Determination of the pion–nucleon coupling constant and scattering lengths

T.E.O. Ericson*
The Svedberg Laboratory, S-75121 Uppsala and CERN, CH-1211 Geneva 23, Switzerland

B. Loiseau†
Laboratoire de Physique Nucléaire et de Hautes Énergies and LPTPE, Université P. & M. Curie, F-75252 Paris Cedex 05, France

A. W. Thomas‡
Department of Physics and Mathematical Physics and Special Research Centre for the Subatomic Structure of Matter, University of Adelaide, Adelaide 5005, Australia

We critically evaluate the isovector GMO sum rule for forward πN scattering using the recent precision measurements of π^-p and π^-d scattering lengths from pionic atoms. We deduce the charged-pion–nucleon coupling constant, with careful attention to systematic and statistical uncertainties. This determination gives, directly from data, $g_c^2(\mathrm{GMO})/4\pi = 14.17 \pm 0.05$ (statistical)±0.19 (systematic) or $f_c^2/4\pi = 0.0786(11)$. This value is intermediate between that of indirect methods and the direct determination from backward np differential scattering cross sections. We also use the pionic atom data to deduce the coherent symmetric and antisymmetric sums of the pion–proton and pion–neutron scattering lengths with high precision, namely $(a_{\pi^-\mathrm{p}} + a_{\pi^-\mathrm{n}})/2 = (-17 \pm 2(\mathrm{statistical}) \pm 8$ (systematic))$\times 10^{-4}\ m_\pi^{-1}$ and $(a_{\pi^-\mathrm{p}} - a_{\pi^-\mathrm{n}})/2 = (900 \pm 3(\mathrm{statistical}) \pm 13$ (systematic))$\times 10^{-4}\ m_\pi^{-1}$.

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I. INTRODUCTION

The pion–nucleon (πNN) coupling constant is of fundamental importance in both nuclear and particle physics. In nuclei it sets the scale of the interaction, together with the pion mass. In particle physics it is of great importance for the Goldberger–Treiman relation [1], one of the most important tests of chiral symmetry. Its experimental error is the main obstacle in the accurate discussion of the corrections to this relation as predicted from chiral symmetry breaking (see, for example, the discussion on page 1086 of Ref. [2]). An accurate test requires a knowledge of the πNN coupling constant to a precision of about 1%, so as to match the experimental precision of the other quantities in the Goldberger–Treiman relation.

The present situation is summarized in Table I with uncertainties as quoted by the authors. In the 1980's, the πNN coupling constant was believed to be well known. In particular, Koch and Pietarinen [3] determined a value of the charged-pion coupling constant, $g_c^2/4\pi = 14.28(18)$, from $\pi^\pm$p scattering data, while Kroll [4] found the neutral-pion coupling constant $g_0^2/4\pi = 14.52(40)$ from a pp forward dispersion relation. This was put in question in the early 1990's, when the Nijmegen group published a series of papers [5,7,8,6] where they reported smaller values on the basis of energy-dependent partial-wave analyses (PWA) of NN scattering data. They obtained $g_0^2/4\pi = 13.47(11)$ and $g_c^2/4\pi = 13.58(5)$. Similarly low values with $g^2/4\pi$ about 13.7 have also been found by the Virginia Tech group [9–12] from an analysis of both $\pi^\pm$N and NN data. Using a similar PWA method in the πp sector, Timmermans [13] found a value of 13.45(14). These more recent analyses often suffer from the drawback that they rely on the joint analysis of large data bases from many experiments with some of the data rejected according to various criteria. The statistical accuracy is high, but the systematic uncertainty is not clear. Exceptions are the Goldberger–Miyazawa–Oehme (GMO) sum rule [14] used by several groups [10,15,16] and the forward scattering sum rule for pp scattering [4], which, in principle, depend directly on physical observables. However, these last determinations have not been subjected to a comprehensive examination of all their ingredients; in the case of Ref. [15] we have corrected their result as given in

*e-mail: torleif.ericson@cern.ch. Also visitor at CSSM, University of Adelaide, Adelaide 5005, Australia
†e-mail: loiseau@in2p3.fr
‡e-mail: athomas@physics.adelaide.edu.au

Table I to account for an erroneous input value according to the Erratum of Ref. [11]. Another direct determination is based on the extrapolation of experimental precision data on single-energy backward differential np cross sections to the pion pole [2,17]. This allows a systematic discussion of statistical and systematic uncertainties, but the uncertainty is so far larger than what can be achieved at present with the use of the GMO sum rule. The extrapolation method gives 14.52(26), a value significantly larger than those deduced by indirect methods. A review of the situation of the πNN coupling constant up to 1997 is found in Ref. [6]. The problems regarding its determination have recently been discussed in a dedicated workshop [18–22] as well as in a recent conference working group [23]. To resolve these

TABLE I. Some deduced values for the πNN coupling constant. The quoted uncertainty are those quoted by the authors and usually do not include systematic uncertainties.

Source	Year	System	$g^2_{\pi NN}/4\pi$
Karlsruhe-Helsinki [3]	1980	πp	14.28(18)
Kroll [4]	1981	pp	14.52(40)
Nijmegen [5]	1993	pp, np	13.58 (5)
VPI [9]	1994	pp, np	13.70
Nijmegen [6]	1997	pp, np	13.54 (5)
Timmermans [13]	1997	π^+p	13.45(14)
VPI [10]	1994	GMO, πp	13.75(15)
Uppsala [2]	1998	np→pn	14.52(26)
Pavan *et al.* [12]	1999	πp	13.73 (9)
Schröder *et al.*, corrected [15,11]	1999	GMO, $\pi^\pm$p	13.77(18)
Present work	1999	GMO, $\pi^\pm$p	14.17(20)

discrepancies it is desirable to have an independent precision determination, directly linked to primary observables and with quantifiable systematic and statistical errors. The purpose of the present article is to demonstrate that recent experimental advances make the GMO relation suitable for this purpose. The GMO is a forward dispersion relation that expresses the charged coupling constant $g_c^2/4\pi$ in terms of the isovector πN scattering length (70% contribution) and a weighted integral, J^-, of the difference between the charged-pion total cross sections (30% contribution). This relation has been repeatedly evaluated in the past [10,11,15,16,24,25]. Since, until recently, there was little information on the scattering lengths available from direct data, these evaluations necessarily relied on scattering lengths extrapolated from semi-phenomenological πN phase-shift analyses, using data from a range of energies above threshold. At the high precision needed, the systematic errors in the extrapolated scattering lengths are unclear and have, to our knowledge, not been estimated. The experimental situation has changed recently. The π^-p and π^-d scattering lengths can, to high precision, be deduced from recent experiments on pionic atoms. As a result, all the major ingredients in the GMO relation can now be discussed as experimentally derived quantities with transparent sources of uncertainty. Further, the approach can be improved by the observation that isospin conservation, which was previously assumed, can be replaced by the weaker assumption of charge symmetry. This avoids the possibility of perturbations from the rather important violation of isospin symmetry expected to be associated with the π^0p and π^0n scattering lengths [26,27]. The GMO relation can now be completely evaluated on the basis of data closely linked to direct experiments and it then determines the charged-pion coupling. It is no longer necessary to rely on the extrapolation of πN scattering data above threshold. We will develop this aspect below and also give a discussion of uncertainties in the dispersion integral.

The paper is organized as follows. In Section II we give a brief review of the GMO sum rule, reorganize it in the most efficient way for the present purpose, and discuss the magnitudes of the main contributions. Section III presents the information on the π^-p and π^-d scattering lengths deduced from data on pionic atoms. We draw the reader's attention to the most critical theoretical point in the present procedure for their extraction. In section IV we critically analyze and improve the theoretical approach to the πd scattering length. In so doing, we pay particular attention to a number of smaller terms and considerably improve the state of the art. We use this understanding to deduce the most accurate values yet for the πN scattering lengths from the experimental data. Practical expressions for the theoretical π^-d scattering length for separable scattering amplitudes are given in Appendix A. In section V we analyze the uncertainties from different sources in the cross section integral J^-. In section VI we summarize the conclusions about the scattering lengths and give the GMO sum rule result for the πNN coupling constant, $g_c^2/4\pi$, with an explicit indication of systematic and statistical uncertainties.

II. THE GMO SUM RULE

The GMO sum rule for charged-pion–nucleon scattering is a very general forward dispersion relation, which assumes only analyticity and crossing symmetry. Contrary to the usual approach to its evaluation [10,11,15,16,24,25], it is not necessary to assume isospin symmetry (for a discussion of the GMO relation assuming isospin symmetry see Eq. (A.6.49) in Ref. [24]). It takes the following form:

$$f_c^2/4\pi = (1-(m_\pi/2M)^2)\left[(1+m_\pi/M)\frac{m_\pi}{4}(a_{\pi^-\mathrm{p}}-a_{\pi^+\mathrm{p}}) - \frac{m_\pi^2}{8\pi^2}\int_0^\infty \frac{\sigma^T_{\pi^-\mathrm{p}}(k')-\sigma^T_{\pi^+\mathrm{p}}(k')}{\sqrt{k'^2+m_\pi^2}}\,dk'\right]. \tag{1}$$

Here m_π is the charged-pion mass and M the proton mass with the neutron–proton mass difference neglected, $a_{\pi^\pm\mathrm{p}}$ the $\pi^\pm$p scattering lengths, $\sigma^T_{\pi^\pm\mathrm{p}}$ the total $\pi^\pm$ proton cross section and k the pion laboratory momentum. The relation gives the charged-pion coupling constant $f_c^2/4\pi = (m_\pi/2M)^2 g_c^2/4\pi$ explicitly in terms of the charged-pion scattering lengths and total cross sections, all directly measurable. In writing Eq. (1) it has been tacitly assumed that Coulomb barrier corrections have been made to sufficient precision both in the extraction of the scattering lengths from pionic atoms and, in particular, in the determination of the total cross sections. We will discuss these issues as well as the effect of mass differences and isospin violation further below.

It is convenient to write the expression (1) in a simplified form with numerical coefficients:

$$g_c^2/4\pi = -4.50\times J^- + 103.3\times\left(\frac{a_{\pi^-\mathrm{p}}-a_{\pi^+\mathrm{p}}}{2}\right). \tag{2}$$

Throughout this paper the scattering lengths are in units of m_π^{-1}, and J^-, given in mb, corresponds to

$$J^- = \frac{1}{4\pi^2}\int_0^\infty \frac{\sigma^T_{\pi^-\mathrm{p}}(k')-\sigma^T_{\pi^+\mathrm{p}}(k')}{\sqrt{k'^2+m_\pi^2}}\,dk'. \tag{3}$$

Everything is in principle measurable to good precision. The relevant scattering lengths in Eq. (1) can be obtained to high precision using the π^-d scattering length as a constraint as will be critically discussed below.

So as to obtain a robust evaluation of the coupling constant in the present context, we rearrange relation (2) in such a way that the most important experimental contributions are explicitly and separately identifiable:

$$g_c^2/4\pi = -4.50\times J^- + 103.3\times a_{\pi^-\mathrm{p}} - 103.3\times\left(\frac{a_{\pi^-\mathrm{p}}+a_{\pi^+\mathrm{p}}}{2}\right). \tag{4}$$

For orientation, and as an initial basis for discussion, we use as a preliminary value $J^- = -1.077(47)$ mb from Koch [28] and the experimental π^-p scattering length 0.0883(8) m_π^{-1} [15]. This gives the following relation, to be improved later: $g_c^2/4\pi = 4.85(22) + 9.12(8) - 103.3\times(a_{\pi^-\mathrm{p}}+a_{\pi^+\mathrm{p}})/2 = 13.97(23) - 103.3\times(a_{\pi^-\mathrm{p}}+a_{\pi^+\mathrm{p}})/2$. We stress that this is not our final result (our best estimate of these terms is given in Eqs. (20), (21) and (24) below). Here the last term is a small quantity. If we use the old Koch–Pietarinen value [3] for $(a_{\pi^-\mathrm{p}}+a_{\pi^+\mathrm{p}})/2 = a^+ = -83(38)\times10^{-4}\,m_\pi^{-1}$ we will find $g_c^2/4\pi = 14.83(45)$, while the SM99 solution [29,30] with $a^+ = 20\times10^{-4}\,m_\pi^{-1}$ will lead to $g_c^2/4\pi = 13.76$. A value for the coupling constant of the order of 13.6 would require either a relatively large positive magnitude for the isoscalar scattering length and/or a substantially less negative value for the cross section integral J^-. It is thus extremely important to obtain an accurate number for the small isoscalar amplitude. This quantity can be evaluated with small statistical and systematic uncertainties from the experimental π^-d scattering length, assuming the validity of charge symmetry, i.e. that the scattering lengths $a_{\pi^+\mathrm{p}}$ and $a_{\pi^-\mathrm{n}}$ are equal. This approximation is expected to be excellent, since the recent estimate of the isospin violation effect in this amplitude, mainly due to virtual photon effects [27], suggests that this leads to an increase of the coupling constant by only 0.2%. The cross section integral J^- is at present becoming the largest source of error. Uncertainties from the small π^-d term will not have a major impact on the result. We now turn to a critical discussion of the different contributions.

III. THE EXPERIMENTAL π^-P AND π^-D SCATTERING LENGTHS

The π^-p scattering length contributes the bulk of the GMO relation and must be very accurately controlled. It is deduced from the energy shift in pionic hydrogen, which (to about 2%) is proportional to the scattering length. The highly accurate value from PSI [15,31] has an uncertainty dominated by systematics in the analysis. The accuracy in

the procedure for extracting the scattering length, with a number of small corrections of electromagnetic origin, has been discussed in detail by Sigg *et al.* [34]. The corrections include those for the finite nucleon and pion size as well as the feedback of the strong interaction shift on the long-ranged vacuum polarization. These can all be calculated to a precision more than an order of magnitude better than the present experimental error. They also include the effect of the proton e.m. polarizability. The crucial step in the analysis is the modeling of the hadronic interaction. Sigg *et al.* have simulated this by using a short-ranged potential for each of the isospin states with the strength tuned to the corresponding free scattering length in the absence of a mass splitting. They then introduce the mass splitting via coupled Klein–Gordon equations and explore the correction for different interaction ranges, with values near 0.7 m_π^{-1} that are considered realistic. The correction and uncertainty are mainly associated with the conversion between charged and neutral pions due to the mass splitting. The final theoretical uncertainty is given as 0.5%, larger than the statistical uncertainty of 0.2%.

