FIELD THEORY AND THE STANDARD MODEL

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Abstract
This is a course of six lectures that was given at the European School of High-Energy Physics, Slovakia, August–September, 1999.

OUTLINE

Prelude: Standard Model (SM) – an example of QFT – a tool for precision calculations in modern High Energy Physics (HEP);

1. QFT BASICS. EXAMPLE OF QED
   - Quantum Fields of the SM and their Properties;
   - Equations of motion;
   - Relation between a Lagrangian $\mathcal{L}$ and equation of motion;
   - $\mathcal{S}$ matrix and amplitude of a process;
   - Cross-sections and decay rates;
   - Input parameters in the Standard Model;
   - QED free Lagrangian;
   - Local gauge transformation and invariance;
   - Feynman rules of QED.
2. STANDARD MODEL LAGRANGIAN BUILDING
   - Yang–Mills sector;
   - The scalar sector;
   - Gauge fixing and Faddeev–Popov ghosts;
   - Propagators in the SM;
   - Interaction Lagrangian;
   - Tadpoles and their role in proving of gauge invariance;
   - Interactions of fermions with gauge fields;
   - Interactions of fermions with scalar fields;
   - Fermion mixing;
   - QCD Lagrangian;
   - Feynman rules for vertices;
   - Summary of two Lectures.
3. DIMENSIONAL REGULARIZATION AND PASSARINO–VELTMAN FUNCTIONS
   - Feynman parametrization and N-point functions;
   - Basics of Dimension regularization;
   - Divergences counting: poles versus powers;
   - One-point integrals, A functions;
   - Two-point integrals, B functions;
   - Three-point integrals, C functions;

PRELUDE: Standard Model – an example of QFT – a tool for precision calculations in modern High Energy Physics (HEP)

In this School, the courses on Quantum Field Theory (QFT) and the Standard Model (SM) are grouped into one course of six lectures.

Maybe, this is not by chance: the SM finally strengthened itself to be the modern QFT capable for precision calculations in HEP. In my opinion, in recent years a new discipline has been born: Precision High-Energy Physics, **PHEP**, both experimentally and theoretically.

Experimentally, this is first of all due to experiments at the Z resonance: LEP1 and SLAC, with their unprecedented statistics, bringing the precision of measurements at the per mil level. However, other facilities, like TEVATRON, also approach PHEP standards. The LHC also expects to be a typical PHEP facility, not speaking about linear collider (LC) where one expects statistics in the Z resonance mode 100 times richer than at LEP1 (GigaZ phase of linear collider).

Theoretically, it is basically the Standard Model (SM), which nowadays represents an example of a *calculable* QFT. This status of the SM was achieved during nearly 40 year's heroic efforts of a large community of theorists' tracing back to pioneering papers by S. L. Glashow, S. Weinberg and A. Salam in the beginning of the sixties, and finally recognized by the decision to award the 1999 Nobel Prize in Physics to G. t'Hooft and M. Veltman "for elucidating of quantum structure of electroweak interactions in physics", and for "having placed this theory on a firmer mathematical foundation".

An important question that I asked myself whilst preparing these lectures was: Which balance between QFT and SM? Presumably, ideally, it should be 50–50. However, eventually a SM dominated course emerged. There were different reasons for this.

Objective reasons:

– At present, we face an impressive success of the SM in the description of the LEP1/SLC data;
– We are at the end of the LEP1/SLC data processing;
– We foresee a bright future for PHEP at the colliders of near future.

However, there were also certain subjective reasons:

– I have worked for about 20 years in the field of PHEP;
– I was deeply involved in the LEP1/SLC analysis within the framework of the ZFITTER project and several CERN Workshops dedicated to precision calculations for the Z resonance;
– Last, but not least a book *The Standard Model in the Making* [1], written together with Giampiero Passarino, and finished in 1999. In this book, we tried to show how the SM works for precision calculations of the Z resonance observables.

Therefore, it is not surprising that this course of lectures is biased to the SM and Z resonance physics. I would like to say a few words as to why it is so biased towards calculations.

Here again I see objective and subjective reasons. Objectively, the precision calculations consume a lot of mathematics and, in my opinion, it is not surprising that the creation of *SCHOONSCHIP* was specially mentioned in the decision to award the 1999 Nobel Prize to Prof. M. Veltman. Nowadays, all the cumbersome diagrammatic calculations are done with algebraic computer systems. However, I am not going to tell you about corresponding algorithms. In my opinion, the underlying mathematics, which the SM physics is based upon, is very simple and everybody may master it. So, I shall dare to tell you about it.

Subjectively, it is our way of understanding physics by means of calculations. When working on the book, we liked to say: "We do not prove Ward identities – we *compute* them." These lectures follow the same approach, although I understand that it may not be appreciated by the majority of the HEP community. Anyway, the first five lectures are self-contained and may be studied.

I would like to say, that these lectures are not a *simple* extraction from the book. I see them as *introductory* and in many respects as *complimentary* to the book. So, in the second lecture I tried to present a more extended discussion of the SM Lagrangian compared to the presentation in the book.

Finally, it should be stressed that both in the book [1] and in these lectures the Pauli metrics is used, i.e. for an on-mass-shell momentum one has: $p^2 = -M^2$. As a result of this, some equations are looking "unnaturally" compared to a more popular choice, the so-called Bjorken–Drell metrics where one has: $p^2 = M^2$.

1. QFT BASICS. EXAMPLE OF QED

In the first lecture, I briefly recall the basics of the Quantum Field Theory (QFT), in particular of Quantum Electrodinamics (QED), which for a very long time represented the only example of a calculable QFT. Nowadays QED is completely absorbed by the Standard Model (SM), which completely inherited the status of QED. We will devote some time to a detailed discussion of the SM theoretical status. In passing, our notation and convention will be introduced.

1.1 Quantum fields of the SM and their properties

We begin with an overview of the SM fields and their properties. The SM involves physical fields (*fermions, gauge bosons and Higgs scalar*) and unphysical fields (*scalars and Faddeev–Popov ghosts*).

Three generation of fermions or matter fields:

$$f = \begin{cases} \begin{pmatrix} \nu \\ l \end{pmatrix} = \begin{pmatrix} \nu_e \\ e^- \end{pmatrix} & \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix} & \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix} \\ \begin{pmatrix} U \\ D \end{pmatrix} = \begin{pmatrix} u \\ d \end{pmatrix} & \begin{pmatrix} c \\ s \end{pmatrix} & \begin{pmatrix} t \\ b \end{pmatrix} \end{cases}$$

possess masses, m_f, charges, Q_f (in units of positron charge), and third projections of weak isospin, $I_f^{(3)}$:

$$m_f, \quad Q_f = \begin{pmatrix} \nu & l & U & D \\ \\ 0 & -1 & +\frac{2}{3} & -\frac{1}{3} \end{pmatrix}, \quad I_f^{(3)} = \begin{pmatrix} \nu & l & U & D \\ \\ +\frac{1}{2} & -\frac{1}{2} & +\frac{1}{2} & -\frac{1}{2} \end{pmatrix}.$$

Gauge fields:

Vector bosons	Unphysical scalars	Faddeev–Popov ghosts
A		Y^A
$Z\ (M_Z)$	ϕ^0	Y^Z
$W^{\pm}(M_W)$	$\phi^{\pm}$	$X^{\pm}$
Gluon possesses strong interaction		
g		Y^G

possess physical charges and physical masses

possess physical charges and unphysical masses

and unphysical charges.

Higgs field:

$H\ (M_H)$ scalar, neutral, massive.

1.2 Equations of motion

Introduce notation for all fields of the SM:

$$\begin{aligned}
\text{scalar, neutral and charged:} \quad & \phi^0(x)\,, \quad \phi^{\pm}(x)\,,\\
\text{spinor:} \quad & \psi(x)\,, \quad \overline{\psi}(x)\,,\\
\text{electromagnetic:} \quad & A_\alpha(x)\,,\\
\text{vector massive, neutral and charged:} \quad & Z_\alpha(x)\,, \quad W^{\pm}_\alpha(x)\,,\\
\text{Faddeev–Popov ghosts:} \quad & X^{\pm},\ Y^A,\ Y^Z,\ Y^G.
\end{aligned} \tag{1}$$

1.2.1 Equations of motion for free fields

All fields in QFT satisfy *equations of motion*, free, or with sources. Here we recall four types of equation of motion for free fields which are met in the SM:

$$\begin{aligned}
\text{Klein–Gordon for scalar fields:} \quad & \left(\Box - M^2\right)\phi^0(x) = 0\,, \quad \text{where} \quad \Box = \partial_\mu\partial_\mu\,,\\
& \partial_\mu\phi^+(x)\,\partial_\mu\phi^-(x) - M^2\phi^+(x)\,\phi^-(x) = 0;\\
\text{Dirac for spinors:} \quad & (\partial\!\!\!/ + m)\,\psi(x) = 0\,, \quad \text{where} \quad \partial\!\!\!/ = \partial_\mu\gamma_\mu\,;\\
\text{Maxwell for photons:} \quad & \partial_\mu F_{\mu\nu} = 0\,, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,;\\
\text{Proca for heavy vector bosons:} \quad & \partial_\mu F_{\mu\nu} - M_{_0}^2 Z_\nu = 0\,, \quad F_{\mu\nu} = \partial_\mu Z_\nu - \partial_\nu Z_\mu\,.
\end{aligned} \tag{2}$$

1.3 Relation between a Lagrangian $\mathcal{L}$ and equation of motion

1.3.1 Euler–Lagrange equation

In QFT there exists a relation between the Lagrangian density $\mathcal{L}(x)$ and equations of motions (I recommend the book in Ref.[2] for a systematic presentation of this subject), namely, a *variation* of the Lagrangian with respect to a field and its derivative gives the corresponding equation of motion:

$$\frac{\partial\mathcal{L}}{\partial\varphi} - \partial_\alpha\frac{\partial\mathcal{L}}{\partial\left(\partial_\alpha\varphi\right)} = 0\,. \tag{3}$$

All fields φ and all their derivatives $\partial_\alpha\varphi$ ($\varphi = \phi^0,\ \phi^{\pm},\ \psi,\ \overline{\psi},\ A_\alpha\,,\ Z_\alpha$, etc.) should be considered as independent variables at variation.

1.3.2 Example of a neutral vector field

Consider the Lagrangian of a free heavy vector field Z_μ:

$$\mathcal{L} \;=\; -\frac{1}{4}F_{\mu\nu}F_{\mu\nu} - \frac{1}{2}M_{_0}^2 Z_\mu Z_\mu\,. \tag{4}$$

Computing the derivatives,

$$\frac{\partial\mathcal{L}}{\partial Z_\nu} = -M_{_0}^2 Z_\nu\,, \qquad \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu Z_\nu\right)} = -F_{\mu\nu}\,, \tag{5}$$

and substituting them into the Euler–Lagrange equation (3), we obtain the Proca equation of motion:

$$\frac{\partial\mathcal{L}}{\partial Z_\nu} - \partial_\mu\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu Z_\nu\right)} = \partial_\mu F_{\mu\nu} - M_{_0}^2 Z_\nu = 0\,. \tag{6}$$

Note the $1/2$ in the Lagrangian for neutral fields contrary to the Lagrangian for charged fields. In the latter case, the fields $W^{\pm}_\alpha$ are independent and the factor of 2 does not arise at variation.

1.3.3 Example of QED

Consider the QED Lagrangian *with interaction:*

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F_{\mu\nu} - \overline{\psi} \left(\partial\!\!\!/ - ieQ_f A\!\!\!/ + m \right) \psi \,. \tag{7}$$

Compute derivatives over all independent fields and derivatives:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial A_\nu} &= \overline{\psi} ieQ_f \gamma_\nu \psi \,, & \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu A_\nu \right)} &= -F_{\mu\nu} \,, \\
\frac{\partial \mathcal{L}}{\partial \overline{\psi}} &= -\left(\partial\!\!\!/ - ieQ_f A\!\!\!/ + m \right) \psi \,, & \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \overline{\psi} \right)} &= 0 \,, \\
\frac{\partial \mathcal{L}}{\partial \psi} &= -\overline{\psi} \left(-ieQ_f A\!\!\!/ + m \right) , & \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \psi \right)} &= -\overline{\psi} \gamma_\mu \,.
\end{aligned} \tag{8}$$

Substituting all these derivatives into Eq. (3), we get *the system of three* Euler–Lagrange equations:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial A_\nu} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu A_\nu \right)} &= \overline{\psi} ieQ_f \gamma_\nu \psi + \partial_\mu F_{\mu\nu} = 0 \,, \\
\frac{\partial \mathcal{L}}{\partial \overline{\psi}} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \overline{\psi} \right)} &= -\left(\partial\!\!\!/ - ieQ_f A\!\!\!/ + m \right) \psi = 0 \,, \\
\frac{\partial \mathcal{L}}{\partial \psi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \psi \right)} &= -\overline{\psi} \left(-ieQ_f A\!\!\!/ + m \right) + \partial_\mu \overline{\psi} \gamma_\mu = 0 \,,
\end{aligned} \tag{9}$$

or equivalently — equations of motion for *interacting* fields, where on the r.h.s we see *the sources* of the fields:

$$\begin{aligned}
\partial_\mu F_{\mu\nu} &= -ieQ_f \overline{\psi} \gamma_\mu \psi, \\
\left(\partial\!\!\!/ + m \right) \psi &= ieQ_f A\!\!\!/ \psi, \\
\overline{\psi} \left(\partial\!\!\!/ - m \right) &= -ieQ_f \overline{\psi} A\!\!\!/.
\end{aligned} \tag{10}$$

The first one is the Maxwell equation with the source and the next two equations are two Dirac equations for the ψ and Dirac-conjugated field $\overline{\psi}$, both with sources. From these equations it is clear, why in QFT language one says that sources *emit/absorb* e^+e^--pairs, γe^- and γe^+, respectively.

1.4 $\mathcal{S}$ matrix and amplitude of a process

Now we recall the notions of the $\mathcal{S}$-matrix and the amplitude of a process. Consider a scattering (annihilation) process:

$$\begin{aligned}
p_1 + p_2 &\to p_1' + p_2' + \cdots \\
P &= p_1 + p_2 \,, && \text{initial momentum} \,, \\
P' &= p_1' + p_2' + \cdots , && \text{final momentum} \,,
\end{aligned} \tag{11}$$

where p_i denotes simultaneously a *particle* and its *4-momentum*.

In QFT, any process is characterized by a matrix element:

$$\langle f | \mathcal{S} - 1 | i \rangle = \langle f | \mathcal{R} | i \rangle \left(2\pi \right)^4 \delta \left(P' - P \right) , \tag{12}$$

of $\mathcal{S}$ matrix,

$$\mathcal{S} = T \left\{ \exp \left[i \int \mathcal{L}_{\rm I} \left(x \right) d^4 x \right] \right\} , \tag{13}$$

which is constructed from an *interaction Lagrangian density*, $\mathcal{L}_{\rm I}\left(x \right)$, with the aid of a *time-ordering* operation T.

Let us summarise our short ex-course into QFT:

- The interaction Lagrangian density, $\mathcal{L}_{\rm I}(x)$, is the primary object of QFT from which the amplitude of a process is derived;
- $\mathcal{L}_{\rm I} \propto$ *coupling constant*, which is usually small and a *perturbation expansion* for a process amplitude may be developed;
- Quantum fields, which a Lagrangian is made of, may **act** on *initial* and *final* states $|i\rangle$ and $\langle f|$, giving rise to plane waves describing *in* and *out* particles, or **contract** with each other, giving rise to *propagators*;
- *Feynman rules* for external lines, vertices and propagators offer a very transparent way of constructing process amplitudes, *order-by-order* in perturbation theory;
- A typical Feynman rule for an external line (scalar, spinor, photon, vector boson) looks like:

$$p \qquad \rightarrow \qquad \frac{1}{(2\pi)^{3/2}}\frac{1}{\sqrt{2p_0}} \times \left[1, \bar{u}\left(p\right), \epsilon_\mu\left(p\right), e_\mu\left(p\right), \ldots\right]$$

where p_0 is the zeroth component of a 4-vector p (energy) and $\bar{u}\left(p\right), \epsilon_\mu\left(p\right), e_\mu\left(p\right)$ are spinors and polarization vectors, respectively.

1.5 Cross-sections and decay rates

Here we recall practical formulae for cross-sections and decay rates constructed from the amplitudes of the corresponding processes.

The total transition probability (in the whole space-time) is

$$dW_{fi} = \mid \langle f|\mathcal{R}|i\rangle \mid^2 (2\pi)^8 \,\delta\left(P' - P\right) \frac{1}{(2\pi)^4} \int e^{i(P'-P)x} d^4x d^3p'_1 d^3p'_2 \cdots \tag{14}$$

The transition probability per unit of time per unit of volume is then

$$dw_{fi} = \lim_{V,T\to\infty} \frac{dW_{fi}}{VT} = \mid \langle f|\mathcal{R}|i\rangle \mid^2 (2\pi)^4 \,\delta\left(P' - P\right) d^3p'_1 d^3p'_2 \cdots \tag{15}$$

The differential <u>cross-section</u> is defined as the ratio

$$d\sigma_{fi} = \frac{dw_{fi}}{j}\,, \tag{16}$$

where j is *the initial flux*

$$j = \rho_1\rho_2 \frac{\sqrt{\left(p_1p_2\right)^2 - m_1^2m_2^2}}{\left(p_1\right)_0\left(p_2\right)_0}\,. \tag{17}$$

Introducing initial densities ρ_1, ρ_2 and normalization factors N_{p_k},

$$\rho_i = \frac{1}{(2\pi)^3}\,, \qquad N_{p_k} = \frac{1}{(2\pi)^{3/2}}\frac{1}{\sqrt{2\left(p_k\right)_0}}\,, \tag{18}$$

the differential cross-section becomes

$$d\sigma_{fi} = \frac{1}{4\sqrt{\left(p_1p_2\right)^2 - m_1^2m_2^2}} \mid \mathcal{M}_{fi} \mid^2 d\Phi_n\,, \tag{19}$$

where $d\Phi_n$ is *the differential phase space*

$$d\Phi_n = (2\pi)^4 \prod_{k=1}^{n} \frac{d^3p'_k}{(2\pi)^3\, 2\left(p'_k\right)_0} \delta\left(\sum_{j=1}^{n} p'_j - P\right). \tag{20}$$

The process matrix element squared, $\mid \mathcal{M}_{fi} \mid^2$,

$$N^2_{p_1} N^2_{p_2} \prod_{k=1}^{n} N^2_{p'_k} \mid \mathcal{M}_{fi} \mid^2 = \overline{\sum_{\text{spins}}} \mid \langle f|\mathcal{R}|i\rangle \mid^2 , \tag{21}$$

is defined without normalization factors N_{p_k} and should be understood as *averaged* over initial and *summed* over final spin degrees of freedom.

For the decay rate of the process:

$$P \quad \rightarrow \quad p'_1 + p'_2 + \cdots \tag{22}$$

one analogously has

$$d\Gamma_{fi} \quad = \quad \frac{dw_{fi}}{\rho} \, , \tag{23}$$

where $\rho = \dfrac{1}{(2\pi)^3}$ is the initial density.

Similarly one obtains,

$$d\Gamma_{fi} \quad = \quad \frac{1}{2P_0} \mid \mathcal{M}_{fi} \mid^2 d\Phi_n \, . \tag{24}$$

Note the difference in the definition of the phase space Eq. (20) with PDG convention, [6]: $(2\pi)^4$ is shifted to the phase space. This is convenient for calculations in n dimensions as will be shown below.

1.6 Input parameters in the Standard Model

1.6.1 Number of independent parameters in the SM

In this section we discuss a very important issue, the notion of *the input parameter set, IPS*. To approach it, let us consider a sequence of theories, ranging from conventional QED to the Extended SM (hereafter ESM). The following Table contains the list of parameters, which a theory Lagrangian depends upon, together with the total number of parameters of the theory N_p:

Theory			List of parameters			N_p
Conventional QED	$\rightarrow$	e	m_e			2
Extended QED	$\rightarrow$	e	m_e	m_μ	m_τ	
			m_u	m_c	m_t	
			m_d	m_s	m_b	10
EW Standard Model	$\rightarrow$	$+$	M_W	M_Z	M_H	
			4 mixing angles			17
Conventional SM	$\rightarrow$	$+\,\alpha_S$				18
Extended SM	$\rightarrow$	$+$	m_{ν_e}	m_{ν_μ}	m_{ν_τ}	
			4 mixing angles			25

One can see that the number of parameters of the ESM is large. However, this is a trivial consequence of a large number of *fundamental fields* and the objective **complexity of Nature**. This Table illustrates that the nature of the parameters in all the considered series of theories is exactly the same. In conventional QED it is 2, but only due to the fact that this theory is limited to the description of the interaction of photons with electrons.

It is important to understand that the number 25 is *a minimal number*. Indeed:
• Three generations is a *minimal* number, necessary to have CP violation, which exists in Nature; remember that the number of complex phases is

$$N_{\rm phases} = \frac{(N_g-1)\,(N_g-2)}{2}\,, \qquad \text{where} \quad N_g - \text{number of generations}, \tag{25}$$

therefore, $N_g = 3$ is a *minimal number* which allows us to have one (*minimal number*) phase.
All nine fundamental fermions are found experimentally.
• Four gauge bosons is a *minimal* number, needed to describe all EW interactions existing in Nature. We have the long range e.m. interaction and CC and NC short-range weak processes, therefore, we need at least four vector carriers — $A,\ W^{\pm},\ Z$ — to mediate these interactions.
All four gauge bosons are found experimentally.
• Fermionic mixing, as is proved in the lectures of Prof. S.M. Bilenky [3] is unavoidable and exists in Nature both in *hadronic* and *leptonic* worlds.
CKM mixing is experimentally well measured, ν-mixing is probably discovered.
• *Only the Higgs boson has not yet been found*. There are *indirect* indications, however. (To be discussed in these lectures.)

The ESM **is not able** to calculate these 25 parameters and in this sense the ESM is not a **predictive** theory. This is why people believe that some day a better theory will be discovered and why they wish to find some experimental indications of new physics *beyond the SM* and build and plan new accelerators, the LHC, LC, etc.

So far, however, neither the experiment has found strong evidence of new physics, (the situation with the description of *all* ν data has to be clarified and I refer to the lectures of S. Bilenky [3] and M. Carena [4] at this School) nor theory proposed the complete explanation of *the whole* mass spectrum of fundamental particles ranging from *fractions of eV* for lightest neutrino to *175 GeV* for heaviest top quark, i.e. **more than 12 orders of magnitude!**

The ESM **is able**, however, to calculate *any experimental observable* $O_i^{\rm exp}$ in terms of its IPS. We define

$$\text{ESM IPS} \qquad \equiv \qquad \text{the 25 parameters of above the Table.} \tag{26}$$

One must emphasize that this set of parameters is not unique. For instance, fermion masses may be replaced by Yukawa coupling constants and one of the gauge boson masses may be substituted by the $SU(2)$ weak coupling constant g. Particle masses seem to be, however, more *natural* to be chosen for IPS, and, moreover, they are more suitable objects for a treatment within the one-mass-shell (OMS) renormalization scheme.

The comparison procedure of experimental measurements with the ESM predictions may be symbolically written as follows

$$O_i^{\rm exp}\,(\text{measured}) \quad \leftrightarrow \quad O_i^{\rm theor}\,(\text{calculated as a function of IPS})\,. \tag{27}$$

We shall now discuss of what is presently known about the IPS. The various parameters are experimentally known with different precision. For instance, precision in measurements of masses ranges from 10^{-7} for m_e to the existence of only lower and upper limits for $M_{_H}$:

$$\begin{array}{rcll} m_e &=& 0.51099907 \pm 0.00000015 \text{ MeV} & \sim \;\; 3\times 10^{-7} \\ M_{_Z} &=& 91.1871 \pm 0.0021 \text{ GeV} & \sim \;\; 2\times 10^{-5} \\ M_{_W} &=& 80.394 \pm 0.042 \text{ GeV} & \sim \;\; 5\times 10^{-4} \\ m_t &=& 174.3 \pm 5.1 \text{ GeV} & \sim \;\; 3\times 10^{-2} \end{array}$$
$$100 \text{ GeV (direct searches)} \;\leq M_{_H} \leq\; 215 \text{ GeV} \quad (95\%\ c.l.\ \text{ indirect limitations}).$$

Precision measurements provide *constraints* on the IPS. This is how one may *extract information* on yet unknown parameters (or improve our knowledge of poorly measured ones). This should not be confused with **prediction** in the above mentioned sense. The story of the discovery of the W and Z bosons and of the t quark is a typical illustration of how information about the masses of yet undiscovered particles was extracted from theory constraints. The same story is now repeated with the H boson. One should clearly understand that the ESM *does not predict* parameters, but *gives hints* about them via *constraints*.

1.6.2 More about IPS

Let us look at typical precisions and scales of various measurements.
The electron anomaly, $a_e = (g_e - 2)/2$, is a typical low-energy phenomenon, where conventional QED is sufficient to give very precise predictions:

$$\begin{aligned} a_e^{\rm exp} &= 1159652193(10) \times 10^{-12}\,, \\ a_e^{\rm th} &= 1159652140(27) \times 10^{-12}\,. \end{aligned}$$

An impressive (**8 digits!**) agreement between the experiment and QED calculations up to fourth order in perturbative expansion, $\mathcal{O}\left(\alpha^4\right)$, illustrates *the calculational power* of QED. It cannot be by chance!
The Z resonance observables are measured at LEP1 (CERN) and SLC (SLAC) with

$$\text{the experimental precision} \le 10^{-3}\,. \tag{28}$$

Therefore, one needs to have

$$\text{the theoretical precision} \sim 2.5 \times 10^{-4}\,. \tag{29}$$

This is the high-energy domain, where QED is not sufficient and one has to apply the conventional SM.

1.6.3 Number of free parameters in fits of Z resonance observables

The number of input parameters, which the Z resonance observables depend upon, is actually much lower than 25. Indeed, all the lepton masses are known very precisely, the worst one,

$$m_\tau = 1777.05^{+0.29}_{-0.26}\ \text{MeV} \quad < 10^{-4}, \tag{30}$$

is known infinitely precisely in the typical LEP1 precision scale 10^{-3}.

Later on, we will see that the Z resonance observables are sensitive to the vacuum polarization:

$\bar{f}$

γ γ

f

which leads to *logarithmic mass singularities:*

$$\sum_f \ln \frac{s}{m_f^2}\,. \tag{31}$$

This represents no problem for leptons, since lepton masses are well defined and well measured. On the contrary, *light quark* masses are *ill-defined* and for this reason they are replaced by the other experimentally well defined and well measured quantity $\sigma\left(e^+e^- \to \text{hadrons}\right)$. This introduces a new parameter $\alpha\left(M_Z^2\right)$ to the theory instead of light quark masses. Next, the Z resonance observables are insensitive to neutrino masses and fermion mixing angles. So, we are left with only six parameters:

$$\alpha\left(M_Z^2\right), \qquad \alpha_S\left(M_Z^2\right), \qquad m_t, \qquad M_Z, \qquad M_W, \qquad M_H\,. \tag{32}$$

Furthermore, one should exploit the precision measurement of the muon lifetime τ_μ. In terms of the Fermi constant, the relevant precision is better than 10^{-5}, which again means infinite precision in our scale. This allows us to derive M_W with a theoretical error of ~ 10 MeV which is much better than the present combined experimental error of ~ 60 MeV. We are therefore left with only 5 parameters:

$$\alpha\left(M_Z^2\right), \qquad \alpha_S\left(M_Z^2\right), \qquad m_t\,, \qquad M_Z\,, \qquad M_H\,. \tag{33}$$

We will call this set **the standard LEP1 IPS**.

With M_Z measured at the Z peak with a precision of $\sim 2\times 10^{-5}$, and with the rich information available from the other measurements for the parameters,

$$\alpha\left(M_Z^2\right), \qquad \alpha_S\left(M_Z^2\right), \qquad m_t, \tag{34}$$

we are approaching a one-parameter fit situation, with the Higgs mass M_H being the only parameter to fit!

1.6.4 More on coupling constants, typical scales

The LEP1/SLC and LEP2 typical scales, $\sqrt{s}$, masses of weak bosons, mass of the top quark, estimated Higgs mass,

$$\begin{aligned}\sqrt{s} &\sim M_Z - 200\ \text{GeV}\,,\\ M_W &\sim 80\ \text{GeV}\,,\\ M_Z &\sim 91\ \text{GeV}\,,\\ m_t &\sim 175\ \text{GeV}\,,\\ M_H &\leq 300\ \text{GeV}\,,\end{aligned} \tag{35}$$

all are of the order of a typical EW scale: 100–300 GeV. Therefore, one cannot construct a small parameter out of $\sqrt{s}\,,\ M_W\,,\ M_Z\,,\ m_t,\ M_H\,,$ and the calculation must, in principle, be exact (*complete*) in all these quantities. In real life, the notion of m_t^2-*enhanced terms* is introduced: $\mathcal{O}\left(G_F m_t^2\right)$. We note that $m_t^2/M_W^2 \approx 4$, therefore this enhancement is not so pronounced. Given a probable interval for the Higgs mass of $100 \leq M_H \leq 300$ GeV, the popular expansions in M_H^2/m_t^2 or m_t^2/M_H^2 may have very bad convergence.

For other than top-quark fermions it is sufficient to keep the first order in $m_f^2/s\,,\, f=\tau, c, b$; higher terms may safely be ignored at LEP energies.

Present codes include the following QED, EW and QCD corrections:

$$\begin{array}{llcll}\text{QED} & \alpha\,(0)\,L & = & 1/137L & \text{up to} \quad \mathcal{O}\left[(\alpha L)^3\right],\\ \text{EW} & \alpha\left(M_Z^2\right) & = & 1/128.9 & \text{up to} \quad \mathcal{O}\left(\alpha^2\right),\\ \text{QCD} & \alpha_S\left(M_Z^2\right) & = & 0.119 & \text{up to} \quad \mathcal{O}\left(\alpha_S^3\right),\end{array} \tag{36}$$

where *big log*

$$L = \ln\frac{s}{m_e^2} - 1 = 23 \quad \text{at} \quad s = M_Z^2\,. \tag{37}$$

Therefore the effective expansion parameter in QED, $\alpha L = 0.169$, is even bigger than the QCD coupling $\alpha_S\left(M_Z^2\right)$.

The neglected terms $\mathcal{O}\left(\alpha^3 L^2\right) \sim 2\times 10^{-4}$ and $\mathcal{O}\left(\alpha_S^4\right) \sim 2\times 10^{-4}$ are qualitatively expected to be at the boundary of importance. However, a test implementation of even more important term $\mathcal{O}\left[(\alpha L)^4\right] \sim 8\times 10^{-4}$ revealed the effect below 10^{-4}. All available mixed corrections $\mathcal{O}\left(\alpha\alpha_S\right)$ and $\mathcal{O}\left(\alpha\alpha_S^2\right)$ are also needed, and they are implemented into the codes.

1.7 QED free Lagrangian

Before discussing at length the ESM, it is worth recalling the basics of QED, because the ESM is very similar to QED as far as the basic principles are concerned.

The QED free (without interaction) Lagrangian reads:

$$\mathcal{L}^0_{\rm QED} = -\frac{1}{4}\,F_{\mu\nu}F_{\mu\nu} - \frac{1}{2}\,(\mathcal{C}^A)^2 - \sum_f \overline{\psi}_f\,\left(\partial\!\!\!/ + m_f\right)\psi_f\,, \tag{38}$$

where the following notations are used:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,, \qquad \partial\!\!\!/ = \partial_\mu\gamma_\mu\,, \qquad \mathcal{C}^A = -\frac{1}{\xi}\,\partial_\mu A_\mu\,. \tag{39}$$

Here $\mathcal{C}^A$ is *the gauge fixing term*, the meaning of which will be fully understood when we will consider the ESM Lagrangian. Here we shall only discuss our notation and convention.

We use the $4\,\times\,4$ representation for the Dirac matrices:

$$\begin{aligned}
\gamma_j &= \begin{pmatrix} O & -i\,\tau_j \\ i\,\tau_j & O \end{pmatrix}, \quad j=1,2,3; \qquad \gamma_4 = \begin{pmatrix} I & O \\ O & -I \end{pmatrix},\\
\gamma_5 &= \gamma_1\gamma_2\gamma_3\gamma_4 = \begin{pmatrix} O & -I \\ -I & O \end{pmatrix}; \qquad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\end{aligned} \tag{40}$$

The basic properties of the γ matrices are

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\delta_{\mu\nu}\,, \qquad \gamma^+_\mu = \gamma_\mu\,, \qquad \gamma^2_\mu = I. \tag{41}$$

The Pauli matrices are as usual:

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{42}$$

and their basic properties are

$$\tau_i = \tau^+_i, \qquad \tau_i\tau^+_i = I\,, \qquad \tau_i\tau_i = I\,, \qquad \left[\frac{1}{2}\tau_i, \frac{1}{2}\tau_j\right] = i\varepsilon_{ijk}\frac{1}{2}\tau_k\,, \qquad \tau_i\tau_j = \delta_{ij} + i\varepsilon_{ijk}\tau_k\,. \tag{43}$$

The quantities $\frac{1}{2}\tau_i$ are $SU(2)$ generators and the general $SU(2)$ transformation reads

$$U = \exp\left\{-i\frac{1}{2}\tau_i\lambda_i\right\}, \qquad UU^+ = I\,, \qquad detU = 1\,. \tag{44}$$

Free-particle spinors satisfy the Dirac equations:

$$\begin{aligned}
(ip\!\!\!/ + m)\;u(p) &= 0\,, \qquad (-ip\!\!\!/ + m)\;v(p) = 0\,,\\
\overline{u}(p)\;(ip\!\!\!/ + m) &= 0\,, \qquad \overline{v}(p)\;(-ip\!\!\!/ + m) = 0\,.
\end{aligned} \tag{45}$$

1.8 Local gauge transformation and invariance

Let us recall the local gauge transformations for all fields entering the QED Lagrangian:

$$\begin{aligned}
\psi'_f(x) &= e^{-ieQ_f\lambda(x)}\psi_f(x)\,,\\
\overline{\psi}'_f(x) &= \overline{\psi}_f(x)e^{ieQ_f\lambda(x)}\,,\\
A'_\mu(x) &= A_\mu(x) - \partial_\mu\lambda(x)\,.
\end{aligned} \tag{46}$$

The full Lagrangian $\mathcal{L}_{\text{QED}}$ will be invariant under local gauge transformations if we replace ∂_μ in Eq. (38) by the covariant derivative,

$$\partial_\mu \to D_\mu = \partial_\mu - ieQ_f A_\mu \,. \tag{47}$$

The Lagrangian with interaction becomes

$$\mathcal{L}_{\text{QED}} = -\frac{1}{4} F_{\mu\nu} F_{\mu\nu} - \frac{1}{2} \left(\mathcal{C}^A\right)^2 - \sum_f \overline{\psi}_f \left(\partial\!\!\!/ - ieQ_f A\!\!\!/ + m_f\right) \psi_f \,. \tag{48}$$

Here e is positive and $e^2 = 4\pi\alpha$, i.e. it is *the positron* charge, and $Q_f =$ (*fraction of charge*)$\times 2I_f^{(3)}$: $Q_l = -1$, $Q_u = +2/3$ and $Q_d = -1/3$.

The gauge invariance may be verified with the aid of identities:

$$F'_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu = \partial_\mu A_\nu - \partial_\mu \partial_\nu \lambda(x) - \partial_\nu A_\mu + \partial_\mu \partial_\nu \lambda(x) = F_{\mu\nu} \,,$$

$$m_f \overline{\psi}'_f(x) \psi'_f(x) = m_f \overline{\psi}_f(x) \psi_f(x) \,,$$

$$\overline{\psi}'_f(x) \left(\partial_\mu - ieQ_f A'_\mu\right) \psi'_f(x) =$$

$$\overline{\psi}_f(x) e^{ieQ_f \lambda(x)} \Big[\partial_\mu - ieQ_f \partial_\mu \lambda(x) - ieQ_f \left(A_\mu(x) - \partial_\mu \lambda(x)\right)\Big] e^{-ieQ_f \lambda(x)} \psi_f(x)$$

$$= \overline{\psi}_f(x) \left(\partial_\mu - ieQ_f A_\mu\right) \psi_f(x) \,. \tag{49}$$

In order to see the invariance of $\mathcal{C}^A$, we have to subject it to our gauge transformation, i.e.

$$-\frac{1}{\xi} \partial_\mu A'_\mu = -\frac{1}{\xi} \partial_\mu A_\mu + \frac{1}{\xi} \partial_\mu \partial_\mu \lambda(x) \,, \qquad \text{where} \quad \partial_\mu \partial_\mu = \Box \,. \tag{50}$$

The gauge invariance will be ensured if one requires

$$\Box \lambda(x) = 0 \,. \tag{51}$$

Therefore, we discover a massless, non-interacting ghost field $\lambda(x) \equiv Y^A(x)$ with the propagator

$$\frac{1}{\xi}\Box \qquad\qquad Y^A \qquad\qquad \frac{\xi}{p^2} \,. \tag{52}$$

1.9 Feynman rules of QED

The Feynman rules could be easily derived from the Lagrangian, Eq. (48). The Feynman rules for electron propagator and QED-vertex could be easily guessed looking at the Lagrangian, Eq. (48), are particularly simple. A complete collection of Feynman rules in QED is

$p \to$

$$\frac{1}{(2\pi)^4 i} \frac{1}{ip\!\!\!/ + m_f} = \frac{1}{(2\pi)^4 i} \frac{-ip\!\!\!/ + m_f}{p^2 + m_f^2 - i\epsilon} \,,$$

μ ν

$$\frac{1}{(2\pi)^4 i} \frac{1}{p^2 - i\epsilon} \left[\delta_{\mu\nu} + \left(\xi^2 - 1\right) \frac{p_\mu p_\nu}{p^2}\right],$$

μ

$$(2\pi)^4 i \; ieQ_f \gamma_\mu \,.$$

Note an appearance of the ξ-dependent term in the photonic propagator, a consequence of the gauge fixing.

Let us recall the expressions for the photon propagators in three frequently used gauges:

- General R_ξ, propagator as given above,
- Feynman gauge, $\xi = 1$, $\quad \dfrac{1}{(2\pi)^4 i} \dfrac{\delta_{\mu\nu}}{p^2 - i\epsilon}$,
- Landau gauge, $\quad \xi = 0$, $\quad \dfrac{1}{(2\pi)^4 i} \dfrac{1}{p^2 - i\epsilon} \left(\delta_{\mu\nu} - \dfrac{p_\mu p_\nu}{p^2} \right)$.

Usually in QED one uses the Feynman gauge. It is well known that the ξ-dependence cancels in the $\mathcal{S}$-matrix for a given physical process. As an example consider any $e^+e^- \to \gamma^*$ sub-process. The corresponding $\mathcal{S}$-matrix element in the R_ξ-gauge will have an additional term

$$-\left(\xi^2 - 1\right) \bar{v}\left(p_+\right) \left(\not{p}_+ + \not{p}_-\right) u\left(p_-\right), \tag{53}$$

which is zero for on-mass-shell fermions by virtue of the Dirac equation. Therefore, the extra term, proportional to $\xi^2 - 1$, may be omitted.

2. STANDARD MODEL LAGRANGIAN BUILDING

This lecture is devoted to SM Lagrangian building. We will proceed in the most general R_ξ gauge with three arbitrary gauge parameters. Let us recall the fields' content in the electroweak sector of the SM:

- triplet of vector bosons, B^a_μ, and singlet, B^0_μ;
- a complex scalar field K, (in the minimal SM we have only one doublet of complex fields);
- Faddeev–Popov ghost-fields $X^\pm, Y^Z, Y^A$;
- fermion families.

The total SM Lagrangian should include all these fields. It may be represented as the sum of the various parts.

2.1 Yang–Mills sector

First part is the standard Yang–Mills Lagrangian:

$$\mathcal{L}_{\rm YM} = -\frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} - \frac{1}{4} F^0_{\mu\nu} F^0_{\mu\nu}, \tag{54}$$

with the usual field-strength tensors

$$F^a_{\mu\nu} = \partial_\mu B^a_\nu - \partial_\nu B^a_\mu + g \varepsilon_{abc} B^b_\mu B^c_\nu, \qquad F^0_{\mu\nu} = \partial_\mu B^0_\nu - \partial_\nu B^0_\mu . \tag{55}$$

We recall that Yang–Mills Lagrangian follows from the requirement of local $SU(2) \times U(1)$ gauge invariance, i.e. if one replaces ∂_μ in the free field Lagrangian by the covariant derivative which general form may be written as a sum of the $SU(2)$ and $U(1)$ parts

$$\partial_\mu \to D_\mu = \partial_\mu - \frac{i}{2} g B^a_\mu \tau^a - \frac{i}{2} g g_i B^0_\mu , \tag{56}$$

where g is the $SU(2)$ bare coupling constant. The Lagrangian, Eq. (54), is therefore invariant under $SU(2) \times U(1)$ gauge transformations. We recall that the $SU(2)$ part of Eq. (56) is totally fixed due to its non-Abelian structure, whilst its Abelian part contains an arbitrary hypercharge g_i, see [2] for more details. The physical fields Z and A are related to the gauge fields B^3_μ and B^0_μ by a well-known rotation involving *the weak mixing angle* θ:

$$\begin{pmatrix} Z \\ A \end{pmatrix} = \begin{pmatrix} c_\theta & -s_\theta \\ s_\theta & c_\theta \end{pmatrix} \begin{pmatrix} B^3 \\ B^0 \end{pmatrix}, \tag{57}$$

where $s_\theta (c_\theta)$ denote the sine and cosine of the weak mixing angle.

2.2 The scalar sector

The second part is the Higgs scalar Lagrangian:

$$\mathcal{L}_{\rm S} = -\left(D_\mu K\right)^+ D_\mu K - \mu^2 K^+ K - \frac{1}{2}\lambda\left(K^+ K\right)^2 , \tag{58}$$

where $\lambda > 0$ is the positive ϕ^4 interaction constant and the mass term has the negative sign, $\mu^2 < 0$, as is required by the spontaneous symmetry breaking.

The complex scalar field doublet in the minimal realization of the SM is

$$K = \frac{1}{\sqrt{2}}\left(\begin{array}{c} \chi \\ \\ \sqrt{2} i\phi^- \end{array}\right), \qquad \text{with} \quad \chi = H + \langle v\rangle + i\phi^0 . \tag{59}$$

It contains four scalar fields: $\phi^\pm$, ϕ^0 and H, where H is the physical Higgs boson field and $\langle v\rangle$ is the vacuum expectation value (v.e.v.).

The covariant derivative for the scalar field in $SU(2)\otimes U(1)$ looks similar to Eq. (56)

$$D_\mu K = \left(\partial_\mu - \frac{i}{2} g B^a_\mu \tau^a - \frac{i}{2} g g_1 B^0_\mu\right) K , \tag{60}$$

where we introduced the hypercharge g_1 which will be fixed below. The scalar field can be conveniently rewritten as

$$K = \frac{1}{\sqrt{2}}\left(H + \langle v\rangle + i\phi^a\tau^a\right)\left(\begin{array}{c} 1 \\ 0 \end{array}\right). \tag{61}$$

Then the covariant derivative becomes

$$\begin{aligned} D_\mu K &= \frac{1}{\sqrt{2}}\left(\partial_\mu - \frac{i}{2} g B^a_\mu \tau^a - \frac{i}{2} g g_1 B^0_\mu\right)\left(H + \langle v\rangle + i\phi^b\tau^b\right)\left(\begin{array}{c} 1 \\ 0 \end{array}\right) \\ &= \frac{1}{\sqrt{2}}\Big\{\partial_\mu H - \frac{i}{2} g g_1 B^0_\mu \left(H + \langle v\rangle\right) + \frac{1}{2} g B^a_\mu \phi^a \\ &\quad + i\left[\partial_\mu \phi^a - \frac{1}{2} g B^a_\mu \left(H + \langle v\rangle\right) - \frac{i}{2} g g_1 B^0_\mu \phi^a + \frac{1}{2} g \varepsilon_{cba} B^c_\mu \phi^b\right]\tau^a\Big\}\left(\begin{array}{c} 1 \\ 0 \end{array}\right). \end{aligned} \tag{62}$$

Similarly, we represent the hermitian conjugate part

$$\begin{aligned} \left(D_\mu K\right)^+ &= (1,0)\,\frac{1}{\sqrt{2}}\Big\{\partial_\mu H + \frac{i}{2} g g_1 B^0_\mu \left(H + \langle v\rangle\right) + \frac{1}{2} g B^a_\mu \phi^a \\ &\quad - i\left[\partial_\mu \phi^a - \frac{1}{2} g B^a_\mu \left(H + \langle v\rangle\right) + \frac{i}{2} g g_1 B^0_\mu \phi^a + \frac{1}{2} g \varepsilon_{cba} B^c_\mu \phi^b\right]\tau^a\Big\}, \end{aligned} \tag{63}$$

and consider their product

$$-\left(D_\mu K\right)^+ D_\mu K . \tag{64}$$

This product, whis is only the first term of $\mathcal{L}_{\rm S}$, Eq. (58), contains 81 terms!

Collecting only terms with $\langle v\rangle^2$, we have

$$\begin{aligned} &-(1,0)\,\frac{1}{\sqrt{2}}\left\{\frac{i}{2} g g_1 B^0_\mu \langle v\rangle + i\frac{1}{2} g B^b_\mu \langle v\rangle \tau^b\right\}\frac{1}{\sqrt{2}}\left\{-\frac{i}{2} g g_1 B^0_\mu \langle v\rangle - i\frac{1}{2} g B^c_\mu \langle v\rangle \tau^c\right\}\left(\begin{array}{c} 1 \\ 0 \end{array}\right) \\ &= -\frac{g^2\langle v\rangle^2}{8}(1,0)\left(g_1 B^0_\mu + B^c_\mu \tau^c\right)\left(g_1 B^0_\mu + B^b_\mu \tau^b\right)\left(\begin{array}{c} 1 \\ 0 \end{array}\right) \to \end{aligned} \tag{65}$$

Using properties of τ^a-matrices, we simplify

$$(1,0)\, B^a_\mu \tau^a \begin{pmatrix} 1 \\ 0 \end{pmatrix} = B^3_\mu\,, \qquad (1,0)\, B^c_\mu \tau^c B^b_\mu \tau^b \begin{pmatrix} 1 \\ 0 \end{pmatrix} = B^a_\mu B^a_\mu\,, \tag{66}$$

and continue Eq. (65) as follows

$$\begin{aligned} &\to \quad -\frac{g^2\langle v\rangle^2}{8}\left(g_1^2 B^0_\mu B^0_\mu + 2g_1 B^0_\mu B^3_\mu + B^a_\mu B^a_\mu\right) \\ &= \quad -\frac{g^2\langle v\rangle^2}{8}\left[\left(g_1 B^0_\mu + B^3_\mu\right)^2 + B^1_\mu B^1_\mu + B^2_\mu B^2_\mu\right] \to \end{aligned} \tag{67}$$

Now we proceed in terms of physical fields:

$$\begin{aligned} W^\pm_\mu &= \frac{1}{\sqrt{2}}\left(B^1_\mu \mp i B^2_\mu\right), \qquad \phi^\pm = \frac{1}{\sqrt{2}}\left(\phi^1 \mp i\phi^2\right), \qquad \phi^0 \equiv \phi^3, \\ Z_\mu &= c_\theta B^3_\mu - s_\theta B^0_\mu\,, \qquad\qquad A_\mu = s_\theta B^3_\mu + c_\theta B^0_\mu\,. \end{aligned} \tag{68}$$

It is seen, that if one chooses $g_1 = -s_\theta/c_\theta$, then Eq. (67) becomes

$$\to -\frac{g^2\langle v\rangle^2}{8}\left[\frac{1}{c^2_\theta}\left(Z_\mu\right)^2 + 2W^+_\mu W^-_\mu\right] = -\frac{1}{2}M^2_0\left(Z_\mu\right)^2 - M^2 W^+_\mu W^-_\mu\,, \tag{69}$$

i.e. it looks like a normal mass-term of a Lagrangian.

Therefore, the Higgs mechanism generates masses of vector bosons:

$$\begin{aligned} &M \ \text{– bare mass of } W \text{ boson, } M = \frac{g\langle v\rangle}{2}\,, \\ &M_0 \text{ – bare mass of } Z \text{ boson, } M_0 = \frac{g\langle v\rangle}{2c_\theta}\,. \end{aligned} \tag{70}$$

The two last equations are equivalent to:

$$c_\theta = \frac{M}{M_0} \quad \text{and} \quad \langle v\rangle = 2\frac{M}{g}\,, \tag{71}$$

and these establish two more relations between the parameters of the Lagrangian. In particular, one can see that the weak mixing angle is no longer a free parameter if one chooses vector boson masses as the free parameters of the theory.

