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Soft Resummation of Quark Anomalous Dimensions and Coefficient Functions in $\overline{\rm MS}$ Factorization

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Abstract

The asymptotic behaviour at large N of the $\overline{\rm MS}$ quark anomalous dimensions is derived to all orders assuming only $\overline{\rm MS}$ factorization and standard results for the exponentiation of soft logarithms in the quark initiated bare cross sections for deep inelastic scattering and Drell-Yan. The result is then used to write the $\overline{\rm MS}$ quark coefficient functions in a form in which all terms of $O(\ln^m N)$ are resummed.

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Quark initiated bare cross sections, whether for deep inelastic or Drell-Yan processes, generally contain logarithmic singularities as $z \to 1$, where z is the longitudinal momentum fraction of the participating partons. These logarithms result from soft and collinear gluon emission, and in Mellin space result in logarithms of the form $\ln^m N$, which can be shown to exponentiate to all orders in perturbation theory [1,2]. Much progress has been made [1]-[5] in resumming these logarithms. Once the general form of the exponent is determined, individual coefficients are fixed through matching to fixed order perturbation theory.

Before comparing to large z data it is first necessary to resum the logarithms in the *factorized* quark initiated cross sections, the quark coefficient functions. The resummation is then expected to improve the convergence of the perturbation series at large N. In order to factorize the cross section it is generally assumed (see for example ref. [7]) that in $\overline{\rm MS}$ factorization no resummation is necessary for the quark anomalous dimensions, i.e. that higher order contributions to the anomalous dimensions grow no faster than the leading order contribution at large N. Considerable support for this assumption was given in ref. [6], where the large N behaviour of the $\overline{\rm MS}$ anomalous dimensions was determined to all orders using eikonal techniques.

In this letter we will show that the all order exponentiation [1,2] and $\overline{\rm MS}$ factorization are by themselves sufficient to determine the all order large N behaviour of the quark anomalous dimensions. More specifically, we take the large N resummed form of the quark initiated bare cross sections obtained in ref. [2], then apply dimensional regularization in order to factorize the infrared singularities in $\overline{\rm MS}$ scheme (ref. [2] uses an explicit infrared cutoff). Matching the bare and factorized cross sections then allows us to simply read off the asymptotic behaviour of the quark anomalous dimensions: we find that the $\overline{MS}$ nonsinglet qq anomalous dimension γ_{qq}^{NS} behaves like $O(\ln N)$ at large N, whereas $\gamma_{q\bar{q}}^{NS}$ behaves like $O\left(1/N\right)$. Finally, we obtain a general form for the $\overline{MS}$ quark coefficient functions at large N to all orders. The results are then illustrated by explicit computation at LL and NLL.

1. We consider DIS and Drell-Yan bare quark initiated cross sections, which after factoring off all electroweak factors and decomposing into Lorentz invariants, may be written as functions of two kinematic variables, Q^2 and z. For DIS Q^2 is the virtuality of the incoming vector boson, while in DY it is the mass of the outgoing dileptons. Similarly, the longitudinal momentum fraction $z = x = Q^2/2p.q$ for DIS, while in DY $z = x_1 x_2 = Q^2/s$. In both cases we will assume all quarks are massless, and use $a_s = \alpha_s/2\pi$ as a perturbative expansion parameter, ignoring all contributions which are suppressed by powers of Q^2.

Now soft and collinear gluons radiated from the incoming quark lines can generate terms of the form $a_s^n \left[\frac{\ln^{m-1}(1-z)}{1-z}\right]_+$, $1 \le m \le 2n$, which diverge as $z \to 1$, and which thus need to be resummed if the perturbative cross section is to be improved at large z. In Mellin

space, these terms become

$$\int_0^1 dz z^{N-1} \left[\frac{\ln^{m-1}(1-z)}{1-z}\right]_+ = \int_0^1 dz \frac{z^{N-1}-1}{1-z} \ln^{m-1}(1-z) = \sum_{i=0}^{m} b_i^m \ln^i N + O\left(\frac{1}{N}\right), \quad (1)$$

so the singularities as $z \to 1$ become logarithmic singularities as $N \to \infty$. Note that in the remainder term we do not distinguish terms of $O\left(\frac{\ln^m N}{N}\right)$ from terms of $O\left(\frac{1}{N}\right)$. In z space there are also terms in the bare cross section from virtual graphs of the form $\delta(1-z)$, which in N space are constant at large N. However all the diagrams in which the initial quark (or at least one of the two initial quarks in the case of Drell-Yan) is not connected to the electroweak boson via a single quark line, i.e. the diagrams contributing to a bare pure singlet cross section, are either non singular or of the form $\ln^m(1-z)$ as $z \to 1$, and are thus of $O(1/N)$ after Mellin transformation. Here we will thus be concerned only with the nonsinglet diagrams, since it is only these which can have large N singularities of the form in eqn. (1). Furthermore, we will only be concerned with the singular parts of these diagrams (or more precisely those parts which at large N are of $O(\ln^m N)$, $m = 0, ..., \infty$), which are independent of the type or polarization of the electroweak boson (and thus for example in DIS will be identical for all three nonsinglet structure functions $F_{1,2,3}$, and in DY for dimuon or W production, after removing electroweak factors).