We have carefully examined the procedure and find no reason to doubt the quoted electromagnetic corrections and their precision, provided the hadronic interaction is tuned to correctly reproduce the experimental energy shift. The treatment of the corrections in the hadronic part, however could be improved, although it is convincing to a level of a few %. We have no major objection to the simulation of the πN s-wave amplitude using potentials, provided these correctly reproduce the low energy expansion of the s-wave scattering amplitude f_0. In terms of the s-wave phase shifts δ we have $\mathrm{Re}f_0 = \sin(\delta)\cos(\delta)/q \; = \; a + bq^2 + ...$ near threshold, where the term b represents a 'range' term or an energy dependence (see Eq. (4.7.9) in Ref. [24]). In an e.m. field, the consequences of such terms must be implemented consistently [32]. This can be made in various ways, e.g. by using potentials without obvious physical meaning, but that generate the terms in question correctly. The procedure in Ref. [34] does not respect the rather well established values for these 'range' terms, which casts doubt on the exact value of their correction term. An alternative approach would be to consider the 'range' terms to reflect an energy dependence and impose gauge invariance in the atom consistently via minimal e.m. coupling. To leading order, the approaches give identical results [32]. An additional correction originates in the mass splitting between the charged and neutral channels. According to the cited study [34], it is closely linked to the interaction ranges. Once again, it is likely that an accurate and consistent description of the corresponding correction requires a simultaneous realistic description of the observed range parameters, which is not the case in the study [34]. We believe that the negative sign of the correction term and its approximate magnitude of -1% are indeed correct, but that the uncertainty must be increased from $\pm 0.5\%$ to $\pm 1.0\%$ at present, i.e. the overall uncertainty is increased from $\pm 8 \times 10^{-4}\ m_\pi^{-1}$ to $\pm 12 \times 10^{-4}\ m_\pi^{-1}$. Here we have not attempted to introduce approximate additional corrections to the deduced scattering length [33], since we believe that this should be made on the basis of a detailed, dedicated study [35,36]. Similar terms also concern the analysis of the π^-p width, but this is not relevant at the present precision.

The experimental π^-d scattering length is derived from the energy shift in the π^-d atom in close analogy to the case of the π^-p scattering length. There is no discussion of the theoretical uncertainties in the experimental papers [33,37], but from the corresponding π^-p case one easily finds that the finite size correction and the change in vacuum polarization due to the strong interaction shift cause no problems, nor does the polarizability of the deuteron: the deuteron electromagnetic corrections can in practice be calculated using a deuteron charge distribution, that correctly reproduces the experimental deuteron charge radius. Further, the deuteron is simpler insofar as the pion mass difference enters in the same way both in the atom and in the free scattering case and to considerable precision. These corrections appear to be well accounted for and the uncertainty is given by the experimental precision. However, as above in the case of a π^-p atom, there is a correction for the energy dependence of the scattering amplitudes, but it appears to be of little importance in the present context. We therefore consider the quoted value [37] as a correctly determined effective value, and the older values by the same group are superseded.

In summary, we have adopted the following deduced scattering lengths with the transition amplitude $a_{\pi^- p \to \pi^0 \mathrm{n}}$ obtained from the width of the 1s state of the π^-p atom with the uncertainty changed from the quoted value to that given above:

$$a_{\pi^- p \to \pi^- \mathrm{p}} = \; (883 \pm 2(\mathrm{statistical}) \pm 10(\mathrm{systematic})) \times 10^{-4}\ m_\pi^{-1} \quad [15, 34], \tag{5}$$

$$a_{\pi^- p \to \pi^0 \mathrm{n}} = \; 1280(60) \times 10^{-4}\ m_\pi^{-1} \quad [15, 34], \tag{6}$$

and

$$a_{\pi^- \mathrm{d}} \; = \; (-261(5) + i63(7)) \times 10^{-4}\ m_\pi^{-1} \quad [37]. \tag{7}$$

We recall that the following relations hold, if isospin symmetry is assumed to be valid: $a_{\pi^- p} \equiv a_{\pi^- p \to \pi^- \mathrm{p}} = a^+ + a^-$; $a_{\pi^- p \to \pi^0 \mathrm{n}} = -\sqrt{2} a^-$, where $a^\pm$ are the symmetric and antisymmetric scattering lengths $a^\pm = \frac{1}{2}(a_{\pi^- \mathrm{p}} \pm a_{\pi^+ \mathrm{p}})$, respectively.

IV. THE THEORETICAL π^-D SCATTERING LENGTH

The part of the GMO relation, Eq. (4), that it has not been possible to determine accurately up to now is the term proportional to the coherent, symmetric combination of the scattering lengths $(a_{\pi^-\mathrm{p}} + a_{\pi^+\mathrm{p}})/2$. Assuming isospin symmetry, this is the isoscalar scattering length a^+. It follows from recent measurements of the hadronic energy shift and width of the pionic hydrogen atom [15] that this gives a directly determined value $a^+ = -22(43) \times 10^{-4}\ m_\pi^{-1}$. However, the accuracy of this direct determination is not sufficient for our present purpose. It is very difficult to determine a^+ with precision, directly from the coherent sum of the individual π^-p and π^+p scattering lengths, because these cancel to a few per cent . On the other hand, assuming only charge symmetry, this quantity is identical to the coherent scattering length for a negative pion on the neutron and proton, $(a_{\pi^-\mathrm{p}} + a_{\pi^-\mathrm{n}})/2$, which is the leading contribution to the accurately known $a_{\pi^-\mathrm{d}}$ scattering length. The accuracy of this approximation is indicated by a recent estimate of the isospin violation effect in the amplitude ratio $R_4 = -0.008(1)$ [27] such that

$$a_{\pi^+\mathrm{p}} - a_{\pi^-\mathrm{n}} = R_4\ a_{\pi^-\mathrm{n}} = 3 \times 10^{-4}\ m_\pi^{-1}. \tag{8}$$

Provided the remaining contributions can be reliably calculated, it is then possible to deduce the relevant coherent combination directly from the deuteron data with only minor assumptions concerning isospin symmetry. The situation is exceptionally favorable for the application of multiple scattering methods. The deuteron is a very loosely bound system and its wave function is accurately known. The nucleons have very little overlap and, consequently, the poorly controlled short range contribution is small. The particular case of the πd scattering length is even a textbook example of multiple scattering (see p. 111 in Ref. [38]), since the expansion parameters are small. The situation has been explored in detail, both within multiple scattering theory and using a three-body Faddeev approach, since it provides a clear-cut testing ground for methods [39–42,44].

In the static (fixed scattering centers) approximation the leading structure and scale of the pion–deuteron scattering length is set by the coherent single scattering term S and the dominant s-wave double scattering term D which is proportional to the inverse deuteron radius $\langle 1/r \rangle$ (p. 111 in Ref. [38]):

$$a_{\pi^-\mathrm{d}}^{\mathrm{static}} \ = \ S \ + \ D \ ...; \tag{9}$$

$$S \ = \ \frac{(1 + m_\pi/M)}{(1 + m_\pi/M_\mathrm{d})}\ (a_{\pi^-\mathrm{p}} + a_{\pi^-\mathrm{n}}); \tag{10}$$

$$D \ = \ 2\ \frac{(1 + m_\pi/M)^2}{(1 + m_\pi/M_\mathrm{d})}\ \left[\left(\frac{a_{\pi^-\mathrm{p}} + a_{\pi^-\mathrm{n}}}{2} \right)^2 - 2 \left(\frac{a_{\pi^-\mathrm{p}} - a_{\pi^-\mathrm{n}}}{2} \right)^2 \right] \langle 1/r \rangle, \tag{11}$$

where M_d is the deuteron mass.

The static double scattering term represents about 90% of the experimental scattering length. We will use this well defined static limit with point interactions as the starting point with respect to which various corrections will be introduced. Numerically, this term has the value

$$D = -256(7) \times 10^{-4}\ m_\pi^{-1}, \tag{12}$$

where we have used the scattering lengths from Eqs. (20–21).

A. Previous approaches to a^+ from the deuteron data

Recently Baru and Kudryatsev (B–K) [44] have investigated the πd scattering length using state-of-the-art multiple scattering methods. We will use the updated and unpublished version of their investigation [45] as the theoretical yardstick for the following discussion. We have numerically reproduced their findings to the same numerical precision, under the same assumptions. This approach is however still incomplete and contains, we believe, one erroneous term. B–K quote a deduced a value equivalent to $a^+ = -15(9)$ in units of $10^{-4}\ m_\pi^{-1}$, but the true systematic error is much larger. In the following we will critically assess the input parameters, corrections and systematics, and introduce substantial theoretical improvements. The classical 3-body approach to the problem is still that of Afnan and Thomas and of Mizutani and Koltun, using separable interactions [39,40]. This approach gives the best picture of the dispersive effects due to absorption and supports the conclusions of the heavy cancelation of unitarity corrections

in the multiple scattering approach. The approach, however, has not been updated in its overall accuracy to match the present high experimental precision and cannot be used directly.

A rather different approach is that of Beane *et al.* [46], based on the nuclear chiral perturbation approach of Weinberg [47] and using phenomenological deuteron wave functions. This approach makes a systematic expansion in the pion 4-momentum, using effective parameters; at present the calculations have been made to $O(q^3)$. The result has the same general structure as the static limit of multiple scattering. Several physical effects discussed in the following are not yet included in this order, such as the Fermi motion term and the dispersive correction from pion absorption. They conclude that $a^+ = -30(5) \times 10^{-4}\ m_\pi^{-1}$ to $O(q^3)$, where the uncertainty represents only the experimental uncertainty in the deuteron scattering length. The magnitude of the present theoretical uncertainty in this approach can be estimated from the multiple scattering value of the terms not yet included, as well as from the 5% uncertainty in the corresponding calculation (to $O(q^3)$) of the isovector scattering length as quoted by the authors [48]. It thus appears that the higher order individual contributions would be about 10% ($25\times10^{-4}\ m_\pi^{-1}$) from the last effect and may even reach 20% ($50\times10^{-4}\ m_\pi^{-1}$) of the deuteron scattering length to judge from the value of the well defined Fermi motion term in multiple scattering descriptions. For these reasons the systematic uncertainty to the present order in the chiral expansion is most likely about one order of magnitude larger than the statistical uncertainty. Consequently, this approach does not yet constrain the isoscalar scattering scattering length to the accuracy desired here. It is better to follow the recommendation of the authors and use the results to constrain undetermined chiral parameters.

B. The inverse deuteron radius

The inverse deuteron radius appearing in Eq. (11) must be evaluated from wave functions. It is essential that the asymptotic normalization be accurately consistent with the experimental np effective range and that the wave functions correspond to an energy-independent interaction. The Paris [49] and Bonn2 [50] wave functions satisfy these criteria and give $\langle 1/r\rangle_{\rm Paris} = 0.449$ fm^{-1} and $\langle 1/r\rangle_{\rm Bonn2} = 0.463$ fm^{-1} with asymptotic normalizations A_S(Paris) = 0.8869 fm$^{-1/2}$ and A_S(Bonn2) = 0.8863 fm$^{-1/2}$, respectively. We have conservatively used the average of these model values $\langle 1/r\rangle = 0.456(7)$ fm^{-1} $= 0.645(10)\ m_\pi$; the uncertainty given is set by their difference. We note that the inverse radius, 0.520 fm^{-1}, of the Hulthén wave function [42], which is often used for explorations of various effects, is nearly 15% larger than these values and should not be used in quantitative studies. The uncertainty in the theoretical πd scattering length from the inverse radius is less than its present experimental precision.

C. Effects of the non-locality of the πN s-wave interaction

The simplest approximation to the double scattering term of Eq. (11) assumes that the πN scattering is point-like. Such an approximation is appropriate if the two scatterers are well separated, as is the case for the bulk of the contributions in the case of the deuteron as a consequence of its loose binding. The rather small non-local correction must, however, be controlled in sign and magnitude at the level of precision aimed for here. However, it is not necessary to describe this effect very accurately. The non-local effects enter mainly in the description of the isovector πN s-wave interaction, which is well known to be closely associated with ρ-meson exchange and which heavily dominates the double scattering term. For calculational convenience it has been conventional to model the non-locality of the scattering amplitude in terms of a separable form, $v(k)v(k')$, with a monopole form factor $v(k) = c^2/(c^2 + k^2)$. Since the initial and final pion are at rest with momentum 0 and the intermediate pion has momentum $\mathbf{q}$, this means that in momentum space the static pion propagator changes from q^{-2} to $v(q)^2 q^{-2}$. In coordinate space this corresponds to a change of the expectation value $\langle 1/r\rangle$ by

$$\delta\langle 1/r\rangle = -\left\langle \left| \frac{1 + c\ r/2}{r} \exp(-cr) \right| \right\rangle . \tag{13}$$

We list in Table II the values of $\delta\langle 1/r\rangle$ and the corresponding contribution to the deuteron scattering length for different values of c as well as the contribution to the scattering length for standard values of the πN scattering lengths.

We conservatively consider that plausible values for the parameter c lie in the interval $3.5 \le c \le 5\ m_\pi$. This is a wide range, which should adequately cover any model dependence of the result. Another choice for the non-locality is to consider each of the scatterings to be associated with a monopole form factor. Since the isovector scattering strongly dominates the double scattering, the natural cut-off parameter is the ρ-meson mass. Such a monopole form factor, characteristic of ρ-meson exchange, would give the same correction as quoted in Table II for $c = 5\ m_\pi$. However, it

TABLE II. Corrections to $\langle 1/r \rangle$ and to the πd scattering length for different cut-off values and wave functions. The πN scattering lengths are from Eqs.(20) and (21).

Model	Paris [49]		Bonn2 [50]	
$\langle 1/r \rangle$	0.449 fm^{-1}		0.463 fm^{-1}	
c	$\delta\langle 1/r \rangle$	$\delta a_{\pi \mathrm{d}}$	$\delta\langle 1/r \rangle$	$\delta a_{\pi \mathrm{d}}$
$[m_\pi]$	$[10^{-3}$ fm$^{-1}]$	$[10^{-4}\ m_\pi^{-1}]$	$[10^{-3}$ fm$^{-1}]$	$[10^{-4}\ m_\pi^{-1}]$
3.0	-50	28	-60	34
3.5	-37	21	-46	26
4.0	-28	16	-36	20
4.5	-21	12	-29	16
5.0	-16	9	-23	13

is well known from the Vector Dominance Model that such ρ-meson exchange must also account for the form factors of the nucleon and the pion. Consequently, the effective overall form factor in each of the pion scatterings is rather a dipole form factor with the ρ-meson mass, corresponding to $c = 3.5\ m_\pi$. It should be observed that the typical modification of $\langle 1/r \rangle$ is a negative contribution by 4 to 8% corresponding to a positive contribution to $\delta a_{\pi^- \mathrm{d}}$ of 9 to 20 $\times 10^{-4}\ m_\pi^{-1}$. We choose the mean of these two approaches as a typical value with the spread setting the scale of the uncertainty, but note that in doing so we may somewhat underestimate the non-local effect, such that our final value of $g_c^2/4\pi$ may be somewhat too low.