Let us continue our study of the product, Eq. (64). At the second step, we substitute $\langle v\rangle$ and look at all terms **without** interaction constant g:

$$\begin{aligned} &-(1,0)\,\frac{1}{\sqrt{2}}\left[\partial_\mu H + iMg_1 B^0_\mu - i\left(\partial_\mu\phi^c - MB^c_\mu\right)\tau^c\right] \\ &\times\frac{1}{\sqrt{2}}\left[\partial_\mu H - iMg_1 B^0_\mu + i\left(\partial_\mu\phi^b - MB^b_\mu\right)\tau^b\right]\begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \end{aligned}$$

Omitting **legal** kinetic terms

$$-\frac{1}{2}\left(\partial_\mu H\right)^2 \qquad \text{etc.,} \tag{72}$$

then terms, which were already considered (mass terms), and observing that $HB^{a,0}_\mu$ transitions cancel identically, we are left with

$$\to \frac{M}{2}(1,0)\left(g_1 B^0_\mu + B^c_\mu \tau^c\right)\left(\partial_\mu \phi^b\right)\tau^b \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$+ \frac{M}{2}(1,0)\left(\partial_\mu \phi^c\right)\tau^c \left(g_1 B^0_\mu + B^b_\mu \tau^b\right)\begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \tag{73}$$

Taking into account, that

$$(1,0)\left(B^c_\mu \tau^c \partial_\mu \phi^b \tau^b + \partial_\mu \phi^c \tau^c B^b_\mu \tau^b\right)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 2\delta_{bc} I B^b_\mu \partial_\mu \phi^c, \tag{74}$$

we arrive at a short expression

$$\to M\left(g_1 B^0_\mu \partial_\mu \phi^0 + B^a_\mu \partial_\mu \phi^a\right) \to \tag{75}$$

which being expressed in terms of physical fields, finally becomes

$$\to M\left(\frac{1}{c_\theta} Z_\mu \partial_\mu \phi^0 + W^+_\mu \partial_\mu \phi^- + W^-_\mu \partial_\mu \phi^+\right). \tag{76}$$

And this term should be ranked as a **criminal** one, since it stands for $Z - \phi^0$ and $W^\pm - \phi^\mp$ *transitions of the zeroth-order* in the coupling constant, and their contribution must be summed up to all orders if one wishes to develop a perturbation theory.

To circumvent this problem one adds a *gauge-fixing* piece to the Lagrangian, $\mathcal{L}_{\rm gf}$, which cancels these mixing terms. However, it breaks the gauge invariance and we must introduce Faddeev–Popov ghost fields to compensate this breaking.

2.3 Gauge fixing and Faddeev–Popov ghosts

We now add to the Lagrangian the gauge-fixing term

$$\mathcal{L}_{\rm gf} = -\mathcal{C}^+\mathcal{C}^- - \frac{1}{2}\left[(\mathcal{C}^Z)^2 + (\mathcal{C}^A)^2\right], \tag{77}$$

where three terms

$$\mathcal{C}^A = -\frac{1}{\xi_A}\partial_\mu A_\mu\,, \qquad \mathcal{C}^Z = -\frac{1}{\xi_Z}\partial_\mu Z_\mu + \xi_Z \frac{M}{c_\theta}\phi^0, \qquad \mathcal{C}^\pm = -\frac{1}{\xi}\partial_\mu W^\pm_\mu + \xi M \phi^\pm, \tag{78}$$

specify the so-called generalized R_ξ gauge with three different gauge parameters associated with three different vector fields: A, Z, $W^\pm$.

Consider, for instance, the term with ξ_Z:

$$\begin{aligned} -\frac{1}{2}\left(\mathcal{C}^Z\right)^2 &= -\frac{1}{2}\left(-\frac{1}{\xi_Z}\partial_\mu Z_\mu + \xi_Z \frac{M}{c_\theta}\phi^0\right)^2 \\ &= -\frac{1}{2}\frac{1}{\xi_Z^2}\left(\partial_\mu Z_\mu\right)^2 + \frac{M}{c_\theta}\left(\partial_\mu Z_\mu\right)\phi^0 - \frac{1}{2}\left(\xi_Z \frac{M}{c_\theta}\phi^0\right)^2. \end{aligned} \tag{79}$$

The first and third terms modify the Z propagator, whilst the second term together with the criminal $Z-\phi^0$ transition of Eq. (76) gives the full derivative

$$\frac{M}{c_\theta}\left(Z_\mu \partial_\mu \phi^0 + \left(\partial_\mu Z_\mu\right)\phi^0\right) = \frac{M}{c_\theta}\partial_\mu\left(Z_\mu \phi^0\right), \tag{80}$$

which does not contribute to the Lagrangian and the problem of zeroth-order terms is solved.

In order to define the Faddeev–Popov ghost Lagrangian we must subject the $\mathcal{C}^{A,Z,\pm}$ to a gauge transformation. This is, in principle, similar to what we did in QED. The relevant derivation will be given below. Contrary to QED, we do have ghost interactions in the SM.

2.4 Propagators in the SM

Consider the sum of the three terms we discussed above,

$$\mathcal{L}_{\rm YM} - (D_\mu K)^+ D_\mu K - \mathcal{C}^+\mathcal{C}^- - \frac{1}{2}(\mathcal{C}^Z)^2 - \frac{1}{2}(\mathcal{C}^A)^2 = \mathcal{L}_{\rm prop} + \mathcal{L}^{\rm bos,I}. \tag{81}$$

The quadratic part (bilinear in fields) of the Lagrangian $\mathcal{L}_{\rm prop}$,

$$\begin{aligned}\mathcal{L}_{\rm prop} \;=\;& -\partial_\mu W^+_\nu \partial_\mu W^-_\nu + \left(1-\frac{1}{\xi^2}\right)\partial_\mu W^+_\mu \partial_\nu W^-_\nu - \frac{1}{2}\partial_\mu Z_\nu \partial_\mu Z_\nu + \frac{1}{2}\left(1-\frac{1}{\xi^2_Z}\right)(\partial_\mu Z_\mu)^2\\ & -\frac{1}{2}\partial_\mu A_\nu \partial_\mu A_\nu + \frac{1}{2}\left(1-\frac{1}{\xi^2_A}\right)(\partial_\mu A_\mu)^2 - \frac{1}{2}\partial_\mu H \partial_\mu H - \partial_\mu \phi^+ \partial_\mu \phi^- - \frac{1}{2}\partial_\mu \phi^0 \partial_\mu \phi^0\\ & -M^2 W^+_\mu W^-_\mu - \frac{1}{2}\frac{M^2}{c^2_\theta} Z_\mu Z_\mu - \xi^2 M^2 \phi^+\phi^- - \frac{1}{2}\xi^2_Z \frac{M^2}{c^2_\theta}\phi^0\phi^0 - \frac{1}{2}M_H H^2, \end{aligned} \tag{82}$$

gives rise to the propagators of bosonic fields.

The scalar field propagators are trivially guessed from Eq. (82)

$$-\partial_\mu \phi^+ \partial_\mu \phi^- - \xi^2 M^2 \phi^+\phi^- \quad\to\quad \frac{1}{p^2+\xi^2 M^2} \qquad \text{etc.} \tag{83}$$

the rule of correspondence for vector fields is more complicated

$$-\frac{1}{2}\partial_\mu Z_\nu \partial_\mu Z_\nu + \frac{1}{2}\left(1-\frac{1}{\xi^2}\right)(\partial_\mu Z_\mu)^2 + \frac{1}{2}M^2_0 Z_\mu Z_\mu \to \frac{\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}}{p^2+M^2_0} + \frac{\frac{p_\mu p_\nu}{p^2}}{\frac{1}{\xi^2}p^2 + M^2_0}\,. \tag{84}$$

It is usually proved in the standard textbooks on QFT, see for example Ref. [2].

2.4.1 Full collection of Feynman rules for propagators

For completeness, we begin with the propagator of a fermion, f, although it was not discussed above:

f $\qquad \dfrac{-i\not p + m_f}{p^2+m^2_f}\,.$

Then, we present three vector boson propagators:

A $\qquad \dfrac{1}{p^2}\left\{\delta_{\mu\nu} + \left(\xi^2_A - 1\right)\dfrac{p_\mu p_\nu}{p^2}\right\},$

Z $\qquad \dfrac{1}{p^2+M^2}\left\{\delta_{\mu\nu} + \left(\xi^2_Z - 1\right)\dfrac{p_\mu p_\nu}{p^2+\xi^2_Z M^2}\right\},$

$W^\pm$ $\qquad \dfrac{1}{p^2+M^2}\left\{\delta_{\mu\nu} + \left(\xi^2 - 1\right)\dfrac{p_\mu p_\nu}{p^2+\xi^2 M^2}\right\}.$

Next, we give propagators of unphysical fields:

$$Y^A \qquad \frac{\xi_A}{p^2}\,,$$

$$\phi^0 \qquad \frac{1}{p^2+\xi_Z^2\dfrac{M^2}{c_\theta^2}}\,, \qquad Y^Z \qquad \frac{\xi_Z}{p^2+\xi_Z^2\dfrac{M^2}{c_\theta^2}}\,,$$

$$\phi^\pm \qquad \frac{1}{p^2+\xi^2 M^2}\,, \qquad X^\pm \qquad \frac{\xi}{p^2+\xi^2 M^2}\,.$$

Finally, the propagator of the physical scalar field, H-boson is

$$H \qquad \frac{1}{p^2+M_H^2}\,.$$

Every propagator should be multiplied by the factor $\frac{1}{(2\pi)^4 i}$. Note that propagators of unphysical fields have a pole at an unphysical mass: $p^2=-\xi^2 M^2$.

2.4.2 *More about propagators in different gauges*

Using partial fraction decomposition, one may present the heavy vector boson, W, propagator (for Z boson we replace $\xi\to\xi_Z$), in three different forms. They are presented below, together with expressions in *the t'Hooft–Feynman*, *unitary* and *Landau* gauges:

$$\begin{aligned}
W^\pm \;\to\;& \frac{1}{p^2+M^2}\left\{\delta_{\mu\nu}+\left(\xi^2-1\right)\frac{p_\mu p_\nu}{p^2+\xi^2M^2}\right\} && R_\xi\text{-gauge,}\\
=\;& \frac{1}{p^2+M^2}\left(\delta_{\mu\nu}+\frac{p_\mu p_\nu}{M^2}\right)-\frac{p_\mu p_\nu}{M^2\left(p^2+\xi^2M^2\right)} &&\\
=\;& \frac{1}{p^2+M^2}\left(\delta_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right)+\frac{\xi^2}{p^2+\xi^2M^2}\frac{p_\mu p_\nu}{p^2} &&\\
=\;& \frac{\delta_{\mu\nu}}{p^2+M^2} \qquad \text{for}\quad \xi=1 && \text{t'Hooft–Feynman gauge,}\\
=\;& \frac{1}{p^2+M^2}\left(\delta_{\mu\nu}+\frac{p_\mu p_\nu}{M^2}\right) \qquad \text{for}\quad \xi=\infty && \text{unitary gauge,}\\
=\;& \frac{1}{p^2+M^2}\left(\delta_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right) \qquad \text{for}\quad \xi=0 && \text{Landau gauge.}
\end{aligned}$$

For the photon propagator, not all of the above cases are possible:

$$\begin{aligned}
A \;\to\;& \frac{1}{p^2}\left\{\delta_{\mu\nu}+\left(\xi_A^2-1\right)\frac{p_\mu p_\nu}{p^2}\right\} && R_\xi\text{-gauge,}\\
=\;& \frac{\delta_{\mu\nu}}{p^2} \qquad \text{for}\quad \xi_A=1 && \text{Feynman gauge,}\\
=\;& \frac{1}{p^2}\left(\delta_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right) \qquad \text{for}\quad \xi_A=0 && \text{Landau gauge.}
\end{aligned}$$

The physical gauge is recovered in the limit $\xi_A\to 1$ and $\xi_Z,\xi\to\infty$. Therefore, the physical gauge is *a mixture* of the unitary and Feynman gauges.

2.5 Interaction Lagrangian

2.5.1 Bosonic Sector

The Lagrangian, describing interactions between vector bosons can be easily derived from Eq. (81) in terms of physical fields using only Eq. (68) on top of the former equation. After trivial but lengthy algebra, we obtain:

$$
\begin{aligned}
\mathcal{L}^{\rm bos,I} \;=\; & -igc_\theta\left\{\partial_\nu Z_\mu W^{[+}_\mu W^{-]}_\nu - Z_\nu W^{[+}_\mu \partial_\nu W^{-]}_\mu + Z_\mu W^{[+}_\nu \partial_\nu W^{-]}_\mu\right\} \\
& -igs_\theta\left\{\partial_\nu A_\mu W^{[+}_\mu W^{-]}_\nu - A_\nu W^{[+}_\mu \partial_\nu W^{-]}_\mu + A_\mu W^{[+}_\nu \partial_\nu W^{-]}_\mu\right\} \\
& +\frac{1}{2}g^2\left\{\left(W^+_\mu W^-_\nu\right)^2 - \left(W^+_\mu W^-_\mu\right)^2\right\} + g^2 c^2_\theta\left\{Z_\mu Z_\nu W^+_\mu W^-_\nu - Z_\mu Z_\mu W^+_\nu W^-_\nu\right\} \\
& +g^2 s^2_\theta\left\{A_\mu A_\nu W^+_\mu W^-_\nu - A_\mu A_\mu W^+_\nu W^-_\nu\right\} + g^2 s_\theta c_\theta\left\{A_\mu Z_\nu W^{[+}_\mu W^{-]}_\nu - 2A_\mu Z_\mu W^+_\nu W^-_\nu\right\} \\
& -gMH\left\{W^+_\mu W^-_\nu + \frac{1}{2c^2_\theta}Z_\mu Z_\mu\right\} \\
& -\frac{i}{2}g\left\{W^+_\mu\left(\phi^0\partial_\mu\phi^- - \phi^-\partial_\mu\phi^0\right) - W^-_\mu\left(\phi^0\partial_\mu\phi^+ - \phi^+\partial_\mu\phi^0\right)\right\} \\
& +\frac{1}{2}g\left\{W^+_\mu\left(H\partial_\mu\phi^- - \phi^-\partial_\mu H\right) - W^-_\mu\left(H\partial_\mu\phi^+ - \phi^+\partial_\mu H\right)\right\} \\
& +\frac{1}{2}\frac{g}{c_\theta}Z_\mu\left(H\partial_\mu\phi^0 - \phi^0\partial_\mu H\right) + ig\left(s_\theta A_\mu - \frac{s^2_\theta}{c_\theta}Z_\mu\right)MW^{[+}_\mu\phi^{-]} \\
& +ig\left(s_\theta A_\mu + \frac{c^2_\theta - s^2_\theta}{c_\theta}Z_\mu\right)\left(\phi^+\partial_\mu\phi^- - \phi^-\partial_\mu\phi^+\right) \\
& -\frac{1}{4}g^2 W^+_\mu W^-_\mu\left(HH + \phi^0\phi^0 + 2\phi^+\phi^-\right) \\
& -\frac{1}{8}\frac{g^2}{c^2_\theta}Z_\mu Z_\mu\left\{HH + \phi^0\phi^0 + 2\left(c^2_\theta - s^2_\theta\right)^2\phi^+\phi^-\right\} \\
& -\frac{1}{2}g^2\frac{s^2_\theta}{c_\theta}Z_\mu\phi^0 W^{[+}_\mu\phi^{-]} - \frac{i}{2}g^2\frac{s^2_\theta}{c_\theta}Z_\mu H W^{[+}_\mu\phi^{-]} + \frac{1}{2}g^2 s_\theta A_\mu\phi^0 W^{[+}_\mu\phi^{-]} \\
& +\frac{i}{2}g^2 s_\theta A_\mu H W^{[+}_\mu\phi^{-]} - g^2\frac{s_\theta}{c_\theta}\left(c^2_\theta - s^2_\theta\right)Z_\mu A_\mu\phi^+\phi^- - g^2 s^2_\theta A_\mu A_\mu\phi^+\phi^-, \qquad (85)
\end{aligned}
$$

where we introduced the anti-symmetrized combination

$$A^{[+}B^{-]} = A^+B^- - A^-B^+. \qquad (86)$$

From Eq. (85) all the relevant Feynman rules for three-linear and four-linear vertices are straightforwardly derived.

2.5.2 FP Ghost Sector

In order to define the FP ghost Lagrangian we must subject $\mathcal{C}^a$ to the gauge transformation. First, we write down the gauge transformations for all bosonic fields of the SM:

$$
\begin{aligned}
B^a_\mu &\;\to\; B^a_\mu + g\varepsilon_{abc}\Lambda^b B^c_\mu - \partial_\mu\Lambda^a, \qquad B^0_\mu \to B^0_\mu - \partial_\mu\Lambda^0, \\
K &\;\to\; \left(1 - \frac{i}{2}g\Lambda^a\tau^a - \frac{i}{2}gg_1\Lambda^0\right)K, \quad \text{with} \quad g_1 = -\frac{s_\theta}{c_\theta}\,. \qquad (87)
\end{aligned}
$$

From the second row we may straightforwardly derive transformations for separate components:

$$H \pm i\phi^0 \;\to\; H \pm i\phi^0 \mp \frac{i}{2}g\left[\left(\Lambda^3 + g_1\Lambda^0\right)\left(H + 2\frac{M}{g} \pm i\phi^0\right) \pm 2i\Lambda^\pm\phi^\mp\right],$$

$$
\begin{aligned}
\phi^0 &\to \phi^0 - \frac{1}{2} g \left(\Lambda^3 + g_1 \Lambda^0\right) \left(H + 2\frac{M}{g}\right) + \frac{i}{2} g \left(\Lambda^- \phi^+ - \Lambda^+ \phi^-\right), \\
\phi^\mp &\to \phi^\mp - \frac{1}{2} g \Lambda^\mp \left(H + 2\frac{M}{g} \pm i\phi^0\right) \mp \frac{i}{2} g \left(-\Lambda^3 + g_1 \Lambda^0\right) \phi^\mp,
\end{aligned} \tag{88}
$$

where $\Lambda^i,\ i = 0,1,2,3$ are *group*, and $\Lambda^j,\ j = \pm, Z, A$ are *physical* gauge transformation parameters, related to each other by means of the usual relations:

$$
\begin{aligned}
\Lambda^1 &= \frac{1}{\sqrt{2}} (\Lambda^+ + \Lambda^-), & \Lambda^2 &= \frac{i}{\sqrt{2}} (\Lambda^+ - \Lambda^-), \\
\Lambda^3 &= c_\theta \Lambda^Z + s_\theta \Lambda^A, & \Lambda^0 &= -s_\theta \Lambda^Z + c_\theta \Lambda^A, \\
\Lambda^3 + g_1 \Lambda^0 &= \frac{1}{c_\theta} \Lambda^Z, & -\Lambda^3 + g_1 \Lambda^0 &= -\frac{c_\theta^2 - s_\theta^2}{c_\theta} \Lambda^Z - 2 s_\theta \Lambda^A.
\end{aligned}
$$

From Eqs. (88)–(89) we derive the gauge transformations of physical fields in terms of physical parameters:

$$
\begin{aligned}
\phi^0 &\to \phi^0 - \frac{1}{2} g \frac{\Lambda^Z}{c_\theta} \left(H + 2\frac{M}{g}\right) + \frac{i}{2} g \left(\Lambda^- \phi^+ - \Lambda^+ \phi^-\right), \\
\phi^\mp &\to \phi^- - \frac{1}{2} g \Lambda^\mp \left(H + 2\frac{M}{g} \pm i\phi^0\right) \pm \frac{i}{2} g \left(\frac{c_\theta^2 - s_\theta^2}{c_\theta} \Lambda^Z + 2 s_\theta \Lambda^A\right) \phi^\mp, \\
W_\mu^\mp &\to W_\mu^\mp \mp i g \Lambda^\mp \left(c_\theta Z_\mu + s_\theta A_\mu\right) \pm i g \left(c_\theta \Lambda^Z + s_\theta \Lambda^A\right) W_\mu^\mp - \partial_\mu \Lambda^\mp, \\
A_\mu &\to A_\mu + i g s_\theta \left(\Lambda^- W_\mu^+ - \Lambda^+ W_\mu^-\right) - \partial_\mu \Lambda^A, \\
Z_\mu &\to Z_\mu + i g c_\theta \left(\Lambda^- W_\mu^+ - \Lambda^+ W_\mu^-\right) - \partial_\mu \Lambda^Z, \\
H &\to H + \frac{1}{2} \left(g \Lambda^3 + g_1 \Lambda^0\right) \phi^0 + \frac{1}{2} \left(\Lambda^+ W_\mu^- + \Lambda^- W_\mu^+\right).
\end{aligned} \tag{89}
$$

General gauge transformations may be written in matrix form:

$$
\mathcal{C}^i \to \mathcal{C}^i + \left(\mathrm{M}^{ij} + g \mathrm{L}^{ij}\right) \Lambda^j, \qquad \text{where} \quad i,j = \pm, Z, A. \tag{90}
$$

For $i = -$, from Eq. (89) we derive the transformation for $\mathcal{C}^-$:

$$
\begin{aligned}
\mathcal{C}^- &= -\frac{1}{\xi} \partial_\mu W_\mu^- + \xi M \phi^- \\
&\to \mathcal{C}^- - \frac{1}{\xi} \partial_\mu \Big\{ -i g \Lambda^- \left(c_\theta Z_\mu + s_\theta A_\mu\right) + i g \left(c_\theta \Lambda^Z + s_\theta \Lambda^A\right) W_\mu^- - \partial_\mu \Lambda^- \Big\} \\
&\quad + g \xi M \Big\{ -\frac{1}{2} \Lambda^- \left(H + 2\frac{M}{g} + i\phi^0\right) + \frac{i}{2} \frac{c_\theta^2 - s_\theta^2}{c_\theta} \Lambda^Z \phi^- + i s_\theta \Lambda^A \phi^- \Big\} \\
&= \mathcal{C}^- + \frac{1}{\xi} \Box \Lambda^- - \xi M^2 \Lambda^- + \frac{i}{\xi} g \partial_\mu \Big\{ \Lambda^- \left(c_\theta Z_\mu + s_\theta A_\mu\right) \Big\} \\
&\quad - \frac{i}{\xi} g \partial_\mu \Big\{ \left(c_\theta \Lambda^Z + s_\theta \Lambda^A\right) W_\mu^- \Big\} - \frac{1}{2} \xi g M \left(H + i\phi^0\right) \Lambda^- \\
&\quad + \frac{i}{2} \xi g M \frac{c_\theta^2 - s_\theta^2}{c_\theta} \Lambda^Z \phi^- + i \xi g s_\theta M \Lambda^A \phi^-,
\end{aligned} \tag{91}
$$

and a similar one for $\mathcal{C}^+$.

In full analogy with conventional gauge transformations and the correspondence of gauge parameters to gauge fields B^a_μ, one may establish the correspondence of *physical* gauge parameters $\Lambda^{+,-,Z,A}$ entering Eq. (91) to *ghost fields* $X^i = X^+, X^-, Y^Z, Y^A$:

$$\Lambda^\pm \to X^\pm, \qquad \Lambda^Z \to Y^Z, \qquad \Lambda^A \to Y^A. \tag{92}$$

The gauge invariance $\mathcal{C}^\pm = -\frac{1}{\xi}\partial_\mu W^-_\mu + \xi M\phi^- \to \mathcal{C}^\pm$ is restored if $\Lambda^\pm$ are identified with ghost fields $X^\pm$ with propagators

$$\frac{1}{\xi}\Box - \xi M^2 \qquad X^\pm \qquad \frac{\xi}{p^2+\xi^2M^2}$$

and interactions

$$g\overline{X}^\pm \mathrm{L}^{\pm j} X^j, \qquad j = \pm, Z, A\,, \tag{93}$$

where we introduced four more fields: $\overline{X}^i = \overline{X}^+, \overline{X}^-, \overline{Y}^Z, \overline{Y}^A$.

Analogously, from Eq. (89), we obtain the transformation of $\mathcal{C}^A$:

$$\begin{aligned}\mathcal{C}^A &= -\frac{1}{\xi_A}\partial_\mu A_\mu \to \mathcal{C}^A - \frac{1}{\xi_A}\partial_\mu\Big[igs_\theta\left(\Lambda^- W^+_\mu - \Lambda^+ W^-_\mu\right) - \partial_\mu\Lambda^A\Big]\\ &= \mathcal{C}^A + \frac{1}{\xi_A}\Box\Lambda^A - \frac{i}{\xi_A}gs_\theta\partial_\mu\left(\Lambda^- W^+_\mu - \Lambda^+ W^-_\mu\right).\end{aligned} \tag{94}$$

The gauge invariance is restored if we require the validity of the equation of motion:

$$\frac{1}{\xi_A}\Box Y^A = \frac{i}{\xi_A}gs_\theta\partial_\mu\left(X^- W^+_\mu - X^+ W^-_\mu\right), \tag{95}$$

i.e. identify Y^A with a field which has the propagator

$$\frac{1}{\xi_A}\Box \qquad Y^A \qquad \frac{\xi_A}{p^2}$$

and interaction

$$g\overline{Y}^A \mathrm{L}^{Aj} X^j, \qquad j = \pm, Z, A\,. \tag{96}$$

Finally, for the transformation of $\mathcal{C}^Z$, from Eq. (89) we derive:

$$\begin{aligned}\mathcal{C}^Z &= -\frac{1}{\xi_Z}\partial_\mu Z_\mu + \xi_Z\frac{M}{c_\theta}\phi^0\\ &\to \mathcal{C}^Z - \frac{1}{\xi_Z}\partial_\mu\Big\{igc_\theta\left(\Lambda^- W^+_\mu - \Lambda^+ W^-_\mu\right) - \partial_\mu\Lambda^Z\Big\}\\ &\quad +\xi_Z\frac{M}{c_\theta}\left\{-\frac{M}{c_\theta}\Lambda^Z - \frac{1}{2}g\frac{\Lambda^Z}{c_\theta}H + \frac{i}{2}g\left(\Lambda^-\phi^+ - \Lambda^+\phi^-\right)\right\}\\ &= \mathcal{C}^Z\frac{1}{\xi_Z} + \Box\Lambda^Z - \xi_Z\frac{M^2}{c^2_\theta}\Lambda^Z - \frac{i}{\xi_Z}gc_\theta\partial_\mu\left(\Lambda^- W^+_\mu - \Lambda^+ W^-_\mu\right)\\ &\quad -\frac{1}{2}\xi_Z g\frac{M}{c^2_\theta}\Lambda^Z H + i\xi_Z g\frac{M}{c_\theta}\left(\Lambda^-\phi^+ - \Lambda^+\phi^-\right),\end{aligned} \tag{97}$$

giving the propagator of Y^Z

$$\frac{1}{\xi_Z}\Box - \xi_Z\frac{M^2}{c_\theta^2} \qquad\qquad Y^Z \qquad\qquad \frac{\xi_Z}{p^2+\xi_Z^2\frac{M^2}{c_\theta^2}}$$

and interaction

$$g\overline{Y}^Z \mathrm{L}^{Zj} X^j, \qquad j=\pm, Z, A. \tag{98}$$

The complete interaction Lagrangian in the FP sector of the SM derives trivially from the above considerations. It reads:

$$\begin{aligned}
\mathcal{L}^{\mathrm{I}}_{\mathrm{gf}} \;=\; & igc_\theta W^+_\mu\left[\frac{1}{\xi_Z}\left(\partial_\mu\overline{Y}^Z\right)X^- - \frac{1}{\xi}\left(\partial_\mu\overline{X}^+\right)Y^Z\right] + igc_\theta W^-_\mu\left[\frac{1}{\xi}\left(\partial_\mu\overline{X}^-\right)Y^Z - \frac{1}{\xi_Z}\left(\partial_\mu\overline{Y}^Z\right)X^+\right] \\
& +igs_\theta W^+_\mu\left[\frac{1}{\xi_A}\left(\partial_\mu\overline{Y}^A\right)X^- - \frac{1}{\xi}\left(\partial_\mu\overline{X}^+\right)Y^A\right] + igs_\theta W^-_\mu\left[\frac{1}{\xi}\left(\partial_\mu\overline{X}^-\right)Y^A - \frac{1}{\xi_A}\left(\partial_\mu\overline{Y}^A\right)X^+\right] \\
& +igc_\theta\frac{1}{\xi}Z_\mu\left(\partial_\mu\overline{X}^+X^+ - \partial_\mu\overline{X}^-X^-\right) + igs_\theta\frac{1}{\xi}A_\mu\left(\partial_\mu\overline{X}^+X^+ - \partial_\mu\overline{X}^-X^-\right) \\
& -\frac{1}{2}gMH\left(\xi\overline{X}^+X^+ + \xi\overline{X}^-X^- + \frac{\xi_Z}{c_\theta^2}\overline{Y}^ZY^Z\right) \\
& -ig\xi M\frac{c_\theta^2-s_\theta^2}{c_\theta}\left(\overline{X}^+Y^Z\phi^+ - \overline{X}^-Y^Z\phi^-\right) + \frac{i}{2}g\xi_Z M\frac{1}{c_\theta}\left(\overline{Y}^ZX^-\phi^+ - \overline{Y}^ZX^+\phi^-\right) \\
& +igs_\theta\xi M\left(\overline{X}^-Y^A\phi^- - \overline{X}^+Y^A\phi^+\right) + \frac{i}{2}g\xi M\left(\overline{X}^+X^+\phi^0 - \overline{X}^-X^-\phi^0\right).
\end{aligned} \tag{99}$$

Note the trivial rules:

- $\overline{Y}^Z$ and $\overline{Y}^A$ are accompanied by ξ_Z and ξ_A, respectively;
- $\overline{X}^\pm$ is accompanied by ξ;
- the terms $\overline{Y}^ZX^-$ and $\overline{Y}^ZX^+$ or $\overline{X}^+X^+$ and $\overline{X}^-X^-$ differ by sign for interactions with all fields but H.

To summarize our findings, we see that ghosts are fields satisfying the Klein–Gordon equation. They possess a charge resembling the fermionic charge. In other words, they are scalar fermions, i.e. have the wrong relation between spin and statistics.

2.5.3 Scalar Sector

The interactions in the scalar sector are given by the scalar potential

$$\mathcal{L}^{\mathrm{I}}_{\mathrm{S}} = -\mu^2K^+K - \frac{1}{2}\lambda\left(K^+K\right)^2, \tag{100}$$

where

$$\begin{aligned}
K &= \frac{1}{\sqrt{2}}\begin{pmatrix} H+\langle v\rangle + i\phi^0, \\ i\sqrt{2}\phi^- \end{pmatrix}, \qquad \langle v\rangle = 2\frac{M}{g}\,, \\
K^+ &= \frac{1}{\sqrt{2}}\left(H+\langle v\rangle - i\phi^0\,, -i\sqrt{2}\phi^+\right).
\end{aligned} \tag{101}$$

For $\mathcal{L}^{\rm I}_{\rm S}$ we derive

$$\begin{aligned}\mathcal{L}^{\rm I}_{\rm S} &= -\frac{1}{2}\mu^2\left[H^2+2\langle v\rangle H+\langle v\rangle^2+\left(\phi^0\right)^2+2\phi^+\phi^-\right]\\ &\quad -\frac{1}{8}\lambda\left\{H^4+4\langle v\rangle H^3+6\langle v\rangle^2H^2+4\langle v\rangle^3H+\langle v\rangle^4+\left(\phi^0\right)^4+4\left(\phi^+\phi^-\right)^2\right.\\ &\quad \left.+2\left(H^2+2\langle v\rangle H+\langle v\rangle^2\right)\left[\left(\phi^0\right)^2+2\phi^+\phi^-\right]+4\left(\phi^0\right)^2\phi^+\phi^-\right\}. \end{aligned} \tag{102}$$

To understand better the physical meaning of the various terms, let us collect some selected terms:

$$\begin{aligned} &\text{constant term:} && -\frac{\langle v\rangle^2}{2}\left(\mu^2+\frac{1}{4}\lambda\langle v\rangle^2\right), \quad \text{irrelevant, may be dropped;}\\ &\text{linear term, } H: && -\langle v\rangle\left(\mu^2+\frac{1}{2}\lambda\langle v\rangle^2\right)=-\langle v\rangle\beta_H, \quad \text{vacuum tadpole;}\\ &\text{quadratic term, } H^2: && -\frac{1}{2}\left(\mu^2+\frac{1}{2}\lambda\langle v\rangle^2+\lambda\langle v\rangle^2\right)=-\frac{1}{2}\left(\beta_H+M_H^2\right);\\ &\left[\left(\phi^0\right)^2+2\phi^+\phi^-\right]: && -\frac{1}{2}\left(\mu^2+\frac{1}{2}\lambda\langle v\rangle^2\right). \end{aligned} \tag{103}$$

Here we introduced, for convenience, the following set of parameters:

$$\beta_H=\mu^2+2\frac{\lambda}{g^2}M^2, \qquad \lambda=\frac{g^2M_H^2}{4M^2}=g^2\alpha_H, \qquad \alpha_H=\frac{1}{4}\frac{M_H^2}{M^2}\,. \tag{104}$$

From these relations one sees, that λ and α_H are not independent. Since M_H is a measurable quantity, λ derives from g, M, M_H and α_H — from M_H, M. On the contrary, μ^2 (or equivalently β_H) should be treated as a new parameter, which has to be adjusted such that the vacuum expectation value of the H field remains zero, order by order in perturbation theory.

Omitting irrelevant constant and mass term, $-\frac{1}{2}M_H^2H^2$, we derive for the interaction Lagrangian:

$$\begin{aligned}\mathcal{L}^{\rm I}_{\rm S} &= -\beta_H\left\{2\frac{M}{g}H+\frac{1}{2}\left[H^2+\left(\phi^0\right)^2+2\phi^+\phi^-\right]\right\}-g\alpha_H M\left[H^3+H\left(\phi^0\right)^2+2H\phi^+\phi^-\right]\\ &\quad -\frac{1}{8}g^2\alpha_H\left[H^4+\left(\phi^0\right)^4+2H^2\left(\phi^0\right)^2+4H^2\phi^+\phi^-+4\left(\phi^0\right)^2\phi^+\phi^-+4\left(\phi^+\phi^-\right)^2\right]. \end{aligned} \tag{105}$$

2.6 Tadpoles and their role in proving gauge invariance

The following 10 diagrams contribute to the vacuum tadpoles:

H

= (1) f + (2) W + (3) Z

+ (4) H + (5) ϕ + (6) ϕ^0

+ (7) X^- + (8) X^+ + (9) Y^Z

+ (10) β_H

In the lowest order, we would have only the tadpole diagram (10) and the constant $\beta_{_H}$ must be set to zero. At the one-loop level, we have to take into account all 10 diagrams and the tadpole constant must be adjusted in such a way that the vacuum expectation value of the H field remains zero.

We describe the correct procedure of such an *adjustment* at the one-loop level. First of all, we have to *renormalize* the vacuum expectation value itself:

$$K = \frac{1}{\sqrt{2}} \begin{pmatrix} \chi \\ \\ \sqrt{2} i \phi^- \end{pmatrix}, \qquad \chi = H + 2\frac{M}{g}\left(1 + g^2 \beta_t\right) + i\phi^0. \tag{106}$$

Now we set $\mu^2 + 2\left(\lambda/g^2\right) M^2 = 0$, as we did in the lowest order, and repeat the same derivation as above. Instead of Eq. (105), for the $\mathcal{L}^{\mathrm{I}}_{\mathrm{S}}$ part of the Lagrangian we derive:

$$\begin{aligned} \mathcal{L}^{\mathrm{I}}_{\mathrm{S}} &= -2\, gM M^2_{_H} \beta_t\, H - \frac{1}{2} M^2_{_H} \left(1 + 3\, g^2 \beta_t\right) H^2 \\ &\quad -\frac{1}{2}\, g^2 M^2_{_H} \beta_t \left[\left(\phi^0\right)^2 + 2\phi^+\phi^-\right] - g\alpha_{_H} M \left[H^3 + H\left(\phi^0\right)^2 + 2H\phi^+\phi^-\right] \\ &\quad -\frac{1}{8} g^2 \alpha_{_H} \left[H^4 + \left(\phi^0\right)^4 + 2H^2\left(\phi^0\right)^2 \right. \\ &\quad \left. +4H^2\phi^+\phi^- + 4\left(\phi^0\right)^2 \phi^+\phi^- + 4\left(\phi^+\phi^-\right)^2\right], \end{aligned} \tag{107}$$

with β_t (instead of $\beta_{_H}$) fixed by the requirement of a zero vacuum expectation value of the H field. Note that the only difference between Eq. (107) and Eq. (105) appears in the H^2 term.

From the renormalization of $\langle v \rangle$ we are automatically led to the addition of tadpoles to the $W - W$ and $Z - Z$ self-energies and to the corresponding vector–scalar transitions:

$$\begin{aligned} &-g^2 \beta_t \left(M^2_{_0} Z_\mu Z_\mu + 2\, M^2 W^+_\mu W^-_\mu\right), \\ &-g^2\, M \beta_t \left(\frac{1}{c_\theta}\, \phi^0 \partial_\mu Z_\mu + \phi^+ \partial_\mu W^-_\mu + \phi^- \partial_\mu W^+_\mu\right). \end{aligned} \tag{108}$$

They are very important for proving that the W, Z and H self-energies are ξ-independent on their mass shells, i.e. at $p^2 = -M^2,\ p^2 = -M^2_{_0}$, and $p^2 = -M^2_{_H}$, respectively.

2.7 Interactions of fermions with gauge fields

Consider a generic fermion isodoublet and decompose it into left (L) and right (R) components:

$$\psi = \begin{pmatrix} u \\ d \end{pmatrix}, \qquad \psi_{_{L,R}} = \frac{1}{2}\left(1 \pm \gamma_5\right)\psi\,. \tag{109}$$

The covariant derivative for the L-fields

$$D_\mu \psi_{_L} \;=\; \left(\partial_\mu + g B^i_\mu T^i\right)\psi_{_L}\,, \qquad i = 0, ..., 3\,, \tag{110}$$

is written in terms of generators of $SU(2) \otimes U(1)$:

$$T^a \;=\; -\frac{i}{2}\tau^a\,, \qquad T^0 \;=\; -\frac{i}{2} g_2 I\,, \tag{111}$$

with arbitrary $U(1)$-hypercharge g_2, whilst for R-fields

$$D_\mu \psi_{_R} \;=\; \left(\partial_\mu + g B^i_\mu t^i\right)\psi_{_R}\,, \qquad i = 0, ..., 3\,, \tag{112}$$

in terms of generators of $U(1)$:

$$t^a \quad = \quad 0, \qquad t^0 = -\frac{i}{2}\begin{pmatrix} g_3 & 0 \\ 0 & g_4 \end{pmatrix}, \tag{113}$$

with hypercharges $g_3,\ g_4$. We stress, that Eq. (112) is written formally similar to Eq. (110) using a diagonal matrix t^0 which should not be confused with the generators of $SU(2)$. Thus, the $\psi_{_L}$ transforms as a doublet under $SU(2)$ and the $\psi_{_R}$ as a singlet. The parameters g_2, g_3 and g_4 are arbitrary hypercharges which will be fixed below. The kinetic part of the Lagrangian can be written as

$$\mathcal{L}_V^{\text{fer,I}} = -\overline{\psi}_{_L} \not{D} \psi_{_L} - \overline{\psi}_{_R} \not{D} \psi_{_R} \,. \tag{114}$$

As an exercise, consider (0+3) components:

$$\begin{aligned}
& -\overline{\psi}_{_L}\gamma_\mu\left(\partial_\mu - \frac{i}{2}gg_2B^0_\mu I - \frac{i}{2}gB^3_\mu\tau^3\right)\psi_{_L} - \overline{\psi}_{_R}\gamma_\mu\left(\partial_\mu - \frac{i}{2}gB^0_\mu\begin{pmatrix} g_3 & 0 \\ 0 & g_4 \end{pmatrix}\right)\psi_{_R} \\
= \; & -(\overline{u},\overline{d})_L\gamma_\mu\left[\partial_\mu - \frac{i}{2}gg_2B^0_\mu\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{i}{2}gB^3_\mu\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right]\begin{pmatrix} u \\ d \end{pmatrix}_{_L} \\
& -(\overline{u},\overline{d})_R\gamma_\mu\left[\partial_\mu - \frac{i}{2}gg_2B^0_\mu\begin{pmatrix} g_3 & 0 \\ 0 & g_4 \end{pmatrix}\right]\begin{pmatrix} u \\ d \end{pmatrix}_{_R} \\
= \; & -\overline{f}_{_L}\not{\partial} f_{_L} - \overline{f}_{_R}\not{\partial} f_{_R} + \frac{i}{2}gg_2B^0_\mu\left(\overline{u}_L\gamma_\mu u_L + \overline{d}_L\gamma_\mu d_L\right) \\
& + \frac{i}{2}gB^3_\mu\left(\overline{u}_L\gamma_\mu u_L - \overline{d}_L\gamma_\mu d_L\right) + \frac{i}{2}gB^0_\mu\left(g_3\overline{u}_R\gamma_\mu u_R + g_4\overline{d}_R\gamma_\mu d_R\right) \to
\end{aligned} \tag{115}$$

Using equations:

$$\overline{f}_{_L}\gamma_\mu f_{_L} = \frac{1}{2}\overline{f}\gamma_\mu\left(1+\gamma_5\right)f\,, \qquad \overline{f}_{_R}\gamma_\mu f_{_R} = \frac{1}{2}\overline{f}\gamma_\mu\left(1-\gamma_5\right)f\,,$$

from Eq. (115) we derive further on

$$\begin{aligned}
\to \quad & -\overline{f}\not{\partial} f + \frac{i}{4}gg_2\left[\overline{u}\gamma_\mu\left(1+\gamma_5\right)u + \overline{d}\gamma_\mu\left(1+\gamma_5\right)d\right]\left(-s_\theta Z_\mu + c_\theta A_\mu\right) \\
& + \frac{i}{4}g\left[\overline{u}\gamma_\mu\left(1+\gamma_5\right)u - \overline{d}\gamma_\mu\left(1+\gamma_5\right)d\right]\left(c_\theta Z_\mu + s_\theta A_\mu\right) \\
& + \frac{i}{4}g\left[g_3\overline{u}\gamma_\mu\left(1-\gamma_5\right)u + g_4\overline{d}\gamma_\mu\left(1-\gamma_5\right)d\right]\left(-s_\theta Z_\mu + c_\theta A_\mu\right).
\end{aligned} \tag{116}$$

First, we collect terms with A_μ, i.e. *the e.m. current*:

$$\frac{i}{4}g\Big\{c_\theta g_2\left[\overline{u}\gamma_\mu\left(1+\gamma_5\right)u + \overline{d}\gamma_\mu\left(1+\gamma_5\right)d\right] \tag{117}$$

$$+s_\theta\left[\overline{u}\gamma_\mu\left(1+\gamma_5\right)u - \overline{d}\gamma_\mu\left(1+\gamma_5\right)d\right] + c_\theta\left[g_3\overline{u}\gamma_\mu\left(1-\gamma_5\right)u + g_4\overline{d}\gamma_\mu\left(1-\gamma_5\right)d\right]\Big\} \to$$

Then, we require the absence of axial currents

$$\begin{aligned}
c_\theta g_2 + s_\theta - c_\theta g_3 = 0\,, \qquad & c_\theta g_2 - s_\theta - c_\theta g_4 = 0\,, \\
g_2 - g_1 - g_3 = 0\,, \qquad & g_2 + g_1 - g_4 = 0\,, \\
g_i \quad = \quad & -\frac{s_\theta}{c_\theta}\lambda_i\,, \\
-\lambda_2 + 1 + \lambda_3 = 0\,, \qquad & -\lambda_2 - 1 + \lambda_4 = 0\,,
\end{aligned} \tag{118}$$

The Eq. (118) defines λ_3 and λ_4 leading to

$$\to \frac{i}{4} g s_\theta \Big\{ -\lambda_2 \left[\overline{u}\gamma_\mu u + \overline{d}\gamma_\mu d \right] + \overline{u}\gamma_\mu u - \overline{d}\gamma_\mu d + (1-\lambda_2)\, \overline{u}\gamma_\mu u - (1+\lambda_2)\, \overline{d}\gamma_\mu d \Big\}$$
$$= \frac{i}{2} g s_\theta \Big\{ (1-\lambda_2)\, \overline{u}\gamma_\mu u - (1+\lambda_2)\, \overline{d}\gamma_\mu d \Big\} = ieQ_u \overline{u}\gamma_\mu u + ieQ_d \overline{d}\gamma_\mu d\,. \tag{119}$$

Thus, three parameters λ_i are fixed by the requirement that the e.m. current has the conventional structure, $iQ_f e \overline{f}\gamma_\mu f$, with the charges $Q_f = 2I_f^{(3)}|Q_f|$. The solution is

$$\lambda_2 = 1 - 2Q_u \;=\; -1 - 2Q_d\,, \qquad \lambda_3 = -2Q_u\,, \qquad \lambda_4 = -2Q_d\,. \tag{120}$$

Having all three hypercharges fixed, we derive the final expression for the interaction Lagrangian $\mathcal{L}_V^{\mathrm{fer,I}}$:

$$\begin{aligned} \mathcal{L}_V^{\mathrm{fer,I}} &= \sum_f \left[igs_\theta Q_f A_\mu \overline{f}\gamma_\mu f + i\frac{g}{2c_\theta} Z_\mu \overline{f}\gamma_\mu \left(v_f + a_f\gamma_5 \right) f \right] \\ &\quad + \sum_d \left[i\frac{g}{2\sqrt{2}} W_\mu^+ \overline{u}\gamma_\mu \left(1+\gamma_5\right) d + i\frac{g}{2\sqrt{2}} W_\mu^- \overline{d}\gamma_\mu \left(1+\gamma_5\right) u \right], \end{aligned} \tag{121}$$

with vector and axial–vector couplings of the Z boson to a fermion f:

$$v_f = I_f^{(3)} - 2Q_f s_\theta^2\,, \qquad a_f = I_f^{(3)}\,. \tag{122}$$

The first sum in Eq. (121) runs over all fermions, f, and the second over all doublets, d, of the SM. We see, that contrary to the Z and A fields, the $W^\pm$ are always coupled to a $V+A$ current. This is a consequence of the $SU(2)\otimes U(1)$ gauge transformation, Eq. (110).

2.8 Interactions of fermions with scalar fields

We now consider the only remaining sector of the SM describing the generation of fermionic masses and the interaction of fermions with scalar fields. We need not only the field K but also its conjugate K^c in order to give masses both to the up and down fermions. In our convention, K gives masses to up fermions, and K^c to down fermions. We recall K and derive K^c using the definition of charge conjugation, $K^c = i\tau^2 K^*$. The relevant set of formulae is

$$\begin{aligned} K &= \frac{1}{\sqrt{2}} \begin{pmatrix} \chi \\ \sqrt{2}i\phi^- \end{pmatrix} = \frac{1}{\sqrt{2}} \left(H + \langle v\rangle + i\phi^a\tau^a \right) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\ K^c &= i\tau^2 K^* = -\frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2}i\phi^+ \\ \chi^* \end{pmatrix} = -\frac{1}{\sqrt{2}} \left(H + \langle v\rangle + i\phi^a\tau^a \right) \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \\ \chi &= H + \langle v\rangle + i\phi^0. \end{aligned} \tag{123}$$

The corresponding part of the Lagrangian may be written as follows:

$$\mathcal{L}_S^{\mathrm{fer}} = -\alpha_f \overline{\psi}_{\scriptscriptstyle L} K u_{\scriptscriptstyle R} - \beta_f \overline{\psi}_{\scriptscriptstyle L} K^c d_{\scriptscriptstyle R} + h.c. \tag{124}$$

In order to prove its gauge invariance, consider four gauge transformations:

$$\begin{aligned} K &\to \left(1 - \frac{i}{2} g\Lambda^a(x)\,\tau^a - \frac{i}{2} gg_1 \Lambda^0(x)\, I \right) K, \\ K^c &\to \left(1 - \frac{i}{2} g\Lambda^a(x)\,\tau^a + \frac{i}{2} gg_1 \Lambda^0(x)\, I \right) K^c, \\ \psi'_{\scriptscriptstyle L} &\to \left(1 - \frac{i}{2} g\Lambda^a(x)\,\tau^a - \frac{i}{2} gg_2 \Lambda^0(x)\, I \right) \psi_{\scriptscriptstyle L}, \\ \psi'_{\scriptscriptstyle R} &\to \left(1 - \frac{i}{2} g \begin{pmatrix} g_3 & 0 \\ 0 & g_4 \end{pmatrix} \Lambda^0(x) \right) \psi_{\scriptscriptstyle R}. \end{aligned} \tag{125}$$

We immediately see that $\mathcal{L}_S^{\rm fer}$ will be gauge-invariant if $g_2 = g_1 + g_3$ and $g_2 = -g_1 + g_4$, the identities already established from the requirement that the e.m. current has a conventional structure.

