Ref.TH.2263-CERN

REPOLARIZATION OF MUONS IN MUONIC ATOMS WITH POLARIZED NUCLEI

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A B S T R A C T

The equation of motion for the density matrix of spin $\frac{1}{2}$ and spin 1 particles with contact interaction is solved. Time-averaging shows that 44% (30%) of the nuclear (muonic) polarization is transferred to the muons (nuclei). Generalizations to higher nuclear spins are given.

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Ref.TH.2263-CERN
14 December 1976

The depolarization of captured muons in the atomic cascade has been given considerable attention [1]. For muonic atoms with spinless nuclei, the main source of depolarization is spin-orbit coupling, resulting in a loss of about five-sixths of the initial muon polarization [2]. If the nucleus has spin the hyperfine contact interaction will give additional depolarization in s orbits [3]. The muon loses part of its polarization to an initially unpolarized nucleus, similar to the Overhauser effect [4]. There is an analogous phenomenon in muonium [5].

If the target is initially polarized, the contact interaction can "repolarize" the muons [6]. The case for spin $\frac{1}{2}$ nuclei is similar to muonium [5], so we treat here the case of spin 1 nuclei in detail. We take the muon to be in a 1s orbit and neglect the effects of the finite muon life-time and of possible conversion between the two hyperfine levels [7]. Initially the density matrix ρ of both muons and nuclei is a direct product of density matrices for spin $\frac{1}{2}$ and spin 1 :

$$\rho = \rho_{\frac{1}{2}} \otimes \rho_1 = \tfrac{1}{2}(1+2\vec{a}_0\cdot\vec{s}_1)\otimes\tfrac{1}{3}(1+\tfrac{3}{2}\vec{b}_0\cdot\vec{s}_2+3\tilde{b}_0:\tilde{T}(\vec{s}_1\vec{s}_2)) \tag{1}$$

where indices 1 and 2 refer to muons and nuclei, respectively. The initial muon and nuclear polarizations are $\vec{a}_0$ and $\vec{b}_0$ and $\tilde{b}_0$ is a symmetric, traceless tensor describing the initial tensor polarization (of rank 2) of the spin 1 nuclei. As usual

$$T_{ij}(\vec{u}\,\vec{v}) = \tfrac{1}{2}(u_i v_j + u_j v_i) - \tfrac{1}{3}(\vec{u}\cdot\vec{v})\delta_{ij}$$

When the contact interaction

$$H = \omega\,\vec{s}_1\cdot\vec{s}_2$$

is switched on, the density matrix develops in time and can no longer be factored. Recoupling $\vec{s}_1$ and $\vec{s}_2$ as well as $\vec{s}_1$ and $\tilde{T}(\vec{s}_2\vec{s}_2)$ to give symmetric, traceless tensors in the product space of $\vec{s}_1$ and $\vec{s}_2$ we obtain a general expression for the density matrix :

$$\rho(t) = \frac{1}{6}\Big\{ 1 + c(t)(\vec{s}_1\cdot\vec{s}_2) + 2\vec{a}(t)\cdot\vec{s}_1 + \tfrac{3}{2}\vec{b}(t)\cdot\vec{s}_2 + \tfrac{3}{2}\vec{c}(t)\cdot(\vec{s}_1\times\vec{s}_2)$$
$$+ \frac{18}{5}\vec{d}(t)\cdot[\vec{s}_1\tilde{T}(\vec{s}_2\vec{s}_2)]^{(1)} + 3\tilde{b}(t):\tilde{T}(\vec{s}_2\vec{s}_2)$$
$$+ 3\tilde{c}(t):\tilde{T}(\vec{s}_1\vec{s}_2) + 4\tilde{e}(t):[\vec{s}_1\tilde{T}(\vec{s}_2\vec{s}_2)]^{(2)}$$
$$+ 6\tilde{f}^{(3)}(t):[\vec{s}_1\tilde{T}(\vec{s}_2\vec{s}_2)]^{(3)}\Big\} \qquad (2)$$

i.e., one scalar, four vectors, three rank two tensors and one rank three tensor, giving a total of 35 time-dependent polarization quantities. The boundary conditions are :

$$c(0) = \vec{a}_0\cdot\vec{b}_0\,,\quad \vec{a}(0) = \vec{a}_0\,,\quad \vec{b}(0) = \vec{b}_0$$
$$\vec{c}(0) = \vec{a}_0\times\vec{b}_0\,,\quad \vec{d}(0) = \vec{a}_0\cdot\tilde{b}_0$$
$$\tilde{c}(0) = \tilde{T}(\vec{a}_0\vec{b}_0)\,,\quad \tilde{e}(0) = [\vec{a}_0\,\tilde{b}_0]^{(2)}$$
$$\tilde{f}^{(3)}(0) = [\vec{a}_0\,\tilde{b}_0]^{(3)} \qquad (3)$$