B–K have previously investigated this effect numerically with different wave functions. They informed us that the values they give in Ref. [44], Table 3, for the realistic Bonn1 and Bonn2 wave functions include no form factor effect (contrary to the statement in the paper). We have received their corrected and extended results [45] for the Bonn1 potential, which we have verified. Note that at the present level of precision it is important to use potentials fully consistently. The Bonn1 potential is energy-dependent; as a consequence, orthonormality can only be respected in matrix elements calculated using this potential if non-trivial weight factors are introduced in the integrands. To eliminate this uncertainty we use here the similar, but energy-independent, Bonn2 potential. B–K consider without arguments cut-off values $c = 2.5,\ 3$ and $3.5\ m_\pi$ in the form factor; this gives positive contributions to the scattering length as compared to the point-like static approximation of 36, 27 and 22 $\times 10^{-4}\ m_\pi^{-1}$, respectively. In our opinion there are good physical reasons to believe that ρ-meson exchange sets the scale for the dominant isovector amplitude with a larger value for the effective c. To be conservative we take $c = 3.5$ and $c = 5\ m_\pi$ for the cut off as limits for this systematic correction from the non-localities and use the central value of these two extremes as the correction. This is smaller than the correction found by B–K. Non-locality is one of the largest theoretical sources of systematic uncertainty in corrections to the point-like static approximation.

D. Corrections to the static approximation

The nature of the leading non-static corrections and the reasons why the static expression (fixed scattering centers) still remains an excellent approximation are well understood. At first sight, even the single scattering amplitudes have rather important non-static modifications, representing about 30% of the total πd scattering length. Such corrections are systematically generated by the multiple scattering description in which physical amplitudes are used, thus guaranteeing the correct behavior of the scattered wave at large distances. The emphasis is thus not on the near-zone behavior of the scattering as in pseudo-potential or effective Lagrangian approaches. In a situation like the present one, this leads to a systematic cancelation of unitary binding corrections between single scattering and double scattering terms, when these are introduced consistently. This phenomenon was first demonstrated in the present context for an analytically soluble model by Fäldt [42]. It has been numerically investigated by B–K [44] using a Hulthén wave function and a separable amplitude with a dipole form factor and a cut-off parameter $3\ m_\pi$. They conclude that the amplitude increases by only $10 \times 10^{-4}\ m_\pi^{-1}$, when the non-static term is included. This is only twice the experimental uncertainty and less than the uncertainty from the form factor. Fäldt evaluated the joint contribution of the non-static and the form factor terms using a dipole form factor with c=3.6 m_π with a Hulthén wave function [42]. The overall contribution corresponds to $34 \times 10^{-4}\ m_\pi^{-1}$. The comparison with our independent evaluation of pure form factor corrections indicates that the non-static term in this case is about $8 \times 10^{-4}\ m_\pi^{-1}$. A detailed calculation of this correction is complicated. Wycech informed us that he is in the process of reevaluating the non-static contributions using a Faddeev approach and separable interactions. At the present moment he has only results using an interaction that reproduces the Hulthén wave function; this gives $+12 \times 10^{-4}\ m_\pi^{-1}$, in excellent

agreement with the previous results [43]. Following B–K we have adopted a value $11(6)\times10^{-4}\ m_\pi^{-1}$, where the liberal uncertainty reflects the lack of verification of the value of non-static effects using high quality deuteron wave functions.

E. Fermi motion

Another well defined correction originates in the nucleon Fermi motion. In the case of s-wave scattering, such contributions cancel systematically to high precision with other binding terms [42]. In addition, the single scattering term from the πN p-wave scattering produces a small, attractive and physically well understood contribution, which can be reliably evaluated as a leading order effect originating in the nucleon momentum distribution and the spin–isospin averaged p-wave threshold scattering amplitude c_0=0.209 m_π^{-3}, which is the value used also in Ref. [44].

$$a(\text{Fermi}) \;=\; 2\; c_0 \frac{m_\pi^2(1+m_\pi/M)}{(M+m_\pi)^2(1+m_\pi/M_\mathrm{d})} \left\langle p^2\; v^2\left(\frac{m_\pi}{M+m_\pi}p\right)\right\rangle . \tag{14}$$

We have calculated this expectation value for two high quality deuteron wave functions. The results are given in Table III. The form factors are manifestly of no importance. The relatively large difference between the Paris potential and the Bonn2 potential arises because of the D-state component, which generates contributions 10 times more effectively than the S-state one. The difference in the correction in the two cases is thus almost entirely a consequence of the well known difference in the D-state probability (P_D=5.7% vs. 4.3%) for the two wave functions. The normalized momentum distributions for the S- and D-wave component, respectively, are very similar in the two models. The D-state probability is not strictly speaking an observable. We therefore treat its effect as a true model dependence. We take the spread in the values of the Fermi motion corrections as a measure of a systematic theoretical uncertainty, although physical arguments for the higher value of P_D exist [51]. Consequently, in the following evaluation, we use the value $a(\text{Fermi}) = 61(7)\times10^{-4}\ m_\pi^{-1}$. This is larger than the value 50 to $53\times10^{-4}\ m_\pi^{-1}$ found by B–K based on the Bonn1 and 2 wave functions.

TABLE III. Estimates of the contribution a(Fermi) to $a_{\pi^-\mathrm{d}}$ from single p-wave scattering as a result of Fermi motion according to Eq. (14) for various deuteron wave functions, different cut-off values and separated into S- and D-state contributions. The last row gives $\langle p^2\rangle$ and the kinetic energy $\langle p^2\rangle$/M.

Model	Paris [49]			Bonn2 [50]		
c	S-state	D-state	Total	S-state	D-state	Total
$[m_\pi]$	[in units of $10^{-4}\ m_\pi^{-1}$]			[in units of $10^{-4}\ m_\pi^{-1}$]		
3	39.6	27.9	67.6	36.7	16.7	53.4
4	39.8	28.1	67.8	36.8	16.9	53.6
5	39.8	28.1	68.0	36.8	16.9	53.7
∞	39.9	28.3	68.2	36.8	17.0	53.9
$\langle p^2\rangle$ [m_π^2]	0.533	0.378	0.912	0.492	0.228	0.720
$\langle p^2\rangle/M$ [MeV]	11.1	7.9	19.0	10.3	4.7	15.0

F. Dispersion contribution

A small repulsive contribution, not described by multiple scattering, is produced by the dispersive term from the absorption reaction $\pi^-\mathrm{d}\to\mathrm{nn}$. This quantity has been repeatedly calculated using Faddeev approaches [39–41]. It typically has a theoretical uncertainty of 20% of its numerical value $-56(14)\times10^{-4}\ m_\pi^{-1}$ [41]. The dispersive contribution is a theoretically calculated correction; a more detailed study of this term is highly desirable. The uncertainties reflect the model dependence of the approach.

G. sp interference

This is the name given by B–K to a term originating in pion p-wave scattering on one of the nucleons due to Galilean invariance [44]. Such Galilean terms generate s-wave scattering contributions even for pion scattering on free nucleons. In the present situation the relevant spin-averaged on-shell scattering volume for charge exchange of a p-wave pion is

well known and the corresponding scattering amplitude on-the-mass-shell depends on the pion momentum in a well defined way. The Galilean correction for nucleon motion involves going off the mass shell and usually depends on the description. B–K advocate that a contribution of about $42 \times 10^{-4}\ m_\pi^{-1}$ originates from p-wave scattering due to the momentum of the intermediate pion when expressed in the πN CM system. However, in the present situation the contribution is almost entirely generated by the isovector πN Born term and it can be evaluated exactly. From the expressions given in Höhler's reference book, Eq. (A.8.2) [24], one finds that this term is proportional to

$$\nu^2 - \frac{(k^2 + k'^2 - t)}{2} = \nu^2 - q \cdot q'. \tag{15}$$

Here, ν is (to order M^{-2}) the Breit frame pion energy, which is proportional to the scalar product of the average 4-vectors of the nucleons (p and p') and pions (q and q'), respectively.

$$\nu = \frac{1}{M}\frac{(p+p')}{2} \cdot \frac{(q+q')}{2} = \frac{(q_0+q_0')}{2} - \frac{1}{M}\frac{(\mathbf{p}+\mathbf{p}')}{2} \cdot \frac{(\mathbf{q}+\mathbf{q}')}{2}. \tag{16}$$

(Eq. (A.1.6) in Ref. [24]). Thus, neglecting terms of order M^{-2}, the pion pole term is proportional to

$$\frac{(q_0-q_0')^2}{4} - \frac{(q_0+q_0')}{M}\frac{(\mathbf{p}+\mathbf{p}')}{2} \cdot \frac{(\mathbf{q}+\mathbf{q}')}{2} + \mathbf{q} \cdot \mathbf{q}'. \tag{17}$$

In the double scattering term, the contribution comes from nucleon 1 with initial (final) momentum $\mathbf{p}$ ($\mathbf{p}-\mathbf{q}'$) and with the initial (intermediate) pion momentum $\mathbf{0}$ ($\mathbf{q}'$), respectively, while for nucleon 2 the initial (final) nucleon momentum is $-\mathbf{p}$ ($-\mathbf{p}+\mathbf{q}'$) with intermediate (final) pion momentum $\mathbf{q}'$ ($\mathbf{0}$), respectively; the pion energies, q_0 and q_0', are unchanged in this term. The sum of these two contributions are

$$\frac{q_0}{M}\frac{\mathbf{q}'^2}{2} - \frac{q_0}{M}\frac{\mathbf{q}'^2}{2} = 0. \tag{18}$$

On the other hand, B–K make the choice of Galilean invariance *for the incoming and outgoing πN systems calculated separately* in the primary amplitude and find in the same limit $0 + q_0\mathbf{q}'^2/M$ in Eq. (18). Instead the exact pole term corresponds, to order M^{-2}, to a Galilean invariant expression using the *average* velocity of the initial and final nucleons, contrary to the B–K assumption. In other words, the pole term is proportional to the scalar product of the pion momenta $\mathbf{q}_B \cdot \mathbf{q}'_B$ in the nucleon Breit frame. We have therefore suppressed this term in the B–K multiple scattering expansion.

We note in passing that, even if the Galilean contributions were of the type proposed by B–K, their importance would most likely be strongly suppressed. The reason is that these terms generate a δ-function interaction in the absence of form factors. We therefore suspect that NN correlations would largely suppress such contributions, in analogy with the Ericson–Ericson–Lorenz–Lorentz effect for p-wave π propagation in the nuclear medium (p. 140ff in Ref. [38]).

H. Isospin and mass difference corrections

In the above expressions, we assumed that isospin holds for the calculation of double scattering and that charge symmetry holds for the single scattering. We now quantify the effect of these approximations. B–K have investigated the consequence of the physical mass difference between π^- and π^0 and between the neutron and the proton in the multiple scattering. They find an increase of the scattering length by about $3.5 \times 10^{-4}\ m_\pi$. The smallness of this term is in part due to a systematic compensation of single and double scattering contributions in analogy to the compensation of unitarity corrections to single and double scattering terms. As an alternative approach we use the recent estimates of the violation of isospin symmetry from light quark mass differences and virtual photon effects in the πN scattering lengths [27]. We maintain only the effects of violations in the amplitudes in the double scattering term in view of the systematic cancelation between single scattering and propagator modifications in the double scattering term. This leads to an increase of the scattering amplitude by $3.5 \times 10^{-4}\ m_\pi$, numerically identical to the previous estimate. It is not clear whether these approaches represent the same physics and this point should be further investigated. However, both results indicate that the effects are small in the present context, although they will become of importance in the future. In view of its smallness and since it is not at present established experimentally, we have not included this correction, which is within experimental uncertainties. It has, however, been included as an uncertainty in our estimate of systematic errors.

I. Higher order multiple scattering corrections

In the present case the multiple scattering expansion is rapidly convergent beyond the double scattering term. In the fixed scattering approximation with separable interactions, these higher order terms can be summed exactly to all orders. B–K calculated these terms approximately, assuming point-like scatterers. We have verified these calculations and reproduce their results. They have since improved the evaluation of this small term, using form factors and find a stable contribution to the scattering length of the order of $+6 \times 10^{-4}\ m_\pi^{-1}$ [44,45]. Our independent evaluation also gives very stable values, but somewhat smaller, in the range of 3 to $4 \times 10^{-4}\ m_\pi^{-1}$ for the form factors considered. We have used the value $4(1) \times 10^{-4}\ m_\pi^{-1}$ for this correction. The effect is much smaller than other uncertainties, for example those due to form factors.

J. Inverse pion photoproduction

Another small electromagnetic correction comes from the physical s-wave photoproduction process $\pi^-\mathrm{p} \to \gamma\mathrm{n}$ acting on one nucleon followed by the inverse reaction on the other one. This double scattering process has nearly the same structure as the corresponding s-wave charge exchange process $\pi^-\mathrm{p} \to \pi^0\mathrm{n}$ in Eq. (11), but for the fact that the intermediate photon now has momentum $\mathrm{k}_\gamma = m_\pi$ in the static limit, such that

$$\mathrm{Re}\ D_\gamma = -2/3\ \frac{(1+m_\pi/M)^2}{2\ (1+m_\pi/M_\mathrm{d})}\ \left[E_{0+}(\gamma\mathrm{n} \to \pi^-\mathrm{p})\right]^2\ \left\langle \frac{\cos(k_\gamma r)}{r} \right\rangle . \tag{19}$$

Here the photoproduction amplitude $E_{0+}(\gamma\mathrm{n} \to \pi^-\mathrm{p}) = -31.4 \times 10^{-3}\ m_\pi$ (Table 8.3 in Ref. [38]). This small term had not been included previously. It is of order $-2 \times 10^{-4}\ m_\pi^{-1}$.

K. Double p-wave scattering

A small correction results from the p-wave scattering due to nucleon motion at both vertices. This effect has been estimated by B–K for an analytically soluble deuteron model with Gaussian wave functions. They find a contribution of about $-3 \times 10^{-4}\ m_\pi^{-1}$. We have included this small effect.

L. Scattering on virtual pions

Finally, one may envisage a contribution from the scattering of the pion on a virtually exchanged pion in the deuteron. However, we are dealing with an isoscalar system, and such a contribution is proportional to virtual isoscalar $\pi\pi$ s-wave scattering and should be very small, from a chiral perspective. In particular, since the deuteron is such a loosely bound system, one expects this term to be small. Robilotta and Wilkin showed that large cancelations in a consistent treatment give only $-5 \times 10^{-4}\ m_\pi^{-1}$ [52]. This is confirmed by a recent chiral estimate of -8 to $-6 \times 10^{-4}\ m_\pi^{-1}$ [46]. We adopt a contribution of $(-6 \pm 2) \times 10^{-4}\ m_\pi^{-1}$ from this effect.