We then substitute K and K^c of Eq. (123) into $\mathcal{L}_S^{\rm fer}$ of Eq. (124) and derive

$$
\begin{aligned}
&-\alpha_f \overline{\psi}_L \frac{1}{\sqrt{2}} \begin{pmatrix} H + \langle v \rangle + i\phi^0 \\ \\ \sqrt{2} i \phi^- \end{pmatrix} u_R + \beta_f \overline{\psi}_L \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} i \phi^+ \\ \\ H + \langle v \rangle - i\phi^0 \end{pmatrix} d_R + h.c. \\
&= -\frac{\alpha_f}{\sqrt{2}} \left[\left(H + \frac{2M}{g} \right) \overline{u}_L u_R + i \overline{u}_L u_R \phi^0 + i\sqrt{2} \overline{d}_L u_R \phi^- \right] \\
&\quad -\frac{\alpha_f}{\sqrt{2}} \left[\left(H + \frac{2M}{g} \right) \overline{u}_R u_L - i \overline{u}_R u_L \phi^0 - i\sqrt{2} \overline{u}_R d_L \phi^+ \right] \\
&\quad +\frac{\beta_f}{\sqrt{2}} \left[\left(H + \frac{2M}{g} \right) \overline{d}_L d_R - i \overline{d}_L d_R \phi^0 + i\sqrt{2} \overline{u}_L d_R \phi^+ \right] \\
&\quad +\frac{\beta_f}{\sqrt{2}} \left[\left(H + \frac{2M}{g} \right) \overline{u}_R u_L + i \overline{d}_R d_L \phi^0 - i\sqrt{2} \overline{d}_R u_L \phi^- \right] \to
\end{aligned} \tag{126}
$$

Using equations:

$$
\overline{u}_R d_L = \overline{u} \frac{1}{2} (1 + \gamma_5) \, d \,, \qquad \overline{u}_L d_R = \overline{u} \frac{1}{2} (1 - \gamma_5) \, d \,, \qquad \text{etc.}, \tag{127}
$$

we obtain

$$
\begin{aligned}
\to \; & -\frac{\alpha_f}{\sqrt{2}} \left[\left(H + \frac{2M}{g} \right) \overline{u} u - i \overline{u} \gamma_5 u \phi^0 + \frac{i}{\sqrt{2}} \overline{d} \, (1 - \gamma_5) \, u \phi^- - \frac{i}{\sqrt{2}} \overline{u} \, (1 + \gamma_5) \, d \phi^+ \right] \\
& +\frac{\beta_f}{\sqrt{2}} \left[\left(H + \frac{2M}{g} \right) \overline{d} d + i \overline{d} \gamma_5 d \phi^0 + \frac{i}{\sqrt{2}} \overline{u} \, (1 - \gamma_5) \, d \phi^+ - \frac{i}{\sqrt{2}} \overline{d} \, (1 + \gamma_5) \, u \phi^- \right] .
\end{aligned} \tag{128}
$$

The second term in each row may be identified with mass terms, giving two solutions for the Yukawa couplings:

$$
\alpha_f = \frac{1}{\sqrt{2}} g \frac{m_u}{M} \,, \qquad \beta_f = -\frac{1}{\sqrt{2}} g \frac{m_d}{M} \,, \tag{129}
$$

in terms of physical fermion masses m_u and m_d of up and down fermions. The Lagrangian may be presented as a sum of two terms

$$
\mathcal{L}_S^{\rm fer} = -\sum_f m_f \overline{f} f + \mathcal{L}_S^{\rm fer,I} \,, \tag{130}
$$

where the second term is the interaction Lagrangian:

$$
\begin{aligned}
\mathcal{L}_S^{\rm fer,I} \; = \; & \sum_d \left\{ i \frac{g}{2\sqrt{2}} \phi^+ \left[\frac{m_u}{M} \overline{u} \, (1 + \gamma_5) \, d - \frac{m_d}{M} \overline{u} \, (1 - \gamma_5) \, d \right] \right. \\
& \left. + i \frac{g}{2\sqrt{2}} \phi^- \left[\frac{m_d}{M} \overline{d} \, (1 + \gamma_5) \, u - \frac{m_u}{M} \overline{d} \, (1 - \gamma_5) \, u \right] \right\} \\
& + \sum_f \left(-\frac{1}{2} g H \frac{m_f}{M} \overline{f} f + i g I_f^{(3)} \phi^0 \frac{m_f}{M} \overline{f} \gamma_5 f \right) .
\end{aligned} \tag{131}
$$

2.9 Fermion mixing

The presentation in the two preceding sections was done neglecting the fermion mixing. Here we present a generalization for the case of mixing. We begin by rewriting the expression for $\mathcal{L}_S^{\rm fer}$, Eq. (124),

$$\mathcal{L}_S^{\rm fer} = -\frac{g}{\sqrt{2}M}\overline{\psi}_L m_u K u_R + \frac{g}{\sqrt{2}M}\overline{\psi}_L m_d K^c d_R + h.c. \tag{132}$$

The latter could be generalized to

$$\mathcal{L}_S^{\rm fer} = -\frac{g}{\sqrt{2}M}\left(\overline{\psi}_L\right)_\alpha (\mathcal{M}^U)_{\alpha,\beta} K\,(u_R)_\beta + \frac{g}{\sqrt{2}M}\left(\overline{\psi}_L\right)_\alpha (\mathcal{M}^D)_{\alpha,\beta} K^c\,(d_R)_\beta + h.c., \tag{133}$$

where we introduced two columns containing all up and down fermions

$$(u_R)_\beta = \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \\ u \\ c \\ t \end{pmatrix}_R = U_R\,, \qquad (d_R)_\beta = \begin{pmatrix} e \\ \mu \\ \tau \\ d \\ s \\ b \end{pmatrix}_R = D_R\,, \tag{134}$$

a string of Dirac-conjugated fields

$$\left(\overline{\psi}_L\right)_\alpha = \left(\overline{\nu}_e, \overline{\nu}_\mu, \overline{\nu}_\tau, \overline{u}, \overline{c}, \overline{t}; \overline{e}, \overline{\mu}, \overline{\tau}, \overline{d}, \overline{s}, \overline{b}\right)_L = \left(\overline{U}, \overline{D}\right)_L\,, \tag{135}$$

and complex matrices $\mathcal{M}_{l,q}^{U,D}$

$$\mathcal{M}^U = \begin{pmatrix} \mathcal{M}_l^U & O \\ O & \mathcal{M}_q^U \end{pmatrix}, \qquad \mathcal{M}^D = \begin{pmatrix} \mathcal{M}_l^D & O \\ O & \mathcal{M}_q^D \end{pmatrix}, \tag{136}$$

with O being a zero-matrix. All these matrices are 3×3 matrices. It is easy to see that the Lagrangian Eq. (133) is also gauge-invariant under transformations of Eq. (125).

Substituting scalar fields K and K^c we obtain the generalized mass term

$$\mathcal{L}_S^{{\rm fer},m} \;=\; -\frac{g}{\sqrt{2}M}\overline{U}'_L\mathcal{M}^U U'_R - \frac{g}{\sqrt{2}M}\overline{D}'_L\mathcal{M}^D D'_R + h.c. \tag{137}$$

In order to reduce it to the usual form, one has to *diagonalize* the four mass matrices. This may be achieved with the aid of *bi-unitary* transformations (see Ref. [2] for a rigorous proof):

$$\mathcal{M}^U = \mathcal{U}_L^+ m_u\,\mathcal{U}_R\,, \qquad \mathcal{M}^D = \mathcal{D}_L^+ m_d\,\mathcal{D}_R\,, \tag{138}$$

where $\mathcal{U}_L$, $\mathcal{U}_R$, $\mathcal{D}_L$, $\mathcal{D}_R$ are four different *unitary* 6×6 matrices:

$$\mathcal{U}_L = \begin{pmatrix} (\mathcal{U}_L)_l & O \\ O & (\mathcal{U}_L)_q \end{pmatrix}, \qquad \text{etc.} \tag{139}$$

Fields with primes U'_L, U'_R, D'_L, D'_R are *weak eigenstates*. Introducing *mass eigenstates* U_L, U_R, D_L, D_R:

$$U_L = \mathcal{U}_L U'_L\,, \qquad U_R = \mathcal{U}_R U'_R\,, \qquad D_L = \mathcal{D}_L D'_L\,, \qquad D_R = \mathcal{D}_R D'_R\,, \tag{140}$$

we arrive at a usual mass term of the Lagrangian, written in matrix form

$$\mathcal{L}_S^{{\rm fer},m} \;=\; -\frac{g}{\sqrt{2}M}\overline{U}m_u U - \frac{g}{\sqrt{2}M}\overline{D}m_d D\,, \tag{141}$$

where m_u and m_d are *diagonal* matrices in 6-dimensional U and D spaces. The interaction part, written in the matrix form, reads

$$\begin{aligned}\mathcal{L}_S^{\text{fer,I}} &= i\frac{g}{2\sqrt{2}}\phi^+\left[\frac{m_u}{M}\overline{U}\left(1+\gamma_5\right)CD-\frac{m_d}{M}\overline{U}\left(1-\gamma_5\right)CD\right]\\&+i\frac{g}{2\sqrt{2}}\phi^-\left[\frac{m_d}{M}\overline{D}\left(1+\gamma_5\right)C^\dagger U-\frac{m_u}{M}\overline{D}\left(1-\gamma_5\right)C^\dagger U\right]\\&+\sum_f\left(-\frac{1}{2}gH\frac{m_f}{M}\overline{f}f+igI_f^{(3)}\phi^0\frac{m_f}{M}\overline{f}\gamma_5 f\right).\end{aligned}\tag{142}$$

The charge boson sector contains the mixing matrix

$$C=\left(\begin{array}{cc}(\mathcal{U}_L)_l\,(\mathcal{D}_L^+)_l & O\\ O & (\mathcal{U}_L)_q\,(\mathcal{D}_L^+)_q\end{array}\right)=\left(\begin{array}{cc}C_l & O\\ O & C_q\end{array}\right),\tag{143}$$

which is non-diagonal because $(\mathcal{U}_L)_l$ and $(\mathcal{D}_L)_l$ [and $(\mathcal{U}_L)_q$ and $(\mathcal{D}_L)_q$] are, in general, different matrices.

The fermionic mixing matrix C involves the usual Cabibbo–Kabajashi–Maskawa (CKM) mixing in the quark sector C_q, and possible leptonic (neutrino) mixing C_l.

Finally, the fermion–vector boson interaction, Eq. (121), in presence of mixing generalizes to

$$\begin{aligned}\mathcal{L}_V^{\text{fer,I}} &= \sum_f\left[igs_\theta Q_f A_\mu\overline{f}\gamma_\mu f+i\frac{g}{2c_\theta}Z_\mu\overline{f}\gamma_\mu\left(v_f+a_f\gamma_5\right)f\right]\\&+i\frac{g}{2\sqrt{2}}W_\mu^+\overline{U}\gamma_\mu\left(1+\gamma_5\right)CD+i\frac{g}{2\sqrt{2}}W_\mu^-\overline{D}\gamma_\mu\left(1+\gamma_5\right)C^\dagger U.\end{aligned}\tag{144}$$

2.9.1 *Some conclusions about fermionic mixing*

To summarize our study of fermion mixing, one may conclude that:

- fermionic mixing arises in the SM very naturally as a consequence of the most general Yukawa interaction compatible with gauge invariance;
- C_q is the usual unitary CKM matrix characterized by 4 real parameters;
- C_l is its analog in lepton sector, also characterized by 4 real parameters, which *are not obliged to be equal* to CKM parameters;
- we therefore have a complete lepton–quark analogy and extended SM (ESM) is a very natural extension of the conventional SM with massless neutrinos;
- no mixing arises in the neutral currents, a consequence of the unitarity of matrices $(\mathcal{U}_L)_l\,(\mathcal{U}_L^+)_l$, etc;
- Eq. (141) involves Dirac mass terms. I refer to the lectures of S. Bilenky [3] and M. Carena [4] at this School for a discussion of Majorana mass terms and fermionic mixing beyond the ESM, as well as whether the simple extension described in this lecture contradicts present experimental data or whether we really do have experimental indications of any new physics beyond the ESM.

2.10 QCD Lagrangian

The SM, besides the electroweak sector described in full detail in this lecture, also contains the QCD sector. For a detailed discussion of the QCD Lagrangian, I refer to the lectures of Prof. J. Stirling [5] at this School.

2.11 Feynman rules for vertices

Here we limit ourselves to presenting the Feynman rules for $Bf\overline{f}$ vertices, where B is a boson field. They may be straightforwardly derived from the Lagrangian of Eqs. (121) and (131). A complete collection of bosonic and FP-ghost Feynman rules may be found in Chapter 3 of Ref. [1].

$\overline{f}$

A μ $\quad ieQ_f\gamma_\mu$

f

$\overline{f}$

Z μ $\quad i\dfrac{g}{2c_\theta}\gamma_\mu\left(v_f + a_f\gamma_5\right)$

f

$\overline{u}$

W^- μ $\quad i\dfrac{g}{2\sqrt{2}}\gamma_\mu\left(1+\gamma_5\right)$

d

$\overline{f}$

H $\quad -\dfrac{1}{2}\,g\dfrac{m_f}{M}$

f

$\overline{f}$

ϕ^0 $\quad igI^{(3)}_f\dfrac{m_f}{M}\gamma_5$

f

$\overline{u}$

ϕ^- $\quad i\dfrac{g}{2\sqrt{2}}\left[\dfrac{m_d}{M}\left(1+\gamma_5\right) - \dfrac{m_u}{M}\left(1-\gamma_5\right)\right]$

d

2.12 Summary of the two lectures

In lectures 1 and 2 we studied:

- The extended Standard Model, its Fields and Lagrangian;
- Gauge transformations and different gauges:
  - general R_ξ, with three parameters, $\xi,\ \xi_Z\ ,\xi_A$;
  - t'Hooft–Feynman with all $\xi_i = 1$;
  - Physical or unitary, $\xi\to\infty,\ \xi_Z\to\infty,\ \xi_A = 1$.
- Gauge invariance, which will lead to ξ-independence of the amplitudes of physical processes;
- Feynman rules.

We are ready to build diagrams. In the following we will distinguish:

– Born or tree level diagrams;

– loop diagrams (one-loop and multi-loop diagrams).

We emphasize again that we are working in Pauli metrics, i.e. for an on-mass-shell momentum one has: $p^2 = -M^2$ and the scalar part of a propagator looks like

$$\sim \frac{1}{p^2+M^2}\ . \tag{145}$$

In our convention for Dirac γ-matrices, the left projector looks like

$$\gamma_L = \frac{1+\gamma_5}{2}\ . \tag{146}$$

We will use the following correspondence between physical and bare parameters

$$M_W \leftrightarrow M\ , \qquad M_Z \leftrightarrow M_0 \equiv \frac{M}{c_\theta}\ , \qquad s_W \leftrightarrow s_\theta\,. \tag{147}$$

At tree-level, we have the following identities for coupling constants and vector boson masses

$$s^2_W = \frac{e^2}{g^2} = 1 - \frac{M^2_W}{M^2_Z}\ , \qquad e^2 = 4\pi\alpha\,. \tag{148}$$

where $\alpha = 1/137.0359895...$ is the fine structure constant. Therefore, s^2_W and g are not independent parameters if M_W and M_Z are considered among input parameters of the theory.

3. DIMENSIONAL REGULARIZATION AND PASSARINO–VELTMAN FUNCTIONS

This lecture is devoted to basic modern tools for the calculation of loop diagrams — dimensional regularization (DR) and Passarino–Veltman functions — which are based on DR and are those most commonly used for the calculation of one-loop diagrams.

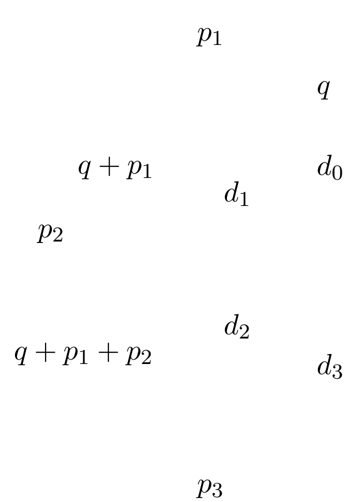
3.1 Feynman parametrization and N-point functions

We begin with a reduction of the propagator products to an integral representation. It makes use of identities, valid for any positive definite $A, B, C, D...$:

$$\begin{aligned}
\frac{1}{AB} &= \int_0^1 dx \frac{1}{\left[Ax + B(1-x)\right]^2}\,,\\
\frac{1}{A^2B} &= \int_0^1 dx \frac{2x}{\left[Ax + B(1-x)\right]^3}\,,\\
\frac{1}{ABC} &= 2! \int_0^1 dyy \int_0^1 dx \frac{1}{\left\{A(1-y) + y[Bx + C(1-x)]\right\}^3}\,,\\
\frac{1}{ABCD} &= \cdots
\end{aligned} \tag{149}$$

with $x, y, z...$ being called *Feynman parameters*.

Define *the N-point function*, i.e. a one-loop diagram with N external legs:



where the arrows indicate the direction of the momentum flow (all external momenta are flowing inwards and the loop momentum flows counter-clockwise). In the figure we also introduced the scalar parts of the propagators d_i,

$$d_i = (q + p_1 + \ldots + p_i)^2 + m_{i+1}^2 - i\epsilon\,. \tag{150}$$

With the aid of identities, Eq. (149), the N-point function can be reduced to a linear combination of N-point integrals

$$\ldots \int_0^1 dyy \int_0^1 dx \frac{\{1, q_\mu, q_\mu q_\nu, \ldots\}}{\left(q^2 - 2q \cdot p_{x,y,\ldots} + m_{x,y,\ldots}^2 - i\epsilon\right)^N}\,. \tag{151}$$

They are called {*scalar, vector, second rank tensor, ...* } integrals corresponding to the type of numerator in Eq. (151).

The quantities $p_{x,y,\ldots}$ are linear combinations of external momenta p_i, and $m_{x,y,\ldots}^2$ — of internal masses m_i^2 and scalar products p_i^2 and $(p_j + \ldots + p_{j+k})^2$.

3.2 Basics of Dimension regularization

All formulae needed in the calculation of N-point functions may be derived from only one integral:

$$\int d^n q \frac{1}{(q^2 + m^2 - i\epsilon)^\alpha} = i\pi^{\frac{n}{2}} \frac{\Gamma\left(\alpha - \frac{n}{2}\right)}{\Gamma(\alpha)} \left(m^2\right)^{\frac{n}{2}-\alpha}\,. \tag{152}$$

For instance, making shift $q \to q - p$, we derive the generic scalar integral:

$$J(p) = \int d^n q \frac{1}{(q^2 - 2q \cdot p + m^2 - i\epsilon)^\alpha} = i\pi^{\frac{n}{2}} \frac{\Gamma\left(\alpha - \frac{n}{2}\right)}{\Gamma(\alpha)} \left(m^2 - p^2\right)^{\frac{n}{2}-\alpha}\,. \tag{153}$$

Differentiating Eq. (153) $\partial_\mu J(p)$, we derive the vector:

$$\int d^n q \frac{q_\mu}{(q^2 - 2q \cdot p + m^2 - i\epsilon)^\alpha} = i\pi^{\frac{n}{2}} \frac{\Gamma\left(\alpha - \frac{n}{2}\right)}{\Gamma(\alpha)} \left(m^2 - p^2\right)^{\frac{n}{2}-\alpha} p_\mu\,. \tag{154}$$

With one more differentiation, we derive the second rank tensor:

$$\begin{aligned}\int d^n q \frac{q_\mu q_\nu}{(q^2 - 2q \cdot p + m^2 - i\epsilon)^\alpha} &= i\pi^{\frac{n}{2}} \frac{1}{\Gamma(\alpha)} \left(m^2 - p^2\right)^{\frac{n}{2}-\alpha} \\ &\times \left[\frac{1}{2}\delta_{\mu\nu} \left(m^2 - p^2\right) \Gamma\left(\alpha - 1 - \frac{n}{2}\right) + p_\mu p_\nu \Gamma\left(\alpha - \frac{n}{2}\right)\right],\end{aligned} \tag{155}$$

etc. We present, for completeness, the third and fourth rank tensors too:

$$\begin{aligned}\int d^n q \frac{q_\mu q_\nu q_\rho}{(q^2 - 2q \cdot p + m^2 - i\epsilon)^\alpha} &= i\pi^{\frac{n}{2}} \frac{1}{\Gamma(\alpha)} \left(m^2 - p^2\right)^{\frac{n}{2}-\alpha} \\ &\times \left[\frac{1}{2}\left(\delta_{\mu\nu} p_\rho + \delta_{\mu\rho} p_\nu + \delta_{\nu\rho} p_\mu\right) \left(m^2 - p^2\right) \Gamma\left(\alpha - 1 - \frac{n}{2}\right)\right. \\ &\left. + p_\mu p_\nu p_\rho \Gamma\left(\alpha - \frac{n}{2}\right)\right],\end{aligned} \tag{156}$$

$$\begin{aligned}\int d^n q \frac{q_\mu q_\nu q_\rho q_\sigma}{(q^2 - 2q \cdot p + m^2 - i\epsilon)^\alpha} &= i\pi^{\frac{n}{2}} \frac{1}{\Gamma(\alpha)} \left(m^2 - p^2\right)^{\frac{n}{2}-\alpha} \\ &\times \left[\frac{1}{4}\left(\delta_{\mu\nu}\delta_{\rho\sigma} + \delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\nu\rho}\delta_{\mu\sigma}\right)\left(m^2 - p^2\right)^2 \Gamma\left(\alpha - 2 - \frac{n}{2}\right)\right. \\ &+ \frac{1}{2}\left(\delta_{\mu\nu} p_\rho p_\sigma + \delta_{\mu\rho} p_\nu p_\sigma + \delta_{\mu\sigma} p_\nu p_\rho + \delta_{\nu\rho} p_\mu p_\sigma\right. \\ &\quad \left. + \delta_{\nu\sigma} p_\mu p_\rho + \delta_{\rho\sigma} p_\mu p_\nu\right) \left(m^2 - p^2\right) \Gamma\left(\alpha - 1 - \frac{n}{2}\right) \\ &\left. + p_\mu p_\nu p_\rho p_\sigma \Gamma\left(\alpha - \frac{n}{2}\right)\right].\end{aligned} \tag{157}$$

In particular, consider Eq. (152) for the case $m^2 = 0$. Since it holds for any n, we may choose $n/2 - \alpha > 0$ and then let $m^2 = 0$. Then, we will have

$$\int d^n q \, \frac{1}{(q^2)^\alpha} = 0 \,. \tag{158}$$

This may be continued for any n. Therefore, in dimension regularization the integral in Eq. (158) is zero for *any* values of α.

3.3 Divergence counting: poles versus powers

3.3.1 Ultraviolet divergences

Consider a vector boson self-energy diagram with an internal W boson loop:

W^+

W^-

In the unitary gauge, the following integral over internal momentum q corresponds to it:

$$\begin{aligned}
\Pi_{\rho\sigma} \quad &\sim \quad \int d^n q \, \frac{\left(\delta_{\mu\nu} + \dfrac{q_\mu q_\nu}{M^2}\right)\left[\delta_{\alpha\beta} + \dfrac{(q+p)_\alpha \, (q+p)_\beta}{M^2}\right]}{(q^2+M^2)\left[(q+p)^2+M^2\right]} \Gamma_{\rho\mu\alpha}\Gamma_{\sigma\nu\beta} \\
&= \quad \int d^n q \, \frac{\left(\delta_{\mu\nu}\delta_{\alpha\beta} + \dfrac{q_\mu q_\nu}{M^2}\delta_{\alpha\beta} + \dfrac{q_\mu q_\nu q_\alpha q_\beta}{M^4} + \ldots\right)\Gamma_{\rho\mu\alpha}\Gamma_{\sigma\nu\beta}}{(q^2+M^2)\left[(q+p)^2+M^2\right]} \,, \\
& \qquad\qquad \ln\Lambda \qquad\quad \Lambda^2 \qquad\qquad\quad \Lambda^4 \\
& \qquad\qquad \frac{1}{n-4} \qquad \frac{1}{n-2} \qquad\quad \frac{1}{n-0} \quad .
\end{aligned} \tag{159}$$

In the second row, we considered selected scalar and two tensor integrals contributing to it. Applying trivial counting of powers of q in the numerator and denominator, it is easy to see that various terms in the third row have indicated ultraviolet cut-off divergences, which correspond to poles in space-time dimensions $n = 4, 2, 0$, respectively, as shown in the fourth row. This is an example of the general rule of correspondence between powers of the cut-off divergences and poles in n.

3.3.2 Infrared divergences

Consider another example, the QED vertex diagram:

p_2

q μ

p_1 Q

with two momentum on-shell $p_1^2 = p_2^2 = -m^2$ and $Q^2 = (p_1 + p_2)^2$. In the Feynman gauge the following integral corresponds to it:

$$\begin{aligned}
\Lambda_\mu &\sim \int d^n q \frac{(-2p_{2\nu} + \gamma_\nu \not{q})\gamma_\mu (2p_{1\nu} + \not{q}\gamma_\nu)}{q^2 \left[(q+p_1)^2 + m^2\right]\left[(q-p_2)^2 + m^2\right]} \\
&= \int d^n q \frac{-4p_1 \cdot p_2 \gamma_\mu + 2\left(\not{p}_1 \gamma_\alpha \gamma_\mu - \gamma_\mu \gamma_\alpha \not{p}_2\right) q_\alpha + (2-n)\,\gamma_\alpha \gamma_\mu \gamma_\beta q_\alpha q_\beta}{q^2 \Big(q^2 + 2q\cdot p_1\Big)\Big(q^2 - 2q \cdot p_2\Big)}\,, \qquad (160) \\
&\qquad\qquad \text{Scalar} \qquad\qquad \text{Vector} \qquad\qquad\qquad \text{Tensor} \\
&\qquad\qquad \text{Infrared} \qquad\quad \text{Finite} \qquad\qquad\quad \text{Ultraviolet}\,.
\end{aligned}$$

Here we have *scalar, vector,* and *tensor* integrals with different types of divergences. The scalar exhibits *infrared* divergence, the vector is finite, and the tensor is *ultraviolet* divergent.

In all the considered cases the type of divergence may be determined by counting the powers of q in the numerator and denominator in the corresponding regimes:
– ultraviolet, when $q \to \infty$;
– infrared, when $q \to 0$.

In what follows, we will use the dimensional regularization while calculating $1, 2, 3, 4$-point one-loop integrals using the language of the A, B, C, D functions by Passarino and Veltman.

3.4 One-point integrals, A functions

One-point integrals are met in the calculation of tadpole diagrams:

m

and in the reduction of higher-order integrals.

3.4.1 Scalar one-point integral

The A_0 function is defined by the integral

$$i\pi^2 A_0(m) = \mu^{4-n} \int d^n q \frac{1}{q^2 + m^2 - i\epsilon}\,, \qquad (161)$$

where we introduced the t'Hooft scale parameter μ in order to prevent changing the dimension of this integral at the variation of the space-time dimension n. The integral is computed using the general formula, Eq. (152) with $\alpha = 1$,

$$A_0(m) = \pi^{n/2-2}\Gamma\left(1 - \frac{n}{2}\right) m^2 \left(\frac{m^2}{\mu^2}\right)^{n/2-2}. \qquad (162)$$

If one introduces $\varepsilon = 4 - n$ and expands around $n = 4$ then

$$A_0(m) = m^2\Big(-\frac{2}{\varepsilon} + \gamma + \ln\pi + \ln\frac{m^2}{\mu^2}\Big) - 1 + \mathcal{O}(\varepsilon)\,. \qquad (163)$$

It is customary to define a quantity $1/\bar{\varepsilon}$ by

$$\frac{1}{\bar{\varepsilon}} = \frac{2}{\varepsilon} - \gamma - \ln\pi\,, \qquad (164)$$

then by dropping higher orders in ε, we get the final answer

$$A_0\left(m\right) = m^2\Big(-\frac{1}{\bar{\varepsilon}} + \ln\frac{m^2}{\mu^2} - 1\Big). \tag{165}$$

Note that the pole has an ultraviolet origin and that it is accompanied by a scale-containing logarithm.

3.4.2 *Tensor one-point integrals*

The vector-like A_1 is zero, since the A functions do not depend on an external momentum and it is impossible to construct a vector, as well as any odd-rank tensor. The even-rank tensors may be constructed and the lowest order, second rank, tensor must be proportional to $\delta_{\mu\nu}$, i.e.

$$\begin{aligned} i\pi^2 A_{\mu\nu}\left(m\right) &= \mu^{4-n}\int d^n q\,\frac{q_\mu q_\nu}{q^2+m^2-i\epsilon}\,, \\ A_{\mu\nu}\left(m\right) &= A_2\left(m\right)\delta_{\mu\nu}\,. \end{aligned} \tag{166}$$

To calculate it, we contract Eq. (166) with $\delta_{\mu\nu}$, use Eqs. (158) and (165) and expand around $n = 4$ again, where one should proceed carefully, namely

$$\frac{1}{n}\frac{1}{\bar{\varepsilon}} = \frac{1}{4-\varepsilon}\left(\frac{2}{\varepsilon}+\dots\right) = \frac{1}{4}\frac{1}{\bar{\varepsilon}} + \frac{1}{8}\,. \tag{167}$$

In this way, we finally derive

$$A_2\left(m\right) = -\frac{1}{4}\,m^2 A_0\left(m\right) + \frac{1}{8}m^4\,. \tag{168}$$

Rank four tensor integral may be reduced in a similar way, see Section 5.1.1.2 of Ref. [1].

3.5 Two-point integrals, B functions

B functions appear when considering self-energies and transitions

m_1

$p \rightarrow$

m_2

Their family is far more reach compared to A functions.

3.5.1 *Scalar two-point integral*

The scalar B_0 function is defined by the integral containing two propagators d_0 and d_1, one of which depends on an arbitrary external momentum p:

$$\begin{aligned} &i\pi^2 B_0\left(p^2;m_1,m_2\right) = \mu^{4-n}\int d^n q\frac{1}{d_0 d_1}\,, \\ &d_0 = q^2+m_1^2-i\epsilon, \qquad d_1 = \left(q+p\right)^2+m_2^2-i\epsilon\,. \end{aligned} \tag{169}$$

Using Eq. (152) for $\alpha = 2$, it is easy to derive the general result for the B_0 function, valid for arbitrary internal masses

$$B_0\left(p^2;m_1,m_2\right) = \frac{1}{\bar{\varepsilon}} - R - \ln\frac{m_1 m_2}{\mu^2} + \frac{m_1^2-m_2^2}{2p^2}\ln\frac{m_1^2}{m_2^2} + 2\,. \tag{170}$$

where $\Lambda^2 = \lambda\left(-p^2, m_1^2, m_2^2\right)$ is the Källen function: $\lambda\left(x,y,z\right) = x^2+y^2+z^2-2xy-2xz-2yz$, and

$$R = -\frac{\Lambda}{p^2}\,\ln\frac{p^2-i\epsilon+m_1^2+m_2^2-\Lambda}{2\,m_1 m_2}\,. \tag{171}$$

Some particular cases are also frequently met:

1. if $m_1 = m_2 = m$

$$B_0\left(p^2;m,m\right) = \frac{1}{\bar{\varepsilon}} - \ln\frac{m^2}{\mu^2} + 2 - \beta\,\ln\frac{\beta+1}{\beta-1}\,, \qquad \beta^2 = 1+\frac{4m^2}{p^2-i\epsilon}\,; \tag{172}$$

2. if one of the internal masses is zero

$$B_0\left(p^2;0,m\right) = \frac{1}{\bar{\varepsilon}} - \ln\frac{m^2}{\mu^2} + 2 - \left(1+\frac{m^2}{p^2}\right)\ln\left(1+\frac{p^2-i\epsilon}{m^2}\right); \tag{173}$$

3. if both internal lines are massless

$$B_0\left(p^2;0,0\right) = \frac{1}{\bar{\varepsilon}} - \ln\frac{p^2-i\epsilon}{\mu^2} + 2\,. \tag{174}$$

The B_0 function develops an imaginary part above the physical threshold, $s = -p^2 \geq (m_1+m_2)^2$:

$$\mathrm{Im}B_0\left(p^2;m_1,m_2\right) = \pi\frac{\sqrt{\lambda\left(s,m_1^2,m_2^2\right)}}{s}\,\theta\left(s-(m_1+m_2)^2\right). \tag{175}$$

3.5.2 Tensor two-point integrals

Here we describe the calculation of the vector and tensor B functions. The calculation exploits the so-called *reduction* to a linear combinations of scalar functions. We begin with the vector:

$$i\pi^2 B_\mu\left(p^2;m_1,m_2\right) = \mu^{4-n}\int d^n q\frac{q_\mu}{d_0 d_1} = i\pi^2 B_1\left(p^2;m_1,m_2\right)\,p_\mu\,. \tag{176}$$

Using the relations

$$q^2 = d_0 - m_1^2\,, \qquad q\cdot p = \frac{1}{2}\left(d_1-d_0+f_1^b\right), \qquad \text{with} \quad f_1^b = -p^2+m_1^2-m_2^2\,, \tag{177}$$

we derive the identity

$$p^2\,B_1\left(p^2;m_1,m_2\right) = \frac{1}{2}\left[A_0\left(m_1\right) - A_0\left(m_2\right) + f_1^b B_0\left(p^2;m_1,m_2\right)\right], \tag{178}$$

which is the required reduction to the scalar integrals. One may easily derive a symmetry property:

$$B_1\left(p^2;m_2,m_1\right) = -B_1\left(p^2;m_1,m_2\right) - B_0\left(p^2;m_1,m_2\right). \tag{179}$$

The rank two tensor integral can be reduced to two functions B_{21} and B_{22}:

$$i\pi^2 B_{\mu\nu}\left(p^2;m_1,m_2\right) = \mu^{4-n}\int d^n q\frac{q_\mu q_\nu}{d_0 d_1} = i\pi^2\left[B_{21}\left(p^2;m_1,m_2\right)p_\mu p_\nu + B_{22}\left(p^2;m_1,m_2\right)\delta_{\mu\nu}\right]. \tag{180}$$

The last relation can be multiplied by $\delta_{\mu\nu}$ and by p_ν, which leads to the system of two linear equations for B_{21} and B_{22}:

$$\begin{aligned} p^2 B_{21}\left(p^2;m_1,m_2\right) + nB_{22}\left(p^2;m_1,m_2\right) &= A_0\left(m_2\right) - m_1^2 B_0\left(p^2;m_1,m_2\right), \\ p^2 B_{21}\left(p^2;m_1,m_2\right) + B_{22}\left(p^2;m_1,m_2\right) &= \frac{1}{2}\left[A_0\left(m_2\right) + f_1^b B_1\left(p^2;m_1,m_2\right)\right]. \end{aligned} \tag{181}$$

The appearance of n in front of a function to be derived prevents the trivial solution of this system. In order to solve it, and similar ones, we have to know the singular parts of all one- and two-point functions. Only then may we properly expand around $n = 4$ (cf. discussion around Eq. (167)). The calculation of poles may be achieved straightforwardly with the aid of the formulae in Section 3.2. We derive:

$$\begin{aligned}
B_0\left(p^2;m_1,m_2\right) &= \frac{1}{\bar{\varepsilon}} - \int_0^1 dx \ln\left(\frac{\chi}{\mu^2}\right) \stackrel{\text{sing}}{\to} \frac{1}{\bar{\varepsilon}}\,, \\
B_1\left(p^2;m_1,m_2\right) &= -\frac{1}{2}\frac{1}{\bar{\varepsilon}} + \int_0^1 dx x \ln\left(\frac{\chi}{\mu^2}\right) \stackrel{\text{sing}}{\to} -\frac{1}{2}\frac{1}{\bar{\varepsilon}}\,, \\
B_{21}\left(p^2;m_1,m_2\right) &= \frac{1}{3}\frac{1}{\bar{\varepsilon}} - \int_0^1 dx x^2 \ln\left(\frac{\chi}{\mu^2}\right) \stackrel{\text{sing}}{\to} \frac{1}{3}\frac{1}{\bar{\varepsilon}}\,, \\
B_{22}\left(p^2;m_1,m_2\right) &= -\frac{1}{2}\left(\frac{1}{\bar{\varepsilon}}+1\right)\int_0^1 dx\chi + \frac{1}{2}\int_0^1 dx\chi\ln\left(\frac{\chi}{\mu^2}\right) \\
&\stackrel{\text{sing}}{\to} -\frac{1}{4}\left(m_1^2+m_2^2+\frac{1}{3}p^2\right)\frac{1}{\bar{\varepsilon}}\,,
\end{aligned} \tag{182}$$

where we used

$$\chi(x) = -p^2x^2 + \left(p^2+m_2^2-m_1^2\right)x + m_1^2 - i\epsilon\,. \tag{183}$$

Using the relations in Eq. (182), we obtain (analogously to the derivation of Eq. (167))

$$\begin{aligned}
n\,B_{22}\left(p^2;m_1,m_2\right) &= 4\,B_{22}\left(p^2;m_1,m_2\right) + \frac{K^2}{6}\,, \\
K^2 &= p^2 + 3\left(m_1^2+m_2^2\right).
\end{aligned} \tag{184}$$

Furthermore, we introduce the matrix

$$X_2 = \begin{pmatrix} p^2 & 4 \\ p^2 & 1 \end{pmatrix}, \tag{185}$$

and the vector b with components

$$\begin{aligned}
b_1 &= A_0\left(m_2\right) - m_1^2 B_0\left(p^2;m_1,m_2\right) - \frac{K^2}{6}\,, \\
b_2 &= \frac{1}{2}\left[A_0\left(m_2\right) + f_1^b B_1\left(p^2;m_1,m_2\right)\right].
\end{aligned} \tag{186}$$

The $B_{2i}\left(p^2;m_1,m_2\right)$ functions can be obtained by inversion

$$B_{2i}\left(p^2;m_1,m_2\right) = \left[X_2\right]^{-1}_{ij}\, b_j\,. \tag{187}$$

3.5.3 List of the final results

$$\begin{aligned}
B_1\left(p^2;m_1,m_2\right) &= \frac{1}{2p^2}\left[A_0\left(m_1\right) - A_0\left(m_2\right) + \left(\Delta m^2 - p^2\right)B_0\left(p^2;m_1,m_2\right)\right], \\
B_{21}\left(p^2;m_1,m_2\right) &= \frac{3\left(m_1^2+m_2^2\right)+p^2}{18p^2} + \frac{\Delta m^2 - p^2}{3p^4}A_0\left(m_1\right) - \frac{\Delta m^2 - 2p^2}{3p^4}A_0\left(m_2\right) \\
&\quad + \frac{\lambda\left(-p^2,m_1^2,m_2^2\right) - 3p^2m_1^2}{3p^4}B_0\left(p^2;m_1,m_2\right), \\
B_{22}\left(p^2;m_1,m_2\right) &= -\frac{3\left(m_1^2+m_2^2\right)+p^2}{18} - \frac{\Delta m^2 - p^2}{12\,p^2}A_0\left(m_1\right) + \frac{\Delta m^2 + p^2}{12\,p^2}A_0\left(m_2\right) \\
&\quad - \frac{\lambda\left(-p^2,m_1^2,m_2^2\right)}{12p^2}B_0\left(p^2;m_1,m_2\right), \quad \text{with} \quad \Delta m^2 = m_1^2 - m_2^2\,.
\end{aligned} \tag{188}$$

3.5.4 Reduction for $p^2=0$

As seen from Eq. (188), the reduction fails at $p^2=0$. In this case, the results should be derived from the integral representations Eq. (182)

$$\begin{aligned}
B_0(0;m_1,m_2) &= \frac{A_0(m_2)-A_0(m_1)}{m_1^2-m_2^2}\,, \\
B_1(0;m_1,m_2) &= -\frac{1}{2}B_0(0;m_1,m_2)+\frac{1}{2}\left(m_1^2-m_2^2\right)B_{0p}(0;m_1,m_2)\,, \\
\text{where}\quad B_{0p}(0;m_1,m_2) &= \left.\frac{\partial B_0(p^2;m_1,m_2)}{\partial p^2}\right|_{p^2=0}, \\
B_{22}(0;m_1,m_2) &= -\frac{1}{4}\left(m_1^2+m_2^2\right)\left(\frac{1}{\bar\varepsilon}-\ln\frac{m_1m_2}{\mu^2}+\frac{3}{2}\right)+\frac{m_1^4+m_2^4}{8\left(m_1^2-m_2^2\right)}\ln\frac{m_1^2}{m_2^2}\,. \qquad (189)
\end{aligned}$$

3.5.5 Derivatives of B functions

In actual calculations one also needs the derivatives of B functions (already seen in Eq. (189)). They appear in the renormalization factors associated with external lines. Again, from the integral representations Eq. (182), we derive:

$$\begin{aligned}
\frac{\partial B_{\{0;1;21\}}}{\partial p^2} &= -\int_0^1 dx\frac{\{x;-x^2;x^3\}(1-x)}{\chi}\,, \\
\frac{\partial B_{22}}{\partial p^2} &= -\frac{1}{12}\frac{1}{\bar\varepsilon}+\frac{1}{2}\int_0^1 dx x(1-x)\ln\left(\frac{\chi}{\mu^2}\right). \qquad (190)
\end{aligned}$$

They all but derivative of B_{22} are finite. The latter contains an UV-pole. For QED diagrams the derivatives are *infrared* divergent and must be regularized. As usual, we use the dimensional regularization, however, now we have to use $n=4+\varepsilon'$. We derive

$$B_0\left(p^2;m,0\right)=\pi^{n/2-2}\Gamma\left(2-\frac{n}{2}\right)\int_0^1 dx\left(\frac{\chi}{\mu^2}\right)^{n/2-2}, \qquad \text{with}\quad \chi(x)=(1-x)\left(p^2x+m^2\right),$$

$$\begin{aligned}
\frac{\partial}{\partial p^2}B_0\left(p^2;m,0\right) &= -\pi^{\varepsilon'/2}\Gamma\left(1-\frac{\varepsilon'}{2}\right)\int_0^1 dx\frac{x(1-x)}{\chi(x)}\left(\frac{\chi(x)}{\mu^2}\right)^{\varepsilon'/2}, \\
\left.\frac{\partial}{\partial p^2}B_0\left(p^2;m,0\right)\right|_{p^2=-m^2} &= -\pi^{\varepsilon'/2}\Gamma\left(1-\frac{\varepsilon'}{2}\right)\frac{1}{m^2}\left(\frac{m^2}{\mu^2}\right)^{\varepsilon'/2}\left(\frac{1}{\varepsilon'}-\frac{1}{1+\varepsilon'}\right). \qquad (191)
\end{aligned}$$

Expanding the various terms in ε', we obtain

$$\left.\frac{\partial}{\partial p^2}B_0\left(p^2;m,0\right)\right|_{p^2=-m^2}=-\frac{1}{2m^2}\left(\frac{1}{\hat\varepsilon}-2+\ln\frac{m^2}{\mu^2}\right), \qquad (192)$$

where we introduced the infrared pole:

$$\frac{1}{\hat\varepsilon}=\frac{2}{\varepsilon'}+\gamma+\ln\pi=\frac{2}{n-4}+\gamma+\ln\pi=-\frac{1}{\bar\varepsilon}\,. \qquad (193)$$

Similarly, one obtains the derivative of B_1:

$$\begin{aligned}
\left.\frac{\partial}{\partial p^2}B_1\left(p^2;m,0\right)\right|_{p^2=-m^2} &= \frac{\pi^{\varepsilon'/2}}{m^2}\Gamma\left(1-\frac{\varepsilon'}{2}\right)\left(\frac{m^2}{\mu^2}\right)^{\varepsilon'/2}\left(\frac{1}{\varepsilon'}-\frac{2}{1+\varepsilon'}+\frac{1}{2+\varepsilon'}\right) \\
&= \frac{1}{2m^2}\left(\frac{1}{\hat\varepsilon}-3+\ln\frac{m^2}{\mu^2}\right). \qquad (194)
\end{aligned}$$

In this section we have presented quite an exhaustive study of B functions and their basic properties. As we have seen, they are much more involved than the simple case of A functions. A similar degree of complication takes place at each step towards the C and D functions. For this reason it would be impossible to cover the subject with the same degree of detail as for the A and B functions in these lectures. Therefore, I will limit myself to definitions and to a minimal amount of information about 3- and 4-point functions. For more details, refer to Sections 5.1.4 and 5.1.5 of the book in Ref. [1].

3.6 Three-point integrals, C functions

C functions appear when considering vertices:

p_3

p_1 m_1

m_3

m_2

p_2

3.6.1 Scalar 3-point function

This is defined by the integral:

$$i\pi^2\, C_0\left(p_1^2, p_2^2, Q^2; m_1, m_2, m_3\right) = \mu^{4-n} \int d^n q \frac{1}{d_0 d_1 d_2}\,, \tag{195}$$

where d_i are

$$d_0 = q^2 + m_1^2 - i\epsilon\,, \qquad d_1 = (q+p_1)^2 + m_2^2 - i\epsilon\,, \qquad d_2 = (q+Q)^2 + m_3^2 - i\epsilon\,. \tag{196}$$

Next, $Q = p_1 + p_2$ and $Q^2 = (p_1+p_2)^2$ is one of the Mandelstam variables, $Q^2 = -s(t \text{ or } u)$, for an arbitrary $2 \to 2$ amplitude. In terms of a particular choice of Feynman parameters C_0 becomes

$$C_0\left(p_1^2, p_2^2, Q^2; m_1, m_2, m_3\right) = \int_0^1 dx \int_0^x dy \left(ax^2 + by^2 + cxy + dx + ey + f\right)^{-1}, \tag{197}$$

with

$$\begin{aligned} a &= -p_2^2\,, \qquad b = -p_1^2\,, \qquad c = p_1^2 + p_2^2 - Q^2, \qquad d = p_2^2 + m_2^2 - m_3^2\,, \\ e &= -p_2^2 + Q^2 + m_1^2 - m_2^2\,, \qquad f = m_3^2 - i\epsilon\,. \end{aligned} \tag{198}$$

The scalar C_0 function is invariant under simultaneous cyclic permutations in the two sets of arguments: $\{p_1^2 p_2^2 Q^2\}$ and $\{m_1 m_2 m_3\}$.

3.6.2 An example of the massive C_0 function

There is only one generic three-point scalar integral which occurs in the calculation of two fermion production when all external fermionic masses are ignored. In this case only one fermion mass has to be kept — the top-quark mass, which appears in the virtual state. To such a C_0 function, the following choice corresponds:

$$p_{1,2}^2 = 0, \quad (p_1+p_2)^2 = Q^2, \quad m_1 = M_1\,, \quad m_2 = M_2\,, \quad m_3 = M_3\,. \tag{199}$$

Then the coefficients in the quadratic form in Eq. (198) become:

$$\begin{aligned} a &= 0\,, & b &= 0\,, & c &= -Q^2\,, \\ d &= M_2^2 - M_3^2\,, & e &= Q^2 + M_1^2 - M_2^2\,, & f &= M_3^2 - i\epsilon\,, \end{aligned}$$

and the integral in Eq. (197) for C_0 reduces to

$$C_0\left(0,0,Q^2;M_1,M_2,M_3\right) = \int_0^1 dx \int_0^x dy \frac{1}{\chi(x,y)} \,, \tag{200}$$

where the quadratic in the x and y form χ in this case linearizes ($a = b = 0$):

$$\chi(x,y) = Q^2 y\,(1-x) + M_1^2 y + M_2^2\,(x-y) + M_3^2\,(1-x)\,. \tag{201}$$

In this particular case we get

$$C_0 = \frac{1}{Q^2}\sum_{i=1}^{3}(-1)^{\delta_{i3}}\left[\mathrm{Li}_2\left(\frac{x_0-1}{x_0-x_i}\right) - \mathrm{Li}_2\left(\frac{x_0}{x_0-x_i}\right)\right], \tag{202}$$

with four different roots

$$\begin{aligned} x_0 &= 1+\frac{M_1^2-M_2^2}{Q^2}\,, \qquad x_3 = \frac{M_3^2}{M_3^2-M_2^2}\,, \\ x_{1,2} &= \frac{Q^2+M_1^2-M_3^2 \mp \sqrt{\lambda\left(-Q^2,M_1^2,M_3^2\right)}}{2Q^2}\,. \end{aligned} \tag{203}$$

The dilogarithm function is defined by:

$$\mathrm{Li}_2\,(x) = -\int_0^1 dy \frac{\ln\,(1-xy)}{y}\,. \tag{204}$$

All masses squared are understood with equal infinitesimal imaginary parts: $M_i^2 \to M_i^2 - i\epsilon$. It is necessary to properly define the analytic continuation at $Q^2 \to -s$.