Consequently, here we need consider only two partonic cross sections $\sigma_q^{[a]}\left(z, Q^2/\kappa^2, a_s(Q^2)\right)$, where the only index a denotes the number of initial state quark lines: $a = 1$ for DIS, and $a = 2$ for DY. Both partonic cross sections are normalized such that $\sigma_q^{[a]} = 1$ for $a_s = 0$. Collinear singularities are regulated by a generic infrared cutoff κ. In general [1,8], the perturbation series for $\sigma_q^{[a]}$ in N space is then (given eqn. (1)) of the form

$$\sigma_q^{[a]}\left(N, \frac{Q^2}{\kappa^2}, a_s(Q^2)\right) = 1 + \sum_{n=1}^{\infty} a_s^n(Q^2) \sum_{m=0}^{2n} c_{nm}^{[a]}\left(Q^2/\kappa^2\right) \ln^m N + O\left(\frac{1}{N}\right), \quad (2)$$

for some coefficients $c_{nm}^{[a]}\left(Q^2/\kappa^2\right)$. Furthermore, the $O(\ln^m N)$ terms, for $m = 0, ..., \infty$, not only factorize: in Mellin space they also exponentiate, that is the bare cross section may be written in the form

$$\ln \sigma_q^{[a]}(N, Q^2/\kappa^2, a_s(Q^2)) = \phi_{-1}^{[a]}(a_s(Q^2) \ln N, Q^2/\kappa^2) \ln N + \sum_{n=0}^{\infty} (a_s(Q^2))^n \phi_n^{[a]}(a_s(Q^2) \ln N, Q^2/\kappa^2) + O\left(\frac{1}{N}\right). \quad (3)$$

This exponentiation provides a convenient framework for organising the expansion: the functions $\phi_{-1}^{[a]}$ contains all the leading logarithms (LL), $\phi_0^{[a]}$ the next-to-leading logarithms (NLL), since there is an extra power of a_s, and so on. It is easy to see that in eqn. (2) the LL are then those terms with $n+1 \le m \le 2n$, NLL, those with $m = n$, etc. Terms of $O\left(\frac{1}{N}\right)$ in $\sigma_q^{[a]}$ will be systematically discarded in what follows, since they are not included in its definition.

To complete the resummation, we must determine the functions $\phi_i^{[a]}$. This may be achieved either by eikonal arguments [1,2,8], or by applying the renormalization group directly to the factorization of the large N singularities [9]. The result is rather simple: the resummation may be performed in closed form essentially through a change in the argument of the running coupling from $a_s(Q^2)$ to $a_s((1-z)^a Q^2)$. Specifically it is found that (see for example ref. [8])

$$\ln \sigma_q^{[a]} = \int_0^1 dz \frac{z^{N-1}-1}{1-z}\Big(a \int_{\kappa^2}^{(1-z)Q^2} \frac{dq^2}{q^2} A(a_s(q^2)) + B^{[a]}(a_s((1-z)Q^2))\Big) + K^{[a]}(a_s(Q^2)), \tag{4}$$

where the $\overline{MS}$ renormalization scheme has been used, and soft singularities are regulated by the infrared cutoff κ. The functions $A(a_s)$, $B^{[a]}(a_s)$, $R^{[a]}(a_s)$ each have perturbative expansions beginning at $O(a_s)$. For DY processes the second term is absent, so $B^{[2]}(a_s) = 0$: in ref. [8] $B^{[1]}(a_s)$ is simply denoted by $B(a_s)$. The remainder terms $K^{[a]}(a_s)$ contain contributions not necessarily related to soft or collinear gluons. To resum LL and NLL logarithms only the first two coefficients in the expansion of A, and the first coefficient in the expansion of $B^{[1]}$ are necessary, and these may be read off by making a direct comparison to the usual fixed order LO and NLO cross sections expanded at large N.