M. Results for the πN scattering lengths

The different contributions from the previous subsections are summarized in Table IV, using the final parameters from Eqs.(20) and (21) whenever appropriate. Consequently, the present energy shift in the π^-d atom leads to the following value for the coherent scattering length from a proton and a neutron:

$$\frac{a_{\pi^-\mathrm{p}} + a_{\pi^-\mathrm{n}}}{2} = (-17 \pm 2(\mathrm{statistical}) \pm 8(\mathrm{systematic})) \times 10^{-4}\ m_\pi^{-1}. \tag{20}$$

In the limit of isospin symmetry, this quantity is the isoscalar scattering length a^+. The main systematic error in Eq. (20) comes from the uncertainty in the dispersive correction term and, to a lesser degree, from the form factor or non-locality in the deuteron double scattering term. The small corrections for isospin violation in the double scattering term and for charge symmetry breaking in the single scattering on the deuteron are well within the stated uncertainties and have no substantial influence on the result.

TABLE IV. Typical contributions to $a_{\pi \mathrm{d}}$ scattering length in units of $10^{-4}\ m_\pi^{-1}$

Contributions	Present work	B–K [44]
$a_{\pi^-\mathrm{d}}$(double scattering; static)	−256 (7)	−252
Fermi motion	61 (7)	50
dispersion correction	− 56(14)	not included
isospin violation	3.5	3.5
$(\pi^-\mathrm{p}, \gamma\mathrm{n})$ double scattering	− 2	not considered
form factor	17 (9)	29(7)
higher orders	4 (1)	6
sp interference	small	− 44
non-static effects	11 (6)	10
p-wave double scattering [44]	− 3	− 3
virtual pion scattering [46,52]	− 7 (2)	not considered
total $= a_{\pi^-\mathrm{d}} - 1.07 \times (a_{\pi^-\mathrm{p}} + a_{\pi^-\mathrm{n}})$	−227(20)	−198
$a_{\pi \mathrm{d}}$(experimental) [37]	−261 (5)	

Combining the information from the experimental π^-p and π^-d scattering lengths with the constraints of the theoretical analysis (20), we obtain a substantially improved determination in the difference $(a_{\pi^-\mathrm{p}} - a_{\pi^-\mathrm{n}})/2$ (this quantity is, in the limit of isospin symmetry, identical to the isovector scattering length a^-):

$$\frac{a_{\pi^-\mathrm{p}} - a_{\pi^-\mathrm{n}}}{2} = (900 \pm 3(\text{statistical}) \pm 13(\text{systematic})) \times 10^{-4}\ m_\pi^{-1}. \tag{21}$$

A graphical determination of these πN scattering lengths is shown in Fig. 1, which also emphasizes that this is a substantial improvement on determinations using only data from pionic hydrogen. The results are in excellent agreement with the central values deduced from the pionic hydrogen shift and width by the experimental PSI group, since it follows from Eqs. (7) and (8) of Ref. [15] that $a^+ = (-22 \pm 43) \times 10^{-4}\ m_\pi^{-1}$; $a^- = (905 \pm 42) \times 10^{-4}\ m_\pi^{-1}$. The PSI group also used the constraint from the pionic deuterium shift, assuming the old calculation of Ref. [41] to be accurate enough. The corresponding result $a^+ = (+16 \pm 13) \times 10^{-4}\ m_\pi^{-1}$; $a^- = (868 \pm 14) \times 10^{-4}\ m_\pi^{-1}$ is superseded by the present far more detailed analysis. From our evaluation here, we have achieved quantitative control of the dominant contribution to the GMO relation from the scattering lengths to about 1% or better in $g_c^2/4\pi$.

It is interesting to compare our results with the extrapolations of scattering amplitudes to threshold as given in Refs. [53,55]. They find the value $a_{\pi^+\mathrm{p}\to\pi^+\mathrm{p}} = (-770 \pm 30) \times 10^{-4}\ m_\pi^{-1}$. This corresponds to $a^+ = (57 \pm 15) \times 10^{-4}\ m_\pi^{-1}$ assuming isospin symmetry invariance and using the experimental value for $a_{\pi^-\mathrm{p}\to\pi^-\mathrm{p}}$ from pionic hydrogen. On the other hand, the charge symmetric scattering length $a_{\pi^- n\to\pi^-\mathrm{n}} = (-917 \pm 18) \times 10^{-4}\ m_\pi^{-1}$ follows from Eqs. (20) and (21) and within charge symmetry the two values should be identical. According to Eq. (8) the estimated effect of charge symmetry breaking in effective chiral theory is $a_{\pi^+\mathrm{p}} - a_{\pi^-\mathrm{n}} = 3 \times 10^{-4}\ m_\pi^{-1}$. The above values give, instead, $(147 \pm 35) \times 10^{-4}\ m_\pi^{-1}$, 50 times larger than the expected value. Thus, unless charge symmetry is unexpectedly badly broken, the scattering length of Refs. [53,55] based on scattering experiments is implausible and should be rejected.

While the extrapolation [53,55] leads to important differences, it cannot, of course, be completely ruled out that other, more constrained, extrapolations from πN scattering data could lead to scattering lengths slightly different from the ones found here. The origin would then most likely be due either to isospin violation in the scattering data or, alternatively, to some unexpected modification of the least controlled part of our deuteron terms, such as the absorption contribution. In the dispersion-relation-constrained extrapolation advocated by Pavan *et al.* [12] they give $a^+ = +20 \times 10^{-4}\ m_\pi^{-1}$ to be compared with $(-17 \pm 10) \times 10^{-4}\ m_\pi^{-1}$ above. Interpreted as a modification of the dispersive term due to deuteron absorption, it would require an increase by a factor of 2.3 in this term in order to make the results compatible, which appears an implausibly large modification. We believe our result to be the preferable one, since it is a more direct determination and fully consistent. The margin for modifications of our theoretical analysis is small.

V. EVALUATION OF THE CROSS SECTION INTEGRAL J^- FROM DATA

The cross section integral represents only one third of the total contribution to the GMO relation. This means that an uncertainty of (say) 3% in the integral would give only 1% uncertainty in the coupling constant. At the present precision, and in spite of this insensitivity, this has now become a major source of uncertainty in the determination of

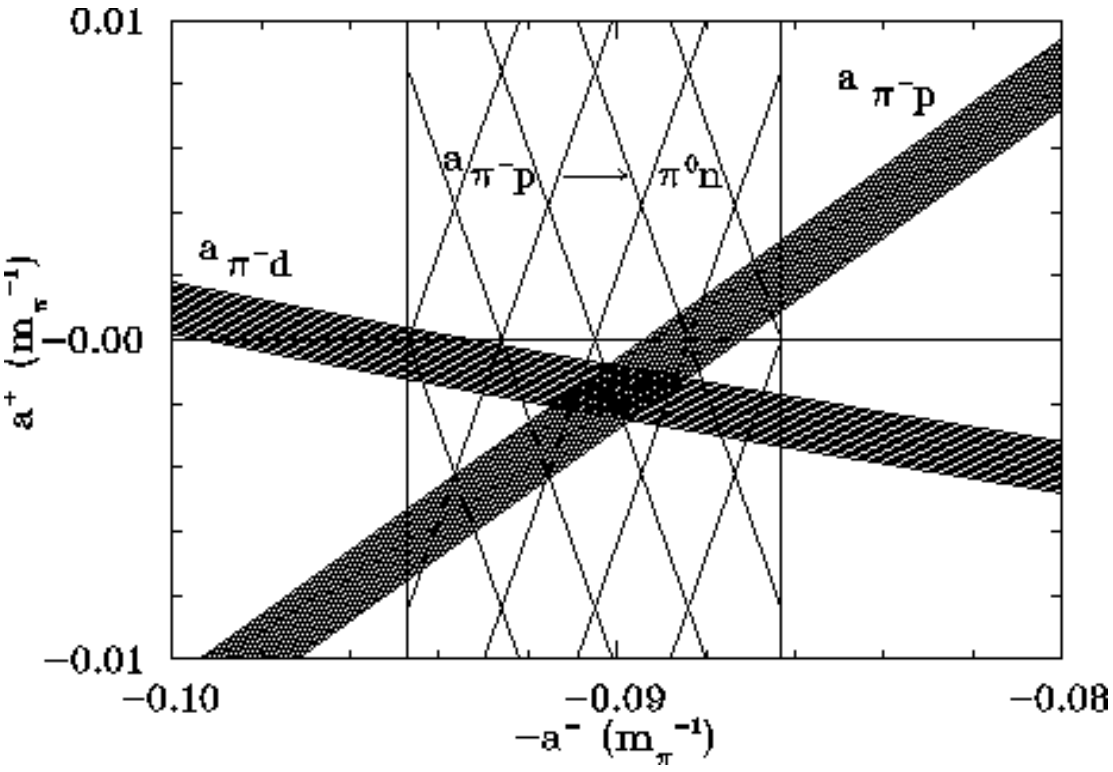

FIG. 1. Graphical determination of the πN scattering lengths a^+ and a^- from the constraints imposed by the pionic atom scattering lengths

the coupling constant. Since total cross sections tend to be inherently accurate, the evaluation can be performed with precision, but for the high-energy region. There exists a vast amount of high quality data up to very high energies (beyond 240 GeV/c) and, in the dominant region below 1 GeV/c, there are detailed results from partial wave analyses. The possibility of systematic effects in the difference must be considered, particularly since Coulomb corrections have opposite signs for $\pi^{\pm}$p scattering. The only previous evaluation with a detailed discussion and clearly stated sources of errors known to us is an unpublished study of 1985 by Koch, which gives $J^- = -1.077(47)$ mb [28]. Later evaluations find values within this band of errors, but the uncertainties are not discussed. In 1992 Workman *et al.* [11] gave the values −1.056 mb and −1.072 mb based on the Karlsruhe–Helsinki and VPI πN amplitudes of the time, respectively. In 1995 the VPI group gave the value −1.05 mb [25]. Gibbs *et al.* give a similar value, $J^- = -1.051$ mb [16]. In this case the dominant contribution below 2 GeV (−1.308 mb) was evaluated using the SM95 phase-shift analysis [54] for the πN cross sections. These values are summarized in Table V.

TABLE V. Values of J^- from the literature

Source	J^- mb
Koch 1985 [28]	−1.077(47)
Workman *et al.* 1992; K–H [11]	−1.056
Workman *et al.* 1992; VPI [11]	−1.072
Arndt *et al.* 1995 [25]	−1.050
Gibbs *et al.* 1998 [16]	−1.051
Present work	−1.083(32)

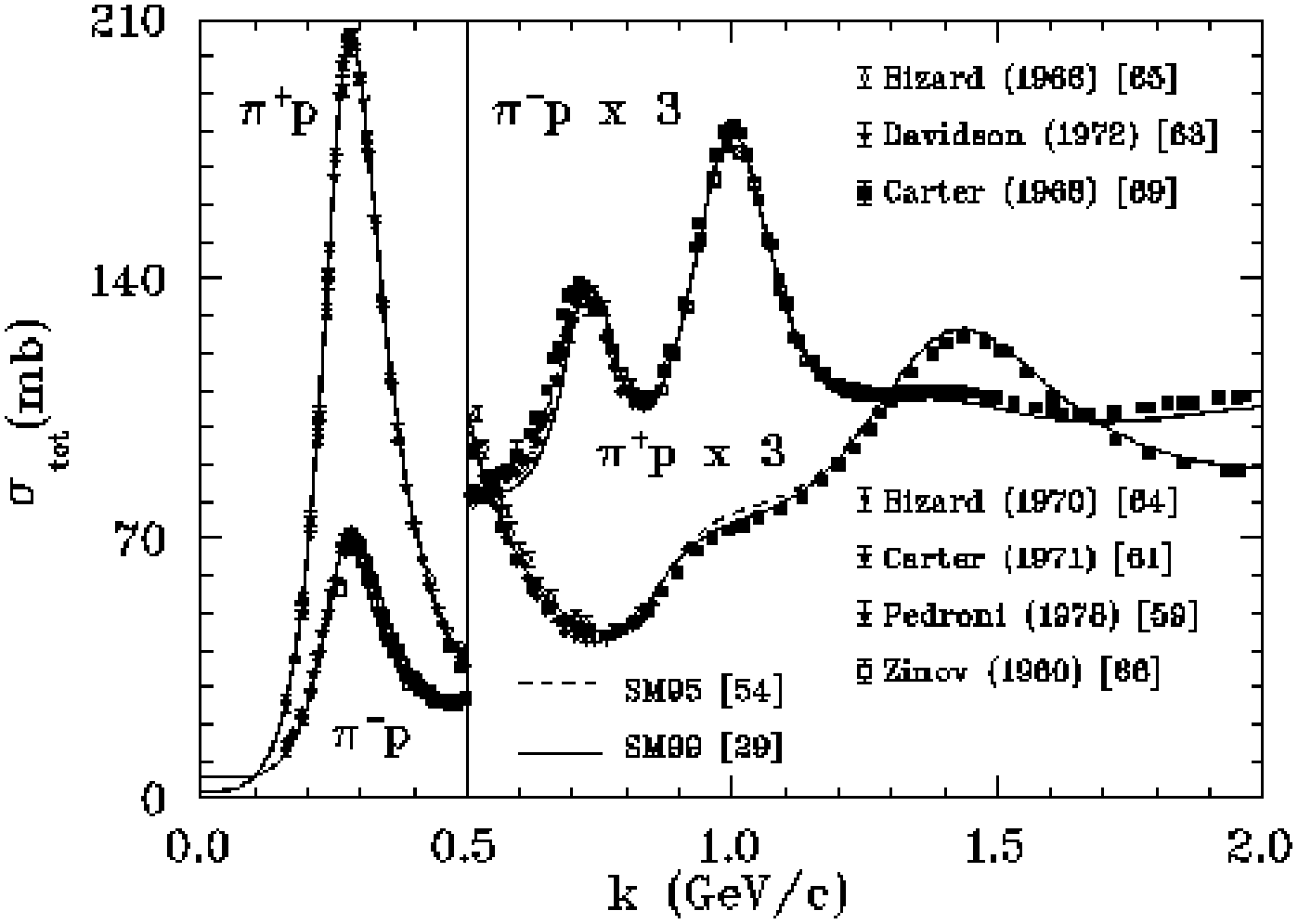

FIG. 2. The experimental total π^+p and π^-p cross sections below 2 GeV/c compared with the SM95 and SM99 PWA hadronic solutions, where Coulomb barrier effects have not been taken into account.