The various symmetric and traceless tensors of rank 1, 2 and 3 are constructed as follows [8] :

$$(\vec{u}\cdot\tilde{v})_i = [\vec{u}\,\tilde{v}]^{(1)}_i = u_j v_{ij}$$
$$[\vec{u}\,\tilde{v}]^{(2)}_{ij} = \tfrac{1}{2}(\varepsilon_{imn}u_m v_{nj} + \varepsilon_{jmn}u_m v_{ni})$$
$$[\vec{u}\,\tilde{v}]^{(3)}_{ijk} = \tfrac{1}{3}(u_i v_{jk} + u_j v_{ik} + u_k v_{ij})$$
$$- \tfrac{2}{15}(\delta_{ij}(\vec{u}\cdot\tilde{v})_k + \delta_{ik}(\vec{u}\cdot\tilde{v})_j + \delta_{jk}(\vec{u}\cdot\tilde{v})_i)$$

if $\tilde{v}$ is traceless and symmetric.

The equations of motion for the polarization quantities follow from the time-dependent Schrödinger equation for ρ [9] :

$$\frac{dP}{dt} = i N_P \, Tr(\rho [H,T])$$

where P is any of the polarization quantities $c, \vec{a}, \ldots, \tilde{f}^{(3)}$ in Eq. (2) and T its associated spin tensor. The normalization N_p follows from

$$P = N_P \, Tr(\rho T)$$

The 35 equations for the various P's are easy to solve because the commutator $[H,T]$ is a linear combination of tensors of the same rank as T itself since H is a scalar in the product space of $\vec{s}_1$ and $\vec{s}_2$. The trace of a product of two-spin tensors, made from either $\vec{s}_1$ or from $\vec{s}_2$ or from both of them [as the nine tensors in (2)] is always zero if their ranks (in product space) are different. This can be verified for all cases in this note, using standard trace theorems [10]. Different tensors of the same rank are also orthogonal ; the nine tensors in (2) are of course orthogonal. For example :

$$Tr\left\{(\vec{s}_1 \times \vec{s}_2)_i [\vec{s}_1 \tilde{T}(\vec{s}_2 \vec{s}_2)]^{(1)}_j\right\} =$$

$$= \varepsilon_{imn} \, Tr(s_{1m} s_{1p}) \, Tr(s_{2n} T_{pj}(\vec{s}_2 \vec{s}_2)) = 0$$

since the last trace, involving only $\vec{s}_2$ operators of different rank, vanishes [8]. When we evaluate $Tr(\rho [H,T])$, the commutator will pick out only terms with the same rank as T. But T itself will not survive since

$$Tr\,(T[H,T]) = 0$$

trivially. There is then a "social hierarchy" in the equations of motion for $c, \ldots, \tilde{f}^{(3)}$; the time derivatives of tensor polarizations will only couple to other tensors of the same rank. Therefore c and $\tilde{f}^{(3)}$ are constant since they have nothing to couple to.

We work out the commutators $[H,T]$ and write them in terms of the spin tensors in (2) as well as other tensors involving the rank two tensor $\tilde{T}(\vec{s}_1 \vec{s}_1)$ and the rank three tensor $\tilde{T}^{(3)}(\vec{s}_2 \vec{s}_2 \vec{s}_2)$. For the vector polarizations we obtain :

$$\dot{\vec{a}} = -2\dot{\vec{b}} = \omega\vec{c}$$
$$\dot{\vec{c}} = -\omega\vec{b} + \tfrac{4}{3}\omega\vec{a} - \omega\vec{d}$$
$$\dot{\vec{d}} = \tfrac{5}{12}\dot{\vec{a}} \qquad (4)$$

which are easily solved subject to the boundary conditions (3) :

$$\vec{a}(t) = \tfrac{1}{9}(8\vec{a}_0 - 6(\vec{b}_0 + \vec{d}_0))\cos\tfrac{3\omega}{2}t - \tfrac{2}{3}(\vec{a}_0 \times \vec{b}_0)\sin\tfrac{3\omega}{2}t + \vec{a}_{av} \qquad (5)$$

and $\vec{b}(t)$ is given by $\vec{b}(t) = \vec{b}_0(\vec{a}_0 - \vec{a}(t))/2$. There are similar expressions for $\vec{c}(t)$ and $\vec{d}(t)$. The last term in Eq. (5), $\vec{a}_{av}$, is what we are left with when we take the time-average of $\vec{a}(t)$. This is the most relevant quantity, physically, since $\vec{a}(t)$ will make of the order of 10^5 oscillations during the lifetime of the muon. We find

$$\vec{a}_{av} = \tfrac{11}{27}\vec{a}_0 + \tfrac{4}{9}(\vec{b}_0 + \vec{d}_0) \qquad (6)$$

This gives the Overhauser effect for spin 1 nuclei. In the absence of any initial nuclear polarization, the muons retain 11/27 of their polarization, and the nuclei are polarized by 8/27 of the polarization the muons had when they entered the 1s orbit.

