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Hamiltonian Formalism for Space-time Non-commutative Theories

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Space-time non-commutative theories are non-local in time. We develop the Hamiltonian formalism for field theories in d space-time dimensions by considering an auxilliary $d+1$ field theory which is local with respect to the extra evolution time. The Hamiltonian path integral quantization is considered. The case of NC ϕ^3 is considered as an example. The non-unitarity of these theories can be deduced from this analysis.

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Space-time non-commutative field theories have peculiar properties due to their acausal behaviour [1] [2] and lack of unitary [3]. In reference [4] it has been shown that there is a relation between lack of unitarity and the obstruction to finding a decoupling limit of string theory in an electromagnetic background [5] [6] [7] [8] [9].

These theories have an infinite number of temporal and spatial derivatives, and therefore are non-local in time and space[1]. The initial value problem requires giving a trajectory or a finite piece of it[2]. The Euler-Lagrange (EL) equation is a constraint in the space on trajectories.

In this work we develop the Hamiltonian formalism of time-like and light-like non-commutative theories [4]. This analysis should shed new light on the structure of these theories.

The Hamiltonian formalism for non-local theories was presented in [10]. In this paper we improve the formalism by clarifying the relation among the Lagrangian and Hamiltonian structures. We first consider an equivalent theory in a space-time of one dimension higher than that of the original theory, in such a way the non-locality in time is replaced by a non-locality in space [10]. For this equivalent theory one can construct the ordinary Hamiltonian. A characteristic feature of the Hamiltonian formalism for non-local theories is that it contains the EL equations as Hamiltonian constraints.

The time dependence of the Hamiltonian and Hamiltonian constraints of the $d+1$ field theory suggests considering a Hamiltonian formalism in d dimensions. This formulation is suitable for making contact with the ordinary Hamiltonian formalism of local theories.

The Hamiltonian path integral for the $d+1$ dimensional field theory is constructed. Integrating out the momenta leads to the Lagrangian path integral formalism for the d dimensional theory.

As an example we consider the case of non-commutative ϕ^3 theory in d dimensions with space-time non-commutativity. The action contains the free Klein-Gordon Lagrangian and the interaction Lagrangian $L_i = -\frac{g}{3!}\int d\vec{x}\ \phi{*}\phi{*}\phi$ where $*$ refers to the Moyal product. We contruct the Hamiltonian in $d+1$ and d dimensions. The path integral Hamiltonian quantization is performed. We get the Feynman rules that coincide with those used in references [1] [2] [3]. The theory is unitary at the classical level (tree level) but not unitary at one loop [3].

The knowledge of the Hamiltonian of light-like and light-like non-commutative field theories could also to be useful to study the energy of their solitons.

Euler-Lagrange equations for non-local theories.

A non-local Lagrangian depends on an infinite number of time derivatives of the position[3]. In other words it depends on a continuous function $q(\lambda)$ for all values of λ, $L^{non}(t) = L[(q(t+\lambda)]$. This means that the analogous of the tanget bundle for Lagrangians depending on positions and velocities is infinite dimensional. Let us indicate this space by $J = \{q(t)\}$. This is the space of all possible trajectories.

The action is

$$S[q] = \int dt L^{non}(t). \tag{1}$$

The EL equation is obtained by taking the functional variation of (1) and is given by

$$\int dt E(t, t'; [q]) = 0, \tag{2}$$

[1] From now on when we will refer to non-local theories as the ones with an infinite number of time derivatives.

[2] For a discussion about general aspects of non-local theories see [11].

[3] In this section we will consider the case of mechanics.

where $E(t,t';[q]) = \frac{\delta L^{non}(t)}{\delta q(t')}$.

For a non-local theory, the initial conditions are already a whole trajectory $q(\lambda)$, or a part of it. The EL equations gives a functional relation for the possible physical trajectories. $F[q] = 0$ is a Lagrangian constraint which defines a submanifold Σ in the space of all trajectories J. The EL equations do not give the evolution of the system in terms of the initial conditions, but simply select the possible allowed physical trajectories. Note the different role of EL equations for local Lagrangians .