In view of the importance of obtaining a clear picture of the origin of present uncertainties, we have re-examined this problem in spite of the approximate consensus. The πN total cross sections below 2 GeV/c are shown in Fig. 2. The typical shape of the integrand J^- is seen in Fig. 3. As might be expected, the main contributions come from the region of the Δ resonance and just above. It would be false, however, to believe that this is the region that produces the main uncertainty of the integral. There are no strong cancelations in the difference between the total $\pi^{\pm}$p cross sections in that region and the cross sections have been very carefully analyzed. Systematic uncertainties of 2–3% or more are very unlikely indeed; if they occur, they will certainly have an important influence on other determinations of the coupling constant as well.

In the following we examine in detail the uncertainties arising from various energy regions with different characteristics:

- in subsection A, the threshold region below 160 MeV/c is dominated by the s- and p-wave threshold parameters (s-wave contribution of about +6%, p-wave one of about −6%);
- in subsection B, the Δ resonance region from 160 MeV/c to 550 MeV/c, in which the major phase shifts are very accurately known (main contribution of about 155%);
- in subsection C, the resonance region from 550 MeV/c to 2 GeV/c, which is partly dominated by higher resonances with mostly high quality data (about −33% contribution);
- in subsection D, the high-energy region and the asymptotic region from 2 GeV/c to ∞ (totally about −22% contribution); about half originates from the asymptotic region beyond 10 GeV/c, for which data are accurately described by asymptotic expressions.

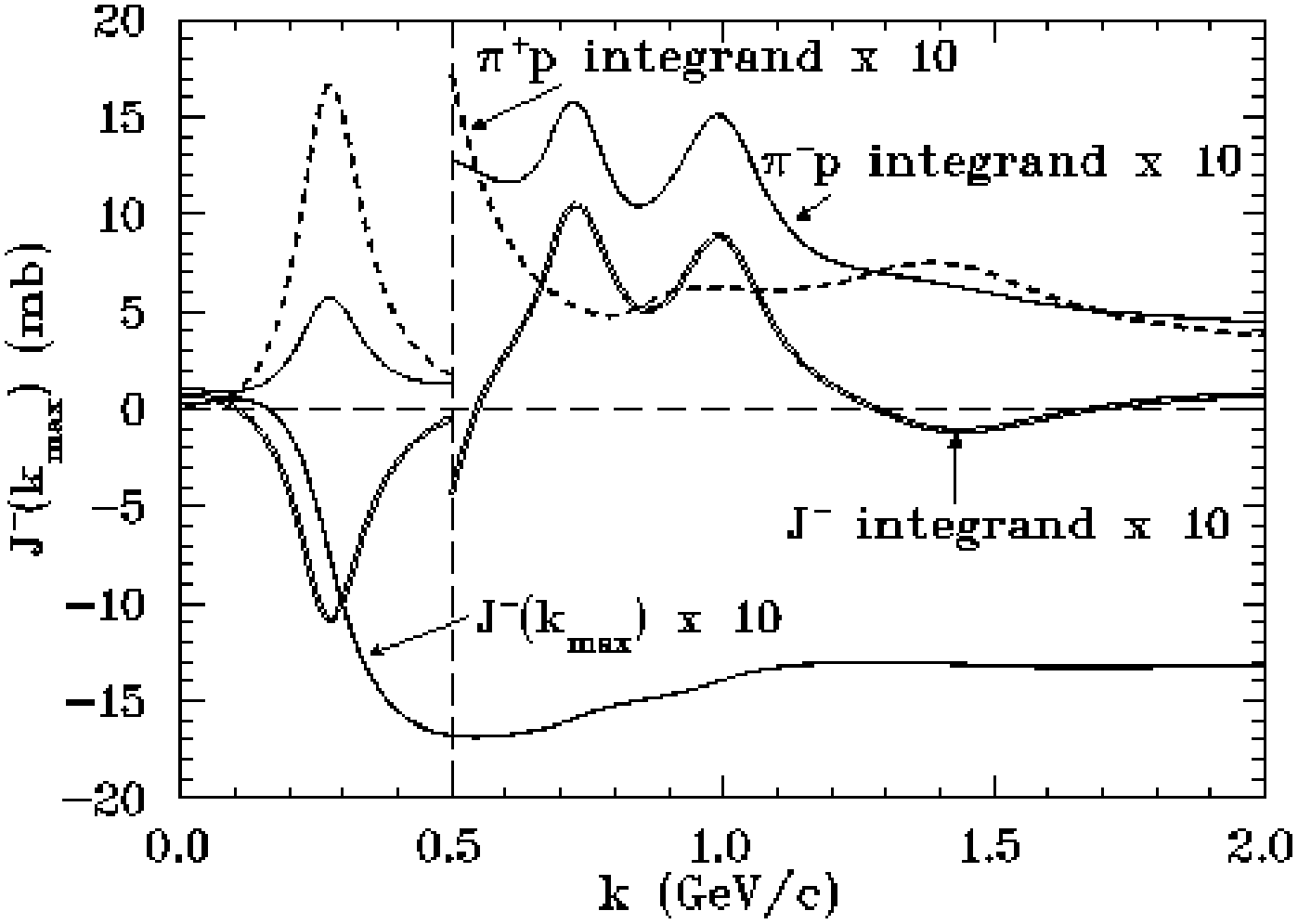

FIG. 3. The separate integrands for $\pi^{\pm}$p, as well as for their difference as a function of laboratory momentum k, together with the cumulative value of the integral $J^-(k_{\rm max})$ integrated from threshold to $k = k_{\rm max}$. The curves are based on the SM99 solution [29]. The integrands are in units of mb GeV/c.

Although we do not need the highest precision in the following, special attention must be paid to the fact that the total cross sections in the integral J^- are the hadronic ones, in the absence of the Coulomb field, and that they differ somewhat from the experimental total cross sections. The π^+p total cross sections are systematically reduced at all energies by the Coulomb repulsion between the particles and, conversely, the π^-p total cross sections are systematically enhanced [56,57]. This effect, which gives a positive contribution to J^-, must corrected for; the coupling constant would be underestimated in the absence of such corrections. Below 500 MeV/c the correction is explicitly applied to the data analysis using the standard Tromborg procedure [58]. At higher energies a rough estimate of the magnitude is readily obtained using ray optics and assuming the total cross section produced by a black diffracting sphere of radius R (or a grey diffracting disk) with the Coulomb potential $V(R)$ evaluated at the surface [56]. This gives an increase in the difference of the experimental cross sections:

$$\sigma^T_{\pi^-{\rm p}} - \sigma^T_{\pi^+{\rm p}} \simeq A_c \, \left(\sigma^T_{\pi^-{\rm p}} + \sigma^T_{\pi^+{\rm p}}\right) \, \omega/k^2, \tag{22}$$

where $A_c = 3.7$ MeV/c for $R = 0.8$ fm. The correction factor is then about 1.5% at 500 MeV/c. The integrated contribution to the integral above a momentum k_1 and assuming a constant total cross section is $2A_c\omega\sigma^T(k)/k^2$, which typically gives $0.007k_1$ mb(GeV/c)$^{-1}$. A similar correction is included in the SM99 Coulomb corrections below 2 GeV/c. As an illustration, the resulting fits for the solution SM99 of Arndt *et al.* are shown in the range $0.5 \leq k_{lab} lab \leq 2$ GeV/c in Figs. 4 and 5. We have not made any correction for this effect at higher energies, which means that we systematically will somewhat underestimate the coupling constant. This effect has little bearing on

the issue of a large versus small coupling constant.

Since the recent VPI/GWU PWA solution up to 2 GeV/c [29], which includes penetration factors, is in good agreement with observations, we will use the hadronic cross sections from this solution as the main guide for the evaluation of the integral in the following and we will investigate possible uncertainties. We also give numbers from the earlier PWA solution SM95 [54] for comparison.

A. The threshold region

There are no direct measurements of total cross sections below 160 MeV/c, but the cross-section difference can be well reconstructed from other considerations to sufficient precision. In this range the low-energy s- and p-wave parameters determine the result. The cross-section difference at threshold is

$$\sigma^T_{\pi^-\mathrm{p}}(0) - \sigma^T_{\pi^+\mathrm{p}}(0) = 8\pi \ \left((a_{\pi^-\mathrm{p}})^2 - (a^+)^2\right), \tag{23}$$

assuming isospin invariance and neglecting the mass differences. Here the first term is accurately known from the π^-p atom, as previously discussed, and the second term is extremely small. With increasing energy the p-wave contributions of opposite sign, governed by the tail of the Δ resonance, take over and compensate the s-wave term beyond 100 MeV/c. These two terms contribute together +0.011 mb [29], but taken individually the s- and p-wave terms represent about 6% each of the total J^-. The uncertainty is dominated by the error in the rather small contribution from the s-wave range terms, while the accurate π^-p scattering length is imposed in the SM99 analysis. The corresponding error in J^-, of about 0.5%, is not a major source of overall uncertainty and even if this uncertainty is underestimated this has little importance.

The 3.3 MeV mass difference in the π^-p and π^0n thresholds breaks the isospin invariance leading to a potentially significant correction, in particular, since the π^-p total cross section diverges at threshold due to the open π^0n channel. The smallness of the contributions from the threshold region hints at a small correction. We have investigated this effect using a simplified model based on the s- and p-wave low-energy parameters only. The dispersion relation must now be evaluated using the imaginary part of the scattering amplitude $\mathrm{Im}F = 4\pi k\sigma^T$, which is well behaved at threshold, but which differs from zero below the physical π^-p threshold. The correction occurs predominantly in the 6% s-wave term. The approximate modification up to the momentum k_1=160 MeV/c is of O(-$\kappa^2/2k_1^2$) $\simeq -0.02$, where $\kappa^2 \simeq 0.045\ m_\pi^2$ is the π^0 squared momentum at threshold. This represents a -0.1% contribution to the integral J^-, which is negligible compared with other uncertainties.

B. The Δ resonance region

This is the main contribution to the integral and it must be accurately evaluated. The resonant 33 wave dominates heavily and its behavior is strongly constrained by other experiments and theory. The main contribution comes from the π^+p cross section, which is approximately three times larger than the π^-p one. The systematic uncertainties are more important than the statistical ones. In order to have an impression of their importance, we first evaluated this contribution directly from the experimental π^+ and π^- data sets taken separately, with statistical errors added in quadrature [62]. The result is $-1.656(6)$ mb. We also evaluated it from hadronic cross sections of the recent phase-shift solution SM99 [29]), which gives -1.696 mb; the latter method should eliminate minor inconsistencies and include the main Coulomb corrections. The difference of 0.040 mb is mainly due to the penetrability correction, which should be adequately included by the standard Tromborg treatment [58]. Even assuming a 30% uncertainty in the correction leads to an uncertainty of only $\pm$ 0.9% in the cross section integral. A modern analysis such as SM99 favors the use of the experimental cross sections dominated by the data of Pedroni *et al.* [59]; the cross sections from Carter *et al.* [61], which dominated the analysis in the 1980's, would lead to a more negative value for J^- and, correspondingly, to a πNN coupling constant larger by about 1%. The hadronic SM99 total cross sections have correctly eliminated the inverse photoproduction cross section, which contributes 1 mb (1.5 %) of the total nuclear π^-p cross section at the resonance peak. This electromagnetic correction has only a marginal influence on the present discussion at present precision.

Another isospin breaking effect is the Δ mass splitting, which may also affect the coupling constant deduced from determinations based on πN data. The magnitude of the splitting is apparent from the textbook difference of the $\pi^\pm$d cross sections by Pedroni *et al.* [59] (see also p. 107ff in Ref. [38]); the dominant Δ^{++} contribution is shifted to lower energies with a corresponding negative contribution to J^-. To leading order, the deuteron gives the sum of the cross sections on the neutron and the proton and thus the Pedroni data give a direct measure of the isovector mass splitting in a null experiment, i.e. it would give zero in the limit of isospin symmetry and no Coulomb effects.

TABLE VI. Evaluation of J^- in the Δ resonance region and up to 2 GeV/c. Here 'Data' refers to experimental cross sections uncorrected for Coulomb penetration and 'Nuclear SM99' to the corresponding PWA cross sections; I_- and I_+ are the corresponding integrals for π^-p and π^+p, respectively, with $J^- = I_- - I_+$.

Input	k [GeV/c]	I_- [mb]	I_+ [mb]	J^- [mb]
Hadronic SM95 [54]	0.00 to 0.16	0.164	0.157	0.007
Hadronic SM99 [29]	0.00 to 0.16	0.162	0.152	0.011
Hadronic SM95 [54]	0.16 to 0.55	1.078	2.763	−1.685
Hadronic SM99 [29]	0.16 to 0.55	1.071	2.767	−1.696
Nuclear SM99 [29]	0.16 to 0.55	1.090	2.726	−1.636
Data [62]	0.16 to 0.55	1.101	2.753	−1.656
Hadronic SM95 [54]	0.55 to 1.20	0.800	0.414	0.386
Hadronic SM99 [29]	0.55 to 1.20	0.789	0.411	0.378
Nuclear SM99 [29]	0.55 to 1.20	0.804	0.400	0.404
Data [62]	0.55 to 1.20	0.816	0.400	0.416
Hadronic SM95 [54]	1.20 to 2.00	0.450	0.460	−0.010
Hadronic SM99 [29]	1.20 to 2.00	0.451	0.458	−0.007
Nuclear SM99 [29]	1.20 to 2.00	0.458	0.450	0.008
Data [62]	1.20 to 2.00	0.459	0.443	0.016

The value of the splitting corresponds to $m_{\Delta^0} - m_{\Delta^+} = 1.3$ MeV. To our knowledge, there exists no information on the isotensor splitting, and we neglect it. With respect to strict isospin symmetry, with an effective position of the Δ resonance chosen as the average between the Δ^{++} and the Δ^0 mass, the correction to the resonance contribution is of the order of $4(m_{\Delta^+} - m_{\Delta^0})/3m_\Delta \simeq -0.6\%$, which would *increase* the coupling constant $g_c^2/4\pi$ by 0.2%. This conclusion is borne out by the analysis of SM99 with the same effective Δ position. Arndt informed us that in this analysis the Δ^0–Δ^{++} splitting was taken to be 0.4 MeV and the effect on J^- is only 0.05 % [60]. If we adjust this result to the observed mass splitting 2.7 MeV, the contribution is 0.4 %, in qualitative agreement with our estimate above. Consequently, we consider that the small systematic effect of the mass splitting will not substantially influence any of our conclusions, provided any good modern analysis of the data is used.