3.6.3 *The special cases which are met in practical calculations*

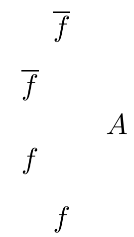
$$\begin{aligned} C_0\left(0,0,Q^2;M_1,0,M_3\right) &= \frac{1}{Q^2}\ln\frac{x_2}{x_2-1}\ln\frac{x_1-1}{x_1}\,, \\ C_0\left(0,0,Q^2;M_1,0,M_1\right) &= \frac{1}{Q^2}\ln^2\frac{\beta_Q+1}{\beta_Q-1}\,, \qquad \beta_Q = \sqrt{1+\frac{4M_1^2}{Q^2}}\,, \\ C_0\left(0,0,Q^2;M_1,M_2,0\right) &= C_0\left(0,0,Q^2;0,M_2,M_1\right) \\ &= \frac{1}{Q^2}\left[\mathrm{Li}_2\left(1-\frac{M_1^2}{M_2^2}\right) - \mathrm{Li}_2\left(1-\frac{Q^2+M_1^2}{M_2^2}\right)\right], \\ C_0\left(0,0,Q^2;0,M_2,0\right) &= \frac{1}{Q^2}\left[\mathrm{Li}_2\,(1) - \mathrm{Li}_2\left(1-\frac{Q^2}{M_2^2}\right)\right], \\ C_0\left(-m^2,-m^2,Q^2;0,m,0\right) &= \frac{1}{m^2\,(y_1-y_2)}\left[2\mathrm{Li}_2\left(\frac{1}{y_1}\right) - 2\mathrm{Li}_2\left(\frac{1}{y_2}\right) + \mathrm{Li}_2\,(y_1) - \mathrm{Li}_2\,(y_2)\right], \end{aligned} \tag{205}$$

with

$$y_{1,2} = -\frac{m^2}{2Q^2}\left(1 \pm \sqrt{1+\frac{4m^2}{Q^2}}\right). \tag{206}$$

3.6.4 *Infrared divergent C_0 function*

In all the cases considered above the C_0 functions were finite. However, the QED vertex contains IRD divergence. It deserves a special study. The corresponding vertex:



is divergent at $p_1^2 = -m^2,\ p_2^2 = -m^2$ and in order to regularize it, in older days people introduced infinitesimal photonic mass, $m_2 = \lambda$, with λ being small with respect to all the other quantities. Although by now the infrared singularities are treated within the dimensional regularization approach, this example is a useful bridge with the mass-regularization method. The defining integral in this case reads:

$$C_0\left(-m^2, -m^2, Q^2; m, \lambda, m\right) = \int_0^1 dy \int_0^1 dx \frac{x}{\chi(x,y)}\ , \tag{207}$$

where

$$\chi(x,y) = x^2\chi(y) + \lambda^2(1-x) - i\epsilon, \qquad \text{with} \quad \chi(y) = m^2 + Q^2 y\,(1-y)\,. \tag{208}$$

Integrating it once and exploiting the infinitesimalness of λ,

$$\int_0^1 dx \frac{x}{\chi(y)x^2 + \lambda^2(1-x)} = \frac{1}{2\chi(y)} \ln\left(\frac{\chi(y)}{\lambda^2}\right) + \mathcal{O}\left(\frac{\lambda}{\sqrt{\chi(y)}}\right), \tag{209}$$

we obtain the following decomposition:

$$\begin{aligned}
C_0 &= \frac{1}{2}\left[F_1 \ln\left(\frac{\mu^2}{\lambda^2}\right) + F_2\right], \\
F_1 &= \int_0^1 dy\, \frac{1}{\chi(y)} = \frac{2}{Q^2\beta_m} \ln\frac{\beta_m+1}{\beta_m-1}\ , \\
F_2 &= \int_0^1 dy\, \frac{1}{\chi(y)} \ln\frac{\chi(y)}{\mu^2} = F_1\, \ln\left(\frac{Q^2 - i\varepsilon}{\mu^2}\right) \\
&\quad + \frac{1}{Q^2\beta_m}\left[\ln\frac{\beta_m+1}{\beta_m-1}\ln\frac{m^2\beta_m^2}{Q^2} - 2\mathrm{Li}_2\left(\frac{\beta_m+1}{2\beta_m}\right) + 2\mathrm{Li}_2\left(\frac{\beta_m-1}{2\beta_m}\right)\right],
\end{aligned} \tag{210}$$

with

$$\beta_m^2 = 1 + \frac{4m^2}{Q^2}\ . \tag{211}$$

In the next lecture, I will present the derivation of this C_0 by the dimensional regularization method. It will be shown that the identification

$$\ln\left(\frac{\mu}{\lambda}\right)^2 \leftrightarrow \frac{1}{\hat{\varepsilon}} \tag{212}$$

establishes the bridge between the two regularizations.

In the most general case the C_0 function contains 12 dilogarithms and several Veltman's η-functions, see Section 5.1.4.3 of Ref. [1].

3.6.5 Tensor three-point integrals

The rank one tensor integral is defined by

$$i\pi^2 C_\mu \left(p_1^2, p_2^2, Q^2; m_1, m_2, m_3\right) = \mu^{4-n} \int d^n q \, \frac{q_\mu}{d_0 d_1 d_2} \,, \tag{213}$$

and its decomposition looks like

$$i\pi^2 \left[C_{11} \left(p_1^2, p_2^2, Q^2; m_1, m_2, m_3\right) p_{1\mu} + C_{12} \left(p_1^2, p_2^2, Q^2; m_1, m_2, m_3\right) p_{2\mu}\right]. \tag{214}$$

The rank two tensor integral,

$$i\pi^2 C_{\mu\nu} = \mu^{4-n} \int d^n q \frac{q_\mu q_\nu}{d_0 d_1 d_2} \,, \tag{215}$$

already contains four structures

$$i\pi^2 \left[C_{21} p_{1\mu} p_{1\nu} + C_{22} p_{2\mu} p_{2\nu} + C_{23} \left\{p_1 p_2\right\}_{\mu\nu} + C_{24} \delta_{\mu\nu}\right], \tag{216}$$

where the symmetrized combination is introduced

$$\left\{p_1 p_2\right\}_{\mu\nu} = p_{1\mu} p_{2\nu} + p_{1\nu} p_{2\mu} \,. \tag{217}$$

The reduction of these tensors is developed using standard technique. All the details may be found in Sections 5.1.4.4 and 5.1.4.5 of Ref. [1].

3.7 Four-point integrals, D functions

They appear in the calculation of box diagrams.

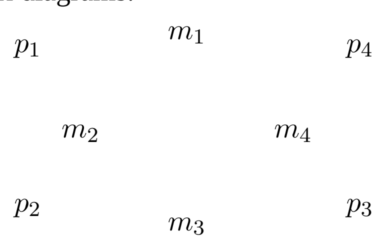


Fig. 1: The box diagram.

The four-point functions are again much more complicated than the previous ones. Only definitions and some particular cases will be presented here.

3.7.1 The scalar four-point integral, D_0 function

The integral defining a D_0 function with 10 arguments is

$$i\pi^2 D_0 \left(p_1^2, p_2^2, p_3^2, p_4^2, (p_1+p_2)^2, (p_2+p_3)^2; m_1, m_2, m_3, m_4\right) = \mu^{4-n} \int d^n q \frac{1}{d_0 d_1 d_2 d_3} \,. \tag{218}$$

The four propagators in this case are

$$\begin{aligned} d_0 &= q^2 + m_1^2 - i\epsilon \,, & d_1 &= (q+p_1)^2 + m_2^2 - i\epsilon \,, \\ d_2 &= (q+p_1+p_2)^2 + m_3^2 - i\epsilon \,, & d_3 &= (q+p_1+p_2+p_3)^2 + m_4^2 - i\epsilon \,, \end{aligned}$$

with all four-momenta flowing inwards as shown in Fig. 1, so that $p_1 + p_2 + p_3 + p_4 = 0$. In terms of Feynman variables x, y and z it reads

$$D_0 = \int_0^1 dx \int_0^x dy \int_0^y dz \frac{1}{\left(ax^2 + by^2 + gz^2 + cxy + hxz + jyz + dx + ey + kz + f\right)^2} \,, \tag{219}$$

with

$$
\begin{array}{rclrcl}
a &=& -p_{23}^2 = -p_3^2\,, & b &=& -p_{12}^2 = -p_2^2\,, \\
g &=& -p_{01}^2 = -p_1^2\,, & c &=& -p_{13}^2 + p_{12}^2 + p_{23}^2\,, \\
h &=& -p_{03}^2 - p_{12}^2 + p_{02}^2 + p_{13}^2\,, & j &=& -p_{02}^2 + p_{01}^2 + p_{12}^2\,, \\
d &=& m_3^2 - m_4^2 + p_{23}^2\,, & e &=& m_2^2 - m_3^2 + p_{13}^2 - p_{23}^2\,, \\
k &=& m_1^2 - m_2^2 + p_{03}^2 - p_{13}^2\,, & f &=& m_4^2 - i\epsilon\,,
\end{array}
$$

and $p_{ij}^2 = (p_i - p_j)^2$.

3.7.2 Reduction of tensor four-point integrals

The 1-, 2- and 3-rank tensors

$$
i\pi^2\Big\{D_\mu; D_{\mu\nu}; D_{\mu\nu\alpha}\Big\} = \mu^{4-n}\int d^n q \frac{\{q_\mu; q_\mu q_\nu; q_\mu q_\nu q_\alpha; q_\mu q_\nu q_\alpha q_\beta\}}{d_0 d_1 d_2 d_3}\,, \tag{220}
$$

contain 3, 7 and 13 structures and tensor functions D_{ij}, respectively:

$$
\begin{array}{rcl}
D_\mu &=& D_{11}p_{1\mu} + D_{12}p_{2\mu} + D_{13}p_{3\mu}\,, \\
D_{\mu\nu} &=& D_{21}p_{1\mu}p_{1\nu} + D_{22}p_{2\mu}p_{2\nu} + D_{23}p_{3\mu}p_{3\nu} \\
&& + D_{24}\{p_1p_2\}_{\mu\nu} + D_{25}\{p_1p_3\}_{\mu\nu} + D_{26}\{p_2p_3\}_{\mu\nu} + D_{27}\delta_{\mu\nu}\,, \\
D_{\mu\nu\alpha} &=& D_{31}p_{1\mu}p_{1\nu}p_{1\alpha} + D_{32}p_{2\mu}p_{2\nu}p_{2\alpha} + D_{33}p_{3\mu}p_{3\nu}p_{3\alpha} \\
&& + D_{34}\{p_2p_1p_1\}_{\mu\nu\alpha} + D_{35}\{p_3p_1p_1\}_{\mu\nu\alpha} + D_{36}\{p_1p_2p_2\}_{\mu\nu\alpha} \\
&& + D_{37}\{p_1p_3p_3\}_{\mu\nu\alpha} + D_{38}\{p_3p_2p_2\}_{\mu\nu\alpha} + D_{39}\{p_2p_3p_3\}_{\mu\nu\alpha} \\
&& + D_{310}\{p_1p_2p_3\}_{\mu\nu\alpha} + D_{311}\{p_1\delta\}_{\mu\nu\alpha} + D_{312}\{p_2\delta\}_{\mu\nu\alpha} + D_{313}\{p_3\delta\}_{\mu\nu\alpha}\,.
\end{array} \tag{221}
$$

For rank-3 tensor an additional symmetrized structure appears

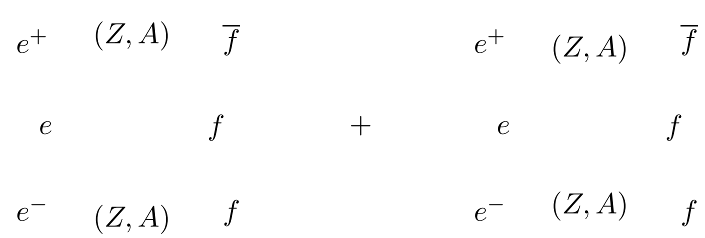
$$
\{pkl\}_{\mu\nu\alpha} = p_\mu\{kl\}_{\nu\alpha} + k_\mu\{pl\}_{\nu\alpha} + l_\mu\{pk\}_{\nu\alpha}\,. \tag{222}
$$

The reduction is performed by making use of standard technique. Details may be found in Section 5.1.5.2 of Ref. [1].

For the e^+e^- annihilation into fermion pairs, SM boxes are met only in two topologies: *direct* or *crossed*. For WW internal lines there is a peculiar aspect due to charge conservation:

only direct box is present for $e^+e^- \to d\overline{d}\,$;
only crossed box is present for $e^+e^- \to u\overline{u}\,$.

The full collection of box diagrams for e^+e^- annihilation into a fermion pair is presented in the Fig. 2.



AA-, ZA-, ZZ-boxes

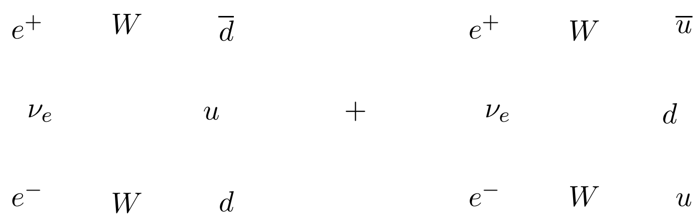


Fig. 2: WW-boxes.

3.7.3 Some particular cases of D_0 functions

Case 1). The most general expression which one encounters when considering ZZ and WW boxes with four massless external fermions is obtained with

$$\begin{aligned} p_i^2 &= 0\,, \qquad (p_1+p_2)^2 = Q^2, \qquad (p_2+p_3)^2 = P^2, \\ m_1 &= M_1\,, \qquad m_2 = 0\,, \qquad m_3 = M_1\,, \qquad m_4 = M_2\,. \end{aligned} \tag{223}$$

With an appropriate choice of Feynman parameters it may be presented and calculated as follows:

$$\begin{aligned} D_0\left(0,0,0,0,Q^2,P^2;M_1,0,M_1,M_2\right) &= \int_0^1 dz \int_0^1 y dy \int_0^1 dx \\ &\quad \times \frac{1}{\left[M_1^2 y + M_2^2(1-y) + P^2(1-y)(1-z) + Q^2 z y^2 x(1-x)\right]^2} \\ &= \frac{1}{Q^2(P^2+M_2^2)\sqrt{d_4}} \sum_{i=1}^{4}\sum_{j=1}^{2} (-1)^{\delta_{i3}+\delta_{j2}} \left[\mathrm{Li}_2\left(\frac{\bar{x}_j}{\bar{x}_j - x_i}\right) - \mathrm{Li}_2\left(\frac{\bar{x}_j - 1}{\bar{x}_j - x_i}\right)\right], \end{aligned} \tag{224}$$

with six roots

$$\begin{aligned} x_{1,2} &= \frac{1}{2}\left(1 \mp \sqrt{1+\frac{4M_1^2}{Q^2}}\right), \qquad \bar{x}_{1,2} = \frac{x_4}{2}\left(1 \mp \sqrt{d_4}\right), \\ x_3 &= \frac{M_2^2}{M_2^2 - M_1^2}\,, \qquad\qquad x_4 = \frac{P^2 + M_2^2}{P^2 + M_2^2 - M_1^2}\,, \end{aligned} \tag{225}$$

and

$$d_4 = 1 + \frac{4M_1^2 P^2\left(P^2 + M_2^2 - M_1^2\right)}{Q^2\left(P^2+M_2^2\right)^2}\,. \tag{226}$$

For $M_2 = 0$ (in practical applications $m_t = 0$), it simplifies to

$$D_0\left(0,0,0,0,Q^2,P^2;M_1,0,M_1,0\right) = \frac{2}{Q^2 P^2 \sqrt{d_4^{(0)}}} \sum_{ij=1}^{2} (-1)^{i+1}\,\mathrm{Li}_2\left(\frac{\tilde{x}_i}{\tilde{x}_i - x_j}\right), \tag{227}$$

with the roots

$$\tilde{x}_{1,2} = \frac{x_4}{2}\left(1 \mp \sqrt{d_4^{(0)}}\right), \qquad d_4^{(0)} = 1 + \frac{4M_1^2\left(P^2 - M_1^2\right)}{Q^2 P^2}\,. \tag{228}$$

Case 2). This case is encountered when considering ZA and AA boxes, where it is useful to introduce three infrared free auxiliary integrals:

$$\begin{aligned} i\pi^2 \bar{J}_{\gamma\gamma}\left(Q^2,P^2;m_e,m_f\right) &= \mu^{4-n}\int d^n q \frac{2q\cdot(q+Q)}{d_0\,(0)\,d_1\,(m_e)\,d_2\,(0)\,d_3\,(m_f)}\,, \\ i\pi^2 \bar{J}_{\gamma Z}\left(Q^2,P^2;m_e,m_f\right) &= \mu^{4-n}\int d^n q \frac{2q\cdot Q}{d_0\,(0)\,d_1\,(m_e)\,d_2\,(M_Z)\,d_3\,(m_f)}\,, \\ i\pi^2 \bar{J}_{Z\gamma}\left(Q^2,P^2;m_e,m_f\right) &= \mu^{4-n}\int d^n q \frac{2Q\cdot(q+Q)}{d_0\,(M_Z)\,d_1\,(m_e)\,d_2\,(0)\,d_3\,(m_f)}\,. \end{aligned} \tag{229}$$

These auxiliary integrals are simple to calculate. Moreover, the following identities are very useful to exhibit and disentengle the infrared behaviour of the scalar functions D_0:

$$
\begin{aligned}
&D_0\left(-m_e^2,-m_e^2,-m_f^2,-m_f^2,Q^2,P^2;0,m_e,0,m_f\right)=\frac{1}{Q^2}\Big[-\bar{J}_{\gamma\gamma}\left(Q^2,P^2;m_e,m_f\right)\\
&\quad+C_0\left(-m_e^2,-m_f^2,P^2;m_e,0,m_f\right)+C_0\left(-m_f^2,-m_e^2,P^2;m_f,0,m_e\right)\Big],\\
&D_0\left(-m_e^2,-m_e^2,-m_f^2,-m_f^2,Q^2,P^2;0,m_e,M_Z,m_f\right)=\frac{1}{Q^2+M_Z^2}\Big[-\bar{J}_{\gamma Z}\left(Q^2,P^2;m_e,m_f\right)\\
&\quad-C_0\left(-m_e^2,-m_f^2,P^2;m_e,M_Z,m_f\right)+C_0\left(-m_f^2,-m_e^2,P^2;m_f,0,m_e\right)\Big],\\
&D_0\left(-m_e^2,-m_e^2,-m_f^2,-m_f^2,Q^2,P^2;M_Z,m_e,0,m_f\right)=\frac{1}{Q^2+M_Z^2}\Big[\bar{J}_{Z\gamma}\left(Q^2,P^2;m_e,m_f\right)\\
&\quad+C_0\left(-m_e^2,-m_f^2,P^2;m_e,0,m_f\right)-C_0\left(-m_f^2,-m_e^2,P^2;m_f,M_Z,m_e\right)\Big].
\end{aligned}
\tag{230}
$$

Here we present the answers for the auxiliary integrals, in terms of one-fold integrals:

$$
\bar{J}_{\gamma\gamma}\left(Q^2,P^2;m_e,m_f\right) = \int_0^1 dx\frac{1}{\chi\left(P^2;m_e,m_f\right)}\ln\frac{\chi\left(P^2;m_e,m_f\right)}{Q^2}\,, \tag{231}
$$

$$
\bar{J}_{\gamma Z}\left(Q^2,P^2;m_e,m_f\right) = -\bar{J}_{Z\gamma}\left(Q^2,P^2;m_e,m_f\right)=\ln\frac{M_Z^2+Q^2}{M_Z^2}\int_0^1 dx\frac{1}{\chi\left(P^2;m_e,m_f\right)}\,,
$$

where $\chi\left(P^2;m_e,m_f\right) = P^2x\left(1-x\right)+m_e^2\left(1-x\right)+m_f^2x$ is the usual quadratic form. The explicit answers for $\bar{J}_{\gamma\{\gamma,Z\}}\left(Q^2,P^2;m_e,m_f\right)$ may be be found in Section 5.1.5.3 of Ref. [1].

3.8 Special PV functions: $a, b, c^{(j)}, d^{(j)}$

The standard Passarino–Veltman (PV) functions, A, B, C, D, considered in this lecture, are sufficient to calculate one-loop corrections in $\xi = 1$ and U gauges. In the R_ξ-gauge additional complications arise. Let us consider a diagram with internal photonic lines, with photon propagators, which contain an additional term $(\xi_A^2-1)q_\mu q_\nu/q^2$, Fig. 3. This leads to a special class of two- (three-, four-) point functions.

W

W

γ

Fig. 3: An example of Feynman diagram leading to special PV functions.

3.8.1 The scalar b_0 function

It is defined by the integral:

$$
i\pi^2\, b_0\left(p^2;m\right)=\mu^{4-n}\int d^nq\frac{1}{\left(q^2\right)^2\left(\left(q+p\right)^2+m^2\right)}\;. \tag{232}
$$

This integral is a badly divergent object in the infrared regime. Using the standard infrared regularization: $n = 4+\varepsilon', \;\; \varepsilon' > 0$, and introducing a dimensionless $\chi = 1+(1-x)p^2/m^2$, we have

$$
b_0\left(p^2;m\right) = \pi^{\varepsilon'/2}\Gamma\left(1-\frac{\varepsilon'}{2}\right)\left(\frac{m^2}{\mu^2}\right)^{\varepsilon'/2}\int_0^1 dx\, x^{-1+\varepsilon'/2}\frac{(1-x)\chi^{-1+\varepsilon'/2}}{m^2}
$$

$$\approx \quad \pi^{\varepsilon'/2}\Gamma\left(1-\frac{\varepsilon'}{2}\right)\left(\frac{m^2}{\mu^2}\right)^{\varepsilon'/2}\int_0^1 dx\, x^{-1+\varepsilon'/2}\, h(x),$$

$$h(x) \quad = \quad \frac{1-x}{m^2\chi}\left(1+\frac{\varepsilon'}{2}\ln\chi\right). \tag{233}$$

By adding and subtracting $h(0)$ and by noticing that $x^{-1}\left[h(x)-h(0)\right]$ is finite for $\varepsilon'\to 0$, we obtain

$$b_0\left(p^2;m\right)=\frac{1}{p^2+m^2}\left[\frac{1}{\hat{\varepsilon}}+\ln\frac{m^2}{\mu^2}+\left(1-\frac{m^2}{p^2}\right)\ln\left(1+\frac{p^2}{m^2}\right)\right]. \tag{234}$$

This integral is defined in the whole n-plane for $p^2\neq -m^2$ and it shows an infrared pole at $n=4$.

3.8.2 Vector b_1 function

This is defined by

$$i\pi^2 b_1\left(p^2;m\right)\, p_\mu \quad = \quad \mu^{4-n}\int d^nq\frac{q_\mu}{\left(q^2\right)^2\left(\left(q+p\right)^2+m^2\right)}$$

$$= \quad -i\pi^{n/2}p_\mu\Gamma\left(3-\frac{n}{2}\right)\int_0^1 dx\frac{x^{n/2-2}\left(1-x\right)}{\left(m^2\chi\right)^{3-n/2}}\,. \tag{235}$$

This function is free of singularities if $p^2\neq -m^2$, where it could be computed at $n=4$. We have

$$b_1\left(p^2;m\right)=-\int_0^1 dx\frac{x}{m^2+p^2x}=-\frac{1}{p^2}\left[1-\frac{m^2}{p^2}\ln\left(1+\frac{p^2}{m^2}\right)\right]. \tag{236}$$

There is an alternative way of evaluating $b_1\left(p^2;m\right)$. With $d=(q+p)^2+m^2$, we derive

$$i\pi^2p^2b_1\left(p^2;m\right) \quad = \quad \frac{1}{2}\int d^nq\left[\frac{1}{\left(q^2\right)^2}-\frac{1}{q^2d}-\frac{p^2+m^2}{\left(q^2\right)^2\, d}\right],$$

$$p^2b_1\left(p^2;m\right) \quad = \quad \frac{1}{2}\left[a_0-B_0\left(p^2;0,m\right)-\left(p^2+m^2\right)\, b_0\left(p^2;m\right)\right]. \tag{237}$$

In the previous derivation we have introduced a new integral,

$$i\pi^2\, a_0=\int d^nq\frac{1}{\left(q^2\right)^2}\,. \tag{238}$$

Since we have,

$$B_0\left(p^2;0,m\right)=\frac{1}{\bar{\varepsilon}}+2-\ln\frac{m^2}{\mu^2}-\left(1+\frac{m^2}{p^2}\right)\ln\left(1+\frac{p^2}{m^2}\right), \tag{239}$$

then from Eqs. (234) and (236), which are valid for any n, we obtain the proper definition of this integral

$$a_0=\frac{1}{\hat{\varepsilon}}+\frac{1}{\bar{\varepsilon}}=0\,, \tag{240}$$

fully consistent with our previous findings, cf. Eq. (158). In this way we derive a typical relation between special and standard PV functions

$$p^2b_1\left(p^2;m\right)=-\frac{1}{2}\left[B_0\left(p^2;0,m\right)+\left(p^2+m^2\right)\, b_0\left(p^2;m\right)\right]. \tag{241}$$

3.8.3 The rank two tensor integral

The tensor is defined by

$$i\pi^2\left(b_{21}p_\mu p_\nu+b_{22}\delta_{\mu\nu}\right)=\mu^{4-n}\int d^nq\frac{q_\mu q_\nu}{\left(q^2\right)^2\left(\left(q+p\right)^2+m^2\right)}\,. \tag{242}$$

Using Feynman parametrization (the second Eq. (149), and Eq. (155)), we derive

$$b_{22}\left(p^2;m\right)=\frac{1}{2}\pi^{n/2-2}\mu^{4-n}\Gamma\left(2-\frac{n}{2}\right)\int_0^1 dx\,x\left(1-x\right)^{n/2-2}\left(m^2+p^2x\right)^{n/2-2}. \tag{243}$$

From which it is easy to obtain the singular part of $b_{22}\left(p^2;m\right)$ and the rule of multiplication by n:

$$n\,b_{22}\left(p^2;m\right)=4\,b_{22}\left(p^2;m\right)-\frac{1}{2}\,. \tag{244}$$

By applying of the usual method one obtains the system:

$$\begin{aligned}p^2b_{21}\left(p^2;m\right)+nb_{22}\left(p^2;m\right)&=B_0\left(p^2;0,m\right),\\ p^2b_{21}\left(p^2;m\right)+b_{22}\left(p^2;m\right)&=\frac{1}{2}\left[B_1\left(p^2;0,m\right)+\left(p^2+m^2\right).b_1\left(p^2;m\right)\right],\end{aligned} \tag{245}$$

and its solution

$$\begin{aligned}b_{22}\left(p^2;m\right)&=\frac{1}{3}B_0\left(p^2;0,m\right)+\frac{1}{6}\left[B_1\left(p^2;0,m\right)+\left(p^2+m^2\right)\,b_1\left(p^2;m\right)+1\right],\\ b_{21}\left(p^2;m\right)&=-4\,b_{22}\left(p^2;m\right)+B_0\left(p^2;0,m\right)+\frac{1}{2}\,.\end{aligned} \tag{246}$$

After *the identification* $1/\hat\varepsilon=-1/\bar\varepsilon$ the following identities may be established:

$$\begin{aligned}\left(p^2+m^2\right)^2b_0\left(p^2;m\right)&=2A_0\left(m\right)+2\,p^2-\left(p^2-m^2\right)B_0\left(p^2;0,m\right),\\ \left(p^2+m^2\right)b_1\left(p^2;m\right)&=-1-\frac{1}{p^2}\left[A_0\left(m\right)+m^2\,B_0\left(p^2;0,m\right)\right],\end{aligned} \tag{247}$$

which give more relations between special and standard PV functions:

$$\begin{aligned}b_{22}\left(p^2;m\right)&=\frac{1}{3}B_0\left(p^2;0,m\right)+\frac{1}{6}\left\{B_1\left(p^2;0,m\right)-\frac{1}{p^2}\left[A_0\left(m\right)+m^2B_0\left(p^2;0,m\right)\right]\right\},\\ b_{21}\left(p^2;m\right)&=\frac{1}{p^2}\left\{-\frac{1}{3}B_0\left(p^2;0,m\right)\right.\\ &\left.-\frac{2}{3}\left(B_1\left(p^2;0,m\right)-\frac{1}{p^2}\left[A_0\left(m\right)+m^2B_0\left(p^2;0,m\right)\right]\right)+\frac{1}{2}\right\}.\end{aligned} \tag{248}$$

3.8.4 One more special series

One more class of functions, $\hat{b}$, are met when calculating $\gamma\gamma$ boxes in the R_ξ gauge. The scalar function $\hat{b}_0$ is defined by:

$$i\pi^2\hat{b}_0\left(Q^2\right)=\mu^{4-n}\int d^nq\frac{1}{\left(q^2\right)^2\left(\left(q+Q\right)^2\right)^2}\,. \tag{249}$$

With infrared regularization, $n = 4 + \varepsilon'$, a straightforward calculation leads to

$$\begin{aligned}
\hat{b}_0\left(Q^2\right) &= \frac{1}{(Q^2)}\pi^{\varepsilon'/2}\Gamma\left(2-\frac{\varepsilon'}{2}\right)\left(\frac{Q^2}{\mu^2}\right)^{\varepsilon'/2}\int_0^1 dx\left[x\left(1-x\right)\right]^{-1+\varepsilon'/2} \\
&= \frac{2}{\left(Q^2\right)^2}\pi^{\varepsilon'/2}\Gamma\left(2-\frac{\varepsilon'}{2}\right)\left(\frac{Q^2}{\mu^2}\right)^{\varepsilon'/2}\mathrm{B}\left(\frac{\varepsilon'}{2},1+\frac{\varepsilon'}{2}\right) \\
&= \frac{2}{\left(Q^2\right)^2}\left(\frac{1}{\hat{\varepsilon}}+\ln\frac{m^2}{\mu^2}-1\right)\equiv\frac{2}{Q^2}\left[b_0\left(Q^2;0\right)-\frac{1}{Q^2}\right].
\end{aligned} \tag{250}$$

3.8.5 Vector and tensor $\hat{b}$ integrals

We present only definitions:

$$\begin{aligned}
i\pi^2\hat{b}_1\left(Q^2\right)Q_\mu &= \mu^{4-n}\int d^nq\frac{q_\mu}{\left(q^2\right)^2\left(\left(q+Q\right)^2\right)^2}, \\
i\pi^2\left[\hat{b}_{21}\left(Q^2\right)Q_\mu Q_\nu+\hat{b}_{22}\left(Q^2\right)\delta_{\mu\nu}\right] &= \mu^{4-n}\int d^nq\frac{q_\mu q_\nu}{\left(q^2\right)^2\left(\left(q+Q\right)^2\right)^2},
\end{aligned} \tag{251}$$

and answers:

$$\hat{b}_1\left(Q^2\right)=-\frac{1}{2}\hat{b}_0\left(Q^2\right),\qquad \hat{b}_{21}\left(Q^2\right)=\frac{1}{Q^2}\left[b_0\left(Q^2;0\right)-\frac{2}{Q^2}\right],\qquad \hat{b}_{22}\left(Q^2\right)=\frac{1}{2Q^2}. \tag{252}$$

Actually only infrared finite objects will appear in the calculation, such as, for instance:

$$\int d^nq\frac{q_\mu\left(q+Q\right)_\nu}{\left(q^2\right)^2\left(\left(q+Q\right)^2\right)^2}=\frac{1}{2Q^2}\delta_{\mu\nu}-\frac{1}{\left(Q^2\right)^2}Q_\mu Q_\nu\,. \tag{253}$$

The full collection of scalar, vectors and tensors is, nevertheless, needed if we wish to develop an automatic computer program for the generation and calculation of all possible one-loop diagrams in the R_ξ gauge.

3.8.6 $c_i^{(j)}$ functions

When considering arbitrary four-fermion processes one encounters additional functions. An example is given by four classes of special functions, called $c_i^{(j)}$ functions, $j = 0, 1, 2, 02$. The function with $j = 02$ is *a pinch* of the $\gamma\gamma$-box diagram. Here we give only defining equations for the scalar functions, referring to Section 5.1.6.2 of Ref. [1] for more details and the reduction:

$$\begin{aligned}
i\pi^2c^{(0)}_{\{1,\mu,\mu\nu\}}\left(p_1^2,p_2^2,Q^2;0,m_2,m_3\right) &= \mu^{4-n}\int d^nq\,\frac{\{1,q_\mu,q_\mu q_\nu\}}{d_0^2d_1d_2}, \\
i\pi^2c^{(1)}_{\{1,\mu,\mu\nu\}}\left(p_1^2,p_2^2,Q^2;m_1,0,m_3\right) &= \mu^{4-n}\int d^nq\,\frac{\{1,q_\mu,q_\mu q_\nu\}}{d_0d_1^2d_2}, \\
i\pi^2c^{(2)}_{\{1,\mu,\mu\nu\}}\left(p_1^2,p_2^2,Q^2;m_1,m_2,0\right) &= \mu^{4-n}\int d^nq\,\frac{\{1,q_\mu,q_\mu q_\nu\}}{d_0d_1d_2^2}, \\
i\pi^2c^{(02)}_{\{1,\mu,\mu\nu\}}\left(p_1^2,p_2^2,Q^2;0,m_2,0\right) &= \mu^{4-n}\int d^nq\,\frac{\{1,q_\mu,q_\mu q_\nu\}}{d_0^2d_1d_2^2}.
\end{aligned} \tag{254}$$

3.8.7 $d_i^{(j)}$ functions

Finally, there are special functions associated with four-point integrals. The class of the $d_i^{(j)}$ functions is richer than that of the $c_i^{(j)}$ functions. As usual, we limit ourself to the functions which appear when considering of $2f \to 2f$ processes. This is why only definitions of three classes of $d_i^{(j)}$ functions with $j = 0, 2, 02$ are given:

$$\begin{aligned}
i\pi^2 d^{(0)}_{\{1,\mu,\mu\nu\}}\left(p_1^2,p_2^2,p_3^2,p_4^2,Q^2,P^2;0,m_2,m_3,m_4\right) &= \mu^{4-n}\int d^n q\,\frac{\{1,q_\mu,q_\mu q_\nu\}}{d_0^2 d_1 d_2 d_3}\,,\\
i\pi^2 d^{(2)}_{\{1,\mu,\mu\nu\}}\left(p_1^2,p_2^2,p_3^2,p_4^2,Q^2,P^2;m_1,m_2,0,m_4\right) &= \mu^{4-n}\int d^n q\,\frac{\{1,q_\mu,q_\mu q_\nu\}}{d_0 d_1 d_2^2 d_3}\,,\\
i\pi^2 d^{(02)}_{\{1,\mu,\mu\nu\}}\left(p_1^2,p_2^2,p_3^2,p_4^2,Q^2,P^2;0,m_2,0,m_4\right) &= \mu^{4-n}\int d^n q\,\frac{\{1,q_\mu,q_\mu q_\nu\}}{d_0^2 d_1 d_2^2 d_3}\,. \qquad (255)
\end{aligned}$$

Their reduction may be found in Section 5.1.6.3 of Ref. [1].

3.9 Summary of the three Lectures

In lectures 1–3 we studied:

- Basics of present QFT
  - Standard Model, its fields, and Lagrangian;
  - Different gauges: $R_\xi, \quad \xi = 1, \quad U$;
  - Gauge invariance;
  - Feynman rules, and *building* of diagrams.
- Dimension regularization and N-point functions;
- Calculation of loop integrals:
  - standard PV functions: $A,\ B,\ C,\ D$;
  - special PV functions: $a,\ b,\ c,\ d$;

It is time to calculate diagrams.

We emphasize that there are Ultraviolet and Infrared dimensional regularizations:

$$\begin{aligned}
\text{Ultraviolet: } n &= 4-\varepsilon \quad \to \quad \frac{1}{\bar{\varepsilon}} = -\frac{2}{n-4} - \gamma - \ln\pi\,,\\
\text{Infrared: } n &= 4+\varepsilon' \quad \to \quad \frac{1}{\hat{\varepsilon}} = +\frac{2}{n-4} + \gamma + \ln\pi\,, \qquad (256)
\end{aligned}$$

which could be *identified* with the aid of *identity*: $$\frac{1}{\bar{\varepsilon}} + \frac{1}{\hat{\varepsilon}} = 0\,. \qquad (257)$$

4. TOWARDS PRECISION PREDICTIONS FOR EXPERIMENTAL OBSERVABLES

In this lecture we will exploit the knowledge mastered in the previous lectures for calculating the simplest QED diagrams. The second half of the lecture will be devoted to a complete calculation of QED radiative corrections for the Z decay for final-state massless fermions using a technique specific to the massless case.

4.1 Calculation of simplest QED diagrams

4.1.1 Photonic self-energy diagram

Photonic self-energy is described by a tensor, $\Pi_{\mu\nu}$:

f

q

p γ γ

$-(p+q)$

$\overline{f}$

Applying the Feynman rules for vertices and propagators, we construct an initial expression to be integrated over an internal momentum q:

$$\begin{aligned}
\Pi_{\mu\nu} &= e^2 Q_e^2 \int d^n q \frac{\mathrm{Tr}\Big[(i\not{q}+m_f)\,\gamma_\mu\,(i\not{p}+i\not{q}+m_f)\,\gamma_\nu\Big]}{\left(q^2+m_f^2\right)\left[(q+p)^2+m_f^2\right]} \\
&= 4e^2 Q_e^2 \int d^n q \frac{\delta_{\mu\nu}(q^2+m_f^2+qp)-(q_\mu p_\nu+q_\nu p_\mu)-2q_\mu q_\nu}{\left(q^2+m_f^2\right)\left[(q+p)^2+m_f^2\right]}\,. \qquad (258)
\end{aligned}$$

Using the definitions of the A_0 function, Eq. (161), and of the vector and tensor B functions, Eqs. (176) and (180), we immediately get the answer:

$$\begin{aligned}
\Pi_{\mu\nu} &= i\pi^2\, 4e^2 Q_e^2 \Big\{ \delta_{\mu\nu}\left[A_0\left(m_f^2\right)+p^2 B_1\left(p^2;m_f,m_f\right)\right] - 2p_\mu p_\nu B_1\left(p^2;m_f,m_f\right) \\
&\quad -2\left[B_{22}\left(p^2;m_f,m_f\right)\delta_{\mu\nu}+B_{21}\left(p^2;m_f,m_f\right)p_\mu p_\nu\right]\Big\} \\
&= i\pi^2\, 4e^2 Q_e^2 \Big\{ \delta_{\mu\nu}\left[A_0\left(m_f\right)+p^2 B_1\left(p^2;m_f,m_f\right)-2B_{22}\left(p^2;m_f,m_f\right)\right] \\
&\quad -2p_\mu p_\nu\left[B_1\left(p^2;m_f,m_f\right)+B_{21}\left(p^2;m_f,m_f\right)\right]\Big\}. \qquad (259)
\end{aligned}$$

It must be *transverse* as a consequence of QED $U(1)$ gauge invariance:

$$\Pi_{\mu\nu} = i\pi^2\, 4e^2 Q_e^2 \left(p^2\delta_{\mu\nu}-p_\mu p_\nu\right)\Pi_f\left(p^2\right). \qquad (260)$$

This property will be satisfied if

$$\begin{aligned}
&A_0\left(m_f\right)+p^2 B_1\left(p^2;m_f,m_f\right)-2B_{22}\left(p^2;m_f,m_f\right) \\
&= 2p^2\left[B_1\left(p^2;m_f,m_f\right)+B_{21}\left(p^2;m_f,m_f\right)\right]. \qquad (261)
\end{aligned}$$

The four functions, $A_0,\ B_1,\ B_{21},\ B_{22}$, may be *reduced* to only two scalar integrals: $A_0,\ B_0$. Therefore, relations among the four are possible. Indeed, from the general result, Eq. (188), in the case of equal masses one has:

$$\begin{aligned}
B_1\left(p^2;m,m\right) &= -\frac{1}{2}B_0\left(p^2;m,m\right), \\
B_{21}\left(p^2;m,m\right) &= \frac{6m^2+p^2}{18p^2}+\frac{1}{3p^2}A_0(m)+\frac{p^2+m^2}{3p^2}B_0\left(p^2;m,m\right), \\
B_{22}\left(p^2;m,m\right) &= -\frac{6m^2+p^2}{18}+\frac{1}{6}A_0(m)-\frac{p^2+4m^2}{6}B_0\left(p^2;m,m\right). \qquad (262)
\end{aligned}$$

and the desired equality is immediately verified.

The final result for $\Pi_f\left(p^2\right)$ deserves careful examination. We give three representations for it:

1) in terms of higher rank functions, $B_{21,1}$;

2) in terms of scalar functions, $A_0,\ B_0$;

3) explicitly, in terms of the separated out UV-pole and a finite logarithm.

All expressions are equally compact:

$$\begin{aligned}\Pi_f\left(p^2\right) &= 2\left[B_{21}\left(p^2;m_f,m_f\right)+B_1\left(p^2;m_f,m_f\right)\right]\\ &= \frac{6m_f^2+p^2}{9p^2}+\frac{2}{3p^2}A_0\left(m_f\right)-\frac{p^2-2m_f^2}{3p^2}B_0\left(p^2;m_f,m_f\right)\\ &= -\frac{1}{3}\left(\frac{1}{\bar{\varepsilon}}-\ln\frac{m_f^2}{\mu^2}\right)+\frac{1}{9}+\frac{1}{3}\left(1-2\frac{m_f^2}{p^2}\right)\left(\beta_f\ln\frac{\beta_f+1}{\beta_f-1}-2\right), \end{aligned} \tag{263}$$

where

$$\beta_f=\sqrt{1+4\frac{m_f^2}{p^2}}\,. \tag{264}$$

Two limiting cases are of practical interest:

$$\begin{aligned} &\text{for}\quad p^2\gg m_f^2, \qquad \Pi_f\left(p^2\right)=-\frac{1}{3}\left(\frac{1}{\bar{\varepsilon}}-\ln\frac{m_f^2}{\mu^2}\right)-\frac{5}{9}+\frac{1}{3}\ln\frac{p^2}{m_f^2}\,,\\ &\text{for}\quad p^2\ll m_f^2, \qquad \Pi_f\left(p^2\right)=-\frac{1}{3}\left(\frac{1}{\bar{\varepsilon}}-\ln\frac{m_f^2}{\mu^2}\right)+\frac{p^2}{15m_f^2}+\cdots \end{aligned} \tag{265}$$

The finite (i.e. free of UV divergence) difference,

$$\Pi_f(p^2)-\Pi_f(0)=-\frac{5}{9}+\frac{4m_f^2}{3p^2}+\frac{1}{3}\left(1-2\frac{m_f^2}{p^2}\right)\beta_f\ln\frac{\beta_f+1}{\beta_f-1}\,, \tag{266}$$

is *the renormalized* photonic self-energy, as will be proved later.

4.1.2 Fermionic self-energy

Fermionic self-energy is a 4×4 matrix, described by the diagram:

Applying the Feynman rules, we derive an initial expression, which again may be immediately written in terms of B functions:

$$\begin{aligned}\Sigma\left(\not{p}\right) &= -e^2Q_e^2\int d^nq\frac{\gamma_\mu\left(i\not{q}+m_f\right)\gamma_\mu}{\left(q^2+m_f^2-i\epsilon\right)\left[\left(q+p\right)^2-i\epsilon\right]}\\ &= i\pi^2\left(-e^2Q_e^2\right)\left[\left(2-n\right)B_1\left(p^2;m_f,0\right)i\not{p}+nm_fB_0\left(p^2;m_f,0\right)\right]. \end{aligned} \tag{267}$$

Furthermore,

$$\frac{1}{\bar{\varepsilon}}=\frac{2}{4-n}+\text{finite terms}\quad\to\quad\frac{n}{\bar{\varepsilon}}=\frac{2n}{4-n}=\frac{4}{\bar{\varepsilon}}-2\,. \tag{268}$$

Therefore, remembering Eq. (182), we derive *multiplication by n rules*

$$nB_0 = 4B_0 - 2\,, \qquad nB_1 = 4B_1 + 1\,, \tag{269}$$

and the final result for the fermionic self-energy becomes

$$\Sigma\left(\not{p}\right) \;=\; i\pi^2\, e^2 Q_e^2 \left\{\left[2B_1\left(p^2; m_f, 0\right) + 1\right] i\not{p} - m_f\left[4B_0\left(p^2; m_f, 0\right) - 2\right]\right\}. \tag{270}$$

The fermionic self-energy on the fermion mass shell is ultraviolet divergent but finite in the infrared regime, whilst its derivative, $\partial\Sigma\left(\not{p}\right)/\partial p^2|_{p^2=-m_f^2}$, develops a singularity due to the zero mass of the photon, which is of infrared origin. We recall, that

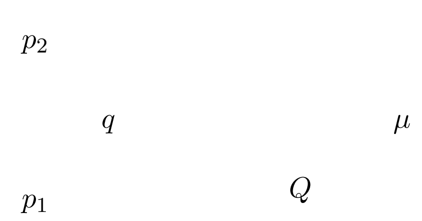
$$\begin{aligned}
\frac{\partial}{\partial p^2}\, B_0\left(p^2; m, 0\right)\Big|_{p^2=-m^2} &= -\frac{1}{2m^2}\left(\frac{1}{\hat{\varepsilon}} - 2 + \ln\frac{m^2}{\mu^2}\right),\\
\frac{\partial}{\partial p^2}\, B_1\left(p^2; m, 0\right)\Big|_{p^2=-m^2} &= \frac{1}{2m^2}\left(\frac{1}{\hat{\varepsilon}} - 3 + \ln\frac{m^2}{\mu^2}\right),
\end{aligned} \tag{271}$$

from which we derive

$$\begin{aligned}
\frac{\partial\Sigma\left(\not{p}\right)}{\partial\not{p}}\Big|_{p^2=-m^2} &= -\pi^2\, e^2 Q_e^2\Bigg[2B_1\left(p^2; m, 0\right) + 1\\
&\quad -4m^2\frac{\partial}{\partial p^2}\, B_1\left(p^2; m, 0\right)\Big|_{p^2=-m^2} - 8m^2\frac{\partial}{\partial p^2}\, B_0\left(p^2; m, 0\right)\Big|_{p^2=-m^2}\Bigg]\\
&= -\pi^2\, e^2 Q_e^2\left(-\frac{1}{\bar{\varepsilon}} + \frac{2}{\hat{\varepsilon}} + 3\ln\frac{m^2}{\mu^2} - 4\right).
\end{aligned} \tag{272}$$

4.1.3 QED vertex

The one-loop QED $f\overline{f}\gamma$ vertex corresponds to the diagram:



For on-mass-shell fermions the most general structure, compatible with both Lorenz and gauge invariances, reads:

$$\Lambda_\mu = (2\pi)^4\, i\, ieQ_e\frac{e^2}{16\pi^2}\left[\gamma_\mu F_1\left(Q^2, m\right) + \sigma_{\mu\nu}\left(p_1 + p_2\right)_\nu mF_2\left(Q^2, m\right)\right]. \tag{273}$$

We note that:

1. The QED vertex dresses the Born expression as
   $(2\pi)^4\, i\, ieQ_e\gamma_\mu \to (2\pi)^4\, i\, ieQ_e\gamma_\mu + \Lambda_\mu\,, \quad Q_e = -1;$
2. F_1 is the Dirac electric form factor, it is ultraviolet and infrared divergent;
3. F_2 is the anomalous magnetic moment of the electron, it is finite.