$1+1$ *field theory description of non-local theories.*

If we insist in constructing a "time" evolution for a given initial physical trajectory $q(\lambda)$, we need to specify a function $Q(t,\lambda)$, where now t is the new "time" variable and λ is regarded as a spatial variable. Physically we should impose that the new trajectory described by $Q(t,\lambda)$ should coincide with the initial trajectory. The constraints on the dynamics for the new time should be such that

$$Q(t,\lambda) = q(t+\lambda) = Q(t+\lambda, 0) = Q(0, t+\lambda). \qquad (3)$$

In this form we associate in a natural way a $1+1$ dimensional field theory to a non-local one in mechanics [10].

The Lagrangian governing the dynamics of this field theory is

$$\tilde{L}(t) = L([Q(t,-)]) + \int d\lambda\ \mu(t,\lambda) \left[\dot{Q}(t,\lambda) - Q'(t,\lambda)\right], \qquad (4)$$

where $L([Q(t,-)])$ is a functional of $Q(t,-)$ at fixed time t.

$$L(t) = L([Q(t,-)]) = \mathcal{L}(t,0)\ =\ \int d\lambda\ \delta(\lambda)\ \mathcal{L}(t,\lambda). \qquad (5)$$

$\mathcal{L}(t,\lambda)$ is constructed from $L^{non}(t)$ by replacing $q(t)$ by $Q(t,\lambda)$, the t derivatives of $q(t)$ by derivatives respect to λ of $Q(t,\lambda)$ and $q(t+\rho)$ by $Q(t,\lambda+\rho)$. Note that $L(t)$ is now local in the time variable t and non-local in the space variable λ. The second term in (4) is made out of the Lagrange multipliers $\mu(t,\lambda)$ and the constraint $\dot{Q}(t,\lambda) - Q'(t,\lambda)$, which is a differential version of (3).

We can follow an ordinary Hamiltonian formalism for this field theory. The momentum is given by

$$P(t,\lambda) \equiv \int d\sigma \chi(\lambda,-\sigma)\ \mathcal{E}(t;\sigma,\lambda), \qquad (6)$$

where $\mathcal{E}(t;\sigma,\lambda)$ and $\chi(\lambda,-\sigma)$ are defined by

$$\mathcal{E}(t;\sigma,\lambda) = \frac{\delta\mathcal{L}(t,\sigma)}{\delta Q(t,\lambda)}, \quad \chi(\lambda,-\sigma)\ =\ \frac{\epsilon(\lambda)-\epsilon(\sigma)}{2}. \qquad (7)$$

where $\epsilon(\lambda)$ is the sign distribution. The phase space of this field theory, for fixed t, is infinite dimensional. We denote it by $T^*J(t) = \{Q(t,\lambda), P(t,\lambda)\}$ and the symplectic form is given by

$$\Omega(t) = \int d\lambda \delta P(t,\lambda) \wedge Q(t,\lambda). \qquad (8)$$

The Hamiltonian is

$$H(t) = \int d\lambda P(t,\lambda) Q'(t,\lambda)\ -\ L(t), \qquad (9)$$

where $Q'(t,\lambda) = \partial_\lambda Q(t,\lambda)$. The Hamilton-Dirac equations in the reduced space, obtained by eliminating the second class constraints (4), are

$$\dot{Q}(t,\lambda) = Q'(t,\lambda), \qquad (10)$$

$$\dot{P}(t,\lambda) = P'(t,\lambda) + \frac{\delta L(t)}{\delta Q(t,\lambda)} = P'(t,\lambda) + \mathcal{E}(t;0,\lambda). \qquad (11)$$