C. The resonance region

This region from 550 MeV/c to 2 GeV/c is, as a whole, well measured and is analyzed in SM99. The contributions to the integral are positive and rather important up to 1.2 GeV/c, partly compensating the contribution from the Δ resonance region. The remaining region contributes little, but is a minor source of uncertainties.

It is interesting to quantify the difference between SM99 and data in more detail (see Table VI). In the region of 550 MeV/c to 1.2 GeV/c the nuclear SM99 solution gives 0.012 mb less contribution to J^- (−1.1%) than the direct experimental cross sections, while from 1.2 to 2 GeV/c the corresponding contribution is 0.008 mb less (−0.7%). The overall Coulomb corrections to the integral in this region represent 0.041 mb or 4% of the total J^-. Even if these corrections are, pessimistically, accurate only to 33%, the overall uncertainty from this source is only ±0.014 mb in J^- or ±0.4% in the coupling constant. The main uncertainty is therefore due to the difference between SM99 and experiments. It comes mainly from the region just above 500 MeV/c, as will now be discussed.

At the low energy end of the region between 550 and 1200 MeV/c, there are long-standing experimental problems of systematic nature with the total cross section data. Those of Davidson *et al.* [63] have an incorrect energy calibration, too low by about 10 MeV/c, and its 72 data points must either be re-calibrated or eliminated from the analysis [12,24,70]. Similarly, the SM99 solution, driven by modern angular distributions, is systematically lower than the π^-p data of Carter *et al.* [69] below 700 MeV/c, a region where data for experimental reasons, are less reliable than at higher energies. These points have been omitted from the PWA analysis [12,70] (see also Fig. 4). This discrepancy, which is larger than the Coulomb penetrability effects, should be resolved. Under the circumstances we have preferred to use the SM99 PWA solution as the best guide, but we use the difference as a measure of the present uncertainty. We therefore use an overall contribution from this region of 0.378 mb and ascribe to it a liberal uncertainty of ± 0.0020 mb.

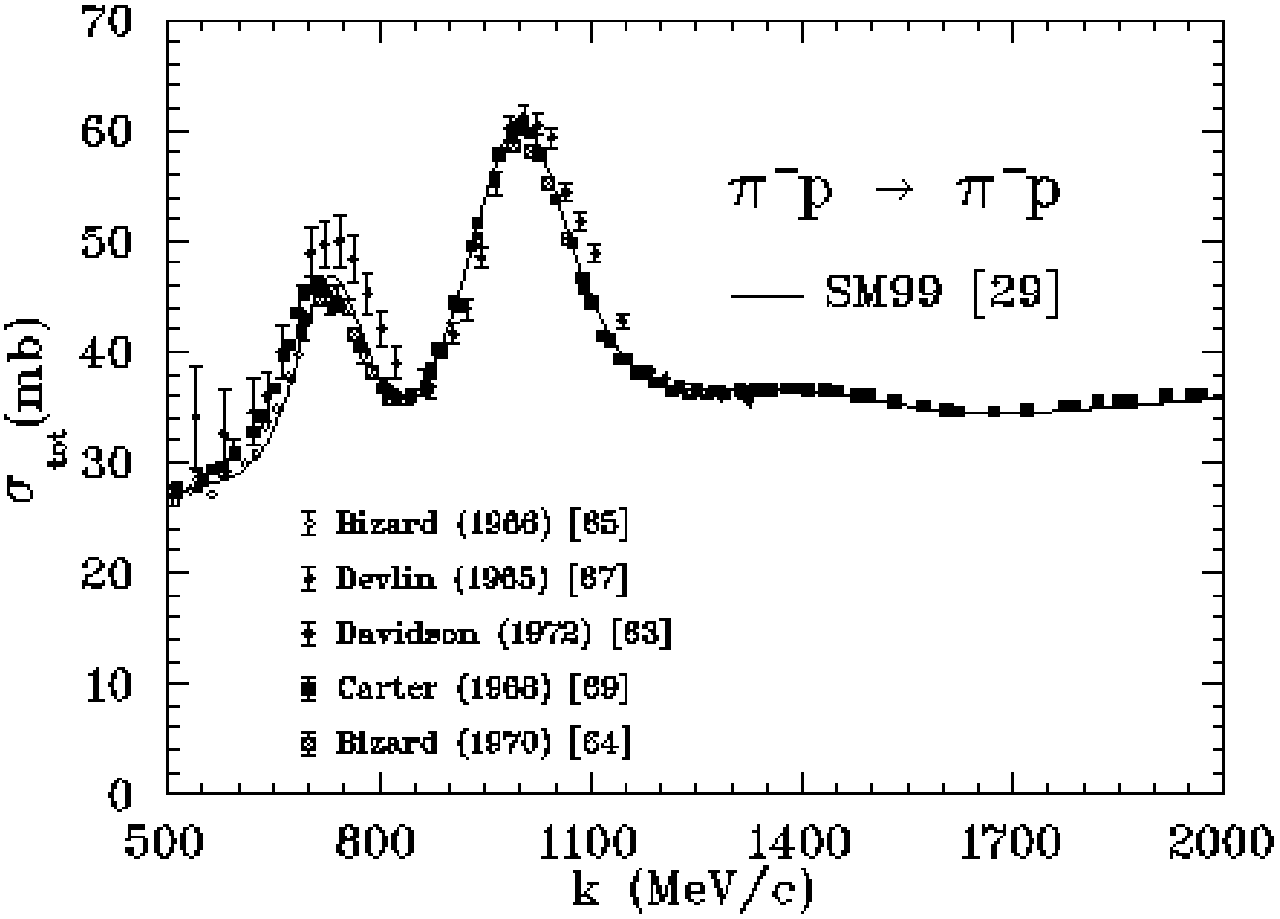

FIG. 4. The π^-p experimental total cross sections in the region $0.5 \leq k \leq 2$ GeV/c compared to SM99 with Coulomb barrier effects accounted for.

D. The high-energy and asymptotic regions

There exists abundant experimental information on $\pi^{\pm}$p cross sections to high precision from 2 GeV/c up to 350 GeV/c. The main uncertainty in J^- in this region is associated with the relatively slow convergence of the integral. At energies beyond 4 GeV/c there has been an important effort to measure and analyze cross sections, since the issue of the rate at which the $\pi^{\pm}$p cross sections become asymptotically equal, is important theoretically for asymptotic theorems. For the moment we neglect the question of Coulomb barrier corrections. The discussion below is summarized in Table VII.

The region $2 \leq k \leq 4.03$ GeV/c has been evaluated from data and gives a moderate contribution of 0.064 mb, with a modest ± 0.007 mb systematic error using the Particle Data Group (PDG) 1998 tables [62]. The statistical uncertainty is small. Beyond this region cross section data with considerable systematic and statistical accuracy exist from $4.03 \leq k \leq 370$ GeV/c and are listed in the PDG tables [62,71]. We first evaluated the contribution directly from the precision data. This gives 0.133 mb in the range $4.03 \leq k \leq 240$ GeV/c, with a small statistical error and a systematic error of about ± 0.022 mb or $\pm 1.8\%$ in J^-.

In addition, the 1994 version of the PDG tables [71] also lists a fit to these data from 4.03 GeV/c (Table 33.3). Using the fitted expression, we have evaluated the contribution in the same interval as above using this expression and find 0.155 mb. This is 0.022 mb higher than the value of 0.133 mb by direct evaluation, but in good general agreement. This larger value has been used in several previous GMO evaluations [9,16]. Since the lower value is more transparently linked to the actual data, we believe it to be somewhat preferable.

Finally, there is a small, but not negligible, contribution from the very high-energy region from 240 GeV/c to ∞. We determine this from the Donnachie–Landshoff Regge fit to the data [72], which describes the observed cross section difference well at the highest energies. This fit is a sum of two Regge terms, one arising from Pomeron exchange and the second from lower-lying resonance exchange. It gives a contribution of 0.030 mb. Alternatively, one might

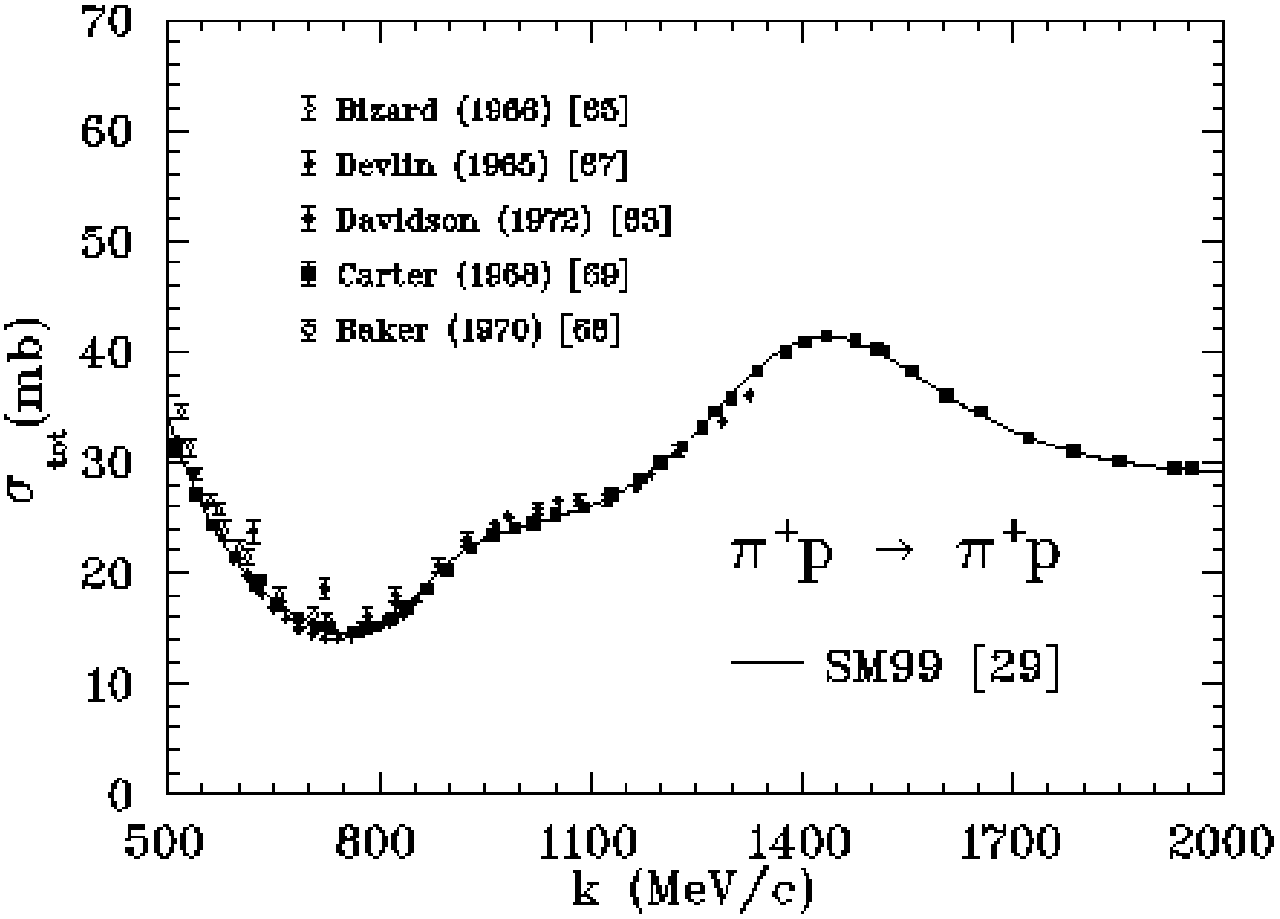

FIG. 5. The π^+p experimental total cross sections in the region $0.5 \leq k \leq 2$ GeV/c compared to SM99 with Coulomb barrier effects accounted for.

consider using the three-term fit (one for the Pomeron and two for the Reggeons) in the 1998 PDG tables (Table 38.2), which gives 0.018 mb. This low value is not surprising, since the 1998 PDG parameterization gives a difference 27% lower than the PDG 1994 one in the region above 200 GeV/c, at variance with the data [74]. At lower energies, this parameterization agrees better with the data in the region of 100 GeV/c [73,74]. We have also used a recent high-energy fit based on a two-Pomeron pole expression fully compatible with universality, Regge factorization, weak Regge exchange degeneracy, and generalized Vector Dominance Model [75]. This parameterization (see Eq. (13) and Table 1 of Ref. [75]) gives a contribution of 0.025 mb. The corresponding uncertainties, given in Table VII, come mainly from the 4% uncertainty in the Regge intercept. This spread of values according to the model considered for the fit introduces an additional systematic uncertainty of 0.006 mb from this high-energy region.

The Coulomb corrections are expected to be negligible in the high-energy region. This is confirmed by an estimate using the approximate Coulomb penetration correction (22) and integrating it from 2 GeV/c to ∞. This gives a 0.003 mb contribution.