Using Dirac equations for on-shell fermions, $\bar{v}(p_2)\not{p}_2 = -i\,m\bar{v}(p_2)\,, \ \not{p}_1 u(p_1) = i\,mu(p_1)\,$, and $p_1^2 = p_2^2 = -m^2\,, \ \ Q^2 = (p_1+p_2)^2 = -2m^2 + 2p_1\cdot p_2$, we arrive at

$$\begin{aligned}
\Lambda_\mu &= i\,(eQ_e)^3\,\mu^{4-n}\int d^nq\frac{1}{q^2\left[(q+p_1)^2+m^2\right]\left[(q-p_2)^2+m^2\right]}N_\mu\,,\\
N_\mu &= -4p_1\cdot p_2\gamma_\mu + 2\left(\not{p}_1\gamma_\alpha\gamma_\mu - \gamma_\mu\gamma_\alpha\not{p}_2\right)q_\alpha + (2-n)\,\gamma_\alpha\gamma_\mu\gamma_\beta q_\alpha q_\beta\,. \qquad (274)
\end{aligned}$$

With the standard Feynman parametrization, and notations: $k_x = xp_2 - (1-x)\,p_1$ and $\chi\left(Q^2,x\right) = Q^2x\,(1-x) + m^2$, we derive further on:

$$\begin{aligned}
\Lambda_\mu &= i\,(eQ_e)^3\,\Gamma(3)\int_0^1 dx\int_0^1 dyy\,\mu^{4-n}\int d^nq\frac{1}{\left(q^2-2\,y\,q\cdot k_x\right)^3}N_\mu\\
&= i\pi^2 i\,(eQ_e)^3\left[-\left(Q^2+2m^2\right)\gamma_\mu S + 2\left(\not{p}_1\gamma_\alpha\gamma_\mu - \gamma_\mu\gamma_\alpha\not{p}_2\right)V_\alpha + \gamma_\alpha\gamma_\mu\gamma_\beta T_{\alpha\beta}\right]. \qquad (275)
\end{aligned}$$

For the **scalar** integral we use *the infrared* regulator ε':

$$\begin{aligned}
S &= 2\Gamma(3)\,\frac{\mu^{-\varepsilon'}}{i\pi^2}\int_0^1 dx\int_0^1 dyy\int\frac{d^nq}{\left(q^2-2yq\cdot k_x\right)^3}\\
&= 2\pi^{\varepsilon'/2}\Gamma\left(1-\frac{\varepsilon'}{2}\right)\int_0^1 dx\int_0^1 dyy^{-1+\varepsilon'}\frac{1}{\chi\left(Q^2,x\right)}\left[\frac{\chi\left(Q^2,x\right)}{\mu^2}\right]^{\varepsilon'/2}. \qquad (276)
\end{aligned}$$

After y-integration, which can be performed for any value of ε

$$\int_0^1 dyy^{-k-\varepsilon} = \frac{1}{1-k-\varepsilon}\,, \qquad k = -1,0,1... \qquad (277)$$

we get an expression in terms of a one-fold integral:

$$S = 2\frac{\pi^{\varepsilon'/2}}{\varepsilon'}\Gamma\left(1-\frac{\varepsilon'}{2}\right)\int_0^1 dx\frac{1}{\chi\left(Q^2,x\right)}\left[\frac{\chi\left(Q^2,x\right)}{\mu^2}\right]^{\varepsilon'/2}. \qquad (278)$$

Finally, expanding around $\varepsilon' = 0$, we have:

$$S = \int_0^1 dx\frac{1}{\chi\left(Q^2,x\right)}\left[\frac{1}{\hat{\varepsilon}} + \ln\frac{\chi\left(Q^2,x\right)}{\mu^2}\right]. \qquad (279)$$

For the **vector** and **tensor**, we use *the ultraviolet* regulator ε. For the vector, we proceed as follows:

$$\begin{aligned}
V_\alpha &= \Gamma(3)\,\frac{\mu^{\varepsilon}}{i\pi^2}\int_0^1 dx\int_0^1 dyy\int\frac{d^nqq_\alpha}{\left(q^2-2yq\cdot k_x\right)^3}\\
&= \pi^{-\varepsilon/2}\Gamma\left(1+\frac{\varepsilon}{2}\right)\int_0^1 dxk_{x,\alpha}\int_0^1 dyy^{-\varepsilon}\frac{1}{\chi\left(Q^2,x\right)}\left[\frac{\chi\left(Q^2,x\right)}{\mu^2}\right]^{-\varepsilon/2}\\
&= \frac{(p_2-p_1)_\alpha}{2}\pi^{-\varepsilon/2}\frac{\Gamma(1+\varepsilon/2)}{1-\varepsilon}\int_0^1 dx\frac{1}{\chi\left(Q^2,x\right)}\left[\frac{\chi\left(Q^2,x\right)}{\mu^2}\right]^{-\varepsilon/2}. \qquad (280)
\end{aligned}$$

We see that the vector is finite, and we may set $\varepsilon = 0$, yielding

$$V_\alpha = \frac{(p_2-p_1)_\alpha}{2}F_2\left(Q^2,m\right)\,, \qquad F_2\left(Q^2,m\right) = \int_0^1 dx\frac{1}{\chi\left(Q^2,x\right)}\,. \qquad (281)$$

Then, we have to perform a Dirac algebra for the vector

$$2\left(\not{p}_1\gamma_\alpha\gamma_\mu - \gamma_\mu\gamma_\alpha\not{p}_2\right)\frac{(p_2-p_1)_\alpha}{2} = 2\left[\left(Q^2+4m^2\right)\gamma_\mu + im\left(p_1-p_2\right)_\mu\right]. \tag{282}$$

For the **tensor** integral we have to consider the full contraction:

$$\begin{aligned}\gamma_\alpha\gamma_\mu\gamma_\beta T_{\alpha\beta} &= \Gamma(3)(2-n)\gamma_\alpha\gamma_\mu\gamma_\beta\frac{\mu^\varepsilon}{i\pi^2}\int_0^1 dx\int_0^1 dyy\int\frac{d^nqq_\alpha q_\beta}{\left(q^2-2yq\cdot k_x\right)^3}\\ &= \int_0^1 dx\int_0^1 dy\left[(2-\varepsilon)^2\gamma_\mu\chi\left(Q^2,x\right)-(2-\varepsilon)\varepsilon\not{k}_x\gamma_\mu\not{k}_x\right]\\ &\quad\times\pi^{-\varepsilon/2}\frac{1}{2}\Gamma\left(\frac{\varepsilon}{2}\right)y^{1-\varepsilon}\frac{1}{\chi\left(Q^2,x\right)}\left[\frac{\chi\left(Q^2,x\right)}{\mu^2}\right]^{-\varepsilon/2}.\end{aligned} \tag{283}$$

After applying y-integration and Dirac algebra

$$\not{k}_x\gamma_\mu\not{k}_x = \gamma_\mu\chi\left(Q^2,x\right)-2imk_{x,\mu}\,, \tag{284}$$

we get

$$\begin{aligned}\gamma_\alpha\gamma_\mu\gamma_\beta T_{\alpha\beta} &= \gamma_\mu(1-\varepsilon)\pi^{-\varepsilon/2}\Gamma\left(\frac{\varepsilon}{2}\right)\int_0^1 dx\left[\frac{\chi\left(Q^2,x\right)}{\mu^2}\right]^{-\varepsilon/2}\\ &\quad -im\left(p_1-p_2\right)_\mu\pi^{-\varepsilon/2}\Gamma\left(1+\frac{\varepsilon}{2}\right)\int_0^1 dx\frac{1}{\chi\left(Q^2,x\right)}\left[\frac{\chi\left(Q^2,x\right)}{\mu^2}\right]^{-\varepsilon/2}.\end{aligned} \tag{285}$$

Therefore, the tensor reduces to the one-fold integral:

$$\gamma_\alpha\gamma_\mu\gamma_\beta T_{\alpha\beta} = \gamma_\mu\left(\frac{1}{\bar{\varepsilon}}-\int_0^1 dx\ln\frac{\chi\left(Q^2,x\right)}{\mu^2}-2\right)-im\left(p_1-p_2\right)_\mu F_2\left(Q^2,m\right). \tag{286}$$

Now we are ready to collect the scalar, vector, and tensor together. Moreover, we use the Gordon identity

$$i\left(p_1-p_2\right)_\mu\bar{v}u = -2m\,\bar{v}\gamma_\mu u+\bar{v}\sigma_{\mu\nu}\left(p_1+p_2\right)_\nu u\,, \tag{287}$$

in order to arrive at the standard parametrization of the QED vertex, Eq. (273), with $F_1\left(Q^2,m\right)$ and $F_2\left(Q^2,m\right)$. The latter is given by Eq. (281), whilst the former is

$$\begin{aligned}F_1\left(Q^2,m\right) &= -\left(Q^2+2m^2\right)\int_0^1 dx\frac{1}{\chi\left(Q^2,x\right)}\left[\frac{1}{\hat{\varepsilon}}+\ln\frac{\chi\left(Q^2,x\right)}{\mu^2}\right]\\ &\quad+\frac{1}{\bar{\varepsilon}}-\int_0^1 dx\ln\frac{\chi\left(Q^2,x\right)}{\mu^2}-2+2\left(Q^2+3m^2\right)\int_0^1 dx\frac{1}{\chi\left(Q^2,x\right)}.\end{aligned} \tag{288}$$

In the derivation presented above we intentionally did not use the formalism of PV functions in order to show that in some cases a direct application of the formulae of Section 3.2 may be profitable. Of course, all the integrals of Eqs. (281) and (288) may be given in terms of PV functions. We have,

$$\begin{aligned}\int_0^1 dx\frac{1}{\chi\left(Q^2,x\right)}\left[\frac{1}{\hat{\varepsilon}}+\ln\frac{\chi\left(Q^2,x\right)}{\mu^2}\right] &= \frac{1}{2}C_0\left(-m^2,-m^2,Q^2;m,0,m\right),\\ \frac{1}{\bar{\varepsilon}}-\int_0^1 dx\ln\frac{\chi\left(Q^2,x\right)}{\mu^2} &= B_0\left(Q^2;m,m\right),\\ \left(Q^2+4m^2\right)\int_0^1 dx\frac{1}{\chi\left(Q^2,x\right)} &= -2\left[B_0\left(Q^2;m,m\right)-B_0\left(-m^2;m,0\right)\right].\end{aligned}$$

Two limiting cases deserve our attention:

1. $s=-Q^2 \gg m^2$,

$$F_1(-s,m) = \frac{1}{\bar{\varepsilon}} - \ln\frac{m^2}{\mu^2} - 2\left(\frac{1}{\hat{\varepsilon}} + \ln\frac{m^2}{\mu^2}\right)\ln\frac{-s-i\epsilon}{m^2} - \ln^2\frac{-s-i\epsilon}{m^2} + \frac{1}{3}\pi^2 + 3\ln\frac{-s-i\epsilon}{m^2}\,, \tag{289}$$

2. $Q^2=0$,

$$F_1(0,m) = \frac{1}{\bar{\varepsilon}} - \frac{2}{\hat{\varepsilon}} - 3\,\ln\frac{m^2}{\mu^2} + 4\,. \tag{290}$$

The quantity of physical interest is F_1 subtracted at zero momentum. From Eqs. (289)–(290), we derive its high-energy limit, $s \gg m^2$:

$$F_1^{\rm sub} = F_1(-s,m) - F_1(0,m) = -2\left(\frac{1}{\hat{\varepsilon}} + \ln\frac{m^2}{\mu^2}\right)\left(\ln\frac{-s-i\epsilon}{m^2} - 1\right) - \ln^2\frac{-s-i\epsilon}{m^2} + \frac{1}{3}\pi^2 + 3\ln\frac{-s-i\epsilon}{m^2} - 4\,. \tag{291}$$

Note that the subtracted vertex is UV-finite but IR-divergent. The latter divergence cancels with the infrared divergence originating from the soft bremsstrahlung contribution.

4.1.4 QED box diagrams

For the annihilation $e^+e^- \to f\overline{f}$ there are two QED box diagrams: the direct (a) and the crossed (b):

(a) (b)

The integration of box diagrams over internal momentum q is rather involved and we will not present it here. However, for completeness, we will give answers since boxes are the last QED one-loop diagrams. In the one-loop approximation, the boxes contribute via interference with the lowest order (Born) γ-exchange diagram. For this reason we give, first of all, the Born amplitude squared, summed over final spins and averaged over initial spins:

$$\mathcal{A}_0 = \frac{1}{4}\overline{\sum_{\rm spins}} \mid \mathcal{M}_0 \mid^2 = 2\,e^4 Q_e^2\, Q_f^2\, \frac{t^2+u^2}{s^2}\,. \tag{292}$$

The corresponding contribution from the interference of the direct box diagram with the Born one reads

$$\mathcal{A}^{\rm dr}_{\rm int} = \frac{1}{4}\overline{\sum_{\rm spins}} 2\,{\rm Re}\,\mathcal{M}_0^*\mathcal{B}^{\rm dr} = -\frac{e^6}{2\,\pi^2}\,Q_e^3 Q_f^3\,\frac{1}{s}\delta^{\rm box}_{\gamma\gamma}(s,t,u)\,, \tag{293}$$

where

$$\delta^{\rm box}_{\gamma\gamma}(s,t,u) = u^2\mathcal{D}^+_{\gamma\gamma}(s,t,u) + t^2\mathcal{D}^-_{\gamma\gamma}(s,t,u)\,. \tag{294}$$

Similarly, the contribution of the crossed box is obtained with the replacement $t \leftrightarrow u$ and the change of overall sign

$$\mathcal{A}^{\rm cr}_{\rm int} = \frac{1}{4}\overline{\sum_{\rm spins}} 2\,{\rm Re}\,\mathcal{M}_0^*\mathcal{B}^{\rm cr} = \frac{e^6}{2\,\pi^2}\,Q_e^3 Q_f^3\,\frac{1}{s}\delta^{\rm box}_{\gamma\gamma}(s,u,t)\,. \tag{295}$$

Only two functions $\mathcal{D}^{\pm}_{\gamma\gamma}(s,t,u)$ are needed to describe boxes

$$\begin{aligned}
t^2\mathcal{D}^{-}_{\gamma\gamma}(s,t,u) &= \frac{t^2}{s}\Big[d_0(s,t)+c_0(s;0,m_e,0)+c_0(s;0,m_f,0)\Big],\\
u^2\mathcal{D}^{+}_{\gamma\gamma}(s,t,u) &= \frac{t^2+u^2}{2\,s}\Big[d_0(s,t)+c_0(s;0,m_e,0)+c_0(s;0,m_f,0)\Big]\\
&\quad+(u-t)\;c_0(t;m_e,0,m_f)+u\Big[B_0(-s;0,0)-B_0(-t;m_e,m_f)\Big],
\end{aligned} \tag{296}$$

where the following *scaled* functions with a reduced list of arguments are introduced:

$$\begin{aligned}
d_0(s,t) &= st\,D_0\Big(-m_e^2,-m_e^2,-m_f^2,-m_f^2,-s,-t;0,m_e,0,m_f\Big),\\
c_0(s;0,m_e,0) &= s\,C_0\Big(-m_e^2,-m_e^2,-s;0,m_e,0\Big),\\
c_0(t;m_e,0,m_f) &= t\,C_0\Big(-m_e^2,-m_f^2,-t;m_e,0,m_f\Big).
\end{aligned} \tag{297}$$

The function d_0 may be split into an infrared divergent function c_0 plus a finite remainder:

$$d_0(s,t)=t\,\bar{J}_{\gamma\gamma}(-s,-t;m_e,m_f)-2\,c_0(t;m_e,0,m_f). \tag{298}$$

With the aid of this expression we prove that the infrared divergences in the boxes factorize into the lowest order

$$\frac{u^2}{s}\mathcal{D}^{+}_{\gamma\gamma}(s,t,u)+\frac{t^2}{s}\mathcal{D}^{-}_{\gamma\gamma}(s,t,u)\Big|_{\rm IR}=-2\,\frac{t^2+u^2}{s^2}\,c_0(t;m_e,0,m_f). \tag{299}$$

The box ingredients are simple in practical cases when external fermion masses are small: $m_e^2,\ m_f^2 \ll -t$ and $m_e^2 \ll s$,

$$\begin{aligned}
\bar{J}_{\gamma\gamma}(-s,-t;m_e,m_f) &= \frac{1}{t}\left[\ln\frac{m_e^2m_f^2}{t^2}\ln\frac{-t}{s}+\frac{1}{2}\ln^2\frac{m_e^2}{-t}+\frac{1}{2}\ln^2\frac{m_f^2}{-t}+\frac{1}{3}\pi^2\right],\\
C_0\Big(-m_e^2,-m_e^2,-s;0,m_e,0\Big) &= -\frac{1}{s}\left(\frac{1}{2}\ln^2\frac{m_e^2}{s}+\frac{1}{6}\pi^2+i\,\pi\ln\frac{m_e^2}{s}\right),\\
C_0\Big(-m_e^2,-m_f^2,-t;m_e,0,m_f\Big) &= \frac{1}{2t}\left[\ln\frac{m_e^2m_f^2}{t^2}\left(\frac{1}{\hat{\varepsilon}}+\ln\frac{-t}{\mu^2}\right)+\frac{1}{2}\ln^2\frac{m_e^2}{-t}+\frac{1}{2}\ln^2\frac{m_f^2}{-t}+\frac{1}{3}\pi^2\right],\\
B_0(-s;0,0)-B_0(-t;m_e,m_f) &= -\ln\frac{s}{-t}+i\,\pi.
\end{aligned} \tag{300}$$

For the total interference terms, the lowest order $\times$ box diagrams we have

$$\begin{aligned}
\mathcal{A}^{\rm box}_{\rm int} &= -\frac{e^6}{2\,\pi^2}\,Q_e^3Q_f^3\,f^{\rm box}_{\gamma\gamma}(s,t,u),\\
f^{\rm box}_{\gamma\gamma}(s,t,u) &= \frac{1}{s}\Big[\delta^{\rm box}_{\gamma\gamma}(s,t,u)-\delta^{\rm box}_{\gamma\gamma}(s,u,t)\Big],
\end{aligned} \tag{301}$$

where

$$\begin{aligned}
{\rm Re}\,f^{\rm box}_{\gamma\gamma}(s,t,u) &= 2\,\frac{t^2+u^2}{s^2}\left(\frac{1}{\hat{\varepsilon}}+\ln\frac{s}{\mu^2}\right)\ln\frac{t}{u}\\
&\quad+\frac{t}{s}\ln\left(-\frac{s}{u}\right)-\frac{u}{s}\ln\left(-\frac{s}{t}\right)+\frac{t-u}{s}\left[\ln^2\left(-\frac{s}{t}\right)+\ln^2\left(-\frac{s}{u}\right)\right].
\end{aligned} \tag{302}$$

Note, that contrary to the vertices, the box diagrams show no mass singularities ($\ln m$ is not present). This is an exhibition of a general property of absence of collinear divergences in interference-like contributions (boxes behave similarly to the initial–final bremsstrahlung interference).

4.2 Massless World

In this section we present an alternative derivation of QED corrections for a simple case of the decay of a neutral heavy particle into massless fermions, avoiding PV functions. We will present a formalism originally proposed in QCD for massless quarks and gluons. It could be applied to QED too. Within this formalism, all the calculations, including kinematics, must be consistently done in n-dimensions. For this reason we begin with a derivation of the two-body phase space in n-dimensions. We will then discuss the calculation of the vertex function for massless fermions, and finally present the three-body phase space in n-dimensions, a calculation of the bremsstrahlung contribution, and of the total correction.

4.2.1 Two-body phase space in n-dimensions

We use the phase space definition:

$$\Phi_2 = (2\pi)^n \mu^{4-n} \int \frac{d^{n-1}p}{(2\pi)^{n-1}\, 2p_0} \int \frac{d^{n-1}q}{(2\pi)^{n-1}\, 2q_0} \delta^{(n)}\left(Q-p-q\right), \tag{303}$$

which differs from a convention of the Particle Data Group (PDG) [6]. In Eq. (303) all the 4-momenta are assumed to be in n-dimensions, and the final state particles — on-shell, i.e. $p^2=0,\ q^2=0$. We then derive:

$$\begin{aligned}
\Phi_2 &= (2\pi)^{2-n} \mu^{4-n} \int d^n p\, \delta^+(p^2) \int d^n q\, \delta^+(q^2) \delta^{(n)}\left(Q-p-q\right) \\
&= (2\pi)^{2-n} \mu^{4-n} \int d^n p\, \delta^+(p^2)\, \delta^+((Q-p)^2)\,,
\end{aligned} \tag{304}$$

where $Q^2=-M^2$ and where $\delta^+(p^2)=\theta\left(p_0\right)\delta\left(p^2\right)$. Furthermore,

$$d^n p = d^{n-1}p\, dp_0\,, \qquad p^2 = \mid \vec{p}\mid^2 - p_0^2\,, \qquad \mid \vec{p}\mid^2 = \sum_{i=1}^{n-1} p_i \cdot p_i\,. \tag{305}$$

Now we go from $n-1$ rectangular coordinates to spherical coordinates involving $\mid\vec{p}\mid$ and $n-2$ angular variables:

$$\begin{aligned}
p_1 &= \mid\vec{p}\mid \cos\theta_1\,, \\
p_2 &= \mid\vec{p}\mid \sin\theta_1 \cos\theta_2\,, \\
p_3 &= \mid\vec{p}\mid \sin\theta_1 \sin\theta_2 \cos\theta_3\,, \\
\cdots &= \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\,, \\
p_{n-2} &= \mid\vec{p}\mid \sin\theta_1 \sin\theta_2 \sin\theta_3 \cdots \sin\theta_{n-3} \cos\theta_{n-2}\,, \\
p_{n-1} &= \mid\vec{p}\mid \sin\theta_1 \sin\theta_2 \sin\theta_3 \cdots \sin\theta_{n-3} \sin\theta_{n-2}\,,
\end{aligned} \tag{306}$$

with limits

$$0 \le \theta_i \le \pi \quad \text{for } i=1,2,\cdots,n-3\,; \qquad 0 \le \theta_{n-2} \le 2\pi\,. \tag{307}$$

Calculating the Jacobian of the transformation Eq. (306),

$$\begin{aligned}
d^{n-1}p &= \mid\vec{p}\mid^{n-2} d\mid\vec{p}\mid \sin^{n-3}\theta_1 d\theta_1 \sin^{n-4}\theta_2 d\theta_2 \cdots \\
&\quad \cdots \sin^2\theta_{n-4} d\theta_{n-4} \sin\theta_{n-3} d\theta_{n-3} d\theta_{n-2}\,,
\end{aligned} \tag{308}$$

and using

$$\int_0^\pi \sin^m\theta d\theta = \sqrt{\pi}\frac{\Gamma\left(\frac{1}{2}(m+1)\right)}{\Gamma\left(\frac{1}{2}(m+2)\right)}\,, \tag{309}$$

one has

$$\Phi_2 = (2\pi)^{2-n} \mu^{4-n} \int |\vec{p}|^{n-3} \frac{1}{2} d|\vec{p}|^2 \, dp_0 \pi^{n/2-2} \frac{\Gamma\left(\frac{1}{2}(n-3)\right)}{\Gamma\left(\frac{1}{2}(n-2)\right)} \frac{\Gamma\left(\frac{1}{2}(n-4)\right)}{\Gamma\left(\frac{1}{2}(n-3)\right)}$$
$$\cdots \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma(2)} \frac{\Gamma(1)}{\Gamma\left(\frac{3}{2}\right)} 2\pi\delta^+(|\vec{p}|^2 - p_0^2)\,\delta^+(-M^2 + 2Mp_0) \sin^{n-3}\theta_1 d\theta_1 \,. \tag{310}$$

After simplification of Eq. (310) we reach an important intermediate result:

$$\Phi_2 = (2\pi)^{4-n} \frac{\pi^{n/2-2}}{8\pi} \frac{|\vec{p}|}{M} \left(\frac{|\vec{p}|}{\mu}\right)^{n-4} \frac{1}{\Gamma\left(\frac{1}{2}n-1\right)} \int_0^\pi \sin^{n-4}\theta_1 d\cos\theta_1 \,. \tag{311}$$

For infrared regularization $n = 4 + \varepsilon'$, with the variable $\cos\theta_1 = y$, and taking into account that $|\vec{p}| = p_0 = M/2$, we continue:

$$\Phi_2 = (2\pi)^{-\varepsilon'} \frac{\pi^{\varepsilon'/2}}{16\pi} \left(\frac{M}{2\mu}\right)^{\varepsilon'} \frac{1}{\Gamma(1+\varepsilon'/2)} \int_{-1}^{1} \left(1-y^2\right)^{\varepsilon'/2} dy \,. \tag{312}$$

Furthermore, introducing one more variable, $z = \frac{1+y}{2}$, we integrate over it,

$$\begin{aligned} \int_{-1}^{1} \left(1-y^2\right)^{\varepsilon'/2} dy &= 2^{1+\varepsilon'} \int_0^1 \left[z(1-z)\right]^{\varepsilon'/2} dz \\ &= 2^{1+\varepsilon'} \mathrm{B}\left(1 + \frac{1}{2}\varepsilon', 1 + \frac{1}{2}\varepsilon'\right) = 2^{1+\varepsilon'} \frac{\Gamma(1+\varepsilon'/2)^2}{\Gamma(2+\varepsilon')} \,, \end{aligned} \tag{313}$$

and get a representation convenient for expansions in ε':

$$\Phi_2 = \frac{1}{8\pi} \left(\frac{M^2}{\mu^2}\right)^{\varepsilon'/2} \frac{(2\pi)^{-\varepsilon'} \pi^{\varepsilon'/2} \Gamma(1+\varepsilon'/2)}{(1+\varepsilon')\Gamma(1+\varepsilon')} \,. \tag{314}$$

For fun of it, using the so-called *duplication Legendre formula*:

$$\sqrt{\pi}\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma\left(z + \frac{1}{2}\right), \tag{315}$$

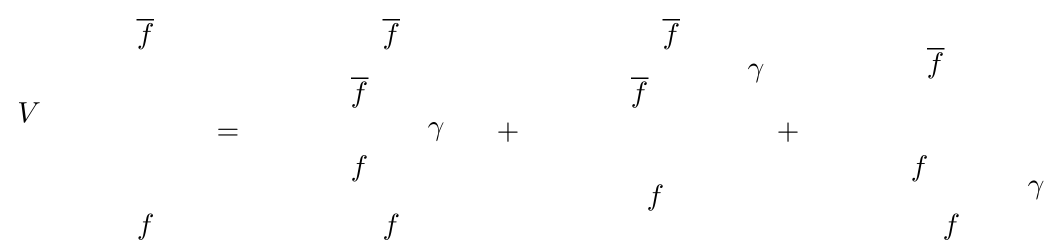
it can be reduced even further

$$\Phi_2 = \frac{1}{16\pi} \left(\frac{M^2}{\mu^2}\right)^{\varepsilon'/2} \frac{2^{-2\varepsilon'} \pi^{1/2-\varepsilon'/2}}{\Gamma(3/2+\varepsilon'/2)} \,, \tag{316}$$

to a representation containing only one Γ function. It is not convenient for expansions, however.

4.2.2 Calculation of Z decay width with QED radiative corrections

In order to calculate $\mathcal{O}(\alpha)$ QED radiative corrections for Z decay, one has to consider *virtual* and *real* corrections. The former originate from all possible insertions of a virtual photon line into the tree-level (Born) diagram and, as will be shown below, only the vertex diagram contributes. The latter are described by two usual bremsstrahlung diagrams. Therefore, in total we have to consider only three diagrams:



4.2.3 QED vertex

I recommend that the same line of calculations for the vertex, as presented in Section 4.1.3 be followed. For on-shell massless fermions and on-shell vector boson, we have: $p_1^2 = p_2^2 = 0$, $\bar{v}(p_2)\,\not{p}_2 = 0$, $\not{p}_1\,u(p_1) = 0$, $Q^2 = 2p_1\cdot p_2 = -M_V^2$ and the expression, with which one has to start, becomes

$$\begin{aligned}\Lambda_\mu &= i\,(eQ_e)^3\,\mu^{4-n}\int d^n q\frac{1}{q^2\,(q+p_1)^2\,(q-p_2)^2}\,N_\mu\,,\\ N_\mu &= -4p_1\cdot p_2\gamma_\mu + 2\,(\not{p}_1\gamma_\alpha\gamma_\mu - \gamma_\mu\gamma_\alpha\not{p}_2)\,q_\alpha + (2-n)\,\gamma_\alpha\gamma_\mu\gamma_\beta q_\alpha q_\beta\,. \qquad (317)\end{aligned}$$

In the massless case: $k_x = xp_2 - (1-x)\,p_1\,,\quad \chi\,(Q^2, x) = Q^2 x\,(1-x)$, and the decomposition into *scalar, vector* and *tensor* simplifies to:

$$\Lambda_\mu = i\,(eQ_e)^3\left[-Q^2 S\gamma_\mu + 2\,(\not{p}_1\gamma_\alpha\gamma_\mu - \gamma_\mu\gamma_\alpha\not{p}_2)\,V_\alpha + \gamma_\alpha\gamma_\mu\gamma_\beta T_{\alpha\beta}\right]. \qquad (318)$$

For *the scalar* we now continue the integration in n dimensions, using, as before, *the infrared regulator* ε':

$$\begin{aligned}-Q^2 S &= -2\frac{\pi^{\varepsilon'/2}}{\varepsilon'}\Gamma\left(1-\frac{\varepsilon'}{2}\right)\left(\frac{Q^2}{\mu^2}\right)^{\varepsilon'/2}\int_0^1 dx x^{\varepsilon'/2-1}\,(1-x)^{\varepsilon'/2-1}\\ &= -2\frac{\pi^{\varepsilon'/2}}{\varepsilon'}\Gamma\left(1-\frac{\varepsilon'}{2}\right)\left(\frac{Q^2}{\mu^2}\right)^{\varepsilon'/2}\mathrm{B}\left(\frac{\varepsilon'}{2},\frac{\varepsilon'}{2}\right)\\ &= -2\frac{\pi^{\varepsilon'/2}}{\varepsilon'}\Gamma\left(1-\frac{\varepsilon'}{2}\right)\left(\frac{Q^2}{\mu^2}\right)^{\varepsilon'/2}\frac{\Gamma^2\,(\varepsilon'/2)}{\Gamma\,(\varepsilon')}\,. \qquad (319)\end{aligned}$$

Similarly for *the vector* we also use the infrared regulator and derive:

$$2\,(\not{p}_1\gamma_\alpha\gamma_\mu - \gamma_\mu\gamma_\alpha\not{p}_2)\,V_\alpha = \gamma_\mu\,2\frac{\pi^{\varepsilon'/2}}{1+\varepsilon'}\Gamma\left(1-\frac{\varepsilon'}{2}\right)\left(\frac{Q^2}{\mu^2}\right)^{\varepsilon'/2}\frac{\Gamma^2\,(\varepsilon'/2)}{\Gamma\,(\varepsilon')}\,. \qquad (320)$$

Note that contrary to the massive case, the massless vector is **not** finite and we may not set $\varepsilon' = 0$. In the massive case we had *mass singularities* which exhibited themselves as $\ln m$, and now we have, instead, *collinear divergences (CD)*, which develop poles in the infrared regulator $1/\varepsilon'$.

For *the tensor* we may also use the infrared regulator, in spite of the fact that it has an UV divergence. It also has CD, and we may use the same infrared regulator for both, remembering the existence of *an identification* of two types of divergences, Eq. (257). For the tensor, we have:

$$\begin{aligned}\gamma_\alpha\gamma_\mu\gamma_\beta T_{\alpha\beta} &= \gamma_\mu\,(1+\varepsilon')\,\pi^{\varepsilon'/2}\Gamma\left(-\frac{\varepsilon'}{2}\right)\left(\frac{Q^2}{\mu^2}\right)^{\varepsilon'/2}\int_0^1 dx x^{\varepsilon'/2}\,(1-x)^{\varepsilon'/2}\\ &= \gamma_\mu\,(1+\varepsilon')\,\pi^{\varepsilon'/2}\Gamma\left(-\frac{\varepsilon'}{2}\right)\left(\frac{Q^2}{\mu^2}\right)^{\varepsilon'/2}\frac{\Gamma^2\,(1+\varepsilon'/2)}{\Gamma\,(2+\varepsilon')}\,. \qquad (321)\end{aligned}$$

Because of the presence of *double poles* all the expansions should be performed up to ε'^2. They are achieved by means of equations:

$$\begin{aligned}\Gamma\,(1+x) &= 1-\gamma\,x+\frac{1}{2}\left[\zeta(2)+\gamma^2\right]x^2+\mathcal{O}\left(x^3\right), \qquad \zeta(2)=\frac{\pi^2}{6}\,,\\ a^x &= 1+(\ln a)\,x+\frac{1}{2}\,(\ln a)^2\,x^2+\mathcal{O}\left(x^3\right). \qquad (322)\end{aligned}$$

Hewe we introduce some new notations and recall some old ones:

$$\frac{1}{\hat{\varepsilon}} = \frac{2}{\varepsilon'} + \gamma + \ln\pi\,, \qquad \overline{\gamma} = \gamma + \ln\pi\,, \qquad \zeta(2) = \frac{\pi^2}{6}\,,$$
$$z_V = \ln\frac{-M_V^2 - i\,\epsilon}{\mu^2} = \ln\frac{M_V^2}{\mu^2} - i\,\pi\,. \tag{323}$$

In the massless case, only F_1 remains:

$$\Lambda_\mu = (2\pi)^4\, i\, ieQ_e \frac{e^2}{16\pi^2}\gamma_\mu F_1\,, \tag{324}$$

with the ingredients

$$\begin{aligned}
\text{Scalar} &= -\frac{2}{\hat{\varepsilon}^2} + \frac{2}{\hat{\varepsilon}}\left(\overline{\gamma} - z_V\right) - \overline{\gamma}^2 - z_V^2 + \zeta(2)\,,\\
\text{Vector} &= \frac{4}{\hat{\varepsilon}} - 8 + 4z_V\,,\\
\text{Tensor} &= -\frac{1}{\hat{\varepsilon}} - z_V\,.
\end{aligned} \tag{325}$$

The complete F_1 reads:

$$F_1 = -2\frac{1}{\hat{\varepsilon}^2} + \frac{2}{\hat{\varepsilon}}\left(\overline{\gamma} - z_V + \frac{3}{2}\right) - \overline{\gamma}^2 - z_V^2 + \zeta(2) + 3z_V - 8\,. \tag{326}$$

To summarize our study of the massless QED vertex we note:

1. F_1 at zero momentum is zero, this is because $\left(\frac{Q^2=0}{\mu^2}\right)^{\varepsilon'/2} = 0$, for $\varepsilon' > 0$; a property of infrared regularization;
2. In the tensor integral we faced *a migration* of the ultraviolet pole into an infrared one;
3. The physical origin of double poles is the product: infrared$\times$collinear divergences.

4.2.4 *Fermionic self-energy in the massless world*

The massive expression for the fermionic self-energy, Eq. (267), in the massless world reduces to:

$$\begin{aligned}
\Sigma\left(\not p\right) &= -e^2Q_e^2\int d^nq \frac{\gamma_\mu i \not q\, \gamma_\mu}{\left(q^2 - i\epsilon\right)\left[\left(q+p\right)^2 - i\epsilon\right]}\\
&= i\pi^2\left(-e^2Q_e^2\right)\left(2-n\right)\pi^{n/2-2}\Gamma\left(\frac{n}{2}-2\right)\int_0^1 dx x\left[\frac{p^2x\left(1-x\right)}{\mu^2}\right]^{\varepsilon'/2} i\not p\\
&= i\pi^2\left(e^2Q_e^2\right)\pi^{\varepsilon'/2}\left(2+\varepsilon'\right)\Gamma\left(-\frac{\varepsilon'}{2}\right)\mathrm{B}\left(2+\frac{\varepsilon'}{2}, 1+\frac{\varepsilon'}{2}\right)\left(\frac{p^2}{\mu^2}\right)^{\varepsilon'/2} i\not p\,.
\end{aligned} \tag{327}$$

We see that the fermionic self-energy in the massless world vanishes on the fermion mass-shell, i.e. at $p^2 = 0$ (for the same reason as $F_1(0) = 0$, see item 1, above).

4.2.5 *Virtual correction in n-dimensions*

Virtual corrections contribute via their interference with the Born amplitude. Recalling Eq. (24),

$$\mathrm{Virtual} = \frac{1}{n-1}\frac{1}{2\,M_V}\overline{\sum_{\text{spins}}} 2\,\mathrm{Re}\left(A^{\mathrm{Born}}A^{1L}\right)\,. \tag{328}$$

The factor $1/(n-1)$ follows from averaging over the V polarizations.

For a correct treatment of the factors $2\,\pi$, we must not forget to replace

$$d^4q \to \frac{(2\pi)^4}{(2\pi)^n} d^nq = (2\pi)^{-\varepsilon'}\, d^nq\,, \tag{329}$$

remembering the integration over an internal momentum q.

Furthermore, it is easy to verify that

$$A^{1L} = \frac{e^2}{16\pi^2} A^{\rm Born} F_1\,, \tag{330}$$

therefore,

$$\text{Virtual} = \left|A^{\rm Born}\right|^2 \frac{\alpha}{\pi}\delta^V. \tag{331}$$

Finally, one has to properly account for n dimensions in the square of the Born amplitude:

$$\overline{\sum_{\rm spins}} \left|A^{\rm Born}\right|^2 \propto \left(1+\frac{\varepsilon'}{2}\right), \tag{332}$$

(this is achieved by means of the trace calculation in n dimensions).

After expanding all the ingredients, we obtain the final expression for the virtual correction

$$\delta^V = -\frac{1}{\hat{\varepsilon}^2} - \frac{2}{\hat{\varepsilon}}\left(L_V - \frac{19}{12}\right) - 2L_V^2 - 2\overline{\gamma}L_V + 5\zeta(2) - \overline{\gamma}^2 + \frac{19}{3}L_V + \frac{19}{6}\overline{\gamma} - \frac{173}{18}\,, \tag{333}$$

with

$$L_V = \ln\frac{M_V^2}{(2\pi\mu)^2}\ . \tag{334}$$

4.2.6 Three-body phase space

For the study of bremsstrahlung in n dimensions, one has to consider the three-body phase space in n-dimensions. We define,

$$\begin{aligned} d\Phi_3 &= (2\pi)^n\,\mu^{8-2n}\frac{d^{n-1}p}{(2\pi)^{n-1}\,2p_0}\,\frac{d^{n-1}q}{(2\pi)^{n-1}\,2q_0}\,\frac{d^{n-1}k}{(2\pi)^{n-1}\,2k_0}\,\delta^{(n)}\left(Q-p-q-k\right) \\ &= M_V^2\,(2\pi)^{3-2n}\,\mu^{8-2n}d^np\,\delta^+(p^2)d^nq\,\delta^+(q^2)d^nk\,\delta^+(k^2) \\ &\quad \times\delta^{(n)}\left(Q-p-q-k\right)d^nP\delta^{(n)}\left(P-p-k\right)d\left(-P^2\right)\delta^+\left(-P^2+(p+k)^2\right). \end{aligned} \tag{335}$$

This parametrization of the phase space corresponds to the *kinematical cascade* (shown in Fig. 4) of the two two-body decays where one of the particles of the first decay is a compound with the invariant mass $-P^2$.

After reordering the terms in Eq. (335), we can use the intermediate result for the two-body phase space directly, Eq. (311):

$$\begin{aligned} d\Phi_3 &= \frac{1}{2\pi}d\left(-P^2\right) \\ &\quad \times(2\pi)^{2-n}\,\mu^{4-n}d^nq\,\delta^+(q^2)d^nP\,\delta^+\left(-P^2+(Q-q)^2\right)\delta^{(n)}\left(Q-P-q\right) \\ &\quad \times(2\pi)^{2-n}\,\mu^{4-n}d^np\,\delta^+(p^2)d^nk\,\delta^+(k^2)\delta^{(n)}\left(P-p-k\right) \\ &= \frac{1}{2\pi}d\left(-P^2\right) \end{aligned}$$

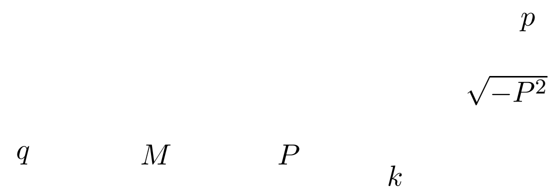


Fig. 4: Kinematical cascade for the $1 \to 3$ decay.

$$
\begin{aligned}
&\times (2\pi)^{-\varepsilon'} \frac{\pi^{\varepsilon'/2}}{8\pi} \left(\frac{M^2+P^2}{2M^2}\right) \left(\frac{M^2+P^2}{2M\mu}\right)^{\varepsilon'} \frac{1}{\Gamma\left(1+\varepsilon'/2\right)} \sin^{\varepsilon'}\theta d\cos\theta \\
&\times (2\pi)^{-\varepsilon'} \frac{\pi^{\varepsilon'/2}}{16\pi} \left(\frac{\sqrt{-P^2}}{2\mu}\right)^{\varepsilon'} \frac{1}{\Gamma\left(1+\varepsilon'/2\right)} \sin^{\varepsilon'}\theta_1 d\cos\theta_1 \, . \qquad (336)
\end{aligned}
$$

The two remaining angular integrations in Eq. (336) should be treated differently. The first one may be taken, since the matrix element squared is independent of the angle of rotation of the whole picture of the cascade in Fig. 4. Therefore,

$$
\int_0^\pi \sin^{\varepsilon'}\theta d\cos\theta = \int_{-1}^1 \left(1-y^2\right)^{\varepsilon'/2} dy = 2^{1+\varepsilon'} \frac{\Gamma\left(1+\varepsilon'/2\right)^2}{\Gamma\left(2+\varepsilon'\right)} \; . \qquad (337)
$$

For the second one we substitute $z = \dfrac{1+y}{2}$

$$
\int_0^\pi \sin\theta_1^{\varepsilon'} d\cos\theta_1 = \int_{-1}^1 \left(1-y^2\right)^{\varepsilon'/2} dy = 2^{1+\varepsilon'}[z\left(1-z\right)]^{\varepsilon'/2} dz \, , \qquad (338)
$$

and keep the integral untaken, since the matrix element squared may depend on it. Substituting two angular integrals, we have

$$
d\Phi_3 = \frac{1}{2^7\pi^3} \frac{(2\pi)^{-2\varepsilon'}\pi^{\varepsilon'}}{\Gamma\left(2+\varepsilon'\right)} d\left(-P^2\right) \frac{M^2+P^2}{M^2} \left(\frac{M^2+P^2}{M\mu}\right)^{\varepsilon'} \left(\frac{\sqrt{-P^2}}{\mu}\right)^{\varepsilon'} [z\left(1-z\right)]^{\varepsilon'/2} dz \, . \qquad (339)
$$

Introducing $-P^2 = xM^2$, we finally get

$$
\Phi_3 \;=\; \frac{M^2}{2^7\pi^3} \left(\frac{M^2}{\mu^2}\right)^{\varepsilon'} \frac{(2\pi)^{-2\varepsilon'}\pi^{\varepsilon'}}{\Gamma\left(2+\varepsilon'\right)} \int_0^1 dx x^{\varepsilon'/2} \left(1-x\right)^{1+\varepsilon'} \int_0^1 dz [z\left(1-z\right)]^{\varepsilon'/2} \, . \qquad (340)
$$

4.2.7 The radiative decay $V \to f\overline{f}\gamma$

For the radiative process, we define 4-momenta, $V(Q) \to f(p) + \overline{f}(q) + \gamma(k)$. Its kinematics may be specified in terms of two invariants, for which it is convenient to choose two dimensionless invariant masses, x and y:

$$
xM_V^2 = -(p+k)^2, \qquad (y+1)M_V^2 = -\left(Q+k\right)^2 . \qquad (341)
$$

All the scalar products may be expressed in terms of x and y:

$$
\begin{aligned}
-2\,p\cdot k &= xM_V^2, & -2\,q\cdot k &= (y-x)M_V^2, \\
-2\,Q\cdot k &= yM_V^2, & -2\,p\cdot q &= (1-y)M_V^2, \\
-2\,Q\cdot q &= (1-x)M_V^2, & -2\,Q\cdot p &= (1-y+x)M_V^2.
\end{aligned}
$$

The bremsstrahlung amplitude has the standard form,

$$\mathcal{M}^{\rm brem} = -i\, e^2\, \bar{u}\,(p) \left[\not{e} \frac{(\not{q}+\not{k})}{(q+k)^2} \not{\epsilon} - \not{\epsilon} \frac{(\not{p}+\not{k})}{(p+k)^2} \not{e} \right] v\,(q)\,, \tag{342}$$

where $e(Q)$ and $\epsilon(k)$ are the V and photon polarization vectors.

The amplitude squared in terms of the invariants $x,\ z$ reads:

$$\overline{\sum_{\rm spins}} \left| \mathcal{M}^{\rm brem} \right|^2 = e^4\, \varepsilon^* \left\{ 2 \left(\frac{1}{zx} - \frac{1}{x} - 1 \right) + \frac{\varepsilon^*}{8} \left[\frac{1}{z}\, \frac{x}{1-x} + 2 + z \left(\frac{1}{x} - 1 \right) \right] \right\}. \tag{343}$$

Here, $\varepsilon^* = 8 + 4\,\varepsilon'$ and $y = (1-x)\,z + x$.

One should also include an extra factor,

$$\frac{1}{n-1}\, \frac{1}{2\, M_V}\,, \tag{344}$$

from averaging over the V boson spin.

The complete bremsstrahlung contribution is the product of the amplitude squared $\times$ the phase-space factor integrated over the $x,\ z$. All the bremsstrahlung integrals can easily be performed in n dimensions and at the very end of calculations one expands around $\varepsilon' = 0$:

$$\int_0^1 dx \int_0^1 dz\, x^{\varepsilon'/2}\, (1-x)^{1+\varepsilon'}\, [z\,(1-z)]^{\varepsilon'/2}\, \mathcal{A}^{\rm brem} = e^4 \left[\left(\frac{8}{\varepsilon'} \right)^2 - \frac{16}{\varepsilon'} + 52 - 48\zeta(2) \right]. \tag{345}$$

If one include phase space and all the relevant factors, one finally gets:

$$\delta^R = \frac{1}{\hat{\varepsilon}^2} + \frac{2}{\hat{\varepsilon}} \left(L_V - \frac{19}{12} \right) + 2L_V^2 + 2\overline{\gamma} L_V - 5\zeta(2) + \overline{\gamma}^2 - \frac{19}{3} L_V - \frac{19}{6} \overline{\gamma} + \frac{373}{36}\ . \tag{346}$$

4.2.8 Total QED correction

The complete expression is the sum of the virtual and real contributions. We define:

$$\Gamma^{\rm QED} = \Gamma^{\rm Born} \left(1 + \frac{\alpha}{\pi} \delta^{\rm QED} \right). \tag{347}$$

Summing up Eqs. (333) and (346), we obtain the total QED correction:

$$\delta^{\rm QED} \quad = \quad \delta^R + \delta^V = \frac{373}{36} - \frac{173}{18} = \frac{3}{4}\ . \tag{348}$$

To summarize our exercises of massless calculations, we conclude:

1. All the double and single poles (infrared and collinear) and all the unphysical terms, like logarithms of the t'Hooft scale and the Euler constant, cancel in the combined expression;
2. The cancellation of the infrared divergences is the consequence of the Blokh–Nordsiek theorem, whilst the cancellation of the collinear divergences — of Kinoshita–Lee–Nauenberg (KLN) theorem for the inclusive set-up (i.e. integrated over the full photonic phase space);
3. No renormalization was needed in this example; we simply computed all the diagrams, summed them up, and got the finite answer. As we will see below, when we study renormalization, this is a property of the massless theory only, where the fermionic self-energy diagrams vanish on-mass-shell. The relevant counter-terms, involving the derivative of the self-energy and the $F_1(0)$ also vanish, and renormalization is effectively not needed.

4.3 Summary of the four lectures

As usual, we summarize what we have learned so far:

- Standard Model, its fields and Lagrangian;
  Feynman rules $\rightarrow$ *building* of diagrams;
- Regularization, N-point functions, PV functions: A, B, C, D functions $\rightarrow$ *calculation* of diagrams.
- QED one-loop diagrams, *building blocks:*
  - photonic and fermionic self energies;
  - vertices and boxes;
- First feeling of renormalization – subtraction at *zero* momentum;
- Example of calculation of RC's for the decay $V \rightarrow f\overline{f}$ in massless QED:
  - well-known correction $\dfrac{3\alpha}{4\pi}$ was recovered;
  - first feeling of divergence cancellation;
  - Why renormalization is needed? Not clear yet...

5. ONE-LOOP DIAGRAMS AND THEIR PROPERTIES

In this lecture we continue our study of one-loop approximation in the SM. We present an overview of the one-loop diagrams and of some simple physics related with them.

Remember that in QED we had only one bosonic self-energy diagram, one fermionic self-energy diagram, one QED vertex and a couple of boxes. In the SM model in the arbitrary gauge the number of diagrams grows drastically. In next two figures we give only two examples of bosonic self-energies; the Z self-energy described by 14 diagrams, see Fig. 5, and W self-energy — by 17 diagrams, Fig. 6. A full collection of all self-energies and transitions occupies many pages, see Chapter 5 of Ref. [1]. The typical number of vertices and boxes in the SM is also of the order of tens instead of 1-2 in case of QED.

5.1 Bosonic self-energy diagrams

Any vector boson self-energy diagram and, therefore the sum too, look like a tensor

$$S_{VV}(p^2)\delta_{\mu\nu} + T_{VV}(p^2)p_\mu p_\mu \, . \tag{349}$$

At the one-loop level the second term does not contribute (see Section 6.5 of Ref. [1]). We will denote by $\Sigma^{\xi}_{AB}(p^2)$ the $\delta_{\mu\nu}$ part of the total V boson self-energy (or transition) related to $S_{AB}(p^2)$:

$$\begin{aligned} S_{\gamma\gamma}(p^2) &= \frac{g^2 s^2_\theta}{16\pi^2}\Sigma^{\xi}_{\gamma\gamma}(p^2)\,, & S_{\gamma Z}(p^2) &= \frac{g^2 s_\theta}{16\pi^2 c_\theta}\Sigma^{\xi}_{\gamma Z}(p^2)\,, \\ S_{ZZ}(p^2) &= \frac{g^2}{16\pi^2 c^2_\theta}\Sigma^{\xi}_{ZZ}(p^2)\,, & S_{WW}(p^2) &= \frac{g^2}{16\pi^2}\Sigma^{\xi}_{WW}(p^2)\,. \end{aligned} \tag{350}$$

Furthermore, $\Sigma^{\xi}_{\gamma\gamma}(0) = 0$ as dictated by QED $U(1)$-invariance. Therefore, one may introduce $\Pi^{\xi}_{\gamma\gamma}(p^2)$ defined by:

$$\Sigma^{\xi}_{\gamma\gamma}(p^2) = p^2\Pi^{\xi}_{\gamma\gamma}(p^2)\,. \tag{351}$$

Every $\Sigma^{\xi}_{AB}(p^2)$ could be represented as a sum of two terms,

$$\Sigma^{\xi}_{AB}(p^2) = \Sigma^{(1)}_{AB}(p^2) + \Sigma^{\rm add}_{AB}(p^2)\,, \tag{352}$$

the first of which corresponds to the $\xi = 1$ gauge and the second contains all ξ dependence and vanishes for $\xi = 1$.

5.1.1 *Bosonic component of bosonic self-energies*

By bosonic component we will understand the sum of all but the first [marked by (1) in both figures] diagrams of Figs. 5–6. It is a gauge-dependent quantity and we will look at Σ_{WW}, as a typical example, in two types of gauges.

Z, A μ Z, A ν $=$ (1) u, d $\overline{u}, \overline{d}$ $+$ (2) W^+ W^- $+$ (3) Z H

$+$ (4) W^+ ϕ^- $+$ (5) ϕ^+ W^-

$+$ (6) ϕ^0 H $+$ (7) ϕ^+ ϕ^-

$+$ (8) X^- X^- $+$ (9) X^+ X^+

(10) W (11) H

$+$ $+$

(12) ϕ^+ (13) ϕ^0

$+$ $+$

(14) β'_t (Z)

$+$

Fig. 5: (Z, A)-boson self-energy; $Z - A$ transition.

W^+ W^-

p μ ν $=$ (1) u $\overline{d}$ $+$ (2) W^+ Z $+$ (3) W^+ A

$+$ (4) W^+ H $+$ (5) ϕ^+ Z $+$ (6) ϕ^+ A

$+$ (7) ϕ^+ H $+$ (8) ϕ^+ ϕ^0

$+$ (9) $Y_{Z,A}$ X^- $+$ (10) X^+ $Y_{Z,A}$

$+$ (11) W $+$ (12) Z $+$ (13) A

$+$ (14) H $+$ (15) ϕ^+ $+$ (16) ϕ^0

$+$ (17) β_t'

Fig. 6: W-boson self-energy.