Note that (6) is a Hamiltonian constraint

$$\varphi(t,\lambda) = P(t,\lambda)\ -\ \int d\sigma \chi(\lambda,-\sigma)\ \mathcal{E}(t;\sigma,\lambda)\ =\ 0. \qquad (12)$$

The stability of (12) requires

$$\dot{\varphi}(t,\lambda) = \delta(\lambda)\ [\int d\sigma\ \mathcal{E}(t;\sigma,0)] = 0. \qquad (13)$$

We should require further consistency conditions of this constraint and so on. We get an infinite set of Hamiltonian constraints which are expressed as

$$\varphi_2(t,\lambda) = \int d\sigma \mathcal{E}(t;\sigma,\lambda) = 0, \quad -\infty < \lambda < \infty. \qquad (14)$$

If we use (10) it reduces to the EL equation for $q(t)$ obtained from $L^{non}(t)$.

Summarizing, the equivalence between the Lagrangian and Hamiltonian formalisms is built in the $1+1$ field theory Hamiltonian formalism of *local* field theories through the Hamiltonian constraints (12) and (14). This type of equivalence between the Hamiltonian and Lagrangian formalism is different from the one in local theories [12].

First order formulation of the EL equation

It is useful to rewrite the Hamiltonian and symplectic form in terms of $Q(t,\lambda)$ using (6)

$$H(t) = \int d\lambda [\int d\sigma \chi(\lambda,-\sigma)\ \mathcal{E}(t;\sigma,\lambda)]\ Q'(t,\lambda) - \int d\lambda \delta(\lambda) \mathcal{L}(t,\lambda) \qquad (15)$$

and

$$\Omega = \frac{1}{2}\int d\lambda\ d\lambda'\ \delta Q(t,\lambda)\omega(t;\lambda,\lambda')\delta Q(t,\lambda'), \qquad (16)$$

where

$$\omega(t;\lambda,\lambda') = \chi(\lambda',-\lambda)\int d\sigma\ \frac{\delta\mathcal{E}(t;\sigma,\lambda)}{\delta Q(t,\lambda')}. \qquad (17)$$

The time evolution vector field is given by

$$X(t) = \int d\lambda\ \dot{Q}(t,\lambda)\frac{\delta}{\delta Q(t,\lambda)}. \qquad (18)$$

Now we will see that $i(X)\Omega+\delta H=0$ gives a first order formulation of the EL equation. In fact,

$$\begin{aligned} i(X)\Omega + \delta H \\ &= \int d\lambda'[\int d\lambda[\dot{Q}(t,\lambda) - Q'(t,\lambda)]\omega(t;\lambda,\lambda') \\ &\quad - \ \delta(\lambda')\ \int d\sigma\mathcal{E}(t;\sigma,\lambda')\ ]\delta Q(t,\lambda'). \end{aligned} \qquad (19)$$

It follows, for any λ'

$$\begin{aligned} &\int d\lambda[\dot{Q}(t,\lambda) - Q'(t,\lambda)]\omega(t;\lambda,\lambda') \\ &- \delta(\lambda')\ \int d\sigma\mathcal{E}(t;\sigma,\lambda') = 0 \end{aligned} \qquad (20)$$

if we use (10) we obtain again the EL equation.

Hamiltonian formalism of non-local theories in terms of mechanical variables.

Now we would like to rewrite the previous Hamiltonian formulation in terms of mechanical variables $q(\lambda)$ and $p(\lambda)$. This can be done from the previous formulation by taking into account the Hamiltonian (9) and the Hamiltonian constraints (12) and (14). The time variable t is fixed to zero. If we further use (10) we have

$$H = \int d\lambda p(\lambda)\dot{q}(\lambda)\ -\ L(0), \qquad (21)$$

$$\varphi = p(\lambda)\ -\ \int d\sigma\chi(\lambda,-\sigma)\ \mathcal{E}(t;\sigma,0)\ =\ 0 \qquad (22)$$

and

$$\int d\sigma\ \frac{\delta L(\sigma)}{\delta q(\lambda)} = 0, \qquad (23)$$

where $q(\lambda) = Q(0,\lambda)$, $p(\lambda) = P(0,\lambda)$ and now $\dot{q}(\lambda)$ means derivate with respect to the argument. This formulation is particularly useful if we want to make contact with the Hamiltonian formalism of local Lagrangians [13].