E. Summary of the results for J^-

We have evaluated the various contributions to J^- with no Coulomb corrections other than those introduced by the experimental authors above 2 GeV or by the theoretical analysis below 2 GeV. The results are summarized in Table VII. The statistical uncertainty in the evaluation is small. The trapezoidal formula has been used to integrate the data and the corresponding statistical errors have been added quadratically. Within the different integration ranges, as given in Table VII, the systematic error has been calculated by varying the experimental results inside the interval defined by the quoted systematic error. This procedure maximizes this error in a given momentum range. The full systematic uncertainty is thus obtained by the quadratic sum of the error in each interval. The systematic

TABLE VII. Different contributions to J^- as function of the k range and of the input data. The first number in the parenthesis is the statistical error, while the second numbers correspond to the systematic uncertainty. The selected data correspond to the world data as given by PDG Tables, where we have suppressed all data with statistical and systematic errors larger than 1%.

i	Input	k(GeV/c)	I_-(mb)	I_+(mb)	J^-(mb) $= I_- - I_+$
1	SM95 [54]	0.00 to 0.16	0.164	0.157	0.007
2	SM99 [29]	"	0.165	0.152	0.013
3	Selected [62]	0.16 to 2.00	2.360 (2) (3)	3.596 (6) (1)	−1.237 (6) (4)
4	Data [59,61]	"	2.377 (3) (2)	3.596 (5) (2)	−1.219 (6) (4)
5	SM95 [54]	0.00 to 2.00	2.491	3.794	−1.302 (6)(20)
6	SM99 [29]	"	2.476	3.804	−1.314 (6)(20)
7	Selected [62]	2.00 to 4.03	0.560 (2) (3)	0.496 (1) (5)	0.064 (2) (7)
8	Data [69,76]	"	0.580 (1) (5)	0.518 (1) (5)	0.063 (1)(10)
9	Selected [62]	4.03 to 240	2.672 (4)(10)	2.539 (3)(12)	0.133 (5)(22)
10	Fit PDG94 [71]	"	2.645	2.489	0.155
11	Regge 94 [71]	240 to ∞	–	–	0.030 (5)
12	Regge 00 [75]	"	–	–	0.025 (4)
13	Regge 98 [62]	"	–	–	0.018 (3)
14	6+7+10+11	0 to ∞	–	–	−1.055(10)(31)
15	6+7+9+11	"	–	–	−1.087 (9)(31)
16	6+7+9+13	"	–	–	−1.099 (8)(31)
17	6+7+9+12	"	–	–	−1.092 (9)(31)

uncertainty has its origin principally in the region above 4 GeV/c. There is also a sizeable systematic uncertainty that is due to the inconsistencies of the π^-p data in the region 550-700 MeV/c, although we have probably overestimated this uncertainty. We find from Table VII, rows 15–17, that three different descriptions, based on the SM99 PWA below 2 GeV/c, give values in a rather narrow range; $-1.087 \pm 0.009 \pm 0.031$ mb, $-1.099 \pm 0.008 \pm 0.031$ mb and $-1.092 \pm 0.009 \pm 0.031$ mb. The difference between these values is smaller than the estimated systematic uncertainty. We also give in row 14 the less negative result obtained with the older SM95 PWA below 2 GeV/c and the fit PDG94 in the momentum range from 4.03 to 240 GeV/c: $J^- = -1.053 \pm 0.010 \pm 0.031$ mb. We have chosen the average of these four values as characteristic of the integral. The systematic uncertainty should provide an adequate band of possible values, so that

$$J^- = -1.083 \pm 0.009 \pm 0.031 \text{ mb}. \tag{24}$$

Our result for J^- is close to the unpublished value of Koch [28], J $^- = -1.077 \pm 0.047$ mb, which is the only previous explicitly documented and detailed evaluation known to us. The main difference with Koch is an updated evaluation of the contributions from the high-energy region and a detailed discussion of the threshold region using modern data. We note that the evaluations of this quantity by different authors quoted in Table V stay within our stated errors.

VI. RESULTS

In conclusion, we summarize our work as follows. We first derived new values for the πN scattering lengths, using the π^-d atomic data analyzed in a substantially improved theoretical approach. The statistical and systematic contributions were carefully examined. By this analysis we have achieved the most accurate and best controlled determination to date of both the isovector and isoscalar scattering lengths. In fact when we examine the basic experimental input of the highly accurately quoted scattering length $a_{\pi^-\mathrm{p}}$, deduced from the π^-p atomic energy shift and quoted to high accuracy [31], we found that there are small inconsistencies in their current procedure at the level of $\pm 1\%$. This should be improved, since the precision is otherwise unsatisfactory for the determination of the πNN coupling constant. In addition, the experimental accuracy is now so high that systematics in the theoretical analysis of the π^-d scattering length is the main source of uncertainty in the disentangling of the isospin components of the πN scattering length. The dominant limitation to higher accuracy is the dispersive contribution from the physical absorption process $\pi^-\mathrm{d} \to \mathrm{nn}$. A thorough modern re-examination of this contribution is highly desirable. Our analysis does not assume strict isospin symmetry, although we do not see any signs of violation at the present level

of precision. We present the results, however, so that they can be directly used in discussions of the validity of this symmetry. The values we find using the empirical π^-p and π^-d scattering lengths from subsection 4.1, Eqs. (20) and (21), are

$$a^+ \simeq \frac{a_{\pi^-\mathrm{p}} + a_{\pi^-\mathrm{n}}}{2} = (-17 \pm 2 \pm 8) \times 10^{-4}\ m_\pi^{-1}. \tag{25}$$

$$a^- \simeq \frac{a_{\pi^-\mathrm{p}} - a_{\pi^-\mathrm{n}}}{2} = (900 \pm 3 \pm 13) \times 10^{-4}\ m_\pi^{-1}. \tag{26}$$

These values are based on a substantial theoretical improvement on previous work. Consequently our values supersede previous values deduced using the data from the πd atom and older approaches. The margin for modification of our theoretical analysis is small.

Our second conclusion concerns the charged πNN coupling constant, which can be derived from the GMO forward dispersion relation, using our new, accurate values for the πN scattering lengths. According to Eq. (4) using $J^- = (-1.083 \pm 0.009 \pm 0.031)$ from Table V and charge symmetry:

$$g_c^2/4\pi = (4.87 \pm 0.04 \pm 0.14) + (9.12 \pm 0.02 \pm 0.10) + (0.18 \pm 0.02 \pm 0.08) = (14.17 \pm 0.05 \pm 0.19). \tag{27}$$

The main uncertainty is no longer dominated by the scattering lengths, but comes largely from the weighted integral J^- of the difference between the charged-pion total cross sections. Its dominant systematic uncertainty comes from the region above 4 GeV/c. Previous determinations using the GMO relation [11,15,16,25] will all give similar results, provided one uses the empirical scattering lengths, which are by now well established. They are superseded unless persuasive arguments are given for the use of a different value of the scattering length a^+.

The value, $g_c^2/4\pi = 14.17$, which we obtain for the coupling constant is intermediate between the low value of 13.6 deduced from the large data banks of NN and πN scattering data using the PWA approach [13,6] and the high value of 14.5 from np charge exchange cross sections [2]. It is compatible with the latter, differing by about one standard deviation. As for the former, the systematic uncertainty of the value deduced from PWA descriptions has not been investigated and the PWA analysis does not single out pion-dominated aspects. To judge from the corresponding situation using the data banks with dispersive constraints [12], the PWA systematic errors are far larger than the small statistical errors, but this is not quantified yet. The modification of the value of J^- required to accommodate a value of 13.6 is about 10%. The major part of such a modification would most likely come from the region above 2 GeV/c, which implies changes in the contributions from that region of the order of 50%. Such large changes appear unlikely to us.

We therefore conclude that the present evaluation of the GMO sum rule, with quantitatively controlled uncertainties in the input values for the πN isoscalar scattering length, as well as for the cross section integral J^-, does not readily support the conclusion of the indirect PWA determinations that the πNN coupling is in the range of 13.6. It should be noted that our value has consistently been evaluated in a conservative way, such that the parameters used in the evaluation systematically lead to a value for the coupling constant, which is somewhat on the low side.

The strongest support for a relatively low value of the coupling constant comes from the careful dispersive analysis by Pavan *et al.* [12], based on the VPI/GWU PWA description of πN scattering. It selectively concentrates on pion-dominated amplitudes. They find a value of 13.73 ± 0.01 ± 0.08, where the first uncertainty is statistical and the second systematic. The authors use a variety of dispersive methods and find $a^+ = +0.0020 \times 10^{-4}\ m_\pi^{-1}$. This value is small, but it has the opposite sign from ours. They evaluate the GMO relation as a consistency check and find a value of $g_c^2/4\pi = 13.75$, in agreement with their dispersion result. Since their evaluation is constrained by the experimental π^-p scattering length and their value for the dispersive integral J^- is nearly the same as ours, which is based to a large extent on their PWA analysis, the difference with our result must be almost entirely ascribed to the difference in the value of a^+, a small quantity, which is difficult to calculate from scattering data. The origin of this difference is not known yet, but it might originate in the treatment of small electromagnetic corrections to the scattering data. The minor inconsistency in their analysis has little importance for most of their discussions, but it becomes highly relevant in the present context.

It is interesting to examine the consequences of our analysis for the Goldberger–Treiman (GT) discrepancy [1]. Following the discussion in Ref. [2] the value for the coupling constant found here corresponds to a discrepancy of $\Delta_{\mathrm{GT}} = (3.8 \pm 2.5)\%$, with Δ_{GT} defined as

$$g_c\ (1 - \Delta_{\mathrm{GT}}) = \mathrm{M}\ g_{\mathrm{A}}/f_\pi.$$

This corresponds to a πNN monopole form factor with a cut-off $\Lambda = 800 \pm 80$ MeV/c. There exists no direct experimental information on this form factor, which is inherently an off-mass-shell quantity. On the other hand,

within the framework of PCAC, it is naturally expected to be similar to the axial form factor of the nucleon, a dipole with a 1 GeV/c cut-off. This expectation has been confirmed in many models, using a variety of approaches [77–81], beginning with Ref. [77]. Such values are fully consistent with our findings for the coupling constant. In contrast to these rather soft form factors, the deuteron properties, and in particular its quadrupole moment, require an effective cut-off of 1.3 GeV/c or more, since the tensor force otherwise becomes too weak [50,82]. It is, however, believed at present that this hard effective form factor is generated by the correlated exchange of an interacting $\pi\rho$-pair, which generates additional tensor strength, when explicitly accounted for: the true one-pion-exchange form factor is softer [84–87]. A low value for the coupling constant should therefore not be considered an advantage in resolving the Goldberger–Treiman discrepancy.

An attempt has been made to use the GT relation generalized to the strange sector (the Dashen–Weinstein relation) to obtain information on the coupling constant [88]. This carries little information on $g_c^2/4\pi$, since the kaon is not soft. The typical corrections of order 30 to 40% make the relation irrelevant as a constraint [89].

Additional support for a relatively large coupling constant comes from the recent measurements by Raichle *et al.* of polarized np total cross sections [90]. From these, the pion-dominated ϵ_1 parameter can be determined. They find that it is systematically larger than the values in the old phase-shift analysis PWA93 of the Nijmegen group [5]. If the discrepancy persists in other PWAs, this observation suggests, as a possible partial explanation, that the PWA coupling constant is too small. In any case, it points to an unexplained discrepancy with those PWA analyses on which the argument for a low coupling constant is based, as is the case also with the Uppsala np elastic data.

In order to facilitate future improvements on the present work, we have presented the various corrections in such a way that modifications of any individual contributions can be readily incorporated without the necessity of a complete re-analysis. We see three main areas in which the present work can be improved. First, theoretical investigations of the relation between the hadronic energy shift of the pionic atom and the scattering length should diminish the present uncertainty in the deduced π^-p scattering length by a factor of at least 2. Second, the measurement to high precision of the width in pionic hydrogen should give a separation of the isospin components in the π^-p scattering lengths to similar precision as that obtained from the deuteron data, but without invoking deuteron structure. Third, studies of the dispersion shift for threshold pion absorption on the deuteron should eliminate a major uncertainty in the theoretical treatment of the π^-d scattering length. This would allow the πNN coupling constant to be determined to 1% precision.

VII. ACKNOWLEDGMENTS

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APPENDIX A: PRACTICAL EXPRESSIONS FOR THE THEORETICAL π^-D SCATTERING LENGTH FOR SEPARABLE SCATTERING INTERACTIONS

We give here full practical expressions for the theoretical π^-d scattering length for separable scattering amplitudes with a dipole form factor $v^2(q) = (1 + q^2/c^2)^{-2}$:

$$a_{\pi^-\mathrm{d}} = \alpha\,(a_{\pi^-\mathrm{p}} + a_{\pi^-\mathrm{n}}) + \beta\left[\left(\frac{a_{\pi^-\mathrm{p}} + a_{\pi^-\mathrm{n}}}{2}\right)^2 - 2\left(\frac{a_{\pi^-\mathrm{p}} - a_{\pi^-\mathrm{n}}}{2}\right)^2\right]\langle f(r)/r\rangle + a(\mathrm{Fermi}) + a(\mathrm{dispersion}) + \delta a, \tag{A1}$$

where

$$\alpha = \frac{(1 + m_\pi/M)}{(1 + m_\pi/M_\mathrm{d})} = 1.0691; \tag{A2}$$

$$\beta = 2\,\frac{(1 + m_\pi/M)^2}{(1 + m_\pi/M_\mathrm{d})} = 2.4560; \tag{A3}$$

$$f(r) = 1 - (1 + cr/2)\exp(-cr) \tag{A4}$$

and with the sum of small correction terms

$$\delta a \;=\; \delta a(\text{multiple}) + \delta a(\text{isospin}) + \delta a(\text{non}-\text{static}) + \delta a(\text{double p}-\text{wave}) + \delta a(\text{virtual pion}). \tag{A5}$$

Assuming isospin symmetry in all terms but the leading order one, and emphasizing the accurate experimental knowledge of $a_{\pi^-\mathrm{p}}$, we have

$$a_{\pi^-\mathrm{d}} \;=\; \alpha\;(a_{\pi^-\mathrm{p}} + a_{\pi^-\mathrm{n}}) \;+\beta\;\left[a^{+2} - 2\;(a_{\pi^-\mathrm{p}} - a^+)^2\right]\langle f(r)/r\rangle \;+\; a(\text{Fermi}) \;+\; a(\text{dispersion}) \;+\; \delta a. \tag{A6}$$

The correction for nucleon motion is, according to Eq. (14):

$$a(\text{Fermi}) \;=\; 2\left(\frac{m_\pi}{M+m_\pi}\right)^2 \alpha\; c_0\;\left\langle p^2\; v^2\left(\frac{m_\pi \mathrm{p}}{M+m_\pi}\right)\right\rangle, \tag{A7}$$

where the form factor correction is negligible and $c_0 = 0.208(3)\; m_\pi^{-3}$ (p.18 in Ref. [38]). The dispersion correction has been taken to be $a(\text{dispersion}) = -56(14)\times 10^{-4}\, m_\pi^{-1}$ [40]. The remaining values of the small terms are taken to be (see text) $\delta a(\text{non}-\text{static}) = 11(6)\times 10^{-4}\; m_\pi^{-1}$, $\delta a(\text{double p}-\text{wave}) = -3\times 10^{-4}\; m_\pi^{-1}$ and $\delta a(\text{virtual pion}) = -7(2)\times 10^{-4}\, m_\pi^{-1}$. In addition it is desirable to control the convergence of the multiple scattering expansion explicitly. We have evaluated the higher order multiple scattering corrections from the expression given by Kolybasov and Kudryatsev for the sum of the multiple scattering series to all orders for point-like scatterers, in the static approximation and neglecting binding and recoil corrections [83]. We have however generalized their expression to include the effect of separable form factors for each scattering:

$$\delta a(\text{multiple}) \;=\; \left\langle\left[2\;\alpha\;a^+ \;+\beta\;\left[a^{+2} - 2\;(a_{\pi^-\mathrm{p}} - a^+)^2\right]\;\frac{f(r)}{r}\right]\left[(1-C)^{-1} - 1\right]\right\rangle, \tag{A8}$$

where $C = (1 + m_\pi/M)^2\;\left[a^{+2} - 2\;(a_{\pi^-\mathrm{p}} - a^+)^2\right]\, f^2(r)/r^2$. In order to extract the value of $(a_{\pi^-\mathrm{p}} + a_{\pi^+\mathrm{p}})/2$ from the experimental $a_{\pi^-\mathrm{d}}$ and $a_{\pi^-\mathrm{p}}$, we now observe that Eq. (A6) is quadratic in a^+ except for higher power terms from the small $\delta a(\text{multiple})$ of Eq. (A8). To check the self-consistency with $\delta a(\text{multiple})$it should be solved iteratively. We have done this with the experimental values and the resulting a^+ is small (about $10^{-3}\; m_\pi^{-1}$). Equation (A6) can then be safely linearized for a fixed value of $\delta a(\text{multiple})$ and the consistency checked by iteration. We have

$$\frac{a_{\pi^-\mathrm{p}} + a_{\pi^-\mathrm{n}}}{2} \;=\left(2\;\alpha + 4\;\beta\;a^{\exp}_{\pi^-\mathrm{p}}\;\langle f(r)/r\rangle\right)^{-1}\left[a^{exp}_{\pi^-\mathrm{d}} +\; 2\;\beta\;{a^{\exp}_{\pi^-\mathrm{p}}}^2\;\langle f(r)/r\rangle - a(\text{Fermi}) \;-\; a(\text{dispersion}) \;-\; \delta a\right]. \tag{A9}$$

Two iterations are sufficient.