In order to express the resummation eqn. (4) in a form more amenable to subsequent discussion, we first rewrite them in $d = 4-2\epsilon$ dimensions. We may remove the collinear regulator by taking $\kappa \to 0$, since bare cross sections are non singular in a non integer number of dimensions. This gives

$$\ln \sigma_q^{[a]} = \int_0^1 dz \frac{z^{N-1}-1}{1-z}\Big(a \int_{0}^{(1-z)Q^2} \frac{dq^2}{q^2} A(a_s(q^2,\epsilon),\epsilon) + B^{[a]}(a_s((1-z)Q^2,\epsilon),\epsilon)\Big) + K^{[a]}(a_s(Q^2),\epsilon) + O(\epsilon), \tag{5}$$

for the DIS and DY cross sections respectively. Here $a_s(\mu^2,\epsilon)$ is the $\overline{\rm MS}$ renormalised coupling in $4-2\epsilon$ dimensions: it satisfies the renormalization group equation

$$\frac{\partial a_s(\mu^2,\epsilon)}{\partial \ln \mu^2} = -\epsilon a_s(\mu^2,\epsilon) - \sum_{n=0}^{\infty} \beta_n a_s^{n+2}(\mu^2,\epsilon), \tag{6}$$

where the β-function coefficients β_n are independent of ϵ. Clearly $a_s(\mu^2,0) = a_s(\mu^2)$, while $a_s(0,\epsilon) = 0$ (by analytic continuation from $\epsilon < 0$). The functions $A(a_s,\epsilon)$, $B^{[a]}(a_s,\epsilon)$ and $K^{[a]}(a_s,\epsilon)$ all have perturbative expansions in powers of a_s, but now the coefficients in these expansions will depend on ϵ. The terms of $O(\epsilon)$ in eqn. (5) are inconsequential and can be dropped.

Now using eqn. (6) for the running of $a_s(q^2,\epsilon)$, we can do the q^2 integrals term in an expansion in powers of a_s:

$$\int_0^{Q^2} d\ln q^2 (a_s(q^2,\epsilon))^n = \int_0^{a_s(Q^2,\epsilon)} da_s \frac{a_s^n}{-\epsilon a_s - \sum_{j=0}^{\infty}\beta_j a_s^{j+2}}. \tag{7}$$

Then from eqn. (5), we find that the quark initiated bare cross sections at large N are of the form

$$\ln \sigma_q^{[a]}(N, a_s(Q^2,\epsilon),\epsilon) = \int_0^1 dz \frac{z^{N-1}-1}{1-z}\left[\sum_{i=1}^{\infty} f_i^{[a]}(\epsilon)(a_s((1-z)^a Q^2,\epsilon))^i\right] + \sum_{i=1}^{\infty} g_i^{[a]}(\epsilon)(a_s(Q^2,\epsilon))^i + O(\epsilon). \tag{8}$$

As advertised above, we can now see explicitly the essential feature of the large N resummation: the change in the argument of the running coupling from Q^2 to $(1-z)^a Q^2$.

2. In this section we will factorize eqn. (8) in the $\overline{\rm MS}$ scheme. The factorization procedure involves separating out the collinear singularities from the bare cross sections $\sigma_q^{[a]}$ into a universal singular factor Γ_q:

$$\ln \sigma_q^{[a]} = \ln C_q^{[a]} + a \ln \Gamma_q. \tag{9}$$

The $C_q^{[a]}$ are process dependent coefficient functions, non singular as $\epsilon \to 0$. Just as the complete bare partonic cross sections for nonsinglet, singlet and valence processes are all proportional to $\sigma_q^{[a]}$ up to terms of $O(1/N)$, so the complete nonsinglet, singlet and valence coefficient functions will all be proportional to $C_q^{[a]}$ up to terms of $O(1/N)$. Of course the same holds true for the singular factor Γ_q, and in particular for the anomalous dimensions from which it is constructed: in $\overline{\rm MS}$ Γ_q satisfies the renormalization group equation

$$\frac{\partial \ln \Gamma_q(N, a_s(\mu^2,\epsilon),\epsilon)}{\partial \ln \mu^2} = \gamma_q(N, a_s(\mu^2,\epsilon)), \tag{10}$$

where $\gamma_q(N,a_s) = \sum_{n=0}^{\infty}\gamma_q^n(N) a_s^n$ is the anomalous dimension, and in the $\overline{\rm MS}$ scheme the coefficients $\gamma_q^n(N)$ are independent of ϵ. This defines the $\overline{\rm MS}$ factorization scheme. This equation has the usual solution

$$\Gamma_q(N, a_s(\mu^2,\epsilon),\epsilon) = \exp\left[\int_0^{\mu^2} d\ln q^2 \gamma_q(N, a_s(q^2,\epsilon))\right], \tag{11}$$

which, when combined with eqn. (6) and expanded in powers of a_s generates all the collinear singularities of the bare cross section, in the form of inverse powers of ϵ.