Path integral quantization.

Let us consider the Hamiltonian path integral quantization of the $1+1$ dimensional field theory associated with the non-local Lagrangian mechanics $L^{non}(t)$. The path integral is given by

$$\int [dP(t,\lambda)][dQ(t,\lambda)]e^{i\int dt d\lambda (P(t,\lambda)[\dot{Q}(t,\lambda)-Q'(t,\lambda)]+L(t)\delta(\lambda))}. \qquad (24)$$

If we integrate out the momenta and use (3) we get

$$\int [dq(t))]e^{i\int dt\ L^{non}(t)}, \qquad (25)$$

which is the Lagrangian path integral formulation for the non-local theory.

Application to space-time non-commutative ϕ^3 theory.

Space-time non-commutative theories have peculiar properties due to their acausal behavior and lack of unitarity. Here we would like to use the previous formalism to study the question of unitarity in these theories.

To fix the ideas we consider a non-commutative ϕ^3 theory with arbitrary non-commutativity in d dimensions. The Lagrangian density is given by

$$\begin{aligned} \mathcal{L}^{non}(x^\mu) = \frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) - \frac{m^2}{2}\phi(x)^2 \\ -\frac{g}{3!}\phi(x)*\phi(x)*\phi(x) \end{aligned} \qquad (26)$$

where $*$ is the star product defined by using a general anti-symmetric background $\theta^{\mu\nu}$,

$$f(x)*g(x) = [e^{i\frac{\theta^{\mu\nu}}{2}\partial^\alpha_\mu\partial^\beta_\nu}f(x+\alpha)g(x+\beta)]_{\alpha=\beta=0}. \qquad (27)$$

The EL equation is

$$(\nabla^2 - \partial_t^2 - m^2)\phi(x) - \frac{g}{2!}\ \phi(x)*\phi(x)\ =\ 0. \qquad (28)$$

In order to construct the Hamiltonian formalism we introduce the field $Q(t,x^\mu)$ and regard t as "time" and x^μ as spatial variables. Now x^0 plays the role of λ in the previous discussion. The Lagrangian density in $d+1$ dimensions for $Q(t,x^\mu)$, (see eq.(5)), is

$$\begin{aligned} \mathcal{L}(t,x^\mu) = -\frac{1}{2}\partial_\mu Q(t,x)\partial^\mu Q(t,x) - \frac{m^2}{2}Q(t,x)^2 \\ -\ \frac{g}{3!}Q(t,x)*Q(t,x)*Q(t,x), \end{aligned} \qquad (29)$$

where now the derivatives in $*$ are with respect to x^μ. Note that this Lagrangian density is local in time.

The momentum constraint (12) is given by

$$\begin{aligned} &\varphi(t,x^\mu) = P(t,x^\mu) - \delta(x^0)\ Q'(t,x) \\ &+\frac{g}{2!}\int dx'\chi(x^0,-x'^0)\int dy_1dy_2 \\ &K(y_1-x',y_2-x',x-x')Q(t,y_1)Q(t,y_2), \end{aligned} \qquad (30)$$

where $Q'(t,x)$ denotes $\partial_{x^0}Q(t,x^\mu)$. K is the symmetric kernel of three star products,

$$\begin{aligned} &f(x)*g(x)*h(x) = \int dy_1dy_2dy_3 \\ &K(y_1-x,y_2-x,y_3-x)f(y_1)g(y_2)h(y_3). \end{aligned} \qquad (31)$$