R_ξ gauge.
Its $\xi = 1$ part is rather short

$$\Sigma^{(1)}_{WW} = \frac{M^2}{12}\left\{-\left[\frac{s_\theta^4}{c_\theta^4}\left(1+8c_\theta^2\right)\frac{M^2}{p^2} - \frac{10}{c_\theta^2} + 54 + 16c_\theta^2 + \left(1-40c_\theta^2\right)\frac{p^2}{M^2}\right] B_0\left(p^2; M, M_0\right)\right.$$
$$-\left[\left(1-\frac{M_H^2}{M^2}\right)^2\frac{M^2}{p^2} - 10 + 2\frac{M_H^2}{M^2} + \frac{p^2}{M^2}\right] B_0\left(p^2; M_H, M\right)$$
$$-8s_\theta^2\left(\frac{M^2}{p^2} + 2 - 5\frac{p^2}{M^2}\right) B_0\left(p^2; 0, M\right)$$

$$+\left[\left(\frac{1}{c_\theta^2} - 2 + \frac{M_H^2}{M^2}\right)\frac{M^2}{p^2} - 14 + 36\frac{M^2}{M_H^2}\right]\frac{A_0\left(M\right)}{M^2}$$
$$-\left[\frac{s_\theta^2}{c_\theta^2}\left(1+8c_\theta^2\right)\frac{M^2}{p^2} - 1 - 18\frac{M_0^2}{M_H^2} + 16c_\theta^2\right]\frac{A_0\left(M_0\right)}{M^2} + \left(\frac{M^2 - M_H^2}{p^2} + 7\right)\frac{A_0\left(M_H\right)}{M^2}$$
$$\left.+12\left(\frac{1}{c_\theta^4} + 2\right)\frac{M^2}{M_H^2} - 2\left(\frac{1}{c_\theta^2} + 18 + \frac{M_H^2}{M^2} - \frac{2}{3}\frac{p^2}{M^2}\right)\right\}, \tag{353}$$

whilst *the additional* part is extremely cumbersome and hardly fits on one page.

$$\Sigma^{\rm add}_{WW} = \frac{M^2+p^2}{12}\left\{\left[s_\theta^4\frac{M^2}{p^2} - 1 + 4c_\theta^2 + c_\theta^4 - c_\theta^2\left(2+c_\theta^2\right)\frac{p^2}{M^2} - c_\theta^4\frac{p^4}{M^4}\right]\right.$$
$$\times\left[B_0\left(p^2; \xi M, \xi_Z M_0\right) - B_0\left(p^2; M_0, \xi M\right)\right]$$
$$+2\left(\frac{s_\theta^4}{c_\theta^2}\frac{M^2}{p^2} - 10 + 8c_\theta^2 - 5c_\theta^2\frac{p^2}{M^2}\right)\left[B_0\left(p^2; M_0, \xi M\right) - B_0\left(p^2; M_0, M\right)\right]$$
$$+\left[s_\theta^4\frac{M^2}{p^2} + 1 - 9c_\theta^4 + c_\theta^2\left(2-9c_\theta^2\right)\frac{p^2}{M^2} + c_\theta^4\frac{p^4}{M^4}\right]$$
$$\times\left[B_0\left(p^2; M, \xi_Z M_0\right) - B_0\left(p^2; M_0, M\right)\right]$$
$$+2s_\theta^2\left(\frac{M^2}{p^2} + 8 - 5\frac{p^2}{M^2}\right)\left[B_0\left(p^2; 0, \xi M\right) - B_0\left(p^2; 0, M\right)\right]$$
$$+\left[\left(\xi_Z^2 - 1\right)\left(\xi_Z^2 + 1 + 2c_\theta^2\frac{p^2}{M^2}\right) + c_\theta^4\left(\xi^2 - 1\right)\left(\xi^2 + 1 + 2\frac{p^2}{M^2}\right)\right.$$
$$\left.-2c_\theta^2\left(\xi_Z^2\xi^2 - 1\right)\right]\left(\frac{M^2}{p^2} - 1\right) B_0\left(p^2; \xi M, \xi_Z M_0\right)$$
$$+\left(\xi_Z^2 - 1\right)\left[\left(\xi_Z^2 + 1 - 2c_\theta^2\right)\frac{M^2}{p^2} + 2c_\theta^2\right]\left(1 + \frac{p^2}{M^2}\right) B_0\left(p^2; M, \xi_Z M_0\right)$$
$$+\left(\xi^2 - 1\right)\left[c_\theta^2\xi^2\left(\left(2-c_\theta^2\right)\frac{M^2}{p^2} + c_\theta^2\right) - \left(2-c_\theta^2\right)^2\frac{M^2}{p^2}\right.$$
$$\left.+c_\theta^2\left(2-c_\theta^2\right) + 2c_\theta^4\frac{p^2}{M^2}\right] B_0\left(p^2; M_0, \xi M\right)$$
$$+2s_\theta^2\left(\xi^2 - 1\right)\left[\left(\xi^2 + 1\right)\frac{M^2}{p^2} + 2\right] B_0\left(p^2; 0, \xi M\right)$$

$$
\begin{aligned}
&+3s_\theta^2\left(\xi_A^2-1\right)\Bigg[\left(1-\xi^2\frac{M^2}{p^2}\right)\left(1-\frac{p^2}{M^2}\right)B_0\left(p^2;0,\xi M\right)\\
&\qquad\qquad\qquad -\left(\frac{M^2}{p^2}+4-\frac{p^2}{M^2}\right)B_0\left(p^2;0,M\right)\Bigg]\\
&+2c_\theta^2\left(s_\theta^2\frac{M^2}{p^2}+5c_\theta^2\right)\frac{A_0\left(\xi_Z M_0\right)-A_0\left(M_0\right)}{M^2}+10\frac{A_0\left(\xi M\right)-A_0\left(M\right)}{M^2}\\
&+\left[2\left(\xi^2-1\right)\frac{M^2}{p^2}-c_\theta^2\left(\xi_Z^2-1\right)\left(\frac{M^2}{p^2}-1\right)\right]\frac{A_0\left(\xi M\right)}{M^2}\\
&-c_\theta^2\left[c_\theta^2\left(\xi^2-1\right)\left(\frac{M^2}{p^2}-1\right)-2\left(\xi_Z^2-1\right)\frac{M^2}{p^2}\right]\frac{A_0\left(\xi_Z M_0\right)}{M^2}\\
&-c_\theta^2\left(\xi^2-1\right)\left[\left(2-c_\theta^2\right)\frac{M^2}{p^2}+c_\theta^2\right]\frac{A_0\left(M_0\right)}{M^2}-c_\theta^2\left(\xi_Z^2-1\right)\left(\frac{M^2}{p^2}+1\right)\frac{A_0\left(M\right)}{M^2}\\
&-3s_\theta^2\frac{\xi_A^2-1}{p^2}\left[A_0\left(\xi M\right)+A_0\left(M\right)-\frac{p^2}{M^2}\Big(A_0\left(\xi M\right)-A_0\left(M\right)\Big)\right]\\
&+4c_\theta^2\left(\xi_Z^2-1\right)+4\left(\xi^2-1\right)+24s_\theta^2\left(\xi_A^2-1\right)\Big\}\\
&+2s_\theta^2\left(\xi_A^2-1\right)\left[M^2B_0\left(p^2;0,M\right)+A_0\left(M\right)-M^2\right]. \qquad (354)
\end{aligned}
$$

As seen, the additional part vanishes not only at $\xi = 1$, but also at $p^2 = -M^2$, i.e. at the W mass shell. This is a property of the R_ξ gauge and is due to a proper treatment of the tadpoles (see discussion in Section 2.6). This ξ-dependent part is bound to cancel with the other ξ-dependent parts coming from the vertices and boxes for each physical amplitude it contributes to. This example teaches us that working in the R_ξ gauge, we mostly produce unphysical terms. This is the price being paid for an explicit control of gauge invariance.

Actually, there is another approach to the calculation of the one-loop amplutudes, which is organized in such a way that all ξ-dependences cancel *before* calculation of integrals over Feynman parameters.

U gauge. The number of diagrams contributing to the total self-energies, as well as the number of total self-energies themselves in the U gauge, is very limited. Below the whole list is presented, where the following short-hand notations are used:

$$
w=\frac{p^2}{M_W^2}\,,\qquad z=\frac{p^2}{M_Z^2}\,,\qquad h=\frac{p^2}{M_H^2}\,,\qquad w_h=\frac{M_H^2}{M_W^2}\,,\qquad h_W^{(i)}=\left(1-w_h\right)^i; \qquad (355)
$$

$$
\begin{aligned}
\frac{\Sigma_{WW}^U(p^2)}{M_W^2} = &\Bigg[-\left(\frac{1}{12c_W^4}+\frac{2}{3}\frac{1}{c_W^2}-\frac{3}{2}+\frac{2}{3}c_W^2+\frac{1}{12}c_W^4\right)\frac{1}{w}+\frac{2}{3}\left(\frac{1}{c_W^2}-4-4c_W^2+c_W^4\right)\\
&+\left(\frac{3}{2}+\frac{8}{3}c_W^2+\frac{3}{2}c_W^4\right)w+\frac{2}{3}c_W^2\left(1+c_W^2\right)w^2-\frac{1}{12}c_W^4w^3\Bigg]B_0\left(p^2;M_Z,M_W\right)\\
&-\frac{s_W^2}{6}\left(\frac{5}{w}+17-17w-5w^2\right)B_0\left(p^2;0,M_W\right)\\
&-\frac{1}{12}\left(\frac{w_h^{(2)}}{w}-10+2\,w_h+w\right)B_0\left(p^2;M_H,M_W\right)\\
&+\Bigg[\frac{1}{12}\left(\frac{1}{c_W^2}-2+c_W^2-c_W^4+w_h\right)\frac{1}{w}-2+\frac{1}{6}c_W^2-\frac{1}{12}c_W^4\\
&+\frac{1}{12}\left(-10+c_W^2+c_W^4\right)w+\frac{1}{12}c_W^4w^2\Bigg]\frac{A_0\left(M_W\right)}{M_W^2}
\end{aligned}
$$

$$
\begin{aligned}
&+\left[-\frac{1}{12}\left(\frac{1}{c_W^2}+9-9c_W^2-c_W^4\right)\frac{1}{w}-\frac{1}{12}-\frac{7}{6}c_W^2-\frac{3}{4}c_W^4\right.\\
&\left.+\frac{1}{12}\left(c_W^2-9c_W^4\right)w+\frac{1}{12}c_W^4w^2\right]\frac{A_0\left(M_Z\right)}{M_W^2}+\frac{1}{12}\left(\frac{1}{w}-\frac{1}{h}-2\right)\frac{A_0\left(M_H\right)}{M_W^2}\\
&-\frac{1}{6}\left(\frac{1}{c_W^2}+22+c_W^2+c_W^4+w_h\right)-\frac{1}{9}\left(2+3c_W^2+\frac{7}{2}c_W^4\right)w\\
&-\frac{1}{9}\left(1+\frac{3}{2}c_W^2+\frac{5}{2}c_W^4\right)w^2-\frac{1}{18}c_W^4w^3\,; \qquad (356)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\Sigma^U_{ZZ}(p^2)}{M_W^2} &= c_W^4\left(-4+\frac{17}{3}w+\frac{4}{3}w^2-\frac{w^3}{12}\right)B_0\left(p^2;M_W,M_W\right)-c_W^2\left(4+\frac{4}{3}w-\frac{w^2}{6}\right)\frac{A_0\left(M_W\right)}{M_Z^2}\\
&+\frac{1}{12}\left[-\left(\frac{1}{c_W^4}-2\,\frac{w_h}{c_W^2}+w_h^2\right)\frac{1}{w}+\frac{10}{c_W^2}-2w_h-w\right]B_0\left(p^2;M_H,M_Z\right)\\
&+\frac{1}{12c_W^2}\left(\frac{1}{h}-\frac{1}{z}+1\right)\frac{A_0\left(M_Z\right)}{M_Z^2}-\frac{1}{12c_W^2}\left(\frac{1}{h}-\frac{1}{z}+2\right)\frac{A_0\left(M_H\right)}{M_Z^2}\\
&-\left[\frac{1}{6c_W^2}+4c_W^4+\frac{w_h}{6}+\left(\frac{1}{18}+\frac{4}{3}c_W^4\right)w+\frac{5}{9}c_W^4w^2+\frac{1}{18}c_W^4w^3\right];\\
\Sigma^U_{AA}(p^2) &= p^2\Pi^U_{\gamma\gamma}(p^2)\,,\qquad \Sigma^U_{ZA}(p^2)=c_W^2p^2\Pi^U_{\gamma\gamma}(p^2)\,,\\
\Pi^U_{\gamma\gamma}(p^2) &= \frac{1}{w}\left[\left(-4+\frac{17}{3}w+\frac{4}{3}w^2-\frac{w^3}{12}\right)B_0\left(p^2;M_W,M_W\right)\right.\\
&\left.+\left(-4-\frac{4}{3}w+\frac{w^2}{6}\right)\frac{A_0\left(M_W\right)}{M_W^2}-4-\frac{4}{3}w-\frac{5}{9}w^2-\frac{w^3}{18}\right],\\
\frac{\Sigma^U_{HH}(p^2)}{M_W^2} &= \left(3+w+\frac{w^2}{4}\right)B_0\left(p^2;M_W,M_W\right)+\frac{9}{8}w_h^2B_0\left(p^2;M_H,M_H\right)\\
&+\frac{1}{2c_W^4}\left(3+z+\frac{z^2}{4}\right)B_0\left(p^2;M_Z,M_Z\right)+\left(3-\frac{w}{2}\right)\frac{A_0\left(M_W\right)}{M_W^2}\\
&+\left(\frac{3}{2c_\theta^2}-\frac{w}{4}\right)\frac{A_0\left(M_Z\right)}{M_W^2}+\frac{3}{4}w_h\frac{A_0\left(M_H\right)}{M_W^2}\,. \qquad (357)
\end{aligned}
$$

Note, that the mixing, $\Sigma^U_{ZA}(p^2)\propto p^2\Pi^U_{\gamma\gamma}(p^2)$. Therefore, $\Sigma^U_{ZA}(0)=0$ which provides far reaching simplifications for a renormalization procedure. This is a property of the U gauge only. In the unitary gauge, we note the appearance of the so-called *non-unitary* terms, growing with $w=p^2/M_W^2$ as powers of $w^{2,3}$, thereby violating the unitary limit. These terms must also cancel in the sum of all one-loop diagrams contributing to a physical amplitude. (Similar terms cancel in Eq. (354), although it is not as easy to see this property.)

5.1.2 *Fermionic components of bosonic self-energies*

By fermionic component of a bosonic self-energy we understand the contribution of the first diagrams in Figs. 5–6. They are gauge-independent contributions and an interesting physics is confined in them. This is why we give all self-energy and transitions for physical fields:

$$
\begin{aligned}
\Pi^{\rm fer}_{\gamma\gamma}(p^2) &= 4\sum_f c_fQ_f^2B_f\left(p^2;m_f,m_f\right),\\
\Sigma^{\rm fer}_{ZA}(p^2) &= 2\sum_f c_fQ_fv_fp^2B_f\left(p^2;m_f,m_f\right),
\end{aligned}
$$

$$\begin{aligned}\Sigma^{\rm fer}_{ZZ}(p^2) &= \sum_f c_f\left[\left(v_f^2+a_f^2\right)p^2 B_f\left(p^2;m_f,m_f\right)-2a_f^2 m_f^2 B_0\left(p^2;m_f,m_f\right)\right],\\ \Sigma^{\rm fer}_{WW}(p^2) &= \sum_{f=d} c_f p^2 B_f\left(p^2;m_{f'},m_f\right)+\sum_f c_f m_f^2 B_1\left(p^2;m_{f'},m_f\right),\\ \Sigma^{\rm fer}_{HH}(p^2) &= \sum_f c_f\frac{m_f^2}{M_W^2}\left[A_0\left(m_f\right)-\frac{p^2+4m_f^2}{2}B_0\left(p^2;m_f,m_f\right)\right],\end{aligned} \tag{358}$$

where c_f denotes *the color factor*, equal to 1 for leptons and to 3 for quarks.

The fermionic component of the $H-V$ transition vanishes, since it is proportional to

$$\propto\left[B_0\left(p^2;m_f,m_f\right)+2B_1\left(p^2;m_f,m_f\right)\right]p_\mu\,, \tag{359}$$

see Eq. (179). In Eq. (358) we have introduced an auxiliary function B_f:

$$B_f\left(p^2;m_{f'},m_f\right)=2\left[B_{21}\left(p^2;m_{f'},m_f\right)+B_1\left(p^2;m_{f'},m_f\right)\right], \tag{360}$$

and $m_{f'}$ stands for the mass of the weak isospin partner of the fermion f.

Then, we will need the pole and finite parts of the B_{ij} functions:

$$B_{ij}\left(p^2;m_1,m_2\right)=c_{ij}\left(\frac{1}{\bar\varepsilon}-\ln\frac{M_W^2}{\mu^2}\right)+B^F_{ij}\left(p^2;m_1,m_2\right), \tag{361}$$

with

$$c_0=1\,,\qquad c_1=-\frac{1}{2}\,,\qquad c_{21}=\frac{1}{3}\,. \tag{362}$$

For equal masses $m_1=m_2=m_f$, one has:

$$p^2 B^F_f\left(p^2;m_f,m_f\right)=\frac{p^2}{9}+\frac{2m_f^2}{3}\ln\frac{m_f^2}{M_W^2}+\frac{1}{3}\left(2m_f^2-p^2\right)B^F_0\left(p^2;m_f,m_f\right), \tag{363}$$

and

$$\begin{aligned}B_1\left(p^2;m_f,m_f\right) &= -\frac{1}{2}B_0\left(p^2;m_f,m_f\right),\\ B^F_0\left(p^2;m_f,m_f\right) &= 2-\ln\frac{m_f^2}{M_W^2}-\beta_f\ln\frac{\beta_f+1}{\beta_f-1},\\ B_{0p}\left(p^2;m_f,m_f\right) &= -\frac{1}{p^2}+\frac{2m_f^2}{p^4}\frac{1}{\beta_f}\ln\frac{\beta_f+1}{\beta_f-1},\qquad \text{with}\quad \beta_f=\sqrt{1+4\frac{m_f^2}{p^2}}\,.\end{aligned} \tag{364}$$

5.2 Heavy top asymptotic behaviour of self-energies; parameter $\Delta\rho$

Here we discuss one example of asymptotic behaviour of fermionic components of some bosonic self-energies. In realistic calculations, say for LEP1/SLC, one may ignore all fermion masses but top quark. Here we will use one more approximation:

$$|p^2|\ \ll\ m_t^2\,, \tag{365}$$

although it is not so good at LEP1/SLC energies and is absolutely untrue at LEP2 energies. We need it for an academic study of asymptotic behaviour when m_t is the largest parameter and is the only scale of the problem. Consider three cases in the asymptotic regime of Eq. (365):

$$
\begin{array}{lllll}
 & & 1) & 2) & 3) \\
 & & m_{f'} = m_t\,, & m_{f'} = 0\,, & m_{f'} = m_t\,, \\
 & & m_f = m_t\,, & m_f = m_t\,, & m_f = 0\,, \\
B_0^F\left(p^2;m_{f'},m_f\right) & \to & -\ln\dfrac{m_t^2}{\mu^2}\,, & 1-\ln\dfrac{m_t^2}{\mu^2}\,, & \text{the same}\,, \\
B_1^F\left(p^2;m_{f'},m_f\right) & \to & \dfrac{1}{2}\ln\dfrac{m_t^2}{\mu^2}\,, & \dfrac{1}{2}\ln\dfrac{m_t^2}{\mu^2}-\dfrac{1}{4}\,, & \dfrac{1}{2}\ln\dfrac{m_t^2}{\mu^2}-\dfrac{3}{4}\,, \\
B_f^F\left(p^2;m_{f'},m_f\right) & \to & \dfrac{1}{2}\ln\dfrac{m_t^2}{\mu^2}\,, & \dfrac{1}{3}\ln\dfrac{m_t^2}{\mu^2}-\dfrac{5}{18}\,, & \text{the same}\,.
\end{array}
\tag{366}
$$

Using this table, one easily derives the heavy top asymptotic for ZZ and WW self-energies:

$$
\begin{aligned}
\Sigma^{\rm fer}_{ZZ}(0) &= \frac{3}{2}m_t^2\ln\frac{m_t^2}{\mu^2}\,, \\
\Sigma^{\rm fer}_{WW}(0) &= \frac{3}{2}m_t^2\left(\ln\frac{m_t^2}{\mu^2}-\frac{1}{2}\right).
\end{aligned}
\tag{367}
$$

Consider now the so-called Veltman's $\Delta\rho$ parameter, which was originally defined as [1]

$$
\Delta\rho_0 = \frac{1}{M_W^2}\left[\Sigma_{WW}(0)-\Sigma_{ZZ}(0)\right].
\tag{368}
$$

Using Eq. (367), we find the asymptotic behaviour of Veltman's $\Delta\rho$ parameter:

$$
\Delta\rho_0^{\rm fer} \approx -\frac{3}{4}\frac{m_t^2}{M_W^2}\,.
\tag{369}
$$

A supscript 'fer' reminds that only fermionic components of the bosonic self-energies contribute in the considered asymptotic regime.

Note, that the higher term $m_t^2\ln\frac{m_t^2}{\mu^2}$ cancelled and therefore the asymptotic is quadratic in the t quark mass. This is why one sometimes says that the $\Delta\rho_0$ is *quadratically enhanced* by the top quark mass.

Consider now another definition of a $\Delta\rho$ parameter, which, as will be seen below, is a very relevant quantity for all electroweak radiative corrections. It is made of *complete* self-energies:

$$
\Delta\rho = \frac{1}{M_W^2}\Big[\Sigma_{WW}(M_W^2)-\Sigma_{ZZ}(M_Z^2)\Big].
\tag{370}
$$

This quantity is gauge-invariant, as is clear from the discussion in the previous section. For this reason it is used for *the re-summation* of large corrections, see below. If one ignores all masses but top quark mass, for its asymptotic we will have the same answer as for $\Delta\rho_0$:

$$
\Delta\rho^{\rm fer} \approx -\frac{3}{4}\frac{m_t^2}{M_W^2}\,.
\tag{371}
$$

It is very important to emphasize that $\Delta\rho$ is the *gauge-invariant but ultraviolet-divergent object*. By the way, the quantity $\Delta\rho_0$, defined by Eq. (368), is neither gauge-invariant nor finite. In the literature, a lot of other ρ's definitions are met. This creates a mess and a Babylon situation. One should always bear in mind which definition of ρ is meant, before making any controversial conclusion.

[1] We do not discuss here the so-called ρ parameter, defined as the ratio of NC/CC effective Fermi couplings, and its relation to parameter $\Delta\rho$. For more detail, see Section 6.11.3 of Ref. [1].

5.3 Ultraviolet behaviour of fermionic components of bosonic self-energies

Other interesting physics is related to the ultraviolet behaviour of fermionic components of bosonic self-energies. Consider two fermionic self-energy diagrams for a vector and a scalar field:

u, d $\overline{f}$

V V H H

$\overline{u}, \overline{d}$ f

We begin with a common initial expression, valid for both cases:

$$\Sigma \quad \propto \quad \frac{\mathrm{Tr}\Big[\left(i\not{q} + m_{f'}\right)\Gamma_1 \left(i\not{p} + i\not{q} + m_f\right)\Gamma_2\Big]}{\left(q^2 + m^2_{f'}\right)\Big[\left(q+p\right)^2 + m^2_f\Big]}\,. \tag{372}$$

For the vector case, e.g. $\Gamma_1 = \gamma_\mu,\ \Gamma_2 = \gamma_\nu$, one has:

$$\left(\Sigma_V\right)_{\mu\nu} \quad \propto \quad 4\frac{\delta_{\mu\nu}\Big[q\left(p+q\right) + m^2_f\Big] - \left(q_\mu p_\nu + q_\nu p_\mu\right) - 2q_\mu q_\nu}{\left(q^2 + m^2_f\right)\Big[\left(q+p\right)^2 + m^2_f\Big]}\,. \tag{373}$$

For the scalar case, $\Gamma_1 = \Gamma_2 = 1$, and we get instead:

$$\Sigma_S \quad \propto \quad 4\frac{q^2 - p\cdot q - m^2_f}{\left(q^2 + m^2_f\right)\Big[\left(q+p\right)^2 + m^2_f\Big]}\,. \tag{374}$$

Let us examine the leading UV divergences in both cases:

$$\begin{aligned}
\left(\Sigma_V\right)_{\mu\nu} &\propto 4\frac{\delta_{\mu\nu}q^2 - 2q_\mu q_\nu}{\left(q^2 + m^2_f\right)\Big[\left(q+p\right)^2 + m^2_f\Big]} \\
&= \delta_{\mu\nu} i\pi^{\frac{n}{2}}\frac{1}{\Gamma\left(\alpha\right)}\Gamma\left(1 - \frac{n}{2}\right)\left(m^2 - p^2\right)^{\frac{n}{2}-1}\frac{1}{2}\left(n-2\right) \to \Gamma\left(2 - \frac{n}{2}\right), \\
\Sigma_S &\propto 4\frac{q^2}{\left(q^2 + m^2_f\right)\Big[\left(q+p\right)^2 + m^2_f\Big]} \\
&= i\pi^{\frac{n}{2}}\frac{1}{\Gamma\left(\alpha\right)}\Gamma\left(1 - \frac{n}{2}\right)\left(m^2 - p^2\right)^{\frac{n}{2}-1}\frac{n}{2} \to \Gamma\left(1 - \frac{n}{2}\right).
\end{aligned} \tag{375}$$

As seen, the UV-behaviour is quite different. From Eq. (372), by counting of powers of q, one could expect quadratic divergences (or poles at $n = 2$) in both cases. However, in the vector case quadratic divergences from the scalar and tensor parts of the diagram cancel, yielding residual logarithmic divergence. In the scalar case the quadratic divergence survives.

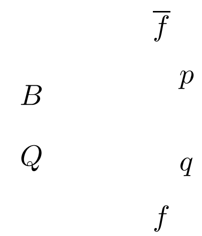
This observation is traded as an exhibition of a *non-naturalness* of the radiative corrections to the mass of a scalar field and is being used as one of the motivations for SUSY, where the quadratic divergences cancel if one adds the contribution from sfermions.

In the framework of the SM, however, this does not represent any problem since the SM needs renormalization anyway, and after renormalization all the divergences, both quadratic and logarithmic, cancel identically. (See the discussion on renormalization below in the next lecture.)

5.4 Calculation of decay rates in the Born approximation

5.4.1 Calculation via tree diagrams

In order to exhibit another interesting property of self-energy diagrams, we have to understand the Born expressions for the partial width of a boson decay into a fermion–antifermion pair. The Born diagram looks like:



where we have indicated all the particles' momenta.

In our convention for phase space, the differential probability is given by (cf. Eq. (24)):

$$d\Gamma = \frac{1}{2M}\overline{\sum_{\text{spins}}} \mid \mathcal{M} \mid^2 d\Phi_2 \,, \tag{376}$$

with the two-body *massive* phase space:

$$\Phi_2 \;=\; (2\pi)^4 \int \frac{d^3p}{(2\pi)^3\, 2p_0} \int \frac{d^3q}{(2\pi)^3\, 2q_0} \delta\left(Q - p - q\right). \tag{377}$$

Below we sketch the calculation of Φ_2 for the case where the final-state fermion masses are not ignored. The calculation proceeds as follows:

$$\begin{aligned}
\Phi_2 \;&=\; \frac{1}{(2\pi)^2} \int \frac{d^3p}{2p_0} \int d^4q\, \delta^+\left(q^2 + m_f^2\right) \delta\left(Q - p - q\right) \\
&\quad \left[\text{with} \quad \delta^+\left(p^2 + m_f^2\right) = \theta\left(p_0\right) \delta\left(p^2 + m_f^2\right)\right] \\
&=\; \frac{1}{(2\pi)^2} \int \frac{\mid \vec{p}\mid^2 \, d\mid \vec{p}\mid}{2p_0} d\Omega_p\, \delta^+\left[(Q-p)^2 + m_f^2\right] \\
&\quad \left[\text{using} \quad \mid \vec{p}\mid d \mid \vec{p}\mid = p_0 dp_0 \quad \text{and} \quad d\Omega_p \to 4\pi\right] \\
&=\; \frac{1}{2\pi} \int \mid \vec{p}\mid dp_0\, \delta\left(-M^2 + 2Mp_0\right) = \frac{1}{2\pi} \frac{\mid \vec{p}\mid}{2M}\,.
\end{aligned} \tag{378}$$

Using further on: $p_0 = \dfrac{M}{2}$, $\quad \mid \vec{p}\mid = \sqrt{\dfrac{M^2}{4} - m_f^2}$, and $\beta_f\left(M\right) = \dfrac{\mid \vec{p}\mid}{p_0} = \sqrt{1 - \dfrac{4m_f^2}{M^2}}$, we finally get

$$\Phi_2 = \frac{1}{8\pi}\beta_f\left(M\right)\,. \tag{379}$$

Next, we calculate of $\overline{\sum_{\text{spins}}} \mid \mathcal{M} \mid^2$ for three decays: $V,\ Z,\ H$.

For the vector and axial–vector cases, we derive:

$$\begin{aligned}
\overline{\sum_{\text{spins}}} \mid \mathcal{M} \mid^2 \;&=\; \frac{1}{3}\left(\delta_{\mu\nu} + \frac{Q_\mu Q_\nu}{M^2}\right) \overline{\sum_{\text{spins}}} \mathcal{M}_\mu^{\text{Born}} \left(\mathcal{M}_\nu^{\text{Born}}\right)^+ , \\
\mathcal{M}_\mu^{\text{Born}} \;&=\; if\bar{u}\left(q\right)\gamma_\mu\left(v_f + a_f\gamma_5\right) v\left(p\right), \\
\left(\mathcal{M}_\mu^{\text{Born}}\right)^+ \;&=\; if\bar{v}\left(p\right)\gamma_\nu\left(v_f + a_f\gamma_5\right) u\left(q\right),
\end{aligned} \tag{380}$$

where the coupling constants for two cases are

$$f = \begin{cases} e\,, & \text{for} \quad V = \text{heavy photon}, \\ \dfrac{g}{2c_\theta}\,, & \text{for} \quad V = Z. \end{cases} \tag{381}$$

For non-polarized fermions the summation over the final spins gives

$$\overline{\sum_{\text{spins}}} u\left(q\right)\bar{u}\left(q\right) = -i\not{q} + m_f\,, \qquad \overline{\sum_{\text{spins}}} v\left(p\right)\bar{v}\left(p\right) = -i\not{p} - m_f\,, \tag{382}$$

and we obtain

$$\overline{\sum_{\text{spins}}} \mid \mathcal{M} \mid^2 = \begin{cases} \dfrac{4}{3}e^2 M_V^2 \left(1 + 2\dfrac{m_f^2}{M_V^2}\right), & \text{for} \quad V = \text{heavy photon}, \\ \\ \dfrac{1}{3}\dfrac{g^2}{c_\theta^2}M_Z^2 \left[\left(v_f^2 + a_f^2\right)\left(1 + 2\dfrac{m_f^2}{M_Z^2}\right) - 6a_f^2\dfrac{m_f^2}{M_Z^2}\right], & \text{for} \quad V = Z\,. \end{cases} \tag{383}$$

Similarly, for the scalar case, we derive

$$\begin{aligned} \overline{\sum_{\text{spins}}} \mid \mathcal{M} \mid^2 &= \overline{\sum_{\text{spins}}} \mathcal{M}^{\text{Born}} \left(\mathcal{M}^{\text{Born}}\right)^+\,, \\ \mathcal{M}_\mu^{\text{Born}} &= -\frac{m_f}{2M_W}\bar{u}\left(q\right)v\left(p\right), \qquad \left(\mathcal{M}^{\text{Born}}\right)^+ = -\frac{m_f}{2M_W}\bar{v}\left(p\right)u\left(q\right), \end{aligned} \tag{384}$$

and

$$\overline{\sum_{\text{spins}}} \mid \mathcal{M} \mid^2 = \frac{g^2 m_f^2 M_H^2}{2M_W^2}\beta_f^2\left(M_H\right). \tag{385}$$

We conclude our exercise with a list of answers for partial widths:

$$\begin{aligned} \Gamma\left(V \to f\overline{f}\right) &= \frac{e^2 M_V}{12\pi}\,\beta_f\left(M_V\right)\left(1 + 2\frac{m_f^2}{M_V^2}\right), \\ \Gamma\left(Z \to f\overline{f}\right) &= 4\Gamma_0\,\beta_f\left(M_Z\right)\left[\left(v_f^2 + a_f^2\right)\left(1 + 2\frac{m_f^2}{M_Z^2}\right) - 6a_f^2\frac{m_f^2}{M_Z^2}\right], \\ \Gamma\left(H \to f\overline{f}\right) &= \frac{G_F m_f^2 M_H}{4\sqrt{2}\pi}\,\beta_f^3\left(M_H\right). \end{aligned} \tag{386}$$

Here we used the notation:

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8M_W^2}\,, \qquad c_W^2 = \frac{M_W^2}{M_Z^2}\,, \qquad \Gamma_0 = \frac{G_F M_Z^3}{24\sqrt{2}\pi}\,. \tag{387}$$

5.4.2 Calculation through self-energy functions

Now we are ready to present another calculation of partial widths and to compare it with what we got in the previous section. From Eq. (358) at the bosonic mass shell, $p^2 = -M^2$, one gets:

$$\begin{aligned} S_{\gamma\gamma} &= \frac{e^2}{16\pi^2}\left[-M_V^2 4B_f\left(-M_V^2; m_f, m_f\right)\right], \\ S_{ZZ} &= \frac{g^2}{16\pi^2 c_\theta^2}\left[-\left(v_f^2 + a_f^2\right)M_Z^2 B_f\left(-M_Z^2; m_f, m_f\right) - 2a_f^2 m_f^2 B_0\left(-M_Z^2; m_f, m_f\right)\right], \\ S_{HH} &= \frac{g^2}{16\pi^2}\left[\frac{m_f^2}{M_W^2}\left(A_0\left(m_f\right) - \frac{-M_H^2 + 4m_f^2}{2}B_0\left(-M_H^2; m_f, m_f\right)\right)\right]. \end{aligned} \tag{388}$$

Let us recall the definition of the $B_f\left(-M^2;m_f,m_f\right)$ function, Eq. (360) and take imaginary parts:

$$\begin{aligned}
\operatorname{Im} A_0\left(m_f\right) &= 0\,,\\
\operatorname{Im} B_1\left(-M^2;m_f,m_f\right) &= -\frac{1}{2}\operatorname{Im} B_0\left(-M^2;m_f,m_f\right),\\
\operatorname{Im} B_{21}\left(-M^2;m_f,m_f\right) &= \frac{1}{3}\left(1-\frac{m_f^2}{M_Z^2}\right)\operatorname{Im} B_0\left(-M^2;m_f,m_f\right),\\
\operatorname{Im} B_0\left(-M^2;m_f,m_f\right) &= \pi\beta_f\left(M\right).
\end{aligned} \tag{389}$$

Substituting the imaginary parts into Eq. (388) and comparing the results with Eq. (386), we immediately verify the validity of identity:

$$\operatorname{Im} S_{BB} = M_B\Gamma\left(B\to f\overline{f}\right), \tag{390}$$

i.e. the imaginary part of the fermionic component of the bosonic self-energy on the bosonic mass shell is equal to the boson mass times the partial bosonic decay width into this fermionic pair. A similar property takes place for the fermionic component of the WW self-energy and for the bosonic component of bosonic self-energies.

5.5 Dispersion relation for $\Pi\left(p^2\right)$

As the last application of bosonic self-energies, we will consider the dispersion relation for $\Pi\left(p^2\right)$. It is being used for the calculation of the hadronic contribution to the running electromagnetic coupling $\alpha(s)$. We begin with a partial contribution to $\Pi\left(p^2\right)$ due to a fermion pair $f\overline{f}$, see the second row of Eq. (263):

$$\Pi_f\left(p^2\right) = \frac{6m_f^2+p^2}{9p^2}+\frac{2}{3p^2}A_0\left(m_f\right)-\frac{p^2-2m_f^2}{3p^2}B_0\left(p^2;m_f,m_f\right). \tag{391}$$

Let us recall its ingredients

$$\begin{aligned}
B_0\left(p^2;m_f,m_f\right) &= \frac{1}{\bar{\varepsilon}}-\ln\frac{m_f^2}{\mu^2}+B_0^F\left(p^2;m_f,m_f\right),\\
B_0^F\left(p^2;m_f,m_f\right) &= 2-\beta_f\ln\frac{\beta_f+1}{\beta_f-1}\,,\\
\beta_f^2 &= 1+4\frac{m_f^2}{p^2-i\epsilon}\,.
\end{aligned} \tag{392}$$

From it we construct the renormalized vacuum polarization ($p^2=-s$):

$$\Pi_f^{\rm ren}(s) = \Pi_f(s)-\Pi_f(0) = \frac{1}{9}-\frac{1}{3}\left(1+2\frac{m_f^2}{s}\right)B_0^F\left(-s;m_f,m_f\right), \tag{393}$$

and take its imaginary part:

$$\operatorname{Im}\Pi_f^{\rm ren}(s) = -\frac{1}{3}\left(1+2\frac{m_f^2}{s}\right)\operatorname{Im} B_0^F\left(-s;m_f,m_f\right) = -\frac{1}{3}\left(1+2\frac{m_f^2}{s}\right)\pi\beta_f\,. \tag{394}$$

Now compute *the dispersion integral*:

$$\begin{aligned}
\frac{s}{\pi}\int_{4m_f^2}^{\infty}d\tau\frac{\operatorname{Im}\Pi_f^{\rm ren}(\tau)}{\tau\left(\tau-s-i\epsilon\right)} &= -\frac{1}{3}\int_{4m_f^2}^{\infty}\frac{sd\tau}{\tau\left(\tau-s-i\epsilon\right)}\left(1+2\frac{m_f^2}{\tau}\right)\sqrt{1-\frac{4m_f^2}{\tau}}\\
&= -\frac{1}{3}\left(1+2\frac{m_f^2}{s}\right)\int_{4m_f^2}^{\infty}\frac{sd\tau}{\tau\left(\tau-s-i\epsilon\right)}\sqrt{1-\frac{4m_f^2}{\tau}}+\frac{2m_f^2}{3}\int_{4m_f^2}^{\infty}\frac{d\tau}{\tau^2}\sqrt{1-\frac{4m_f^2}{\tau}}\,.
\end{aligned} \tag{395}$$

The above two integrals could easily be taken:

$$\int_{4m_f^2}^{\infty} \frac{s d\tau}{\tau\left(\tau - s - i\epsilon\right)} \sqrt{1 - \frac{4m_f^2}{\tau}} \;=\; \beta \ln \frac{\beta+1}{\beta-1} - 2\,,$$
$$\int_{4m_f^2}^{\infty} \frac{d\tau}{\tau^2} \sqrt{1 - \frac{4m_f^2}{\tau}} \;=\; \frac{1}{6m_f^2}\,. \qquad (396)$$

Substituting these two integrals, we verify the identity

$$\Pi^{\rm ren}(s) = \frac{s}{\pi} \int_{4m_f^2}^{\infty} d\tau \frac{{\rm Im}\,\Pi^{\rm ren}(\tau)}{\tau\left(\tau - s - i\epsilon\right)}\,. \qquad (397)$$

The result for the hadronic contribution to the running electromagnetic coupling, $\Delta\alpha_h^{(5)}(s)$, is obtained in the literature by making use of a similar dispersion relation:

$$\Delta\alpha_h^{(5)}(s) = -\frac{\alpha}{3\pi} s\, {\rm Re} \int_{4m_\pi^2}^{\infty} ds' \frac{R_\gamma\left(s'\right)}{s'\left(s' - s - i\epsilon\right)}\,, \qquad (398)$$

with the ratio

$$R_\gamma(s) = \frac{\sigma\left(e^+e^- \to \gamma^* \to \text{hadrons}\right)}{\sigma\left(e^+e^- \to \gamma^* \to \mu^+\mu^-\right)}\,, \qquad (399)$$

as an experimental input.

For the hadronic contribution at M_Z it gives:

$$\Delta\alpha_h^{(5)}(M_Z^2) = 0.0280398\,. \qquad (400)$$

For more details about this subject see Section 1.5 of Ref. [1].

5.6 Fermion self-energies in the Standard Model

In the R_ξ gauge there are six fermion self-energy diagrams, shown in Fig. 7.

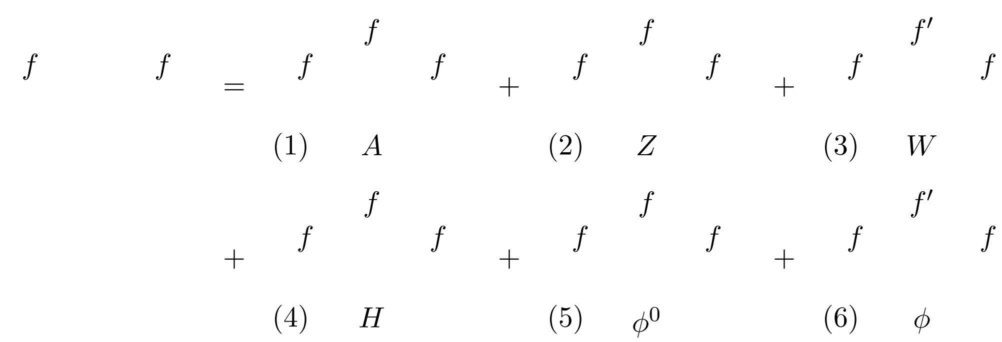


Fig. 7: Fermionic self-energy diagrams

where f' is the weak isospin partner of the f-fermion, and the couplings to the Z boson are

$$v_f = I_f^{(3)} - 2s_\theta^2 Q_f\,, \qquad a_f = I_f^{(3)}\,, \qquad (401)$$

We will also use combinations of couplings:

$$\sigma_f \;=\; v_f + a_f, \qquad \sigma_f^{(2)} = v_f^2 + a_f^2, \qquad \sigma_f^i = \Big(v_f + a_f\Big)^i,$$
$$\delta_f \;=\; v_f - a_f, \qquad \delta_f^{(2)} = v_f^2 - a_f^2, \qquad \delta_f^i = \Big(v_f - a_f\Big)^i. \qquad (402)$$

Each self-energy diagram containing a boson B-line is denoted by $\Sigma_B(\not p)$ and has the structure:

$$\Sigma_B(\not p) = (2\pi)^4\, i\frac{g^2}{16\,\pi^2} A_B\,, \qquad \not p = p_\alpha \gamma_\alpha\,. \tag{403}$$

There are six A_B functions in the R_ξ gauge, and only four [from (1) to (4) in Fig. 7] in the U gauge. As an example we give QED contribution in the R_ξ gauge:

$$\begin{aligned} A^\xi_A &= s^2_\theta\, Q^2_f \Big\{ i\not p\Big[2B_1\left(p^2; m_f, 0\right) + 1\Big] - 2m_f\Big[2B_0\left(p^2; m_f, 0\right) - 1\Big] \\ &\quad - (i\not p + m_f)\left(\xi^2_A - 1\right)\Big[B_0\left(p^2; m_f, 0\right) + m_f\,(i\not p - m_f)\, b_1\left(p^2; m_f\right)\Big]\Big\}, \end{aligned} \tag{404}$$

where one sees the presence of the special PV function $b_1(p^2; m_f)$.

In the U gauge, the two diagrams with heavy vector bosons may be expressed as

$$\begin{aligned} A^U_Z &= -\frac{1}{4c^2_\theta}\Big\{ i\not p\left(\sigma^{(2)}_f + 2v_f a_f \gamma_5\right)\left[\frac{p^2 + m^2_f}{M^2_0} B_1\left(p^2; M_0, m_f\right) + A^U_{\rm w}\left(p^2; M_0, m_f\right)\right] \\ &\quad + m_f \delta^{(2)}_f \Big[3B_0\left(p^2; M_0, m_f\right) + \frac{1}{M^2_0} A_0(m_f) - 2\Big]\Big\}, \\ A^U_W &= -\frac{1}{4} i\not p\,(1 + \gamma_5)\left[\frac{p^2 + m^2_{f'}}{M^2} B_1\left(p^2; M, m_f\right) + A^U_{\rm w}\left(p^2; M, m_f\right)\right], \end{aligned} \tag{405}$$

i.e. by means of a common *auxiliary function*:

$$A^U_{\rm w}\left(p^2; M, m\right) = 2B_1\left(p^2; M, m\right) + B_0\left(p^2; M, m\right) + \frac{1}{M^2} A_0(m) - 1\,. \tag{406}$$

Self-energy diagrams, both bosonic and fermionic, are *universal* in the sense that they depend only on the type of propagating particle. On the contrary, vertices and boxes depend on the process, and in this sense are termed to be *non-universal*.

5.7 The Standard Model vertices

I will limit myself to only one example of a vertex shown in Fig. 8. The following classification is useful:

- (1) is the QED diagram;
- (2) and (12) form the Z *Abelian cluster*;
- (3) and (8) are similarly the W *Abelian cluster*;
- (4) and (9)–(11) form the W *non-Abelian cluster*;
- remaining (5)–(7) and (13)–(14) form the H *cluster*.

Only diagrams (1)–(7) remain in the unitary gauge; Only diagrams (1)–(4) contribute in the case of massless fermions.

As an example, consider the W Abelian cluster with virtual (W, ϕ) exchange for the case of the $Vb\bar b$ vertex. Even for the massless b-quark, the diagrams 8.(8-11) will contribute, since in this case $f' = t$ and m_t cannot be neglected.

The vertex is a vector, $V^{Wn}_\mu(Q^2)$, which, in turn, is different for two cases:

$$\text{1) } \gamma f\overline{f} \text{ vertex,} \qquad V^{Wn}_\mu\left(Q^2\right) = (2\pi)^4\, i\frac{ig^3}{16\pi^2}\frac{s_\theta}{2}\left(-I^{(3)}_f\right)\gamma_\mu\,(1 + \gamma_5)\, G^g_{Wn}(Q^2)\,, \tag{407}$$

$$\text{2) } Zf\overline{f} \text{ vertex,} \qquad V^{Wn}_\mu\left(Q^2\right) = (2\pi)^4\, i\frac{ig^3}{16\pi^2}\frac{c_\theta}{2}\left(-I^{(3)}_f\right)\gamma_\mu\,(1 + \gamma_5)\, Z^g_{Wn}(Q^2)\,. \tag{408}$$

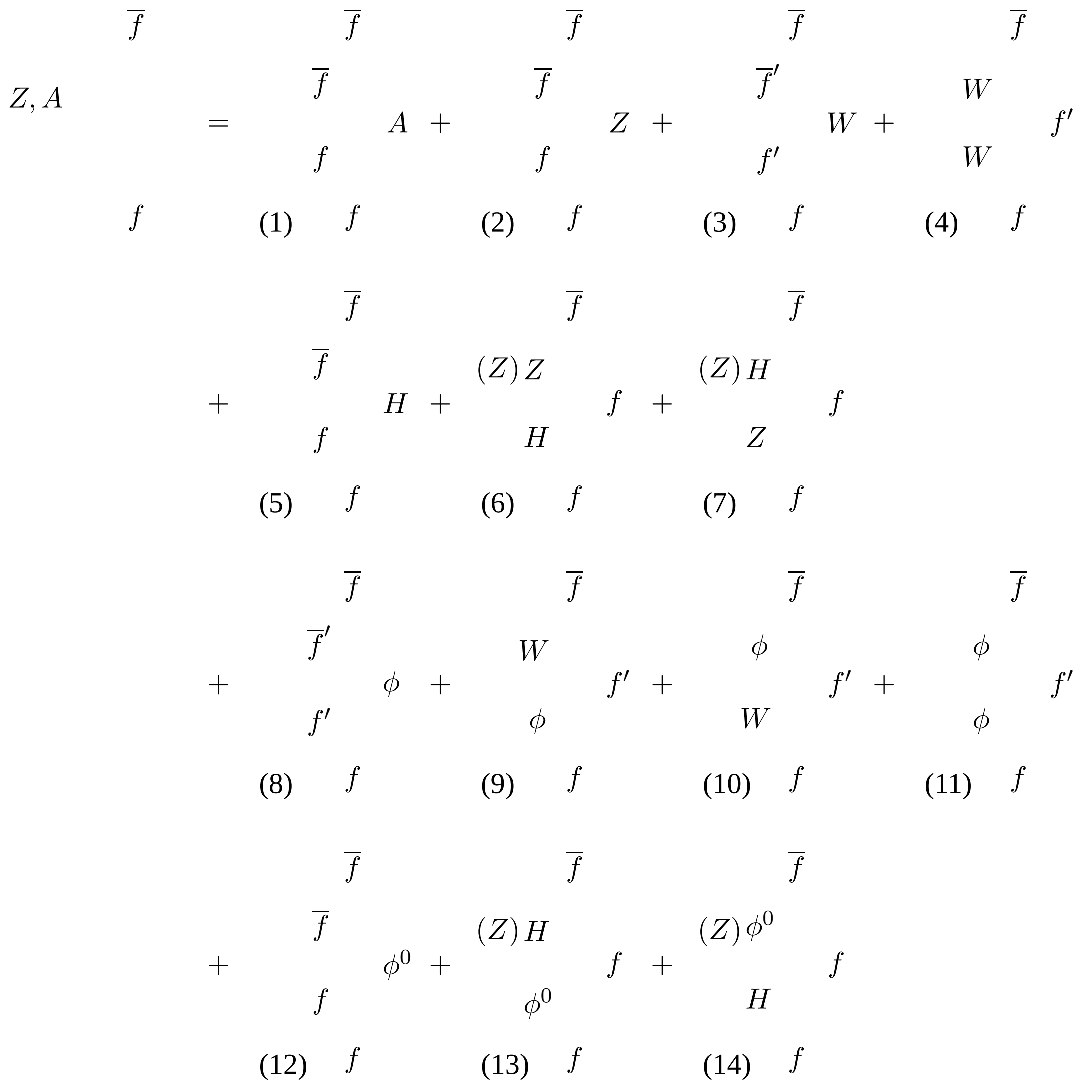


Fig. 8: $(Z, A) \to f\overline{f}$ vertices. The symbol (Z) in some graphs indicates that it contributes only to the Z vertex.