We first use eqn. (9) to determine the degree of divergence of the coefficients $f_i^{[a]}(\epsilon)$ and $g_i^{[a]}(\epsilon)$ as $\epsilon \to 0$. Differentiating eqn. (9) with respect to $\ln Q^2$, and using eqn. (10), we find

$$\frac{\partial \ln \sigma_q^{[a]}}{\partial \ln Q^2} = \frac{\partial \ln C_q^{[a]}}{\partial \ln Q^2} + a\gamma_q(N, a_s(Q^2, \epsilon)). \tag{12}$$

Since both terms on the right hand side are nonsingular as $\epsilon \to 0$, it follows that $\partial \ln \sigma_q^{[a]}/\partial \ln Q^2$ is non singular. Differentiation of eqn. (8) then leads to the conclusion that

$$f_i^{[a]}(\epsilon) = \sum_{t=0}^{i} f_{i,t}^{[a]} \epsilon^{-t} + O(\epsilon), \qquad g_i^{[a]}(\epsilon) = \sum_{t=0}^{i} g_{i,t}^{[a]} \epsilon^{-t} + O(\epsilon), \tag{13}$$

as well as various relations among the $f_{i,t}^{[a]}$, and $g_{i,t}^{[a]}$.

Using eqn. (11), we can rewrite $\ln \Gamma_q$ in the form

$$\ln \Gamma_q(N, a_s(Q^2, \epsilon), \epsilon) = \int_0^1 dz z^{N-1} \int_0^{Q^2} \frac{dq^2}{q^2} P_q(z, a_s(q^2, \epsilon)). \tag{14}$$

where $P_q(z, a_s) = \sum_{n=1}^{\infty} P_q^n(z) a_s^i$ is the quark splitting function (so the $P_q^n(z)$ Mellin transform to the anomalous dimensions γ_q^n). Since the $\gamma_q^n(N)$ are independent of ϵ, so too are the $P_q^i(z)$. Substituting eqns. (14,8) into eqn. (9) we then find that

$$\begin{aligned}
\ln C_q^{[a]} = \int_0^1 dz (z^{N-1} - 1) \Bigg[& \frac{1}{1-z} \sum_{i=1}^{\infty} \sum_{t=1}^{i} f_{i,t}^{[a]} \epsilon^{-t} (a_s((1-z)^a Q^2, \epsilon))^i \\
& - a \int_0^{(1-z)^a Q^2} \frac{dq^2}{q^2} P_q(z, a_s(q^2, \epsilon)) \Bigg] \\
& + \left[\sum_{i=1}^{\infty} \sum_{t=1}^{i} g_{i,t}^{[a]} \epsilon^{-t} (a_s(Q^2, \epsilon))^i - a \int_0^1 dz \int_0^{Q^2} \frac{dq^2}{q^2} P_q(z, a_s(q^2, \epsilon)) \right] \\
& + \Bigg[\int_0^1 dz (z^{N-1} - 1) \frac{1}{1-z} \sum_{i=1}^{\infty} f_{i,0}^{[a]} (a_s((1-z)^a Q^2, \epsilon))^i + \sum_{i=1}^{\infty} g_{i,0}^{[a]} (a_s(Q^2, \epsilon))^i \\
& - a \int_0^1 dz (z^{N-1} - 1) \int_{(1-z)^a Q^2}^{Q^2} \frac{dq^2}{q^2} P_q(z, a_s(q^2, \epsilon)) \Bigg] + O\left(\frac{1}{N}\right) + O(\epsilon).
\end{aligned} \tag{15}$$

The motivation for organisation of the terms in this equation is that, as we will now show, each of the three pairs of square brackets is finite as $\epsilon \to 0$.

