^\dagger(S)] \\
&\quad - \sqrt{[\Sigma_{11}^\dagger(S) - \Sigma_{33}^\dagger(S)]^2 + 4\Sigma_{13}^\dagger(S)^2}\Big\};
\end{aligned} \tag{40}
$$

with

$$
\begin{aligned}
\Sigma_{11}^\dagger(S) &= \Sigma_{11}^\dagger(0) + 2\Sigma_{12}^\dagger(0)S + \Sigma_{22}^\dagger(0)S^2 \\
&= \Sigma_{11}(0) + 2\Sigma_{12}(0)\varphi + \Sigma_{22}(0)\varphi^2 \\
&\equiv \Sigma_{11}(\varphi); \\
\Sigma_{33}^\dagger(S) &= \Sigma_{33}^\dagger(0) + 2\Sigma_{34}^\dagger(0)S + \Sigma_{44}^\dagger(0)S^2 \\
&= \Sigma_{33}(0) + 2\Sigma_{34}(0)\varphi + \Sigma_{44}(0)\varphi^2 \\
&\equiv \Sigma_{33}(\varphi); \\
\Sigma_{13}^\dagger(S) &= \Sigma_{13}^\dagger(0) + [\Sigma_{14}^\dagger(0) + \Sigma_{23}^\dagger(0)]S + \Sigma_{24}^\dagger(0)S^2 \\
&= \Sigma_{13}(0) + [\Sigma_{14}(0) + \Sigma_{23}(0)]\varphi + \Sigma_{24}(0)\varphi^2 \\
&\equiv \Sigma_{13}(\varphi)
\end{aligned} \tag{41}
$$

where $\varphi = \dfrac{S}{\cos\phi}$. Thus:

$$
\begin{aligned}
\hat{\Sigma}_{11}^\dagger(S) &= \hat{\Sigma}_{11}(\varphi); \\
\hat{\Sigma}_{33}^\dagger(S) &= \hat{\Sigma}_{33}(\varphi); \\
\Theta^\dagger(S) &= \Theta(\varphi)
\end{aligned} \tag{42}
$$

with $\hat{\Sigma}_{11}$, $\hat{\Sigma}_{33}$ and Θ given by 5 and 9.

Furthermore, applying the synchro-beam mapping (SBM) [1] for the test particles we get:

$$
\begin{aligned}
\bar{x}^* &= x^* + p_x^* S - X^\dagger(Z^\dagger); \\
\bar{y}^* &= y^* + p_y^* S - Y^\dagger(Z^\dagger); \\
\bar{z}^* &= z^*
\end{aligned} \tag{46}
$$

and

$$
\begin{aligned}
\bar{p}_x^* &= p_x^*; \\
\bar{p}_y^* &= p_y^*; \\
\bar{p}_z^* &= p_z^* - \frac{(p_x^*)^2 + (p_y^*)^2}{4}.
\end{aligned} \tag{47}
$$

Here we have assumed a (virtual) drift space.

The SBM within a solenoid field can be found in Appendix B of [8].

The particle-slice interaction at the CP finally leads to:

$$
(\bar{x}^*, \bar{y}^*, \bar{z}^*) \to (\bar{x}^*, \bar{y}^*, \bar{z}^*) \tag{48}
$$

and

$$
\begin{aligned}
\bar{p}_x^* &\to \bar{p}_x^* - n^* F_x^*; \\
\bar{p}_y^* &\to \bar{p}_y^* - n^* F_y^*; \\
\bar{p}_z^* &\to \bar{p}_z^* - n^* F_z^*,
\end{aligned} \tag{49}
$$

whereby n^* is the number of particles in the slice and

$$
\begin{aligned}
F_x^* &= \frac{\partial}{\partial \bar{x}^*}\hat{U}(\bar{x}^*, \bar{y}^*; \hat{\Sigma}_{11}(\varphi), \hat{\Sigma}_{33}(\varphi); \Theta(\varphi)); \\
F_y^* &= \frac{\partial}{\partial \bar{y}^*}\hat{U}(\bar{x}^*, \bar{y}^*; \hat{\Sigma}_{11}(\varphi), \hat{\Sigma}_{33}(\varphi); \Theta(\varphi)); \\
F_z^* &= \frac{\partial}{\partial \bar{z}^*}\hat{U}(\bar{x}^*, \bar{y}^*; \hat{\Sigma}_{11}(\varphi), \hat{\Sigma}_{33}(\varphi); \Theta(\varphi)) \\
&= \frac{1}{2}\frac{\partial}{\partial S}\hat{U}(\bar{x}^*, \bar{y}^*; \hat{\Sigma}_{11}(\varphi), \hat{\Sigma}_{33}(\varphi); \Theta(\varphi))
\end{aligned} \tag{50}
$$

[1] The SBM as described in detail in Ref. [1] can be represented by a Hamiltonian

$$
\mathcal{H} = \mathcal{H}_{bb}(\vec{x}^*)\delta(s^*) \tag{43}
$$

with $\mathcal{H}_{bb}$ defined implicitly by

$$
\exp(:\mathcal{H}_{bb}:) = \prod_{Z^\dagger} \exp(:F(\vec{x}^*, Z^\dagger):) \tag{44}
$$

and with

$$
F(\vec{x}^*, Z^\dagger) = n^* U(\hat{x}^*, \hat{y}^*; \hat{\Sigma}_{11}, \hat{\Sigma}_{33}) \tag{45}
$$

describing the interaction of a test particle in the weak bunch with a slice represented by $Z^\dagger$

with $\hat{U}$ given by Eq. 3.