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[1] M. L. Goldberger and S. B. Treiman, Phys. Rev. **110**, 1178 (1958).
[2] J. Rahm, J. Blomgren, H. Condé, S. Dangtip, K. Elmgren, N. Olsson, T. Rönnqvist, R. Zorro, A. Ringbom, G. Tibell *et al*, Phys. Rev. C **57**, 1077 (1998).
[3] R. Koch and E. Pietarinen, Nucl. Phys. **A336**, 331 (1980).
[4] P. Kroll, Physics Data **22-1**, Eds. H. Behrens and G. Ebel (Fachinformationszentrum, Karlsruhe, 1981).
[5] V. Stoks, R. Timmermans and J. J. de Swart, Phys. Rev. C **47**, 512 (1993).
[6] V. Stoks, R. Timmermans and J.J. de Swart, IIN Newslett. **13**, 96 (1997).
[7] J. R. Bergervoet, P. C. van Campen, R. A. M. Klomp, J.-L. de Kok, T. A. Rijken, V. G. J. Stoks and J. J. de Swart, Phys. Rev. C **41**, 1435 (1990).
[8] R. A. M. Klomp, V. G. J. Stoks and J. J. de Swart, Phys. Rev. C **44**, R1258 (1991).
[9] R. A. Arndt, I. I. Strakovsky and R. L. Workman, Phys. Rev. C **50**, 2731 (1994).
[10] R. A. Arndt, R. L. Workman and M. M. Pavan, Phys. Rev. C **49**, 2729 (1994).
[11] R. L. Workman, R. A. Arndt and M. M. Pavan, Phys. Rev. Lett. **68**, 1653 (1992); Erratum **68**, 2712 (1992).
[12] M. M. Pavan and R. A. Arndt, IIN Newslett. **15**, 171 (1999).
[13] R. G. E. Timmermans, IIN Newslett. **13**, 80 (1997).
[14] M. L. Goldberger, H. Miyazawa and R. Oehme, Phys. Rev. **99**, 986 (1955).

[15] H. C. Schröder, A. Badertscher, P.F.A. Goudsmit, M. Janousch, H.J. Leisi, E. Matsinos, D. Sigg, Z.G. Zhao, D. Chatellard, J.P. Egger *et al.*, Phys. Lett. B **469**, 25 (1999).
[16] W. R. Gibbs, Li Ai and W. B. Kaufmann, Phys. Rev. C **57**, 784 (1998).
[17] T. E. O. Ericson, B. Loiseau, J. Nilsson, N. Olsson, J. Blomgren, H. Condé, K. Elmgren, O. Jonsson, L. Nilsson, P. Renberg, *et al.*, Phys. Rev. Lett. **75**, 1046 (1995).
[18] M. M. Pavan, R. A. Arndt, I. I. Strakovsky and R. L. Workman, nucl-th/9910040, Physica Scripta **87**, 65 (2000).
[19] R. A. Arndt, I. I. Strakovsky, R. L. Workman and M. M. Pavan, nucl-th/9908057, Physica Scripta **87**, 62 (2000).
[20] T. E. O. Ericson, B. Loiseau and A. W. Thomas, Physica Scripta **87**, 71 (2000).
[21] B. Loiseau and T. E. Ericson, hep-ph/0002085, Physica Scripta **87**, 53 (2000).
[22] R. Machleidt, nucl-th/9909036, Physica Scripta **87**, 47 (2000).
[23] M. E. Sainio, IIN Newslett. **15**, 156 (1999).
[24] G. Höhler in *Pion–Nucleon Scattering*, Ed. H. Schopper, in Landolt-Börnstein, New Series, Vol. **9b** (Springer, New York 1983).
[25] R. A. Arndt, I. I. Strakovsky, R. Workman and M. M. Pavan, Phys. Rev. C **52**, 2120 (1995).
[26] S. Weinberg, Trans. N. Y. Acad. Sci. **38**, 185 (1977).
[27] N. Fettes, U.-G. Meissner and S. Steininger, Phys. Lett. B **451**, 233 (1999).
[28] R. Koch, Karlsruhe preprint TKP 85-5 (unpublished) (1985).
[29] R. A. Arndt, M. M. Pavan, R. L. Workman, and I. I. Strakovsky, Scattering Interactive Dial-Up (SAID), VPI, Blacksburg, The VPI/GWU πN solution SM99 (1999); http://said.phys.vt.edu/analysis/pin_analysis.html.
[30] M. M. Pavan, R. A. Arndt, I. I. Strakovsky and R. L. Workman, IIN Newslett. **15**, 118 (1999).
[31] D. Sigg, A. Badertscher, M. Bogdan, P. F. A. Goudsmit, H. J. L. Leisi, H. C. Schroeder, Z. G. Zhao, D. Chatellard, J. P. Egger, E. Jeannet *et al.*, Nucl. Phys. **A609**, 269 (1996).
[32] T. E. O. Ericson and L. Tauscher, Phys. Lett. B **112**, 425 (1982).
[33] D. Chatellard, J.-P. Egger, E. Jeannet, A. Badertscher, M. Bogdan, P. F. A. Goudsmit, M. Janousch, H. J. Leisi, E. Matsinos, H. C. Schroeder *et al.*, Nucl. Phys. **A625**, 855 (1997).
[34] D. Sigg, A. Badertscher, P. F. A. Goudsmit, H. J. L. Leisi and G. C. Oades, Nucl. Phys. **A609**, 310 (1996).
[35] J. Gasser, V. E. Lyubitskij and A. Russetsky, in preparation and private communication (2000).
[36] T. E. O. Ericson, B. Loiseau and S. Wycech, in progress.
[37] P. Hauser, K. Kirch, L. M. Simons, G. Borchert, D. Gotta, T. Siems, P. El-Khoury, P. Indelicato, M. Augsburger, D. Chatellard *et al.*, Phys. Rev. C **58**, R1869 (1998).
[38] T. E. O. Ericson and W. Weise, *Pions and Nuclei*, Clarendon Press, Oxford (1988).
[39] I. R. Afnan and A. W. Thomas, Phys. Rev. C **10**, 109 (1974).
[40] T. Mizutani and D. Koltun, Ann. Phys. (NY) **109**, 1 (1977).
[41] A. W. Thomas and R. H. Landau, Phys. Rep. **58**, 121 (1980).
[42] G. Fäldt, Physica Scripta **16**, 81 (1977).
[43] S. Wycech, private communication (2000).
[44] V. V. Baru and A. E. Kudryatsev, Phys. Atom. Nucl. **60**, 1475 (1997).
[45] V. V. Baru and A. E. Kudryatsev, private communication (1999).
[46] S. R. Beane, V. Bernard, T.-S. Lee and U.-G. Meissner, Phys. Rev. C **57**, 424 (1998).
[47] S. Weinberg, Nucl. Phys. **B295**, 114 (1992).
[48] N. Fettes, U.-G. Meissner and S. Steininger, Nucl. Phys. **A460**. 199 (1998).
[49] M. Lacombe, B. Loiseau, R. Vinh Mau, J. Côté, P. Pirés, and R. de Tourreil, Phys. Lett. B **101**, 139 (1981).
[50] R. Machleidt, K. Holinde and Ch. Elster, Phys. Rep. **149**, 1 (1987), tables 11 and 13.
[51] T. E. O. Ericson and M. Rosa-Clot, Annu. Rev. Nucl. Part. Sci. **35**, 271 (1985).
[52] M. R. Robilotta and C. J. Wilkin, J. Phys.:G **4**, L115 (1978).
[53] N. Fettes and E. Matsinos, Phys. Rev. C **55**, 464 (1997).
[54] R. A. Arndt, M.M. Pavan, R. L. Workman, and I. I. Strakovsky, Scattering Interactive Dial-Up (SAID), VPI, Blacksburg, πN solution SM95 (1995): http://said.phys.vt.edu/analysis/pin_analysis.html.
[55] E. Matsinos, Phys. Rev. **C56**, 3014 (1997).
[56] D. V. Bugg and A. A. Carter, Phys. Lett. **44B**, 67 (1974).
[57] D. Bugg, IIN Newslett. **15**, 319 (1999), and private communication (1999).
[58] B. Tromborg, S. Waldenstrom and I. Overbo, Helv. Phys. Acta **51**, 584 (1978).
[59] E. Pedroni, K. Gabathuler, J. J. Domingo, W. Hirt, P. Schwaller, J. Arvieux, C.H.Q. Ingram, P. Gretillat, J. Piffaretti, N.W. Tanner, and C. Wilkin, Nucl. Phys. **A300**, 321 (1978).
[60] A. Arndt, private communication (1999).
[61] A. A. Carter, J. R. Williams, D. V. Bugg, P. J. Bussey and D. R. Dance, Nucl. Phys. **B26**, 445 (1971).
[62] Review of Particle Physics, C. Caso, S: C. Caso, G. Conforto, A. Gurtu, M. Aguilar-Benitez, C. Amsler, R.M. Barnett, P.R. Burchat, C.D. Carone, O. Dahl, M. Doser *et al.*, Eur. Phys. J. **C3**, 205 (1998); http::://pdg.lbl.gov/xsect/contents.html.
[63] D. Davidson, T. Bowen, P. K. Caldwell, E. W. Jenkins, R. M. Kalbach, D. V. Petersen, A. E. Pifer, and R. E. Rotschild, Phys. Rev. D **6**, 1199 (1972).

[64] G. Bizard, Y. Déclais, J. L. Duchon, J. L. Laville, J. Séginot, C. Bricman, J. M. Pereau, and C. Valadas, Phys. Lett. **31B**, 481 (1970).
[65] G. Bizard, J. Duchon, J. Séguinot, J. Yonnet, P. Bareyre, C. Bricman, G. Valadas, and G. Villet, Nuovo Cimento **44**, 999 (1966).
[66] V. G. Zinov, A. D. Konin, S. M. Korenchenko, and B. Pontecorvo, Sov. Phys.–JETP **11**, 1233 (1960).
[67] T. J. Devlin, J. Solomon, and G. Bertsch, Phys. Rev. Lett. **14**, 1031 (1965).
[68] S. L. Baker, P. J. Dornan, G. P. Gopal, G. J. Webster, V. A. Bull, P.V. March, V. Tayler, and D. W. Townsend, Nucl. Phys. **B18**, 29 (1970).
[69] A. A. Carter, K. F. Riley, R. J. Tapper, D. V. Bugg, R. S. Gilmore, K. M. Knight, D. C. Salter, G. H. Stafford, E. J. N. Wilson, J. D. Davies *et al.*, Phys. Rev. **168**, 1457 (1968).
[70] M. Pavan, private communication (1999).
[71] Review of Particle Properties, Phys. Rev. D **50**, 1335 (1994).
[72] A. Donnachie and P. V. Landshoff, Phys. Lett. B **296**, 227 (1992).
[73] A. S. Carroll, I-H. Chiang, T. F. Kycia, K. K. Li, P. O. Mazur, P. Mockett, D. C. Rahm, R. Rubinstein, W. F. Baker, D. P. Eartly *et al.*, Phys. Rev. Lett. **33**, 928 (1974).
[74] A. S. Carroll, I. H. Chiang, T. F. Kycia, K. K. Li, M. D. Marx, D. C. Rahm, W. F. Baker, D. P. Eartley, G. Giacomelli, A. M. Jonckheere *et al.*, Phys. Lett. **80B**, 423 (1979).
[75] P. Gauron, and B. Nicolescu, Phys. Lett. B **486**, 71 (2000), and private communication (2000).
[76] A. Citron, W. Galbraith, T. F. Kycia, B. A. Leontic, R. H. Phillips, A. Rousset and P. H. Sharp, Phys. Rev. **144**, 1101 (1966).
[77] P. A. Guichon, G. A. Miller and A. W. Thomas, Phys. Lett. B **124**, 109 (1983).
[78] H. Böckman, C. Hanhart, O. Krehl, S. Krewald and J. Speth, Phys. Rev. C **60**, 055212 (1999).
[79] K. F. Liu, S. J. Dong, T. Draper and W. Wilcox, Phys. Rev. Lett. **74**, 2172 (1995).
[80] U. G. Meissner, N. Kaiser, A. Wirzba and W. Weise, Phys. Rev. Lett. **57**, 1676 (1986).
[81] N. Kaiser, U. G. Meissner and W. Weise, Phys. Lett. B **198**, 319 (1987).
[82] T. E. O. Ericson and M. Rosa-Clot, Nucl. Phys. **A405**, 497 (1983).
[83] V. M. Kolybasov and A. F. Kudryatsev, Nucl. Phys. **B41**, 510 (1980).
[84] K. Holinde and A. W. Thomas, Phys. Rev. C **42**, 1195 (1990).
[85] J. Haidenbauer, K. Holinde and A. W. Thomas, Phys. Rev. C **45**, 952 (1992).
[86] G. Janssen, J. W. Durso, K. Holinde, B. C. Pearce and J. Speth, Phys. Rev. Lett. **71**, 1978 (1993).
[87] G. Janssen, K. Holinde and J. Speth, Phys. Rev. Lett. **73**, 1332 (1994).
[88] J.L. Goity, R. Lewis, M. Schvellinger L. Zhang, Phys. Lett. B **454**, 115 (1999).
[89] R. E. Stuckey and M. C. Birse, J. Phys.:G **23**, 29 (1997).
[90] B. W. Raichle, C. R. Gould, D. G. Haase, M. L. Seely, J. R. Walston, W. Tornow, W. S. Wilburn, S. I. Penttilä, and G. W. Hoffmann, Phys. Rev. Lett. **83**, 2711 (1999).