We first work with the first pair of square brackets. Using eqn. (6), we can perform the integration of the $a_s^n(q^2)$ in the second term as

$$\int_0^{(1-z)^a Q^2} \frac{dq^2}{q^2}(a_s(q^2,\epsilon))^n = \int_0^{a_s((1-z)^a Q^2,\epsilon)} da_s \frac{a_s^n}{-\epsilon a_s - \sum_{j=0}^{\infty}\beta_j a_s^{j+2}}$$
$$= \sum_{i=n}^{\infty} p_{i,n}(\epsilon)(a_s((1-z)^a Q^2,\epsilon))^i. \tag{16}$$

It follows that

$$\int_0^{(1-z)^a Q^2} \frac{dq^2}{q^2} P_q(z, a_s(q^2,\epsilon)) = \sum_{n=1}^{\infty} \int_0^{(1-z)^a Q^2} \frac{dq^2}{q^2}(a_s(q^2,\epsilon))^n P_q^n(z)$$
$$= \sum_{n=1}^{\infty}\sum_{i=n}^{\infty} P_q^n(z) p_{i,n}(\epsilon)(a_s((1-z)^a Q^2,\epsilon))^i$$
$$= \sum_{i=1}^{\infty}\sum_{n=1}^{i} P_q^n(z) p_{i,n}(\epsilon)(a_s((1-z)^a Q^2,\epsilon))^i. \tag{17}$$

We now expand eqn. (17) in ϵ. In eqn. (16), the $p_{i,n}(\epsilon)$ may be expanded as

$$p_{i,n}(\epsilon) = \sum_{s=1}^{i-n+1} p_{i,n}^s \epsilon^{-s}. \tag{18}$$

Note that $p_{i,n}^s = 0$ if $s = 1$ and $i \neq n$, but we do not explicitly show this in order to simplify the notation. Substitution in eqn. (17) then gives finally

$$\int_0^{(1-z)^a Q^2} \frac{dq^2}{q^2} P_q(z, a_s(q^2,\epsilon)) = \sum_{i=1}^{\infty}\sum_{n=1}^{i}\sum_{s=1}^{i-n+1} P_q^n(z) p_{i,n}^s \epsilon^{-s}(a_s((1-z)^a Q^2,\epsilon))^i$$
$$= \sum_{i=1}^{\infty}\sum_{t=1}^{i}\sum_{n=1}^{i-t+1} P_q^n(z) p_{i,n}^t \epsilon^{-t}(a_s((1-z)^a Q^2,\epsilon))^i. \tag{19}$$

The second term in the second pair of square brackets in eqn. (15) may be expanded similarly, using eqn. (19) with $a = 0$:

$$\int_0^{Q^2} \frac{dq^2}{q^2} P_q(z, a_s(q^2,\epsilon)) = \sum_{i=1}^{\infty}\sum_{t=1}^{i}\sum_{n=1}^{i-t+1} P_q^n(z) p_{i,n}^t \epsilon^{-t}(a_s(Q^2,\epsilon))^i. \tag{20}$$

Finally, the last term in the third pair of brackets in eqn. (15) is given by the difference

of eqns. (19) and (20), and is thus finite as $\epsilon \to 0$. Indeed, we can write it as

$$\int_{(1-z)^a Q^2}^{Q^2} \frac{dq^2}{q^2} P_q(z, a_s(q^2, \epsilon)) = \int_{(1-z)^a Q^2}^{Q^2} \frac{dq^2}{q^2} \sum_{n=1}^{\infty} (a_s(q^2))^n P_q^n(z) + O(\epsilon). \tag{21}$$

Putting all this together, substituting eqns. (19, 20,21) into eqn. (15) we find

$$\begin{aligned}
\ln C_q^{[a]} = \int_0^1 dz (z^{N-1} - 1) \Bigg[& \frac{1}{1-z} \sum_{i=1}^{\infty} \sum_{t=1}^{i} f_{i,t}^{[a]} \epsilon^{-t} (a_s((1-z)^a Q^2, \epsilon))^i \\
& - a \sum_{i=1}^{\infty} \sum_{t=1}^{i} \sum_{n=1}^{i-t+1} P_q^n(z) p_{i,n}^t \epsilon^{-t} (a_s((1-z)^a Q^2, \epsilon))^i \Bigg] \\
& + \left[\sum_{i=1}^{\infty} \sum_{t=1}^{i} g_{i,t}^{[a]} \epsilon^{-t} (a_s(Q^2, \epsilon))^i - a \int_0^1 dz \sum_{i=1}^{\infty} \sum_{t=1}^{i} \sum_{n=1}^{i-t+1} P_q^n(z) p_{i,n}^t \epsilon^{-t} (a_s(Q^2, \epsilon))^i \right] \\
& + \Bigg[\int_0^1 dz (z^{N-1} - 1) \frac{1}{1-z} \sum_{i=1}^{\infty} f_{i,0}^{[a]} (a_s((1-z)^a Q^2, \epsilon))^i + \sum_{i=1}^{\infty} g_{i,0}^{[a]} (a_s(Q^2, \epsilon))^i \\
& - a \int_0^1 dz (z^{N-1} - 1) \int_{(1-z)^a Q^2}^{Q^2} \frac{dq^2}{q^2} \sum_{n=1}^{\infty} (a_s(q^2))^n P_q^n(z) \Bigg] + O(\epsilon).
\end{aligned} \tag{22}$$