Introducing the variables

$$\begin{aligned} \underline{x}^* = &\quad w_1\bar{x}^* + w_2\bar{y}^*; \\ \underline{y}^* = &\quad -w_2\bar{x}^* + w_1\bar{y}^* \end{aligned} \tag{51}$$

(see Eq. 5) with

$$w_1 = \cos\Theta; \quad w_2 = \sin\Theta, \tag{52}$$

we can also write:

$$\begin{aligned} F_x^* &= \frac{\partial}{\partial\bar{x}^*}U(\underline{x}^*,\underline{y}^*;\hat{\Sigma}_{11}(\varphi),\hat{\Sigma}_{33}(\varphi)) \\ &= w_1(\varphi)\frac{\partial}{\partial\underline{x}^*}U(\underline{x}^*,\underline{y}^*;\hat{\Sigma}_{11}(\varphi),\hat{\Sigma}_{33}(\varphi)) \\ &\quad -w_2(\varphi)\frac{\partial}{\partial\underline{y}^*}U(\underline{x}^*,\underline{y}^*;\hat{\Sigma}_{11}(\varphi),\hat{\Sigma}_{33}(\varphi)); \\ F_y^* &= \frac{\partial}{\partial\bar{y}^*}U(\underline{x}^*,\underline{y}^*;\hat{\Sigma}_{11}(\varphi),\hat{\Sigma}_{33}(\varphi)) \\ &= w_2(\varphi)\frac{\partial}{\partial\underline{x}^*}U(\underline{x}^*,\underline{y}^*;\hat{\Sigma}_{11}(\varphi),\hat{\Sigma}_{33}(\varphi)) \\ &\quad +w_1(\varphi)\frac{\partial}{\partial\underline{y}^*}U(\underline{x}^*,\underline{y}^*;\hat{\Sigma}_{11}(\varphi),\hat{\Sigma}_{33}(\varphi)); \end{aligned} \tag{53}$$

$$\begin{aligned} F_z^* &= \frac{1}{2}\frac{\partial}{\partial S}U(\underline{x}^*,\underline{y}^*;\hat{\Sigma}_{11}(\varphi),\hat{\Sigma}_{33}(\varphi)) \\ &= \frac{\partial U}{\partial\underline{x}^*}\left[w_1'(\varphi)\bar{x}^* + w_2'(\varphi)\bar{y}^*\right]\frac{1}{2\cos\phi} \\ &\quad +\frac{\partial U}{\partial\underline{y}^*}\left[-w_2'(\varphi)\bar{x}^* + w_1'(\varphi)\bar{y}^*\right]\frac{1}{2\cos\phi} \\ &\quad +\frac{\partial U}{\partial\hat{\Sigma}_{11}}\hat{\Sigma}_{11}'(\varphi)\frac{1}{2\cos\phi} \\ &\quad +\frac{\partial U}{\partial\hat{\Sigma}_{33}}\hat{\Sigma}_{33}'(\varphi)\frac{1}{2\cos\phi} \end{aligned}$$

with U defined in Eq. 1; the prime denotes differentiation with respect to s. Expressions for the terms $\partial U/\partial\underline{x}^*$, $\partial U/\partial\underline{y}^*$, $\partial U/\partial\hat{\Sigma}_{11}$ and $\partial U/\partial\hat{\Sigma}_{33}$ appearing in Eq. 53 can be found in Ref. [1] (see Eqs. 21, 22, 86, 87) for a tri-gaussian distribution.

The terms $\hat{\Sigma}_{11}'(\varphi)$ and $\hat{\Sigma}_{33}'(\varphi)$ in Eq. 53 may be obtained by using Eqs. 9 and 10 and by taking into account, that the eigenvectors $\vec{v}_k(s)$ ($k = I, II, III$) obey the equations of motion. A drift space reads:

$$\begin{aligned} \frac{d}{ds}v_{k1} = &\quad v_{k2}; \\ \frac{d}{ds}v_{k3} = &\quad v_{k4}; \\ \frac{d}{ds}v_{k2} = &\quad \frac{d}{ds}v_{k4} = 0. \end{aligned} \tag{54}$$

The quantities w_1 and w_2 are determined by Eqs. 5 and 52.

Lastly, in order to calculate the derivatives $w_1'(s)$ and $w_2'(s)$ we use the relations

$$\begin{aligned} \cos 2\Theta &= \cos^2\Theta - \sin^2\Theta \\ &= 2\cos^2\Theta - 1 \\ &= 1 - 2\sin^2\Theta \end{aligned} \tag{55}$$

$$\Longrightarrow \begin{cases} w_1'(s) \equiv \frac{d}{ds}\cos\Theta &= \frac{1}{4\cos\Theta}\frac{d}{ds}\cos 2\Theta; \\ w_2'(s) \equiv \frac{d}{ds}\sin\Theta &= -\frac{1}{4\sin\Theta}\frac{d}{ds}\cos 2\Theta, \end{cases} \tag{56}$$

whereby $\cos 2\Theta$ has to be taken from Eq. 5.

3 SUMMARY

We have studied the beam-beam interaction for coupled motion in the framework of the weak-strong formalism taking into account a tilted cross section of the strong beam induced by linear coupling. This coupling has been included in the 6D beam-beam formalism of Hirata, Moshammer and Ruggiero.

The extended formalism also allows for an arbitrary crossing plane. Furthermore, a SBM-solution for solenoid fields is derived which allows to investigate the beam-beam kick within a solenoid.

The equations derived in this paper shall be incorporated into the tracking codes MAD and SIXTRACK.

4 REFERENCES

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