The Hamiltonian (9) is

$$H(t) = \int dx \; [\; P(t,x) \; Q'(t,x) \; - \; \mathcal{L}(t,x) \; \delta(x^0)]$$
$$= \int dx[\; P(t,x)Q'(t,x)$$
$$+\delta(x^0)\{-\frac{1}{2}(\partial_{x^0}Q(t,x))^2 + \frac{1}{2}(\nabla Q(t,x))^2 + \frac{m^2}{2}Q(t,x)^2$$
$$+ \frac{g}{3!}Q(t,x) * Q(t,x) * Q(t,x)\}]. \tag{32}$$

The Hamilton equations are

$$\dot{Q}(t,x) = Q'(t,x), \tag{33}$$
$$\dot{P}(t,x) = P'(t,x) - \delta'(x^0)[Q'(t,x)]_{x^0=0}$$
$$+ \delta(x^0)\{\nabla^2 Q(t,x) - m^2 Q(t,x)\}$$
$$- \frac{g}{2!}\int dx' dy_1 dy_2 \delta(x'^0)$$
$$K(y_1 - x', y_2 - x', x - x')Q(t,y_1)\; Q(t,y_2). \tag{34}$$

The stability of the constraint implies the new constraints

$$\varphi_2(t,x) \equiv (\nabla^2 - \partial_{x^0}^2 - m^2)Q(t,x) - \frac{g}{2!}\; Q(t,x) * Q(t,x)$$
$$= 0, \quad at \quad x^0 = 0. \tag{35}$$

By requiring further consistency we have an infinite number of constraints which can be written as

$$\varphi_2(t,x) = 0 \qquad for \quad -\infty < x^0 < \infty. \tag{36}$$

Using Hamilton equation for Q (33), (36) becomes the EL equation

$$(\nabla^2 - \partial_t^2 - m^2)Q(t,x) - \frac{g}{2!}\; Q(t,x) * Q(t,x) \;\; = \;\; 0, \tag{37}$$

where ∂_{x^0} on Q is replaced by ∂_t both in the first term and in the $*$ product. It is the original non-local EL equation (28).

If we write the symplectic form and the Hamiltonian in terms of $Q(t,x)$, eqs. (15) and (16), we have

$$\Omega = \int dx \delta(x^0)\; \delta Q'(t,x) \wedge \delta Q(t,x)$$
$$- \frac{g}{4}\int dx \delta(Q(t,x) * Q(t,x))\epsilon(x^0)\delta Q(t,x) \tag{38}$$

and

$$H = \int dx \frac{\delta(x^0)}{2}\{Q'(t,x)^2 + (\nabla Q(t,x))^2 + m^2 Q(t,x)^2\}$$
$$-\frac{g}{4}\int dx(Q(t,x) * Q(t,x))\epsilon(x^0)Q'(t,x). \tag{39}$$

These expressions can be rewritten in terms of $\phi(x)$ using (33) *i.e* (3). In particular the interaction Hamiltonian becomes

$$H_i = -\frac{g}{4}\int dx(\phi(x) * \phi(x))\epsilon(x^0)\dot{\phi}(x). \tag{40}$$

Note that the appearance of temporal derivatives of any order in the interaction Hamilonian is not forbidden in a non-local theory. This property is clearly not fulfilled by a local theory.

Now we can perform the path integral quantization using (25) to obtain

$$\int [d\phi(x)] e^{\int dx (\frac{1}{2}\partial_\mu \phi(x) \partial^\mu \phi(x) - \frac{m^2}{2}\phi(x)^2 - \frac{g}{3!}\phi(x)*\phi(x)*\phi(x))} \tag{41}$$

From which we read the Feynman rules. They coincide with the ones used in used in [3]. Therefore, it follows from our analysis that noncommutative ϕ^3 with time-like non-commutativity is not unitary while noncommutative ϕ^3 with light-like non-commutativity is unitary [4].

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