Now, consider only the coefficients of ϵ^{-t} for $t = 1, ..., \infty$ in eqn. (22), which must vanish so that $C_q^{[a]}$ is finite in the limit $\epsilon \to 0$. In the first pair of square brackets, these coefficients are all functions of N, in the second pair of square brackets they are all independent of N, while all terms in the third pair of square brackets are nonsingular. Thus each of the three pairs of square brackets is separately nonsingular as $\epsilon \to 0$. Moreover the cancellation of singularities in the first pair of square brackets implies that, for $i \geq t \geq 1$ and $z \neq 1$,

$$\frac{1}{1-z} f_{i,t}^{[a]} = a \sum_{n=1}^{i-t+1} P_q^n(z) p_{i,n}^t, \tag{23}$$

We can treat $p_{i,n}^t$ in eqn. (23) as a set of matrices with indices i and n, but vanishing for $n > i - t + 1$; each matrix is then triangular and thus invertible. Then multiplication of both sides of eqn. (23) by the inverse of this matrix gives an expression with just P_q^n on the right hand side, with the left hand side proportional to $(1/(1-z))$. Thus we find that for $z \neq 1$

$$P_q(z, a_s) = Q(a_s) \frac{1}{1-z}. \tag{24}$$

where $Q(a_s) = \sum_{i=1}^{\infty} a_s^i Q_i$, and the Q_i may be determined by substitution into eqn. (23). Note that for consistency we also require that $f_{i,t}^{[a]}$ is proportional to a for $t > 0$: this is the

essence of universal factorization. Similarly the cancellation of singularities in the second bracket implies that

$$g_{i,t}^{[a]} = a \sum_{n=1}^{i-t+1} \int_0^1 dz P_q^n(z) p_{i,n}^t, \tag{25}$$

and thus that $\int_0^1 dz P_q^n(z)$ is finite. It follows that for all z, we may write

$$P_q(z, a_s) = Q(a_s) \left[\frac{1}{1-z}\right]_+ + R(a_s)\delta(1-z), \tag{26}$$

where again $R(a_s) = \sum_{i=1}^{\infty} a_s^i R_i$, and the R_i may be determined by substituting eqn. (26) into eqn. (25) and inverting.

Thus we find that, in Mellin space, as $N \to \infty$, the anomalous dimension

$$\gamma_q(N, a_s) = -Q(a_s)(\ln N + \gamma_E) + R(a_s) + O(1/N), \tag{27}$$

where γ_E is Euler's constant. Remembering that all the nonsinglet, singlet and valence anomalous dimensions are equal to γ_q up to terms which vanish as $1/N$ in the large N limit, then since in the usual decomposition

$$\gamma_{q_i q_j} = \delta_{ij}\gamma_{qq}^{NS} + \gamma_{qq}^{PS}, \qquad \gamma_{q_i \bar{q}_j} = \delta_{ij}\gamma_{q\bar{q}}^{NS} + \gamma_{q\bar{q}}^{PS}, \qquad \gamma_{\pm}^{NS} = \gamma_{qq}^{NS} \pm \gamma_{q\bar{q}}^{NS}, \tag{28}$$

we must have

$$\begin{aligned} &\gamma_{qq}^{NS}(N) = \gamma_q(N) + O(1/N), && \gamma_{q\bar{q}}^{NS}(N) = O(1/N), \\ &\gamma_{qq}^{PS}(N) = O(1/N), && \gamma_{q\bar{q}}^{PS}(N) = O(1/N), \end{aligned} \tag{29}$$

consistent with our remarks earlier that diagrams in which the quark evolves into a gluon are suppressed by $1/N$.

It remains to take the limit $\epsilon \to 0$ in eqn. (22) to give an explicit expression for the resummed large N $\overline{\rm MS}$ coefficient function: we find

$$\begin{aligned} \ln C_q^{[a]} = \int_0^1 dz \frac{z^{N-1}-1}{1-z} \Big[a \int_{Q^2}^{(1-z)^a Q^2} \frac{dq^2}{q^2} \sum_{n=1}^{\infty} Q_n (a_s(q^2))^n + \sum_{i=1}^{\infty} f_{i,0}^{[a]}(a_s((1-z)^a Q^2))^i \Big] \\ + \sum_{i=1}^{\infty} g_{i,0}^{[a]} (a_s(Q^2))^i + O\left(\frac{1}{N}\right). \end{aligned} \tag{30}$$

We immediately recognise this as eqn. (4), with

$$A(a_s) = \sum_{n=1}^{\infty} Q_i a_s^i, \qquad B^{[a]}(a_s) = \sum_{i=1}^{\infty} f_{i,0}^{[a]} a_s^i, \qquad K^{[a]}(a_s) = \sum_{i=1}^{\infty} g_{i,0}^{[a]} a_s^i, \tag{31}$$

and with the infrared regulator set equal to the factorization scale, i.e. $\kappa^2 = Q^2$.