The muon polarization can never exceed 100%. Take the muons to be 100% polarized initially in the z direction, $\vec{a}_0 = \vec{k}$, and let the nuclei be prepared in a pure state with magnetic quantum number +1. The initial nuclear density matrix is then :

$$\rho_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

from which we obtain [see Eq. (1)] :

$$\vec{b}_0 = \vec{k} \quad , \quad \tilde{b}_0 = \tfrac{1}{2}(\vec{k}\vec{k} - \tfrac{1}{3}\tilde{I})$$

which gives $\vec{d}_o = \vec{k}_o/3$. So when both muons and nuclei are in pure "spin up" states, Eq. (6) gives $\vec{a}_{av} = \vec{k}$, again 100% muon polarization.

The equations of motion for $\tilde{b}$, $\tilde{c}$ and $\tilde{e}$ are even simpler than Eq. (4). We find :

$$\dot{\tilde{b}} = -2\dot{\tilde{c}} = 4\omega\tilde{e}$$

$$\dot{\tilde{e}} = \frac{3}{8}\omega(\tilde{c} - \tilde{b})$$

which are trivial to solve. The frequency of the oscillations are again $3\omega/2$ and

$$\tilde{b}_{av} = \frac{1}{3}\left(\tilde{b}_0 + 2\tilde{T}(\vec{a}_0\vec{b}_0)\right)$$

So we obtain a tensor polarization (alignment) if the nuclei have initial vector polarization even if they have no tensor polarization to start with, provided the muons are polarized.

The corresponding results for pin $\frac{1}{2}$ nuclei are

$$\vec{a}_{av} = \vec{b}_{av} = \frac{1}{2}(\vec{a}_0 + \vec{b}_0) \qquad (7)$$

and the polarization oscillations, similar to Eq. (5), have frequency ω and not $3\omega/2$. Spin $\frac{1}{2}$ nuclei are about 10% more effective for repolarizing muons than spin 1 nuclei, since they transfer half their polarization.

One can generalize the results (5) and (6) to nuclei with any spin I. There are never more than four vector polarizations, so one gets a system of equations similar to (4), but with coefficients which are complicated functions of I. The re-coupled (time-averaged) results for general spin when $\vec{a}_o$ and $\vec{b}_o$ are parallel (or anti-parallel) can be obtained by doing an incoherent average over the two hyperfine states :

$$\langle a \rangle = \sum_{m=-F_+}^{F_+} N_+(m)\langle F_+ m|\sigma_3|F_+ m\rangle + \sum_{m=-F_-}^{F_-} N_-(m)\langle F_- m|\sigma_3|F_- m\rangle$$

where $F_\pm = I \pm \frac{1}{2}$ and σ_z is a Pauli matrix. The magnetic sub-level populations $N_\pm(m)$ have been given in a previous note [6]. Rank two and higher nuclear polarizations have been ignored. The result is :

$$\langle a \rangle = \frac{1}{3} a_0 \left(1 + \frac{2}{(2I+1)^2}\right) + \frac{4 b_0 I}{(2I+1)^2} \qquad (8)$$

which agrees with (6) for $I = 1$ and with (7) for $I = \frac{1}{2}$. For $I = 0$ we get no muon depolarization, of course. The result (8) was derived by Überall for $b_0 = 0$ [3]. The same procedure for b gives :

$$\langle b \rangle = \frac{4}{3} a_0 \frac{I+1}{(2I+1)^2} + b_0 \left(1 - \frac{2}{(2I+1)^2}\right)$$

This checks with the constancy of $\vec{a} + 2I\vec{b}$, which follows from the fact that the total spin $\vec{J} = s_1 + s_2$ commutes with H. In the limit of infinite nuclear polarization the muons retain one third of their original polarization and the nuclear polarization does not change. It is clear from Eq. (8) that spin $\frac{1}{2}$ nuclei will be the most effective for repolarizing the muons with a polarized target.

Our results are of practical importance for muonic atoms with light nuclei with spin, where the hyperfine conversion is absent or negligible. Applications may be in several areas of physics : (1) Muon spin relaxation (μSR) with hyperfine coupling. The initial nuclear polarization could be adjusted to give a zero final muon polarization. (2) Parity-violating effects in nuclear muon capture experiments such as the gamma-neutrino correlation, neutron or photon asymmetry, etc. [1] can be enhanced. (3) The coupling of the nuclear polarization to the muon polarization can possibly be used to detect neutral currents [11].

We have explicitly solved the equation of motion for the density matrix for spin 1 and spin $\frac{1}{2}$ particles interacting by a hyperfine contact Hamiltonian. Tensor techniques reduced the equations for the 35 polarization quantities (sometimes referred as Wangsness-Bloch equations [12] in the context of muonium [5]) to, effectively, a set of four equations and a set of three equations. The time-averaged solution gave the polarizations after they have been recoupled by the contact interaction. The explicit time-dependence of the muon polarization, Eq. (5), will be needed to calculate the depolarization in higher s orbits with lifetimes of the order of the oscillation period ω^{-1}.

The dynamics treated here, in particular the regeneration of muon polarization, are essential features in any muon capture experiment with a polarized target.

ACKNOWLEDGEMENTS

We are grateful for discussions with J. Deutsch, M. Locher and P.F. Meyer.

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