The $G^g_{Wn}(Q^2)$ and $Z^g_{Wn}(Q^2)$ are scalar form factors, bearing the sup-index g = *gauge*. In general, they are different for $\gamma f\overline{f}$ and $Zf\overline{f}$ vertices. In the U gauge, however, one has:

$$G^U_{Wn}(Q^2) = Z^U_{Wn}(Q^2) = F^U_{Wn}(Q^2)\,, \tag{409}$$

with

$$\begin{aligned}
F^U_{Wn}(Q^2) = &\left[-\left(1-\frac{m^2_{f'}}{M^2}\right)^2\left(2+\frac{m^2_{f'}}{M^2}\right)\frac{M^2}{Q^2}+4-\frac{5}{2}\frac{m^2_{f'}}{M^2}+2\frac{m^4_{f'}}{M^4}-\frac{m^6_{f'}}{2M^6}\right.\\
&\left.-\frac{m^2_{f'}}{M^2}\left(2-\frac{m^2_{f'}}{2M^2}\right)\frac{Q^2}{M^2}\right]M^2C_0\left(0,0,Q^2;M,m_{f'},M\right)\\
&-\left[\frac{2}{3}-\frac{m^2_{f'}}{2M^2}-\left(\frac{3}{2}-\frac{m^2_{f'}}{4M^2}\right)\frac{Q^2}{M^2}+\frac{Q^4}{12M^4}\right]B_0\left(Q^2;M,M\right)\\
&-\left[\left(1-\frac{m^2_{f'}}{M^2}\right)\left(2+\frac{m^2_{f'}}{M^2}\right)\frac{M^2}{Q^2}-3+\frac{3}{2}\frac{m^2_{f'}}{M^2}-\frac{m^4_{f'}}{2M^4}\right]\left[B_0\left(Q^2;M,M\right)-B_0\left(0;m_{f'},M\right)\right]
\end{aligned}$$

$$-\left(\frac{2}{3}-\frac{Q^2}{6M^2}\right)\frac{1}{M^2}A_0\left(M\right)-\frac{1}{M^2}A_0\left(m_{f'}\right)-\frac{2}{3}-\frac{m_{f'}^2}{2M^2}-\left(\frac{4}{9}-\frac{m_{f'}^2}{4M^2}\right)\frac{Q^2}{M^2}-\frac{Q^4}{18M^4}\,. \quad (410)$$

This vertex, as well as the Abelian diagrams of Fig. 8.(3,8) with virtual (W,ϕ) exchange, are one more source of m_t^2/M_W^2 enhanced terms. These terms are also called *non-universal*.

The world of vertices and boxes is much more rich than that of self-energies. Many more examples may be be found in Sections 5.9–5.12 and 14.13–14.14 of Ref. [1]. We also would like to emphasize that nowadays, one-loop diagrams are usually calculated using the methods of computer algebra. For instance, all calculations in Ref. [1] are achieved by a set of codes written in `form`. These codes automatically generate all the possible on-loop diagrams, substitute the Feynman rules and make the tensorial reduction up to the scalar PV functions. In principle, they are accessible from the authors upon request.

5.8 Summary of five Lectures

Let us briefly summarize what we have studied and learned in the five lectures:

- Standard Model, its fields and Lagrangian;
  Feynman rules $\rightarrow$ *building* of diagrams;
- Regularization, N-point functions;
  PV functions $\rightarrow$ *calculation* of diagrams;
- Groups of diagrams, *building blocks:*
  - Tadpoles reduce to one-point functions;
  - Self-energies reduce to two- and one-point functions;
    while studying them, we discussed:
    - ρ-parameter;
    - m_t^2-enhanced terms;
    - problem of quadratic divergences;
  - Vertices reduce to 3,2,1 point functions;
  - Boxes (direct/crossed) reduce to 4,3,2,1 functions.

We are approaching:

- Calculation of *amplitudes* for physical observables;
- Understanding *the inevitability* of renormalization.

6. RENORMALIZATION, ONE-LOOP AMPLITUDES, PRECISION TESTS OF THE SM

In the five previous lectures, our presentation was rather complete and consequent. Approaching the most interesting subject, we face a lack of time and impossibility to continue with the same degree of comprehension. This is why the following presentation will be unavoidably brief and fragmentary.

6.1 Renormalization for pedestrians

We begin with an explanation of the main principles of renormalization. However, first of all, we have to devote some time to the Dyson re-summation.

6.1.1 Dyson re-summation

Consider a *bare* propagator. Turning to *the dressed* one, we have to sum up all the one-loop insertions. This may be schematically depicted in the following figure:

$$= \quad \frac{1}{(2\pi)^4\, i}\, \frac{1}{(p^2+M^2)}\,,$$

$p \qquad p$

$$= \qquad + $$

$$+$$

$$+ \qquad + \quad \cdots$$

$$= \quad \frac{1}{(2\pi)^4\, i}\, \frac{1}{\left[\;p^2+M^2 \quad - \quad \dfrac{1}{(2\pi)^4\, i} \qquad\right]}\,.$$

This procedure is known as *the Dyson re-summation*.

In the case of conventional QED, we have the well-known result:

$$S_{\mu\nu} \;=\; \frac{1}{(2\pi)^4\, i}\, i\pi^2 e^2 \left(p^2\delta_{\mu\nu} - p_\mu p_\nu\right) 4\Pi(p^2). \tag{411}$$

The $p_\mu p_\nu$ part does not contribute whenever one considers $S_{\mu\nu}$ as being coupled to a conserved fermionic currents. Therefore, Dyson re-summation results in the substitution:

$$\frac{1}{(2\pi)^4\, i}\, \frac{\delta_{\mu\nu}}{p^2} \;\to\; \frac{1}{(2\pi)^4\, i}\, \frac{\delta_{\mu\nu}}{p^2}\, \frac{1}{1 - \dfrac{e^2}{4\pi^2}\Pi\left(p^2\right)}\,, \tag{412}$$

with $\Pi(p^2)$ given by Eq. (263). This equation describes the running electromagnetic coupling.

Similarly, for the Z boson propagator in the $\xi = 1$ gauge we obtain:

$$S_{\mu\nu} \;=\; \frac{1}{(2\pi)^4\, i}\, i\pi^2 \frac{g^2}{c_W^2} \left(\delta_{\mu\nu}\Sigma_{ZZ}(p^2) + p_\mu p_\nu T_{ZZ}(p^2)\right), \tag{413}$$

$$\frac{1}{(2\pi)^4\, i}\, \frac{\delta_{\mu\nu}}{p^2+M_Z^2} \;\to\; \frac{1}{(2\pi)^4\, i}\, \frac{\delta_{\mu\nu}}{p^2 + M_Z^2 - \dfrac{g^2}{16\pi^2 c_W^2}\Sigma_{ZZ}\left(p^2\right)}\,. \tag{414}$$

There is a big difference between Eqs. (412) and (414). The former does not change *the position of the pole* of the photon propagator, which was at $p^2 = 0$ before summation (bare propagator) and remained at $p^2 = 0$ after. We must emphasize however that it does change *the residue* of the photon propagator, which was equal to one before summation. On the contrary, Eq. (414) drastically changes the position of the pole of the Z propagator. The bare propagator had the pole at $p^2 = -M_Z^2$. Let us recall now Eqs. (388) and (390). We see that the pole of the re-summed propagator shifts into the complex plane because $\Sigma_{ZZ}(p^2)$ has an imaginary part. Therefore, the Dyson re-summation results in the Breit–Wigner form of the propagator of an unstable particle. However, it is not a full story. The quantity $\Sigma_{ZZ}(p^2)$ also possesses *a divergent* real part and the re-summed expression is meaningless. To continue, we must learn more about *renormalization procedure*.

6.1.2 *Renormalization in QED*

We come back to QED describing the interaction of spin-$\frac{1}{2}$ particles with photons. We recall the QED Lagrangian in the Feynman gauge:

$$\mathcal{L}_{\text{QED}} = -\frac{1}{4}\,F_{\mu\nu}F_{\mu\nu} - \frac{1}{2}\,(\mathcal{C}^{A})^2 - \sum_f \overline{\psi}_f\,\left(\partial\!\!\!/ - ieQ_f A\!\!\!/ + m_f\right)\psi_f\,, \tag{415}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,, \qquad \mathcal{C}^{A} = -\partial_\mu A_\mu\,, \tag{416}$$

and the sum runs over the fermion fields f (with charge eQ_f, and mass m_f). We also recall the Feynman rules of QED:

$p \rightarrow$ $$\frac{1}{(2\pi)^4\,i}\,\frac{-ip\!\!\!/ + m_f}{p^2 + m_f^2 - i\epsilon}\,,$$

μ ν $$\frac{1}{(2\pi)^4\,i}\,\frac{1}{p^2 - i\epsilon}\,\delta_{\mu\nu}\,,$$

μ $$(2\pi)^4\,i\;ieQ_f\,\gamma_\mu\,.$$

There are many alternative ways to describe renormalization. Here we use the language of the so-called *on-mass-shell renormalization (OMS).*

The QED Lagrangian is unambiguous at tree level. Moving to higher orders, we face problems because both the individual diagrams and their sum contain UV and IR divergences, and one has to modify something in the procedure of the calculations in order to get a meaningful answer.

A natural question might be raised: Which are the fields and parameters that the Lagrangian of Eq. (415) is made of? We assume that it is made of some *bare* fields and parameters labelled with indices 0, and specify the renormalization constants for both fields — A_μ and ψ — and parameters — the mass m and the charge e — as follows:

$$\begin{aligned} A_{0\mu} &= Z_A^{1/2}\,A_\mu\,, \qquad \psi_0 = Z_\psi^{1/2}\,\psi\,, \\ e_0 &= Z_e\,e\,, \qquad\quad m_0 = Z_m\,m = m + e^2\delta m + \mathcal{O}\left(e^4\right). \end{aligned} \tag{417}$$

The renormalization constants, as everything else within a perturbative approach, are assumed to be representable as Taylor expansions in the coupling constant e^2, i.e.

$$Z_i = 1 + e^2\delta Z_i + \mathcal{O}\left(e^4\right). \tag{418}$$

The Lagrangian can now be re-written, up to terms $\mathcal{O}\left(e^2\right)$

$$\mathcal{L}_{\text{QED}} \quad\rightarrow\quad \mathcal{L}^{\text{R}}_{\text{QED}} = \mathcal{L}_{\text{QED}} + \mathcal{L}_{\text{ct}}\,, \tag{419}$$

with a *counter-term* Lagrangian

$$\begin{aligned} \mathcal{L}_{\text{ct}} &= e^2\,\mathcal{L}^{(2)}_{\text{ct}} + \mathcal{O}\left(e^4\right), \\ \mathcal{L}^{(2)}_{\text{ct}} &= -\frac{1}{4}\,\delta Z_A\,F_{\mu\nu}F_{\mu\nu} - \frac{1}{2}\,\delta Z_A\,(\partial_\mu A_\mu)^2 - \delta Z_\psi\,\overline{\psi}\,\partial\!\!\!/\,\psi \\ &\quad - (\delta Z_\psi m + \delta m)\,\overline{\psi}\psi - i\left(\delta Z_e + \delta Z_\psi + \frac{1}{2}\delta Z_A\right) eA_\mu\overline{\psi}\gamma_\mu\psi\,. \end{aligned} \tag{420}$$

The first part of the Lagrangian, $\mathcal{L}_{\rm QED}$, generates the standard set of diagrams and Feynman rules which were shown in the previous figure. The counter-term Lagrangian generates a new set of diagrams with Feynman rules:

A $\quad\rightarrow\quad$ $e^2 \delta Z_A\,,$

e $\quad\rightarrow\quad$ $-e^2\left(\delta Z_\psi i\not{p} + \delta Z_\psi m + \delta m\right)\,,$

e^+

A $\quad\rightarrow\quad$ $-ie\gamma_\mu e^3\left(\delta Z_e + \delta Z_\psi + \frac{1}{2}\delta Z_A\right)\,,$

μ

e^-

and we have to take into account contributions generated by both parts.

The crucial moment in the above modification is an assumption that we have two kind of fields and parameters, bare and physical ones, and that they are related by the simplest kind of transformation, *a multiplicative scale* transformation Eq. (417) with some yet unknown renormalization constants. In this way, we introduced into the theory a set of new parameters (degrees of freedom) which should somehow be fixed. We will see that there is a very physical way of their *fixation*, after which all UV-divergences do automatically cancel. In order to understand better the meaning of the fixation procedure, we will consider once again diagrams of a different kind.

The photon propagator.

With the new Lagrangian Eq. (419) after Dyson re-summation, instead of Eq. (412), we will have:

$$\frac{1}{(2\pi)^4 i}\,\frac{\delta_{\mu\nu}}{p^2}\,\frac{1}{1 + e^2\delta Z_A - \dfrac{e^2}{4\pi^2}\Pi\left(p^2\right)}\,. \tag{421}$$

The essence of the on-mass-shell renormalization scheme is to preserve the meaning of the original parameters of the Lagrangian. For *the dressed* photonic propagator, we require that its residue should be unchanged at the photonic mass shell, $p^2 = 0$, i.e.

$$\frac{1}{1 + e^2\delta Z_A - \dfrac{e^2}{4\pi^2}\Pi\left(0\right)} = 1\,. \tag{422}$$

This requirement guarantees that the wave function for external photonic lines does not change due to one-loop radiative corrections (for the proof see Section 1.4 of Ref. [1]) and simultaneously fixes $e^2\delta Z_A$:

$$e^2\delta Z_A = \frac{e^2}{4\pi^2}\Pi\left(0\right). \tag{423}$$

Recalling Eq. (265), we substitute $\Pi\left(0\right)$ and obtain an explicit answer for one of the counter-terms:

$$\delta Z_A = \frac{1}{12\pi^2}\left(-\frac{1}{\bar{\varepsilon}} + \ln\frac{m^2}{\mu^2}\right). \tag{424}$$

In other words, one can say that we used the first fixation condition and fixed the counter-term δZ_A.

The electron propagator.

With the Lagrangian Eq. (419), we have

$$S \;=\; \frac{1}{(2\pi)^4 i}\left[\left(1 + e^2\delta Z_\psi\right)\left(i\not{p} + m\right) + e^2\delta m - \frac{1}{(2\pi)^4 i}\Sigma\left(\not{p}\right)\right]^{-1}. \tag{425}$$

The second fixation condition. For the dressed electron propagator we also require residue $= 1$ (*residue one*) at the electron mass shell, $i\not{p} = -m$, i.e. the on-shell propagator should be equal to

$$S = \frac{1}{(2\pi)^4 i \; (i\not{p} + m)} \; . \tag{426}$$

In order to exploit this fixation condition, we have to expand $\Sigma(\not{p})$ around the physical electron mass $i\not{p} = -m$ (this point is sometimes called *the subtraction point*). It is sufficient to take into account only the first two terms in the Taylor expansion

$$\Sigma(\not{p}) = \Sigma(im) + (i\not{p} + m)\,\Sigma_{\rm WF} + \mathcal{O}\left((i\not{p} + m)^2\right), \tag{427}$$

where the coefficient of the linear term is called *the wave function renormalization factor*

$$\Sigma_{\rm WF} \;\; = \;\; \left.\frac{\partial\Sigma(\not{p})}{\partial(i\not{p})}\right|_{i\not{p}=-m} . \tag{428}$$

For the re-summed propagator we derive

$$\begin{aligned} S \;\; = \;\; & \frac{1}{(2\pi)^4 i}\Biggl\{\left(1 + e^2\delta Z_\psi\right)(i\not{p} + m) + e^2\delta m \\ & - \frac{1}{(2\pi)^4 i}\left[\Sigma(im) + (i\not{p} + m)\,\Sigma_{\rm WF} + \mathcal{O}\left((i\not{p} + m)^2\right)\right]\Biggr\}^{-1} . \end{aligned} \tag{429}$$

the residue one requirement will be fulfilled if

$$e^2\delta m = \frac{\Sigma(im)}{(2\pi)^4 i}\,, \qquad e^2\delta Z_\psi = \frac{\Sigma_{\rm WF}}{(2\pi)^4 i}\; . \tag{430}$$

The first equation is *mass renormalization*, whilst the second is *wave function renormalization*. *the residue one* requirement preserves the external line electron wave function from being renormalized by the one-loop radiative corrections and simultaneously fixes two more counter-terms.

By straightforward calculations in dimensional regularization, we derive

$$\begin{aligned} \Sigma(im) \;\; &= \;\; i\pi^2 e^2 m\left(-\frac{3}{\bar{\varepsilon}} + 3\ln\frac{m^2}{\mu^2} - 4\right), \\ \Sigma_{\rm WF} \;\; &= \;\; i\pi^2 e^2\left\{2B_1\left(-m^2; m, 0\right) + 1 - 4\,m^2\left[B_{1p}\left(-m^2; m, 0\right) + 2\,B_{0p}\left(-m^2; m, 0\right)\right]\right\} \\ &= \;\; i\pi^2 e^2\left(-\frac{1}{\bar{\varepsilon}} + \frac{2}{\hat{\varepsilon}} + 3\ln\frac{m^2}{\mu^2} - 4\right). \end{aligned} \tag{431}$$

Substituting these results into Eq. (430), we obtain explicit answers for two more counter-terms:

$$\delta m = \frac{m}{16\pi^2}\left(-\frac{3}{\bar{\varepsilon}} + 3\ln\frac{m^2}{\mu^2} - 4\right), \qquad \delta Z_\psi = \frac{1}{16\pi^2}\left(-\frac{1}{\bar{\varepsilon}} + \frac{2}{\hat{\varepsilon}} + 3\ln\frac{m^2}{\mu^2} - 4\right). \tag{432}$$

The γe^+e^- vertex. Consider the γe^+e^- vertex with both fermions on mass shell. Collect again all contributions to the γ_μ-part of the γe^+e^- vertex in the one-loop approximation. In terms of $F_1\left(Q^2, m\right)$, introduced in Subsection 4.1.3, we have

$$-(2\pi)^4 i\, ie\left\{1 + e^2\left[\delta Z_e + \frac{1}{2}\delta Z_A + \delta Z_\psi + \frac{1}{16\pi^2}F_1\left(Q^2, m\right)\right]\right\}\gamma_\mu\,. \tag{433}$$

Third fixation condition.
In the spirit of on-mass-shell renormalization we have to preserve the meaning of the parameters of the original Lagrangian. For the one-loop corrected vertex we require it to be

$$-\left(2\pi\right)^{4} i\, ie\gamma_{\mu}\,, \tag{434}$$

at $Q^2 = 0$, which preserves the Thompson limit of the electric charge from being renormalized by one-loop radiative corrections, i.e.

$$\delta Z_e + \frac{1}{2}\delta Z_{A} + \delta Z_{\psi} + \frac{1}{16\pi^2}F_1\left(0, m\right) = 0\,. \tag{435}$$

Substituting the already fixed counter-term δZ_{ψ}, and the derived expression for $F_1\left(0, m\right)$, we observe the famous QED Ward identity

$$\delta Z_{\psi} + \frac{1}{16\pi^2}F_1\left(0, m\right) \equiv 0\,, \tag{436}$$

that fixes the last counter-term

$$\delta Z_e \equiv -\frac{1}{2}\delta Z_{A}\,. \tag{437}$$

So, all the counter-terms in the Lagrangian are fixed and one may calculate any QED process at the one-loop level.

Let us summarize our findings:

- The one-loop and the counter-term contributions for any external on-shell line compensate each other identically; this is known as the principle of non-renormalizability for external lines;
- For any $2 \rightarrow 2$ fermion process, at the one-loop level, we encounter only two building blocks:
  1) The effective (running) electric charge, $e^2\left(p^2\right)$, entering the photonic propagator,

$$e^2 D_{\mu\nu} = \frac{e^2\left(p^2\right)}{\left(2\pi\right)^4 i}\,\frac{\delta_{\mu\nu}}{p^2}\,, \qquad e^2\left(p^2\right) = \frac{e^2}{1 - \frac{e^2}{4\pi^2}\Pi^{\rm ren}\left(p^2\right)}\,, \tag{438}$$

  the evolution of which is governed by *the renormalized* quantity

$$\Pi^{\rm ren}\left(p^2\right) = \Pi\left(p^2\right) - \Pi\left(0\right); \tag{439}$$

  2) The renormalized vertex, $F_1^{\rm ren}\left(Q^2, m\right)$, entering the complete $\gamma e^+ e^-$ vertex,

$$\Lambda_{\mu} = \left(2\pi\right)^4 i\,\frac{ie^3}{16\pi^2}\left[\gamma_{\mu}F_1^{\rm ren}\left(Q^2, m\right) + \sigma_{\mu\nu}\left(p_1 + p_2\right)_{\nu} mF_2\left(Q^2, m\right)\right]. \tag{440}$$

  The renormalized vertex is again the difference

$$F_1^{\rm ren}\left(Q^2, m\right) = F_1\left(Q^2, m\right) - F_1\left(0, m\right). \tag{441}$$

Integral representations, limiting cases.
At the end of our study of renormalization in QED, we present the integral representation of two renormalized quantities and discuss some of their properties.

We recall the expression for $\Pi^{\rm ren}\left(p^2\right)$:

$$\Pi^{\rm ren}\left(p^2\right) = \frac{1}{9} + \frac{1}{3}\left(1 - 2\frac{m^2}{p^2}\right)\int_0^1 dx \ln\frac{\chi\left(p^2, x\right)}{m^2}\,, \tag{442}$$

with

$$\chi\left(p^2,x\right)=p^2x\left(1-x\right)+m^2. \tag{443}$$

For very low p^2, one has

$$\Pi^{\rm ren}\left(p^2\right)=\frac{p^2}{15\,m^2}\,,\qquad \text{for}\quad p^2\to 0\,, \tag{444}$$

which is the well-known contribution to the Uehling effect, i.e. the modification of Coulomb law due to vacuum polarization.
Alternatively for large $s=-p^2$, we have

$$\Pi^{\rm ren}\left(p^2\right)=\frac{1}{3}\left(\ln\frac{s}{m^2}-i\,\pi\right),\qquad \text{for}\quad s=-p^2\to\infty. \tag{445}$$

The $F_1^{\rm ren}\left(Q^2,m\right)$ in an integral form reads:

$$\begin{aligned}F_1^{\rm ren}\left(Q^2,m\right) &= 2\left(\frac{1}{\hat\varepsilon}+\ln\frac{m^2}{\mu^2}\right)\left[1-\frac{Q^2+2m^2}{2}\int_0^1 dx\frac{1}{\chi\left(Q^2,x\right)}\right]\\ &\quad -\left(Q^2+2m^2\right)\int_0^1 dx\frac{1}{\chi\left(Q^2,x\right)}\ln\frac{\chi\left(Q^2,x\right)}{m^2}\\ &\quad -\int_0^1 dx\ln\frac{\chi\left(Q^2,x\right)}{m^2}+2\left(Q^2+3m^2\right)\int_0^1 dx\frac{1}{\chi\left(Q^2,x\right)}-6\,.\end{aligned} \tag{446}$$

The last expression still contains a pole and a scale-dependent logarithm,

$$\frac{1}{\hat\varepsilon}+\ln\frac{m^2}{\mu^2}\,, \tag{447}$$

which has an infrared origin and which will be compensated in any realistic calculation by the contribution of the real *soft photons* emission and also by *the box* diagrams which are ultraviolet finite by themselves.

6.2 Non-minimal OMS renormalization scheme in the U gauge

Now we briefly discuss the on-mass-shell renormalization in the SM. In the spirit, it is absolutely analogous to that we have considered in QED. Moreover, in the U gauge, we are dealing only with physical fields, and the renormalization procedure is particularly simple.

6.2.1 Multiplicative renormalization in the SM

In the SM, the independent quantities of the scheme are: the electric charge, the masses of all particles and all fields. They undergo *a multiplicative renormalization*.
For fields:

$$\begin{array}{rclrcl}\psi^i_{0L} &=& \left(Z_L^{1/2}\right)_{ij}\psi^j_L, & \psi^i_{0R} &=& \left(Z_R^{1/2}\right)_{ij}\psi^j_R,\\ W_{0\mu} &=& Z_W^{1/2}W_\mu\,, & Z_{0\mu} &=& Z_Z^{1/2}Z_\mu\,,\\ H_0 &=& Z_H^{1/2}H\,, & A_{0\mu} &=& Z_A^{1/2}A_\mu+Z_M^{1/2}Z_\mu\,.\end{array} \tag{448}$$

For bosonic masses:

$$M^2=Z_{M_W}Z_W^{-1}M_W^2\,,\qquad M_0^2=Z_{M_Z}Z_Z^{-1}M_Z^2\,,\qquad M_{0H}^2=Z_{M_H}Z_H^{-1}M_H^2\,. \tag{449}$$

Fermionic mass renormalization is more involved, due to the mixing. We introduce *the matrices* of the renormalization constants Z_{m_f} and $Z^+_{m_f}$:

$$\mathcal{L}_{\rm ct}\sim-\left(\overline{\psi}_L Z_{m_f}\psi_R+\overline{\psi}_R Z^+_{m_f}\psi_L-\overline{\psi}m_f\psi\right). \tag{450}$$

All but one of the renormalization constants are fixed by requiring that the residue of *all* the propagators are 1. This remaining renormalization constant is associated with the renormalization of the electric charge

$$e_0 = Z_e Z_A^{-1/2} e\,. \tag{451}$$

Alternatively, one may use an *additive renormalization* of the electric charge

$$\begin{aligned} e_0^2 &= e^2 + \delta e^2, \\ \frac{\delta e^2}{e^2} &= 2\,(Z_e - 1) - (Z_A - 1)\,. \end{aligned} \tag{452}$$

If the electric charge renormalization is defined by Eq. (451), then the relevant Ward identity implies

$$Z_e \equiv 1\,. \tag{453}$$

Within the OMS renormalization scheme, one has to adopt two definitions, valid *to all orders* in the perturbation theory.

1. The OMS weak mixing angle, θ_W ($c_W = \cos\theta_W$):

$$M_Z^2 c_W^2 = M_W^2\,; \tag{454}$$

2. The OMS weak charge, g:

$$g^2 = \frac{e^2}{s_W^2}\,, \qquad \left(s_W^2 = 1 - c_W^2 = 1 - \frac{M_W^2}{M_Z^2} \right). \tag{455}$$

The necessity to adopt them as definitions follows from the fact that s_W and g are not independent quantities in this framework.

6.2.2 *Counter-term Lagrangian*

With the aid of the same procedure used in the case of QED, it is rather easy to derive the counter-terms Lagrangian from an original one, using multiplicative renormalization Eqs. (448)–(451). Here we present only the final result.

The kinetic and mass terms for bosonic fields are

$$\begin{aligned} \mathcal{L}_{\rm ct}^{{\rm kin},A} &= -\frac{1}{4}\,(Z_A - 1)\,(A_{\mu\nu})^2\,, \\ \mathcal{L}_{\rm ct}^{{\rm kin},Z} &= -\frac{1}{4}\,(Z_Z + Z_M - 1)\,(Z_{\mu\nu})^2 - \frac{1}{2}\left(Z_{M_Z} - 1\right) M_Z^2\,(Z_\mu)^2 - \frac{1}{2} Z_A^{1/2} Z_M^{1/2} A_{\mu\nu} Z_{\mu\nu}\,, \\ \mathcal{L}_{\rm ct}^{{\rm kin},W} &= -\frac{1}{2}\,(Z_W - 1)\,|W_{\mu\nu}|^2 - \left(Z_{M_Z} - 1\right) M_W^2\,|W_\mu|^2, \\ \mathcal{L}_{\rm ct}^{{\rm kin},H} &= -\frac{1}{2}\,(Z_H - 1)\,(\partial_\mu H)^2 - \frac{1}{2}\left(Z_{M_H} - 1\right) M_H^2 H^2, \end{aligned} \tag{456}$$

with

$$V_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu\,. \tag{457}$$

The fermionic kinetic term reads,

$$\mathcal{L}_{\rm ct}^{{\rm kin},f} = -\frac{1}{2}\,\overline{\psi}\partial\!\!\!/ \left[\left(\sqrt{Z_L}^{\,\dagger}\sqrt{Z_L} - I\right)\gamma_+ + \left(\sqrt{Z_R}^{\,\dagger}\sqrt{Z_R} - I\right)\gamma_-\right]\psi. \tag{458}$$

Here we introduced:

$$\gamma_\pm = 1 \pm \gamma_5\,. \tag{459}$$

Since $\sqrt{Z_L}$ and $\sqrt{Z_R}$ are understood to be the matrices acting in the full fermionic-flavour space, the equation,

$$\left|\sqrt{Z_L}\right|^2 = \sqrt{Z_L}^{\,\dagger}\sqrt{Z_L}\,, \tag{460}$$

should be understood as a notation. In general, these matrices are non-diagonal and even non-Hermitian, due to the mixing induced by loop corrections. The renormalization requirement fixes the combinations,

$$\left|\sqrt{Z_L}\right|^2 - I\,, \qquad \left|\sqrt{Z_R}\right|^2 - I\,, \tag{461}$$

which directly enter the kinetic term.

In the one-loop approximation we may consistently accept that $\sqrt{Z_{L,R}}$ are Hermitian matrices, then

$$\sqrt{Z_{L,R}} - I = \frac{1}{2}\left(\left|\sqrt{Z_{L,R}}\right|^2 - I\right), \tag{462}$$

and all the combinations entering the interaction Lagrangian become known.

For the $V(H)f\overline{f}$ interaction parts of the Lagrangian one obtains:

$$\begin{aligned}
\mathcal{L}^{\gamma f\overline{f}}_{\rm ct} &= \frac{i}{2}eQ_f\overline{\psi}\gamma_\mu\left[\left(\left|\sqrt{Z_L}\right|^2 - I\right)\gamma_+ + \left(\left|\sqrt{Z_R}\right|^2 - I\right)\gamma_- + 2\left(Z_e - 1\right)\right]\psi A_\mu\,,\\
\mathcal{L}^{Zf\overline{f}}_{\rm ct} &= \frac{i}{2}\frac{e}{s_W c_W}\overline{\psi}\gamma_\mu\Bigg\{\left[\left|\sqrt{Z_L}\right|^2\left(\frac{Z_{M_Z}Z_W}{Z_A Z_{M_W}Z_c}\right)^{1/2} - I\right]I^{(3)}_f\gamma_+\\
&\quad -2Q_f s^2_W\left[\frac{1}{2}\left(\left|\sqrt{Z_L}\right|^2\gamma_+ + \left|\sqrt{Z_R}\right|^2\gamma_-\right)\left(\frac{Z_{M_Z}Z_W}{Z_A Z_{M_W}Z_c}\right)^{1/2} - I\right]\\
&\quad -2Q_f s_W c_W\left(\frac{1}{2}\left|\sqrt{Z_L}\right|^2\gamma_+ + \frac{1}{2}\left|\sqrt{Z_R}\right|^2\gamma_-\right)\left(\frac{Z_M}{Z_A}\right)^{1/2}\Bigg\}\psi Z_\mu\,,\\
\mathcal{L}^{Wf\overline{f}'}_{\rm ct} &= \frac{i}{2\sqrt{2}}\frac{e}{s_W}\overline{\psi}^u\gamma_\mu\gamma_+\left[\sqrt{Z_{uL}}^{\,\dagger}C\sqrt{Z_{dL}}\left(\frac{Z_W}{Z_A Z_c}\right)^{1/2} - C\right]\psi^d + h.c.\\
\mathcal{L}^{Hf\overline{f}}_{\rm ct} &= -\frac{e}{2M_W s_W}\overline{\psi}\left[\frac{1}{2}\left(Z_{m_f}\gamma_- + Z^+_{m_f}\gamma_+\right)\left(\frac{Z_H Z_W}{Z_A Z_{M_W}Z_c}\right)^{1/2} - m_f\right]\psi,
\end{aligned} \tag{463}$$

where

$$Z_c = 1 - \frac{\delta c^2_W}{s^2_W}\,, \qquad s^2_W = 1 - \frac{M^2_W}{M^2_Z}\,, \qquad \frac{\delta c^2_W}{c^2_W} = \frac{\delta M^2_W}{M^2_W} - \frac{\delta M^2_Z}{M^2_Z}\,, \tag{464}$$

with M_W and M_Z being the physical masses of the vector bosons, and C being the CKM mixing matrix.

The full list of bosonic renormalization constants, which is derived after their fixation by *residue one* requirements, looks as follows: (we note that an unnatural looking of the first three rows is an artifact of the definition Eq. (449)):

$$\begin{aligned}
Z_{M_W} - Z_W &= \frac{\delta M^2_W}{M^2_W} = \frac{g^2}{16\pi^2 M^2_W}\Sigma_{WW}\left(M^2_W\right),\\
Z_{M_Z} - Z_Z &= \frac{\delta M^2_Z}{M^2_Z} = \frac{g^2}{16\pi^2 c^2_\theta M^2_Z}\Sigma_{ZZ}\left(M^2_Z\right),\\
Z_{M_H} - Z_H &= \frac{\delta M^2_H}{M^2_H} = \frac{g^2}{16\pi^2 M^2_H}\Sigma_{HH}\left(M^2_H\right),
\end{aligned}$$

$$
\begin{aligned}
Z_M^{1/2} &= \frac{g^2 s_W}{16\pi^2 c_W M_Z^2}\Sigma_{ZA}\left(M_Z^2\right), \\
Z_A - 1 &= \frac{e^2}{16\pi^2}\Pi_{\gamma\gamma}(0), \\
Z_Z - 1 &= \frac{g^2}{16\pi^2 c_\theta^2}\frac{\partial\Sigma_{ZZ}(p^2)}{\partial p^2}\bigg|_{p^2=-M_Z^2}, \\
Z_W - 1 &= \frac{g^2}{16\pi^2}\frac{\partial\Sigma_{WW}(p^2)}{\partial p^2}\bigg|_{p^2=-M_W^2}, \\
Z_H - 1 &= \frac{g^2}{16\pi^2}\frac{\partial\Sigma_{HH}(p^2)}{\partial p^2}\bigg|_{p^2=-M_H^2}.
\end{aligned} \tag{465}
$$

It should be noted that we use a convention for arguments. For every self-energy function: Σ_{VV}, $\Pi_{\gamma\gamma}$,... if $p^2 = -s$ or $p^2 = -M^2$, we will omit the minus sign, i.e. we will write $\Sigma_{VV}(s)$... On the contrary, in the argument list of every $B_k,\ C_k$... function, we will explicitly maintain the sign.

6.2.3 *Linearized form of the counter-term Lagrangian*

Since we are working within the perturbation theory, where all renormalization constants are a power series in the coupling constant e^2 (cf. Eq. (418)), we may simplify a little the counter-term interaction Lagrangian Eq. (464) and rewrite it as

$$
\begin{aligned}
\mathcal{L}_{\rm ct}^{Zf\overline{f}} &= \frac{i}{2}\frac{e}{s_W c_W}\overline{\psi}\gamma_\mu\Bigg\{\Bigg[\left|\sqrt{Z_L}\right|^2 - I + \frac{1}{2}\left((Z_Z-1)-(Z_A-1)+\frac{c_W^2-s_W^2}{s_W^2}\frac{\delta c_W^2}{c_W^2}\right)\Bigg] I_f^{(3)}\gamma_+ \\
&\quad -2Q_f s_W^2\left[\frac{1}{2}\left(\left|\sqrt{Z_L}\right|^2 - I\right)\gamma_+ + \frac{1}{2}\left(\left|\sqrt{Z_R}\right|^2 - I\right)\gamma_-\right. \\
&\quad \left.+\frac{1}{2}\left((Z_Z-1)-(Z_A-1)-\frac{1}{s_W^2}\frac{\delta c_W^2}{c_W^2}\right)+\frac{c_W}{s_W}Z_M^{1/2}\right]\Bigg\}\psi Z_\mu\,, \\
\mathcal{L}_{\rm ct}^{Wf\overline{f}'} &= \frac{i}{2\sqrt{2}}\frac{e}{s_W}\overline{\psi}^u\gamma_\mu\gamma_+\Bigg\{\left(\sqrt{Z_{uL}}-I\right)C + C\left(\sqrt{Z_{dL}}-I\right) \\
&\quad +C\left[\frac{1}{2}(Z_W-1)-\frac{1}{2}(Z_A-1)+\frac{\delta c_W^2}{2s_W^2}\right]\Bigg\}\psi^d + h.c., \\
\mathcal{L}_{\rm ct}^{Hf\overline{f}} &= -\frac{e}{2M_W s_W}\overline{\psi}\Bigg\{\left(Z_{m_f}-m_f\right)+m_f\left[\frac{1}{2}(Z_H-1)-\frac{1}{2}\left(Z_{M_W}-1\right)\right. \\
&\quad \left.+\frac{1}{2}(Z_W-1)-\frac{1}{2}(Z_A-1)+\frac{1}{2}\frac{\delta c_W^2}{s_W^2}\right]\Bigg\}\psi.
\end{aligned} \tag{466}
$$

This is the so-called linearized form of the counter-term Lagrangian from which one easily derives additional Feynman rules for vertices involving renormalization constants.

6.2.4 *Fermionic renormalization constants*

In previous sections we calculated all the renormalization constants associated with bosonic fields and masses. We still need to fix fermionic renormalization constants, Eq. (461), and fermionic mass renormalization, Z_{m_f}, Eq. (450).

The procedure of fixation is very similar to that of QED, although it has some peculiar features due to the presence of γ_5. Below we briefly sketch the procedure.

Consider a fermionic self-energy diagram:

f

f f

B

The most general expression for such a diagram looks like:

$$\Sigma(i\not{p}) = (2\pi)^4\, i \left[a_1(p^2) + a_2(p^2)\gamma_5 + \left(a_3(p^2) - a_4(p^2)\gamma_5\right) i\not{p}\right]. \tag{467}$$

However, in the Standard Model we have always $a_2(p^2) \equiv 0$ (see the proof in Section 6.6 of Ref. [1]). Therefore, it reduces to

$$\Sigma(i\not{p}) = (2\pi)^4\, i \left[a_1(p^2) + a_3(p^2) i\not{p} + a_4(p^2) i\not{p}\gamma_5\right]. \tag{468}$$

The kinetic and mass terms of the counter-term Lagrangian may be symbolically depicted as:

f f

and their contribution, derived from Eqs. (450) and (458), as:

$$-\frac{1}{2} i\not{p} \left[\left(\left|\sqrt{Z_L}\,\right|^2 - I\right)\gamma_+ + \left(\left|\sqrt{Z_R}\,\right|^2 - I\right)\gamma_-\right] - \left(Z_{m_f} - m_f\right). \tag{469}$$

From the requirement that the sum vanishes on the fermion mass shell, one derives all the fermionic renormalization constants:

$$\begin{aligned}
\left|\sqrt{Z_L}\,\right|^2 - I &= a_3(m^2) - 2m^2 a_3'(m^2) + 2m a_1'(m^2) + a_4(m^2)\,,\\
\left|\sqrt{Z_R}\,\right|^2 - I &= a_3(m^2) - 2m^2 a_3'(m^2) + 2m a_1'(m^2) - a_4(m^2)\,,\\
Z_{m_f} &= m + a_1(m^2) + 2m^2 a_1'(m^2) - 2m^3 a_3'(m^2)\,,
\end{aligned} \tag{470}$$

where a_i' denotes the derivatives, $a_i'(m^2) = \partial a_i(p^2)/\partial p^2|_{p^2=-m^2}$ and where we have used the expansion:

$$a_i(p^2) = a_i(m^2) + 2m(i\not{p} + m)a_i'(m^2)\,, \tag{471}$$

assuming that from the left side of Eq. (468) the Dirac equation holds, i.e. $i\not{p} = -m$. Equation (470) is obtained by means of re-shuffling the terms as follows, $A + B(i\not{p} + m) + \mathcal{O}\left((i\not{p} + m)^2\right)$, and requiring $A = 0,\ B = 0$. Higher-order terms $\mathcal{O}\left((i\not{p} + m)\right)^2$ may be neglected on the mass shell.

To summarize our study of the renormalization procedure, we recall the important steps:

- Dyson re-summation;
- Invention of the renormalization constants;
- Construction of the counter-term Lagrangian;
- Fixation of the renormalization constants in the spirit of the OMS scheme;
- Physical meaning of *the residue one* requirement.

We recall that *the residue one* requirement means that we preserve the physical meaning of the parameters of the original Lagrangian. This means in turn, that renormalization has nothing to do with the cancellation of divergences. We obtain the cancellation of UV-divergences *for free* as a byproduct of a procedure aimed at preserving the physical meaning of the parameters from being changed by radiative corrections. This means in turn, that within a renormalizable theory, the problem of UV-divergences simply does not present. We need, of course, to exploit a well-defined *regularization*, nowadays dimensional regularization, in order to parametrize the divergences of individual terms. However, after proper treatment of the Lagrangian parameters, all the UV-divergences cancel identically, in other words the theory becomes UV-finite.

It is necessary to understand, however, one important difference of the SM from usual QED. In QED we were able to introduce the notion of *the renormalized diagram* for every individual diagram, see Eqs. (442) and (446). In the SM it is, in general, impossible. As an example, consider Z self-energy (Fig. 5) in general R_ξ gauge. It could be subdivided into a fermionic component, Fig. 5.(1), and bosonic one, Fig. 5.(2-14).

Define *the renormalized* self-energy by means of the expression:

$$\Sigma^{\rm ren}_{ZZ}(p^2) = \Sigma_{ZZ}(p^2) - \Sigma_{ZZ}(M^2_Z) - (p^2 + M^2_Z)\frac{\partial \Sigma_{ZZ}(p^2)}{\partial p^2}\bigg|_{p^2=-M^2_Z}. \tag{472}$$

It is easy to verify that the fermionic component of $\Sigma^{\rm ren}_{ZZ}(p^2)$, which is known to be gauge-invariant, is free of UV-pole, and therefore, full analogy with QED holds. However, the bosonic component of Eq. (472), although also UV-free, does depend on ξ, and therefore the notion of the renormalized self-energy diagram is meaningless. In the unitary gauge, the quantity $\Sigma^{\rm ren}_{ZZ}(p^2)$ even contains UV-divergences. The gauge-dependent terms cancel in the sum of self-energy, vertex and box diagrams for a physical amplitude. The same is true for UV-poles in the unitary gauge.

With this minimal knowledge about the renormalization procedure, we are ready to discuss the amplitudes for some physical processes.

6.3 One-loop amplitudes

6.3.1 The Born amplitude and diagrams

To approach the discussion of the amplitudes and to introduce more notions, we begin with the Born approximation of the amplitude of the process $e^+e^- \to f\overline{f}$. It is described by the two tree-level diagrams with γ and Z exchanges:

e^+ $\overline{f}$ e^+ $\overline{f}$

A $+$ Z

e^- f e^- f

The photon exchange amplitude has a unique *vector* $\otimes$ *vector* structure, whilst the Z exchange amplitude may be written in two *basises*, VA or LQ:

$$\mathcal{A}^{\rm Born}_{\gamma} = \frac{e^2 Q_e Q_f}{s}\gamma_\mu \otimes \gamma_\mu\,,$$

$$\mathcal{A}^{\rm Born}_{Z} = \frac{e^2}{4s^2_W c^2_W}\chi_Z(s)\,\gamma_\mu(v_e + a_e\gamma_5)\otimes\gamma_\mu(v_f + a_f\gamma_5)\ -\ \text{VA-basis},$$

$$\mathcal{A}^{\rm Born}_{Z} = \frac{e^2}{4s^2_W c^2_W}\chi_Z(s)\,\gamma_\mu\left[I^{(3)}_e\gamma_+ - 2Q_e s^2_W\right]\otimes\gamma_\mu\left[I^{(3)}_f\gamma_+ - 2Q_f s^2_W\right]\ -\ \text{LQ-basis}. \tag{473}$$

Here $\gamma_{\pm} = 1 \pm \gamma_5$ and the symbol $\otimes$ is used in the following short-hand notation:

$$\gamma_\mu \left(v_1 + a_1\gamma_5\right) \otimes \gamma_\nu \left(v_2 + a_2\gamma_5\right) = \bar{u}\left(p_+\right)\gamma_\mu \left(v_1 + a_1\gamma_5\right) v\left(p_-\right) \bar{v}\left(q_-\right)\gamma_\nu \left(v_2 + a_2\gamma_5\right) u\left(q_+\right). \quad (474)$$

Furthermore, the $\chi_Z(s)$ denotes the Z boson propagator:

$$\chi_Z(s) = \frac{1}{s - M_Z^2 + is\Gamma_Z/M_Z}\,. \quad (475)$$

From the basic relations between the parameters,

$$\frac{g^2}{8M_W^2} = \frac{G_F}{\sqrt{2}}\,, \qquad s_W^2 = \frac{e^2}{g^2}\,, \qquad c_W^2 = 1 - s_W^2 = \frac{M_W^2}{M_Z^2}\,, \quad (476)$$

one easily derives

$$\frac{e^2}{4s_W^2 c_W^2} = \sqrt{2}G_F M_Z^2\,, \quad (477)$$

or, using $e^2 = 4\pi\alpha$, we define *the conversion factor*:

$$f = \frac{\sqrt{2}G_F M_Z^2 s_W^2 c_W^2}{\pi\alpha}\,, \quad (478)$$

which is equal to one in the lowest order. Of course, it may differ from one due to radiative corrections. This will be the subject of next section.

6.4 Muon decay, Sirlin's parameter Δr

As already mentioned in the first lecture, one has to exploit somehow the precise measurement of the muon lifetime, since in terms of the Fermi coupling constant, $G_F = 1.16639(2) \times 10^{-5}$ GeV^{-2}, the relevant accuracy is $\mathcal{O}\left(10^{-5}\right)$. In this section we briefly discuss the relevant issues. For a complete presentation, see Chapter 4 and Section 7.13 of Ref. [1].

6.4.1 Muon lifetime

The process being considered is

$$\mu \to e + \nu_\mu + \overline{\nu}_e\,. \quad (479)$$

If one includes lowest-order QED corrections and W boson propagator effects, then for the inverse muon life-time one obtains, (cf. with the standard presentation in Ref. [6]):

$$\frac{1}{\tau_\mu} = \frac{G_F^2 m_\mu^5}{192\,\pi^3} F\left(\frac{m_e^2}{m_\mu^2}\right)\left(1 + \frac{3}{5}\frac{m_\mu^2}{M_W^2}\right)\left[1 + \frac{\alpha\left(m_\mu^2\right)}{2\,\pi}\left(\frac{25}{4} - \pi^2\right)\right], \quad (480)$$

where

$$F(r) = 1 - 8\,r + 8\,r^3 - r^4 - 12\,r^2 \ln r \quad (481)$$

is the phase space factor, and $\alpha^{-1}\left(m_\mu^2\right) \approx 136$ is the QED running coupling constant at the scale m_μ.

This low-energy decay process may be described with the effective four-fermion Fermi Lagrangian

$$\mathcal{L}_F = \frac{G_F}{\sqrt{2}}\,\overline{\psi}_e \gamma_\mu \gamma_+ \psi_\mu \overline{\psi}_{\nu_\mu} \gamma_\mu \gamma_+ \psi_{\nu_e} + h.c. \quad (482)$$

One usually calculates the observable distribution, $dI^0(x)$, in terms of a kinematical variable $x = \frac{2E_e}{m_\mu}$, where E_e is the electron energy in the muon rest frame.

If the electron mass is neglected, x varies from 0 to 1. Then, in the lowest order (tree level), one has

$$dI^0(x) = \frac{G_F^2 m_\mu^5}{96\,\pi^3}\, x^2\,(3-2\,x)dx \;\rightarrow\; \frac{1}{\tau_\mu} = \frac{G_F^2 m_\mu^5}{192\,\pi^3}\,. \tag{483}$$

Since the electron mass m_e is very small, it is sufficient to calculate the real and virtual QED radiative corrections ignoring the electron mass.

6.4.2 *Real corrections in μ-decay*

The bremsstrahlung, or *real photon* emission, in μ-decay, i.e. the process

$$\mu \rightarrow e + \nu_\mu + \overline{\nu}_e + \gamma\,, \tag{484}$$

is described in Fermi theory by two Feynman diagrams:

e $\quad$ e

μ $\quad$ $\overline{\nu}_e$ $\quad$ μ $\quad$ $\overline{\nu}_e$

ν_μ $\quad$ ν_μ

The quantity of experimental interest is the transition probability summed over the full photonic phase space. After lengthy calculations of the decay probability of the bremsstrahlung process, one derives:

$$\begin{aligned}
dI^{\rm r}(x) &= \frac{G_F^2 m_\mu^5}{96\,\pi^3}\,\frac{\alpha}{2\,\pi}\,\mathcal{I}(x)dx,\\
\mathcal{I}(x) &= 2x^2\,(3-2x)\left\{\left[\frac{1}{\hat{\varepsilon}} + \ln\frac{m_\mu m_e}{\mu^2} + \ln\frac{(1-x)^2}{x}\right](L-1) + {\rm Li}_2\,(x) - {\rm Li}_2\,(1)\right\}\\
&\quad -x\,(3-2x)\,(1-x)\ln(1-x) + \frac{1}{3}\,(1-x)\left[\left(5+17x-34x^2\right)L - 22x + 34x^2\right],\\
\text{where}\quad L &= \ln\left(x\frac{m_\mu}{m_e}\right).
\end{aligned} \tag{485}$$

Note the appearance of the IR-pole term, which is due to the soft photon emission.