To calculate the coefficients $f_{i,0}^{[a]}$ and the $g_{i,0}^{[a]}$, we simply expand eqn. (30) in $a_s(Q^2)$, i.e. undo the resummation, and compare coefficients of $a_s^i(Q^2)$ with those in the fixed order coefficient functions. The universal coefficients Q_i of the $O(\ln N)$ terms in the anomalous dimension serve as a consistency check. This procedure will be illustrated explicitly for all LL and NLL terms in the next section.

3. We now show explicitly that the factorized coefficient functions eqn. (30) correctly resum all LL and NLL terms, by comparing then to the fixed order NLO and NNLO coefficient functions and thereby deduce the leading behaviour at large N of the LO and NLO quark anomalous dimensions.

To NLO, eqn. (30) reads

$$\begin{aligned}\ln C_q^{[a]}(N, a_s(Q^2)) = \int_0^1 dz \frac{z^{N-1}-1}{1-z}\Big[&- \frac{aQ_1}{\beta_0} \ln\left(\frac{a_s((1-z)^a Q^2)}{a_s(Q^2)}\right) \\ &+ a\left(\frac{Q_1\beta_1}{\beta_0^2} - \frac{Q_2}{\beta_0}\right)\left(a_s((1-z)^a Q^2) - a_s(Q^2)\right) + f_{1,0}^{[a]} a_s((1-z)^a Q^2)\Big] \\ &+ O(a_s(a_s \ln N)^m).\end{aligned} \tag{32}$$

Performing the z integral, we find

$$\begin{aligned}\ln C_q^{[a]}(N, a_s) = &\frac{Q_1}{\beta_0}\frac{1}{a_s\beta_0}\left[(1-a\lambda_s)\ln(1-a\lambda_s) + a\lambda_s\right] \\ &+ \left(\frac{f_{1,0}^{[a]}}{a\beta_0} - \frac{aQ_1\gamma_E}{\beta_0} + \frac{Q_1\beta_1}{\beta_0^3} - \frac{Q_2}{\beta_0^2}\right)\ln(1-a\lambda_s) \\ &+ \frac{Q_1\beta_1}{2\beta_0^3}\ln^2(1-a\lambda_s) - \left(\frac{Q_2}{\beta_0^2} - \frac{Q_1\beta_1}{\beta_0^3}\right)a\lambda_s + O(a_s(a_s\ln N)^m),\end{aligned} \tag{33}$$

where $\lambda_s \equiv a_s\beta_0 \ln N$. This result should be compared to the general expression eqn. (3): the first line gives the LL terms ϕ_{-1}, while the rest gives the NLL ϕ_0.

To determine the large N behaviour of the LO and NLO anomalous dimensions, i.e. the coefficients Q_1 and Q_2, and also the nonuniversal coefficients $f_{1,0}^{[a]}$, it is sufficient to expand eqn. (33) in $a_s(Q^2)$ to NNLO and compare to NL0 and NNLO coefficient functions at large N. The result, after exponentiating, is

$$C_q^{[a]}\left(N,a_s\right) = 1 + a_s\left(\frac{1}{2}a^2Q_1\ln^2 N + \left(a^2Q_1\gamma_E - f_{1,0}^{[a]}\right)\ln N\right)$$
$$+a_s^2\Big(\frac{1}{8}a^4Q_1^2\ln^4 N + \left[\frac{1}{6}a^3Q_1\beta_0 + +a^2Q_1\left(a^2Q_1\gamma_E - f_{1,0}^{[a]}\right)\right]\ln^3 N$$
$$+\Big[\frac{1}{2}a^3\beta_0Q_1\gamma_E - \frac{1}{2}a\beta_0 f_{1,0}^{[a]} + \frac{1}{2}a^2Q_2 + \frac{1}{2}\left(a^2Q_1\gamma_E - f_{1,0}^{[a]}\right)^2\Big]\ln^2 N\Big) + O(a_s^3). \quad (34)$$

Now, for large N, to NLO [10] and NNLO [11] the DIS quark coefficient function

$$C_q^{[1]}\left(N,a_s\right) = 1 + a_s\Big[C_F\ln^2 N + C_F\left(2\gamma_E + \frac{3}{2}\right)\ln N + \gamma_E^2 + \frac{3}{2}\gamma_E - \zeta(2) + O\left(\frac{1}{N}\right)\Big]$$
$$+a_s^2\Big[\left(2C_F^2\gamma_E - \frac{2}{9}C_FT_Rn_f + \frac{3}{2}C_F^2 + \frac{11}{18}C_FC_A\right)\ln^3 N$$
$$+\Big(\frac{11}{6}C_FC_A\gamma_E + \frac{9}{2}C_F^2\gamma_E - C_FC_A\zeta(2) - \frac{27}{8}C_F^2 - C_F^2\zeta(2) + \frac{367}{72}C_FC_A$$
$$+3C_F^2\gamma_E^2 - \frac{2}{3}C_FT_Rn_f\gamma_E - \frac{29}{18}C_FT_Rn_f\Big)\ln^2 N + O(\ln N)\Big]. \quad (35)$$

Comparing this with eqn. (34) with $a = 1$, we find

$$Q_1 = 2C_F, \qquad Q_2 = C_FC_A\left(\frac{67}{9} - 2\zeta(2)\right) - \frac{20}{9}C_FT_Rn_f, \quad (36)$$

consistent with the large N behaviour of the LO [12] and NLO [13] anomalous dimensions. Furthermore the coefficient

$$f_{1,0}^{[1]} = -\frac{3}{2}C_F, \quad (37)$$

as in [2,3]. For Drell-Yan at large N, to NLO [14] and NNLO [15]

$$C_q^{[2]}\left(N,a_s\right) = 1 + a_s\Big[4C_F\ln^2 N + 8C_F\gamma_E\ln N + 4C_F\gamma_E^2 + 8C_F\zeta(2) - 8C_F + O(1)\Big]$$
$$+a_s^2\Big[8C_F^2\ln^4 N + \left(\frac{44}{9}C_FC_A - \frac{16}{9}C_FT_Rn_f + 32C_F^2\gamma_E\right)\ln^3 N$$
$$+\Big(-\frac{40}{9}C_FT_Rn_f - \frac{16}{3}C_FT_Rn_f\gamma_E + \frac{44}{3}C_FC_A\gamma_E + \frac{134}{9}C_FC_A - 4C_FC_A\zeta(2)$$
$$+32C_F^2\zeta(2) + 48C_F^2\gamma_E^2 - 32C_F^2\Big)\ln^2 N + O(\ln N)\Big], \quad (38)$$

which is again consistent with eqn. (34), this time with $a = 2$, provided eqns. (36) for the large N anomalous dimensions are satisfied, and $f_{1,0}^{[2]} = 0$. With this result and eqn. (37), eqn. (33) agrees with the results of [7] at NLL. Moreover, our general proof of eqn.(26) now places the NNLL results of [7] on a firmer footing.

4. We have used the dimensionally regularized form of the large N resummed bare quark initiated cross sections for DIS and Drell-Yan, eqn. (8), together with the fact that the bare quark initiated cross sections are independent of the species of the electroweak boson,

and the definition of $\overline{MS}$ factorization, to obtain to all orders the $O(\ln N)$ behaviour of the nonsinglet $\overline{\rm MS}$ anomalous dimensions P_{qq}^{NS} and the $O\left(\frac{1}{N}\right)$ behaviour of $P_{q\bar{q}}^{NS}$ and the pure singlet. This has interesting implications for the large x evolution of $\overline{\rm MS}$ quark parton distribution functions: at large N the evolution factor

$$\Gamma(Q^2/\mu^2) = N^{-\int_{\mu^2}^{Q^2}\frac{dq^2}{q^2}Q(a_s(q^2))}(1+O(1/N)). \tag{39}$$

so that if $q(x,\mu^2) \sim (1-x)^{b(\mu^2)}$ as $x \to 1$, this behaviour persists at higher scales with

$$b(Q^2) = b(\mu^2) + \int_{\mu^2}^{Q^2}\frac{dq^2}{q^2}Q(a_s(q^2)) \tag{40}$$

order by order in perturbation theory.

We then used the result (26) to construct a general large N resummed expression for the $\overline{MS}$ quark coefficient functions for DIS and DY, eqn. (30), and in particular showed that all the large N singularities in the DY case can be deduced from the $O(\ln N)$ terms in the quark anomalous dimension. We verified these results explicitly at LL and NLL. For the future, the large N behaviour of NNLO anomalous dimensions and NNNLO DIS and DY coefficient functions will provide a useful consistency check on new calculations.

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