6.4.3 *Virtual QED corrections for μ-decay*

There are three diagrams that contribute to the $\mathcal{O}\,(\alpha)$ QED corrections in the Fermi theory, which are due to *the virtual photon* exchange (actually only the third one, see Section 6.1):

e

μ $\quad$ $\overline{\nu}_e$

ν_μ

The effect of the virtual diagrams may be seen as *dressing* the lowest-order interaction $\bar{u}_e\gamma_\alpha\gamma_+u_\mu$ with QED corrections, resulting in the appearance of a more complicated structure:

$$-\frac{\alpha}{4\,\pi}\left(\mathcal{F}_1\gamma_\alpha\gamma_+ + \frac{i}{m_\mu}F_2\,q_{\mu,\alpha}\gamma_- + \frac{i}{m_\mu}F_3\,q_{e,\alpha}\gamma_-\right). \tag{486}$$

Note that only the first term has the Born-like structure, the second and third ones are new and this is why we use the notion of *induced structures* for them. The result of calculating the diagrams reads:

$$\begin{aligned}\mathcal{F}_1 &= 2\left[\left(\frac{1}{\hat{\varepsilon}}+\ln\frac{m_\mu m_e}{\mu^2}\right)(L-1)+\mathrm{Li}_2\,(1)-\mathrm{Li}_2\,(x)\right]\\ &\quad+\left[2L-2\ln(1-x)+\frac{1}{1-x}\right]\ln x-3L+4\,,\\ F_2 &= \frac{2}{(1-x)^2}\Big[x\ln x+1-x\Big],\\ F_3 &= \frac{2}{(1-x)^2}\Big[(1-2\,x)\ln x-1+x\Big].\end{aligned} \tag{487}$$

The virtual corrections contribute via interference of the amplitude Eq. (486) with the Born amplitude. After calculating the traces one derives:

$$dI^{0+\mathrm{v}}(x)=\left(1-\frac{\alpha}{2\,\pi}\mathcal{F}_1\right)dI^{(0)}(x)+\frac{G_F^2m_\mu^5}{96\,\pi^3}\,\frac{\alpha}{4\,\pi}\,x^3\,(F_2+F_3)\,dx\,. \tag{488}$$

The lowest-order result is multiplied by a correction factor, $\mathcal{F}_1$, which is ultraviolet finite (after renormalization), but infrared divergent; the induced form factors, F_2 and F_3, are finite. The latter should be the case, since there are no other sources to compensate any divergence of induced form factors. The infrared divergence must cancel when we combine the contribution of the virtual photons, Eq. (488), with real photons contribution, $dI^{\mathrm{r}}(x)$, Eq. (485).

6.4.4 *Total QED corrections for μ-decay*

The experimentally observable quantity is the sum of the two transition probabilities for real and virtual processes, which is free of infrared divergences. For the sum, we obtain:

$$\begin{aligned}dI(x) &= \frac{G_F^2m_\mu^5}{96\,\pi^3}\left[x^2\,(3-2\,x)+\frac{\alpha}{2\,\pi}\Delta I(x)\right]dx\,,\\ \Delta I(x) &= 2x^2\,(3-2\,x)\left[\left(2\,\ln\frac{1-x}{x}+\frac{3}{2}\right)(L-1)+2\,\mathrm{Li}_2\,(x)-2\,\mathrm{Li}_2\,(1)+\ln(1-x)\ln x-\frac{1}{2}\right.\\ &\quad\left.-\frac{1-x}{x}\ln(1-x)\right]-3x^2\ln x+\frac{1-x}{3}\left[\left(5+17\,x-34\,x^2\right)\,L-22\,x+34\,x^2\right].\end{aligned} \tag{489}$$

The total QED correction is derived by integrating $dI(x)$ over x from 0 to 1, yielding

$$\frac{1}{\tau_\mu}=\frac{G_F^2m_\mu^5}{192\,\pi^3}\left[1+\frac{\alpha}{2\,\pi}\left(\frac{25}{4}-\pi^2\right)\right]. \tag{490}$$

Let us emphasize again that this result was calculated within QED $\otimes$ effective 4-fermion Fermi theory. Of course, the calculation could be performed exclusively within the Standard Model framework. This would give something like

$$\frac{1}{\tau_\mu}=\frac{m_\mu^5}{192\pi^3}\,\frac{g^4}{32\,M_W^4}\,\left(1+\frac{\alpha}{2\pi}\delta_\mu\right). \tag{491}$$

However, the Fermi constant was *historically defined* by Eq. (490). This is why we sketched, first of all, a derivation within the Fermi theory. Now we turn to a complete calculation within the SM.

6.4.5 EW corrections for muon decay, Sirlin's parameter Δr

Turning to the discussion of complete one-loop corrections to μ decay, we note first of all that the QED corrections, discussed in the previous section, form a gauge-invariant result. Moreover, both the QED and remaining EW corrections are infrared- and ultraviolet-finite and gauge-invariant, therefore they can be treated separately and we may write:

$$\delta_\mu = \delta^{em}_\mu + \delta^{ew}_\mu . \tag{492}$$

or, equalizing Eqs. (490) and (491), we obtain

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8M^2_W}\left[1+\frac{\alpha}{4\pi}\left(\delta_\mu - \delta^{em}_\mu\right)\right] = \frac{g^2}{8M^2_W}\left(1+\frac{\alpha}{4\pi}\delta^{ew}_\mu\right). \tag{493}$$

Recalling the basic definitions of the OMS scheme

$$M^2_Z c^2_W = M^2_W\,, \qquad g^2 = \frac{e^2}{s^2_W}\,, \tag{494}$$

and using them, we derive

$$s^2_W c^2_W = \frac{\pi\alpha}{\sqrt{2}G_F M^2_Z}\left(1+\Delta r\right), \qquad \text{with} \quad \Delta r = \frac{\alpha}{4\pi}\delta^{ew}_\mu\,. \tag{495}$$

Alberto Sirlin (1980) suggested that the last equation be rewritten as

$$s^2_W c^2_W = \frac{\pi\alpha}{\sqrt{2}G_F M^2_Z}\,\frac{1}{1-\Delta r}\,, \tag{496}$$

as if it could be *re-summed* to all orders (similar to the Dyson re-summation) as would be true for $\Delta\alpha$ (see Eq. (502) below).

After lengthy calculations (see Section 7.13 of Ref. [1]), one derives the finite result for Δr in the one-loop approximation

$$\Delta r = \frac{\alpha}{4\pi}\frac{1}{s^2_W}\left\{s^2_W\left[-\frac{2}{3}-\Pi^{\mathrm{fer},F}_{\gamma\gamma}(0)\right]+\frac{c^2_W}{s^2_W}\Delta\rho^F+\Delta\rho^F_W+\frac{11}{2}-\frac{5}{8}c^2_W\left(1+c^2_W\right)+\frac{9c^2_W}{4s^2_W}\ln c^2_W\right\}, \tag{497}$$

where *finite* parts of the $\Delta\rho^F$ factors defined by,

$$\Delta\rho^F = \Delta\rho^{\mathrm{bos},F}+\Delta\rho^{\mathrm{fer},F}, \qquad \Delta\rho^F_W = \Delta\rho^{\mathrm{bos},F}_W+\Delta\rho^{\mathrm{fer},F}_W, \tag{498}$$

have *fer*mionic and *bos*onic contributions

$$\begin{aligned}
\Delta\rho^{\mathrm{bos(fer)},F} &= \frac{1}{M^2_W}\left[\Sigma^{\mathrm{bos(fer)},F}_{WW}\left(M^2_W\right)-\Sigma^{\mathrm{bos(fer)},F}_{ZZ}\left(M^2_Z\right)\right],\\
\Delta\rho^{\mathrm{bos(fer)},F}_W &= \frac{1}{M^2_W}\left[\Sigma^{\mathrm{bos(fer)},F}_{WW}(0)-\Sigma^{\mathrm{bos(fer)},F}_{WW}\left(M^2_W\right)\right].
\end{aligned} \tag{499}$$

The bosonic contributions, written down explicitly, are

$$\begin{aligned}
\Delta\rho^{\mathrm{bos},F}_W = &-\left(\frac{1}{12c^4_W}+\frac{4}{3c^2_W}-\frac{17}{3}-4c^2_W\right)B^F_0\left(-M^2_W;M_Z,M_W\right)\\
&-\left(1-\frac{1}{3}w_h+\frac{1}{12}w^2_h\right)B^F_0\left(-M^2_W;M_H,M_W\right)\\
&+\left[\frac{3}{4\left(1-w_h\right)}+\frac{1}{4}-\frac{1}{12}w_h\right]w_h\ln w_h+\left(\frac{1}{12c^4_W}+\frac{17}{12c^2_W}-\frac{3}{s^2_W}+\frac{1}{4}\right)\ln c^2_W
\end{aligned}$$

$$
+\frac{1}{12c_W^4}+\frac{11}{8c_W^2}+\frac{139}{36}-\frac{177}{24}c_W^2+\frac{5}{8}c_W^4-\frac{1}{12}w_h\left(\frac{7}{2}-w_h\right),
$$

$$
\begin{aligned}
\Delta\rho^{\mathrm{bos},F} = & -\left(\frac{1}{12c_W^2}+\frac{4}{3}-\frac{17}{3}c_W^2-4c_W^4\right)\left[B_0^F\left(-M_Z^2;M_W,M_W\right)-\frac{1}{c_W^2}B_0^F\left(-M_W^2;M_Z,M_W\right)\right] \\
& +\left(1-\frac{1}{3}w_h+\frac{1}{12}w_h^2\right)B_0^F\left(-M_W^2;M_H,M_W\right) \\
& -\left(1-\frac{1}{3}z_h+\frac{1}{12}z_h^2\right)\frac{1}{c_W^2}B_0^F\left(-M_Z^2;M_H,M_Z\right) \\
& +\frac{1}{12}s_W^2 w_h^2(\ln w_h-1)-\left(\frac{1}{12c_W^4}+\frac{1}{2c_W^2}-2+\frac{1}{12}w_h\right)\ln c_W^2 \\
& -\frac{1}{12c_W^4}-\frac{19}{36c_W^2}-\frac{133}{18}+8c_W^2\,, \qquad (500)
\end{aligned}
$$

where we introduced two ratios

$$
w_h=\frac{M_H^2}{M_W^2}\,,\qquad z_h=\frac{M_H^2}{M_Z^2}\,, \qquad (501)
$$

and the finite parts of the B_0 function as in Eqs. (361)–(362).

6.4.6 *Re-summation of large corrections*

In order to reach a high precision of theoretical predictions, one has to improve upon the one-loop expression. We begin with the extraction of $\Delta\alpha^{\mathrm{fer}}\left(M_Z^2\right)$ from Δr. From the definition of $\Delta\alpha^{\mathrm{fer}}\left(M_Z^2\right)$,

$$
\alpha^{\mathrm{fer}}\left(M_Z^2\right)=\frac{\alpha}{1-\Delta\alpha^{\mathrm{fer}}\left(M_Z^2\right)}\,, \qquad (502)
$$

and the definition of the e.m. running coupling

$$
\alpha(s)=\frac{\alpha}{1-\dfrac{\alpha}{4\pi}\Pi^F(s)}\,,\qquad \text{with}\qquad \Pi^F(s)=\Pi^{\mathrm{ren}}(s)=\Pi_{\gamma\gamma}(s)-\Pi_{\gamma\gamma}(0), \qquad (503)
$$

we derive the following representation for Δr

$$
\begin{aligned}
\Delta r = & \ \Delta\alpha^{\mathrm{fer}}\left(M_Z^2\right)+\frac{\alpha}{4\pi s_W^2}\Bigg\{s_W^2\left[-\frac{2}{3}-\Pi_{\gamma\gamma}^{t,F}(0)-\Pi_{\gamma\gamma}^{l+5q,F}\left(M_Z^2\right)\right] \\
& +\frac{c_W^2}{s_W^2}\Delta\rho^F+\Delta\rho_W^F+\frac{11}{2}-\frac{5}{8}c_W^2\left(1+c_W^2\right)+\frac{9c_W^2}{4s_W^2}\ln c_W^2\Bigg\}, \qquad (504)
\end{aligned}
$$

where the superscript $l+5q$ stands for a summation over leptons and five light quarks.

Note, that the running QED coupling, $\Delta\alpha^{\mathrm{fer}}\left(M_Z^2\right)$, is defined at the scale $\mu=M_Z$. The two quantities in Eq. (499) are defined at the scale $\mu=M_W$ as an artifact of the definition in Eq. (361). It is reasonable to re-scale all the relevant quantities to *the natural* value of the scale $\mu=M_Z$. The quantity $\Delta\rho^F$, evaluated at $\mu=M_Z$, is a gauge-invariant object, and therefore a good candidate for a re-summation. Define the leading and remainder contributions to Δr:

$$
\begin{aligned}
\Delta r_L = & -\frac{\alpha}{4\pi}\frac{c_W^2}{s_W^4}\Delta\rho^F\Big|_{\mu=M_Z}\,, \\
\Delta r_{\mathrm{rem}} = & \ \frac{\alpha}{4\pi s_W^2}\Bigg\{s_W^2\left[-\frac{2}{3}-\Pi_{\gamma\gamma}^{t,F}(0)-\Pi_{\gamma\gamma}^{l+5q,F}\left(M_Z^2\right)\right]+\left(\frac{1}{6}N_f-\frac{1}{6}-7c_W^2\right)\ln c_W^2 \\
& +\Delta\rho_W^F+\frac{11}{2}-\frac{5}{8}c_W^2\left(1+c_W^2\right)+\frac{9c_W^2}{4s_W^2}\ln c_W^2\Bigg\}\Bigg|_{\mu=M_Z}. \qquad (505)
\end{aligned}
$$

The re-summed one-loop representation reads

$$\frac{\sqrt{2}G_F M_Z^2 s_W^2 c_W^2}{\pi\alpha} = \frac{1}{\left(1-\Delta\alpha^{\rm fer}\left(M_Z^2\right)-\Delta r_{\rm rem}\right)\left(1+\frac{\sqrt{2}G_F M_Z^2 s_W^2 c_W^2}{\pi\alpha}\Delta r_L\right)}\,. \tag{506}$$

The re-summation of $\Delta\alpha^{\rm fer}\left(M_Z^2\right)$ is dictated by renormalization group arguments. The re-summation of terms containing $\Delta\rho^{{\rm fer},F}$ (Δr_L) finds its roots in the two-loop EW calculations (G. Degrassi et al. 1996–1999). The Eq. (506) is therefore an improved version of the one-loop result Eq. (496).

Higher orders, in particular QCD corrections of $\mathcal{O}\left(\alpha\alpha_S\right)$ and second-order electroweak corrections $\mathcal{O}\left(G_F^2 m_t^4\right)$ and $\mathcal{O}\left(G_F^2 m_t^2 M_Z^2\right)$, are applied by means of modifications of the leading and remainder terms:

$$\Delta r_L \to \Delta r_L + \Delta r_L^{\rm ho}, \qquad \Delta r_{\rm rem} \to \Delta r_{\rm rem} + \Delta r_{\rm rem}^{\rm ho}\,. \tag{507}$$

Equation (506) formally looks like an equation for conversion factor f, cf. Eq. (478). If all radiative corrections are switched off, $f=1$, and f differs from 1 due to non-zero radiative corrections. We may consider the Eq. (506) as an equation with respect to M_W. The results of *an iterative* solution of this equation for the M_W which incorporate second-order electroweak corrections, without and with QCD correction $\mathcal{O}\left(\alpha\alpha_S\right)$, are shown in the Tab. 1. This Table is shown not only to give some taste of the

Table 1: The W-boson mass, M_W [GeV] in OMS scheme, $\alpha_S=0$ — first entry, $\alpha_S=0.120$ — second entry.

m_t [GeV]	M_H [GeV]		
	65	300	1000
170.1	80.445	80.349	80.256
	80.375	80.279	80.186
175.6	80.482	80.386	80.291
	80.409	80.312	80.219
181.1	80.521	80.423	80.329
	80.444	80.346	80.252

numbers. It shows that the two-loop corrections of $\mathcal{O}\left(\alpha\alpha_S\right)$ shift the predicted mass of the M_W boson by about 80 MeV, which is **bigger** than the present experimental error of direct measurements of M_W! It is a nice illustration of the importance of precision calculations.

6.5 Z resonance observables at one loop

Before discussing Z resonance observables, we have to give two definitions in order to understand the terminology that has arisen in the depths of the LEP community.

Definition 1 *Realistic Observables. They are the cross-sections $\sigma^f(s)$ and asymmetries $A^f(s)$ of the reactions,*

$$e^+e^- \to (\gamma, Z) \to f\overline{f}(n\gamma)\,, \tag{508}$$

calculated for a given value of $s=4E^2$ with all available higher-order corrections (QCD, EW), including real and virtual QED photonic corrections, possibly accounting for kinematical cuts.

Definition 2 *Pseudo-Observables. They are related to measured cross-sections and asymmetries by a de-convolution or unfolding procedure (i.e. undressing of QED corrections). The concept of the pseudo-observability itself is rather difficult to define. One way to introduce it is to say that the experiments measure some primordial (basically cross-sections and thereby also asymmetries) quantities which are then reduced to secondary quantities under a set of specific assumptions. Within these assumptions, the secondary quantities, the pseudo-observables, also deserve the label of observability.*

6.5.1 *The Z partial widths*

The Z partial widths represent a typical example of pseudo-observables, i.e. they have to be *defined*. At the Born level, we define the partial width of the $Z \to f\overline{f}$ decay as a quantity described by the square of one diagram:

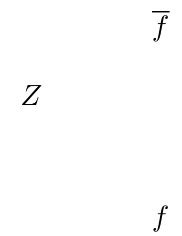


Fig. 9: Process $Z \to f\overline{f}$; Born approximation.

Its amplitude is written by the direct application of the Feynman rules of Section 2.11. Like Eq. (473), the amplitude of the process $Z \to f\overline{f}$ decay amplitude may be written in two *basises*:
VA-basis,

$$V_\mu^{Zf\overline{f}} = (2\pi)^4\, i\, \frac{ig^3}{16\,\pi^2 c_W}\,\gamma_\mu\left[v_f + a_f\,\gamma_5\right], \tag{509}$$

LQ-basis,

$$V_\mu^{Zf\overline{f}} = (2\pi)^4\, i\, \frac{ig^3}{16\,\pi^2 c_W}\,\gamma_\mu\left[I_f^{(3)}\gamma_+ - 2Q_f s_W^2\right]. \tag{510}$$

Both expressions are identical and we write both for didactic reasons only. The partial width of the $Z \to f\overline{f}$ decay in the Born approximation is given by Eq. (386) which we recall here:

$$\Gamma\left(Z \to f\overline{f}\right) \;=\; \frac{G_F M_Z^3}{6\sqrt{2}\pi}\,\beta_f\left(M_Z\right)\left[\left(v_f^2 + a_f^2\right)\left(1 + 2\frac{m_f^2}{M_Z^2}\right) - 6a_f^2\frac{m_f^2}{M_Z^2}\right]. \tag{511}$$

Here the Z–fermion couplings are defined by Eq. (401).

6.5.2 *QED diagrams and corrections*

QED corrections in the massless approximation are described by three diagrams in Fig. 10.

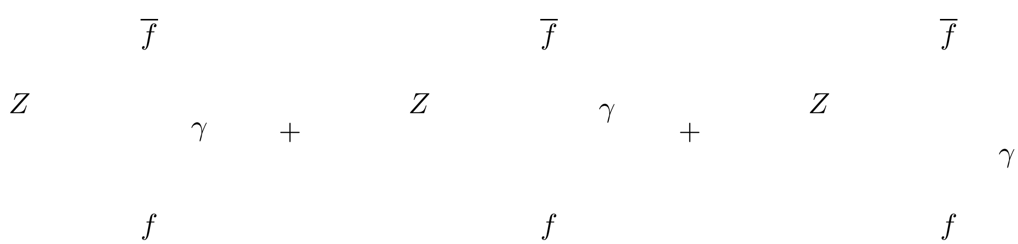


Fig. 10: Process $Z \to f\overline{f}$; QED corrections.

QED diagrams are separately gauge-invariant and finite. Their contribution integrated over the full bremsstrahlung photon phase space is (see derivation in Section 4.2)

$$\Gamma_f^{\rm QED} = \Gamma_f^{(0)}\left(1 + \frac{3}{4}\frac{\alpha}{\pi}\,Q_f^2\right). \tag{512}$$

6.5.3 *The $Z \to f\overline{f}$ decay amplitude*

All remaining one-loop diagrams refer to EW corrections. They form another gauge-invariant subset of diagrams. Recall that all the counter-terms were fixed in such a way that all external lines remain unchanged by radiative corrections, therefore only vertex diagrams and vertex-type counter-terms contribute:

$\overline{f}$ $\overline{f}$

Z $+$ Z

f f

Fig. 11: Process $Z \to f\overline{f}$; fermion vertex and its counter-terms.

The effect of radiative corrections may be parametrized in terms of *amplitude form factors*. In the massless approximation, the amplitude has a Born-like structure with only two form factors and again two basises might be used:

VA-basis,

$$V_\mu^{Zf\overline{f}} = (2\pi)^4\, i\, \frac{ig^3}{16\,\pi^2 c_W}\, \gamma_\mu \left[F_V(M_Z^2) + F_A(M_Z^2)\,\gamma_5\right], \tag{513}$$

LQ-basis,

$$V_\mu^{Zf\overline{f}} = (2\pi)^4\, i\, \frac{ig^3}{16\,\pi^2 c_W}\, \gamma_\mu \left[I_f^{(3)} F_L\left(M_Z^2\right)\gamma_+ - 2Q_f s_W^2 F_Q\left(M_Z^2\right)\right]. \tag{514}$$

We see that the only difference from the Born case is the replacement $1 \to F_{L,Q}\left(M_Z^2\right)$. With the aid of this amplitude one constructs the Z partial widths, Γ_f, which can be compared, in principle, with the experimental data.

6.5.4 *The Z width in the one-loop approximation*

Consider the sum of the Born and of the one-loop corrected amplitudes for the Z boson decay

$$\begin{aligned} V_\mu^{\rm cor}\left(M_Z^2\right) &\propto \frac{ie}{2s_W c_W}\,\gamma_\mu\left[I_f^{(3)} f_{Z,L}\gamma_+ - 2Q_f s_W^2 f_{Z,Q}\right] \\ &= \frac{ie f_{Z,L}}{2s_W c_W}\gamma_\mu\Big[I_f^{(3)}\gamma_+ - 2Q_f s_W^2\left(1 + f_{Z,Q} - f_{Z,L}\right)\Big], \end{aligned} \tag{515}$$

where

$$f_{Z,L(Q)} = 1 + \frac{\alpha}{4\pi s_W^2} F_{Z,L(Q)}\left(M_Z^2\right). \tag{516}$$

Using the definition of Δr, Eq. (496), rewritten as follows,

$$\frac{e}{s_W c_W} = 2\sqrt{\sqrt{2} G_F M_Z^2}\left(1 - \frac{1}{2}\Delta r\right), \tag{517}$$

we eliminate the ratio $e/(s_W c_W)$ in favour of the Fermi constant G_F, and $F_{Z,L}\left(M_Z^2\right)$ receive shifts of $-\Delta r/2$. This procedure eliminates running QED coupling $\Delta\alpha(M_Z^2)$ from $F_{Z,L}\left(M_Z^2\right)$ and to an extent *minimizes* the radiative correction, since $\Delta\alpha(M_Z^2)$ contains big logs. Define the two *effective couplings*

ρ_Z^f (one more ρ!, this time *finite and gauge-invariant*, see comments below) and κ_Z^f ,

$$\begin{aligned}\rho_Z^f &= 1+\frac{\alpha}{4\pi s_W^2}\Big[2F_{Z,L}\left(M_Z^2\right)-s_W^2\delta_\mu^{ew}\Big],\\ \kappa_Z^f &= 1+\frac{\alpha}{4\pi s_W^2}\Big[F_{Z,Q}\left(M_Z^2\right)-F_{Z,L}\left(M_Z^2\right)\Big],\end{aligned}\tag{518}$$

where δ_μ^{ew} was discussed in Section 6.4.5. In terms of these effective couplings the one-loop improved expression for the partial width of $Z\to f\overline{f}$ decay becomes

$$\Gamma_f=\frac{G_F M_Z^3}{6\sqrt{2}\pi}c_f\rho_Z^f\;\left[\left(v_{\text{eff}}^f\right)^2R_V^f+\left(I_f^{(3)}\right)^2R_A^f\right],\tag{519}$$

where

$$\begin{aligned}v_{\text{eff}}^f &= I_f^{(3)}-2Q_f\sin^2\theta_{\text{eff}}^f\,,\\ \sin^2\theta_{\text{eff}}^f &= \kappa_Z^f s_W^2\,.\end{aligned}\tag{520}$$

In Eq. (519), we included factors R_V^f and R_A^f, which accumulate final state (FSR) QED and QCD corrections. The lowest-order QED $\otimes$ QCD result may be obtained from Eq. (512) if one remembers the colour trace QCD factor 4/3:

$$R_V^f=R_A^f=1+\frac{3\alpha}{4\pi}Q_f^2+\frac{\alpha_S}{\pi}\,.\tag{521}$$

Now many more terms have been are computed and really needed to match the high precision of the experiment. The factors $R_{V,A}^f$ look like a series in $\alpha(M_Z^2)$ and $\alpha_S(M_Z^2)$:

$$\begin{aligned}R_V^f&=1+\frac{3\alpha(M_Z^2)}{4\pi}Q_f^2+\frac{\alpha_S(M_Z^2)}{\pi}-\frac{\alpha(M_Z^2)}{4\pi}\frac{\alpha_S(M_Z^2)}{\pi}Q_f^2+C_V^{(2)}\left(\frac{\alpha_S(M_Z^2)}{\pi}\right)^2+\dots\\ R_A^f&=1+\frac{3\alpha(M_Z^2)}{4\pi}Q_f^2+\frac{\alpha_S(M_Z^2)}{\pi}-\frac{\alpha(M_Z^2)}{4\pi}\frac{\alpha_S(M_Z^2)}{\pi}Q_f^2+C_A^{(2)}\left(\frac{\alpha_S(M_Z^2)}{\pi}\right)^2+\dots\end{aligned}\tag{522}$$

The discussion of FSR QED $\otimes$ QCD corrections deserves a separate lecture.

At the end of this section I would like to emphasize:

1. We met an important notion of *the amplitude form factors*. Since they describe a physical amplitude (or another physical quantity, like the anomalous magnetic moment), they are gauge-invariant and divergence-free functions (or constants). Moreover, we may rephrase slightly the definition of pseudo-observables given above, as follows:
2. *Definition: In a very general sense, the pseudo-observable is a construction made of gauge-invariant form factors of an amplitude of a process.*
   Therefore, Sirlin's parameter Δr or the ρ-parameter of Eq. (518), ρ_Z^f, are typical pseudo-observables, whilst Veltman's parameter $\Delta\rho$ is not.

6.5.5 Re-summation of large corrections

In a similar way to what has been done for Δr, one has to improve upon the one-loop approximation for ρ_Z^f and κ_Z^f. Define *the leading* (enhanced) and *remainder* contributions to ρ_Z^f and κ_Z^f:

$$\begin{aligned}\rho_Z^f &= 1+\rho_L^f+\rho_{\text{rem}}^f\,,\\ \kappa_Z^f &= 1+\kappa_L^f+\kappa_{\text{rem}}^f\,.\end{aligned}\tag{523}$$

When we eliminate Δr and normalize amplitudes to the Fermi constant G_F, all large corrections containing $\alpha^{\rm fer}\left(M_Z^2\right)$ are automatically accounted for.

Therefore, in contrast to what happened in the re-summation of Δr, here one has to re-sum only the m_t^2-enhanced terms. As for Δr, one derives:

$$\rho_Z^f = \frac{1+\rho^f_{\rm rem}}{1+\dfrac{\sqrt{2}G_F M_Z^2 s_W^2 c_W^2}{\pi\alpha}\rho_L^f}\,. \tag{524}$$

For κ one has to follow a slightly different procedure,

$$\kappa_Z^f = \left(1+\kappa^f_{\rm rem}\right)\left(1+\frac{\sqrt{2}G_F M_Z^2 s_W^2 c_W^2}{\pi\alpha}\kappa_L^f\right)+\frac{1}{s_W^2}\text{Im-parts}\,, \tag{525}$$

where some ${\rm Im}$-parts are added (see Section 6.11.6.3 of Ref. [1]). These are second-order terms, enhanced by $\pi^2 N_f$ (where N_f is the total number of fermions in the SM) which have to be taken into account as soon as the leading two-loop corrections are added.

The leading contributions are made of the gauge-invariant quantity $\Delta\rho^F$ as follows

$$\rho_L^f = -\frac{\alpha}{4\pi}\frac{1}{s_W^2}\Delta\rho^F\,, \qquad \kappa_L^f = -\frac{\alpha}{4\pi}\frac{c_W^2}{s_W^4}\Delta\rho^F = \Delta r_L\,. \tag{526}$$

The inclusion of higher-order irreducible effects, is achieved by means of the modification of the leading and of the reminder terms. As for Δr, we have:

$$\begin{array}{rcl} \Delta r_L & \to & \Delta r_L + \Delta r_L^{\rm ho}\,, \\ \rho^f_{\rm rem} & \to & \rho^f_{\rm rem} + \rho^{f,{\rm ho}}_{\rm rem}\,, \\ \kappa^f_{\rm rem} & \to & \kappa^f_{\rm rem} + \kappa^{f,{\rm ho}}_{\rm rem}\,. \end{array} \tag{527}$$

The numerical results for $\sin^2\theta^e_{\rm eff}$, derived including the re-summation of the leading corrections and the leading and sub-leading two-loop irreducible electroweak corrections $\mathcal{O}\left(G_F^2 m_t^4\right)$ and $\mathcal{O}\left(G_F^2 m_t^2 M_Z^2\right)$, are shown in Tab. 2.

Table 2: The OMS $\sin^2\theta^e_{\rm eff}$.

m_t [GeV]	M_H [GeV]		
	65	300	1000
170.1	0.23109	0.23187	0.23253
175.6	0.23090	0.23168	0.23234
181.1	0.23070	0.23149	0.23215

This table illustrates that the $\sin^2\theta^e_{\rm eff}$ is quite sensitive to variations of both m_t and M_H. It is instructive to compare a typical variation due to Higgs mass ~ 0.00045 with the present combined experimental error ~ 0.00026. This illustrates why the present precision already ensures a sensitivity to the mass of the Higgs boson.

6.6 Realistic observables in the process $e^+e^- \to f\overline{f}$

For this process we may also consider a gauge-invariant subset of QED diagrams: QED vertices, $\gamma\gamma$ and $Z\gamma$ boxes. It has to be considered together with four QED bremsstrahlung diagrams, Fig. 12. The sum of all the QED diagrams is free of infrared divergences.

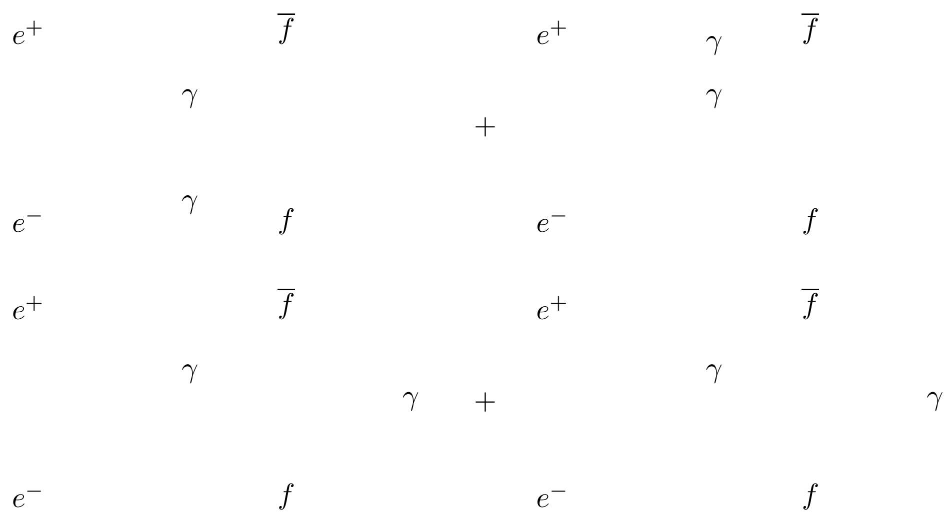


Fig. 12: Bremsstrahlung process $e^+e^- \to f\overline{f}\gamma$.

6.6.1 *One-loop diagrams and corrections for* $e^+e^- \to f\overline{f}$

The remaining one-loop diagrams form *the non-QED* or *weak* corrections. The total weak amplitude may be represented as the sum of *dressed* γ and Z exchange amplitudes plus the contribution from *weak box* diagrams, i.e. ZZ and WW boxes. The ZZ boxes are separately gauge-invariant.

Fermionic loops are also separately gauge-invariant and may be re-summed. Bosonic loops have to be expanded to the first order. Dressed γ and Z exchanges may be symbolically depicted as:

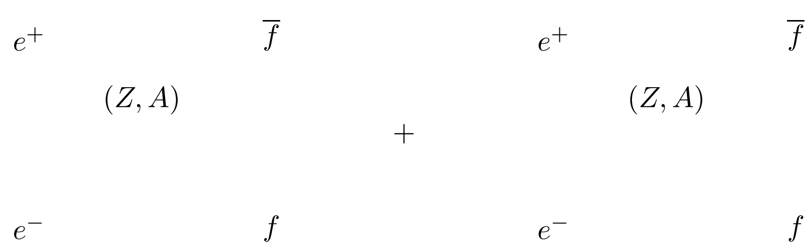


Fig. 13: Process $e^+e^- \to (Z, A) \to f\overline{f}$; final fermion vertex and its counter-terms.

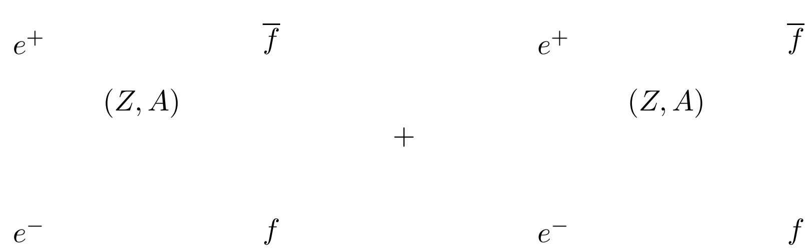


Fig. 14: Process $e^+e^- \to (Z, A) \to f\overline{f}$; electron vertex and its counter-terms.

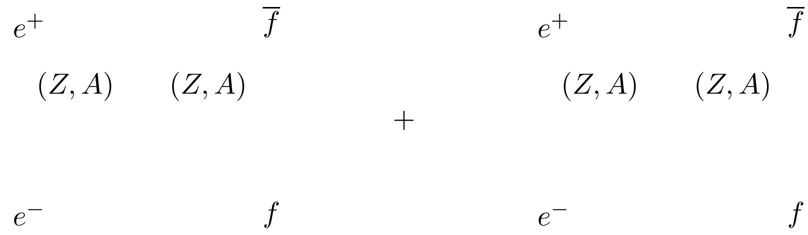


Fig. 15: Process $e^+e^- \to (Z,A) \to f\overline{f}$; self energies and kinetic counter-terms.

If external fermion masses are neglected, then the complete one-loop amplitude (OLA) can be described by only four scalar functions and by the running electromagnetic constant $\alpha^{\rm fer}(s)$.

There are two ways of representing the dressed amplitude:

1) In terms of four scalar form factors, $F_{ij}\,(s,t)$,

$$\begin{aligned}\mathcal{A}^{\rm OLA}_{Z+A} &= \frac{e^2 I^{(3)}_e I^{(3)}_f}{4s^2_{_W} c^2_{_W}}\chi_Z(s)\Big\{\gamma_\mu\gamma_+\otimes\gamma_\mu\gamma_+ F_{LL}\,(s,t) - 4|Q_e|s^2_{_W}\gamma_\mu\otimes\gamma_\mu\gamma_+ F_{QL}\,(s,t)\\ &\quad -4|Q_f|s^2_{_W}\gamma_\mu\gamma_+\otimes\gamma_\mu F_{LQ}\,(s,t) + 16|Q_eQ_f|s^4_{_W}\gamma_\mu\otimes\gamma_\mu F_{QQ}\,(s,t)\Big\}; \end{aligned} \tag{528}$$

2) In terms of the effective couplings $\rho_{ef}(s,t)$ and $\kappa_{ij}(s,t)$, which in this case are s,t-dependent, contrary to the Z decay where they were constants. (The t-dependence is due to the weak boxes.)

$$\begin{aligned}\mathcal{A}^{\rm OLA}_{Z+A} &= \sqrt{2}G_F I^{(3)}_e I^{(3)}_f M^2_Z\chi_Z(s)\rho_{ef}(s,t)\Big\{\gamma_\mu\gamma_+\otimes\gamma_\mu\gamma_+ - 4|Q_e|s^2_{_W}\kappa_e(s,t)\gamma_\mu\otimes\gamma_\mu\gamma_+\\ &\quad -4|Q_f|s^2_{_W}\kappa_f(s,t)\gamma_\mu\gamma_+\otimes\gamma_\mu + 16|Q_eQ_f|s^4_{_W}\kappa_{ef}(s,t)\gamma_\mu\otimes\gamma_\mu\Big\}. \end{aligned} \tag{529}$$

On top of the $\mathcal{A}^{\rm OLA}_{Z+A}$ there is the corrected γ-exchange amplitude, which contains, by construction, only the QED running coupling $\alpha^{\rm fer}(s)$:

$$\mathcal{A}^{\rm OLA}_{A} = \frac{4\pi\alpha^{\rm fer}(s)}{s}\gamma_\mu\otimes\gamma_\mu\,. \tag{530}$$

There are residual corrections to the photon exchange diagram but it is always possible to assign them to the Z exchange amplitude, since both contain the same Dirac structure $\gamma_\mu\otimes\gamma_\mu$.

The effective couplings ρ and κ's are related to the form factors $F_{ij}\,(s,t)$ and to the quantity Δr (or δ^{ew}_μ, see Section 6.4.5) by the following equations:

$$\begin{aligned}\rho_{ef}(s,t) &= 1+\frac{\alpha}{4\pi s^2_{_W}}\Big[F_{LL}\,(s,t) - s^2_{_W}\delta^{ew}_\mu\Big],\\ \kappa_e(s,t) &= 1+\frac{\alpha}{4\pi s^2_{_W}}\Big[F_{QL}\,(s,t) - F_{LL}\,(s,t)\Big],\\ \kappa_f(s,t) &= 1+\frac{\alpha}{4\pi s^2_{_W}}\Big[F_{LQ}\,(s,t) - F_{LL}\,(s,t)\Big],\\ \kappa_{ef}(s,t) &= 1+\frac{\alpha}{4\pi s^2_{_W}}\Big[F_{QQ}\,(s,t) - F_{LL}\,(s,t)\Big]. \end{aligned} \tag{531}$$

Here 1 is due to the Born amplitude which has also been included.

6.6.2 *Convolution with QED radiation*

Here we briefly discuss the subsequent **chain of calculations**. Having constructed OLA amplitudes, Eqs. (529)–(531), which may also be called *the Improved Born Approximation* (or IBA) amplitudes, we **may calculate the corresponding IBA cross-section.** The latter may also be called *doubly de-convoluted cross-section*, i.e. prior to subsequent *convolution* with Initial State (ISR) and Final State (FSR) radiations. It is convenient to introduce the notion of a *singly de-convoluted cross-section*, i.e. with FSR and without ISR; the latter being a function of the reduced c.m.s. energy s' and possible kinematical cuts in the final state.

So, the natural next step would be: **From IBA → IBA ⊕ FSR cross-section**. This would give us a *kernel cross-section* for a subsequent convolution with the ISR.

$$\sigma^{\rm dec} = \hat{\sigma}(s', \text{cuts}). \tag{532}$$

The final step would be: **From IBA ⊕ FSR cross-section → *complete QED convoluted* cross-section**, which would account for multiple real bremsstrahlung in the ISR, virtual ISR corrections and corrections due to the emission of real and virtual unobserved pairs, shown symbolically in Fig. 16.

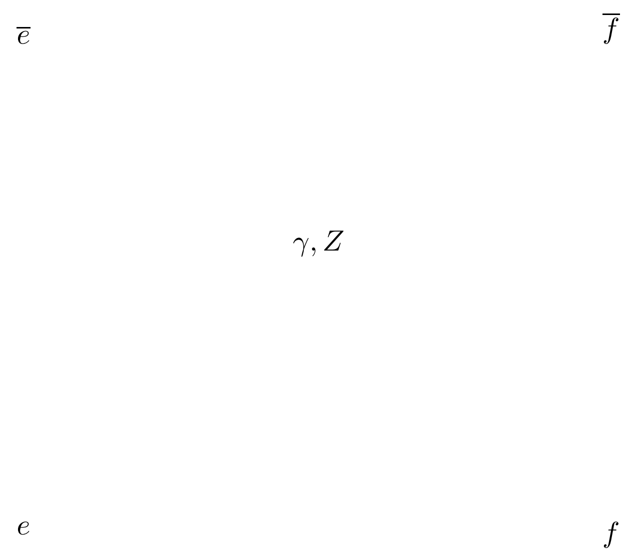


Fig. 16: ISR⊕FSR QED corrections for $e\overline{e} \to (Z,\gamma) \to f\overline{f}$.

The ISR corrections are accounted for by means of the structure functions (SF), $D\,(z;s)$, or the flux function (FF), $H\,(x;s)$. The QED convoluted cross-section $\sigma\,(s)$ is related to the kernel cross-section by *the convolution integral*,

$$\sigma\,(s) = \int_0^{1-s_0/s} dx H\,(x;s)\,\hat{\sigma}\Big((1-x)\,s\Big), \tag{533}$$

where the flux function H is related to the structure functions by:

$$H\,(x;s) = \int_{1-x}^{1} \frac{dz}{z}\,D\,(z;s)\,D\left(\frac{1-x}{z};s\right). \tag{534}$$

The FF may be presented as a sum of virtual + soft photon (V+S) and hard photon (H) contributions:

$$\begin{aligned} H\,(x;s) &= \beta\,x^{\beta-1}\,\delta^{\rm V+S} + \delta^{\rm H}, \\ \beta &= \frac{2\,\alpha}{\pi}\left(\ln\frac{s}{m_e^2} - 1\right), \end{aligned} \tag{535}$$

with the virtual + soft photon part being *exponentiated*.

The flux function is known up to $\mathcal{O}\left(\alpha^2\right)$ *completely*, and up to $\mathcal{O}\left(\alpha^3 L^3\right)$ in *the leading log* approximation (LLA). These issues deserve, indeed, a separate lecture.

6.7 Experimental status of the SM

I shall present only two plots taken from Ref. [7], courtesy of M. Grünewald, referring for a comprehensive experimental review of this field to Ref. [7] and to his work Ref. [8].

The overall status of the SM might be well illustrated by the so-called *pulls*, Fig. 17. Although there are several points where deviations between the theory and experiment approach two standards, the average situation should be ranked as extremely good. We note that the level of precision reached is of the order of $\sim 10^{-3}$, and that it is extremely non-trivial to control all the experimental systematics at this level. In the second figure, Fig. 18, we present the famous *blue-band* showing the $\Delta\chi^2_{\rm min}(M^2_H)$ distribution derived from a combined fit of all the world experimental data to the SM exploiting the best knowledge of precision theoretical calculations which is realized in computer codes `ZFITTER` and `TOPAZ0`. It illustrates what we call an *indirect discovery* of the Higgs boson made via the study of *constraints*, provided by PHEP, as discussed in the first lecture.

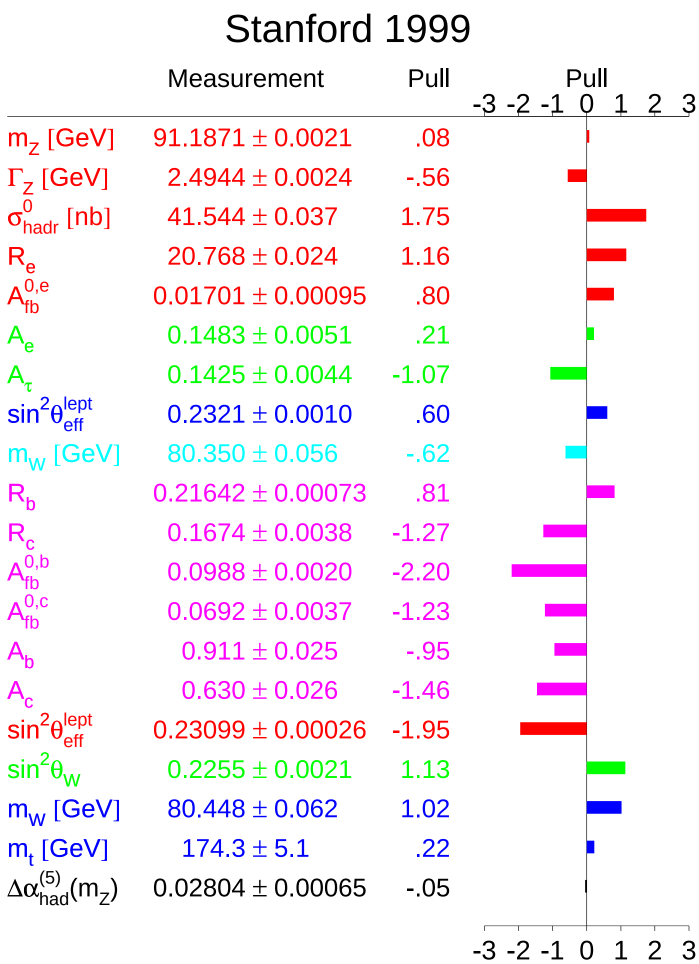


Fig. 17: *Pulls for pseudo-observables.* The pull is defined as the difference between the measurement and the SM prediction calculated for the central values of the fitted SM IPS $[\alpha(M^2_Z) = 1/128.878,\ \alpha_S(M^2_Z) = 0.1194,\ M_Z = 91.1865\,\text{GeV},\ m_t = 171.1\,\text{GeV}]$ divided by the experimental error.

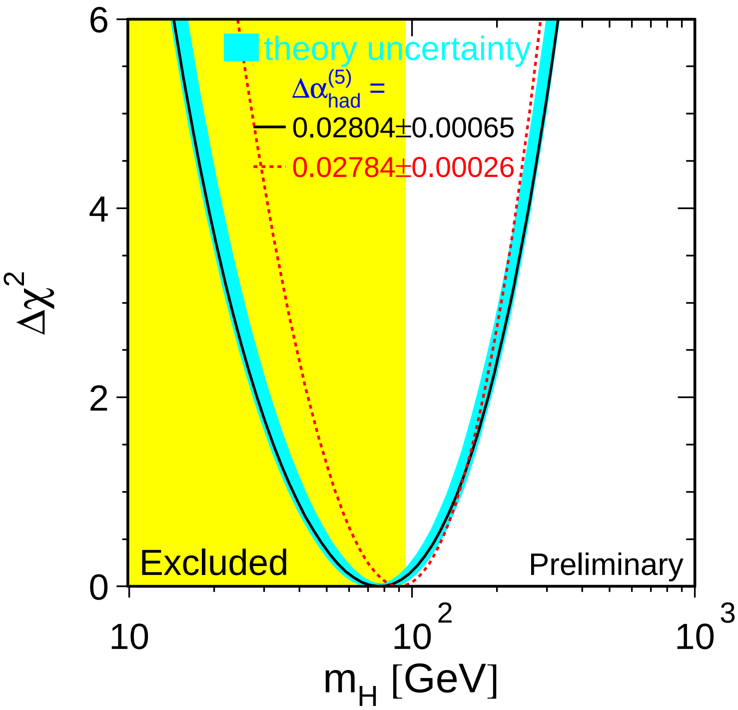


Fig. 18: *The Blue-Band.* Curve showing $\Delta\chi^2_{\min}(M_H^2) = \chi^2_{\min}(M_H^2) - \chi^2_{\min}$ as a function of M_H. The width of the shaded band around the curve shows the theoretical uncertainty. The two lines correspond to different calculations of $\Delta\alpha^{(5)}(M_Z^2)$, namely $\Delta\alpha^{(5)}(M_Z^2) = 0.02804 \pm 0.00065$ (*Eidelman, Jegerlehner*) and $\Delta\alpha^{(5)}(M_Z^2) = 0.02784 \pm 0.00026$ (*theory-driven analyses*). Also shown is the region excluded at 95% CL by the negative direct search for the Higgs boson at LEP2, $\sim 100\,$GeV.

These figures, as well as many more proofs of the correctness of the SM collected in recent experiments, convinces us to conclude these lectures with:

7. CONCLUSION

- The Standard Model has been completed theoretically and must be ranked as The Standard Theory, which should completely replace QED.
- The Standard Theory has not been completed experimentally.
  The Higgs boson is the only ingredient still waiting to be discovered, and it will inevitably be discovered. However, it is very difficult to predict *where and when?*

ACKNOWLEDGEMENTS

Since these lectures are heavily based on the book [1], written together with Giampiero Passarino, my first and pleasant duty is to thank him for several years of fruitful scientific collaboration, both within the book project and on the theoretical support of precision measurements in HEP.

I am very much obliged to Penka Christova, Lida Kalinovskaya, Gizo Nanava, and Maxim Nekrasov for many useful discussions whilst preparating the transparencies for these lectures and this written contribution. These discussions influenced considerably the content of the lectures.

I am thankful to Martin Grünewald for numerous discussions on experimental issues and for providing me with two plots shown at the end of this contribution.

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