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The Shape Function in Field Theory

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Abstract

We consider the decay of a heavy flavour into an inclusive hadronic state X of invariant mass m_X small with respect to its energy E_X, $m_X \ll E_X$. The electron spectrum and the hadronic mass distribution in semileptonic $b \to u$ decays, or the photon spectrum in $b \to s\gamma$ decays, all require, close to their endpoints, a control over this region. This region is affected both by non-perturbative phenomena related to the Fermi motion of the heavy quark and by perturbative soft gluon radiation in the final state (Sudakov form factor). Fermi motion can be described by the shape function $f(m_*)$, which represents the distribution of the effective mass m_* of the heavy quark at disintegration time. We perform a factorization with a simple technique in order to consistently separate perturbative from non-perturbative effects. We find that the shape function, contrary to naive expectations, is not a physical distribution, as it is affected by substantial regularization scheme effects, controlling even the leading, double-logarithmic term. It factorizes, however, the bulk of non-perturbative effects in lattice-like regularizations. Some non-perturbative effects are present in the coefficient function even at leading twist, but they are expected to be suppressed on physical grounds. Finally, we clarify a controversial factor of 2 in the evolution kernel of the shape function.

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1 Introduction

Nowadays there are many facilities that allow an accurate experimental study of heavy flavour decays. It is therefore becoming more and more important to improve the accuracy and the reliability of the theoretical calculations. In this paper, we study the properties of the decays of heavy flavour hadrons into inclusive hadronic states X "tight" in mass, i.e. with an invariant mass m_X small with respect to the energy E_X:

$$m_X \ll E_X. \tag{1}$$

More specifically, we consider the situation where

$$m_X^2 \sim E_X \, \Lambda_{QCD}, \tag{2}$$

so that

$$\frac{m_X^2}{E_X^2} \sim \frac{\Lambda_{QCD}}{E_X} \ll 1 \qquad (E_X \gg \Lambda_{QCD})\,. \tag{3}$$

Heavy flavour decays are characterized by three mass or energy scales: the mass of the heavy flavour m_Q, the energy E_X, and the invariant mass m_X of the final hadronic state. A formal definition of kinematics (2) is the limit, in the heavy quark rest frame:

$$\begin{aligned} E_X &\rightarrow \infty, \\ m_X^2 &\rightarrow \infty \end{aligned}$$

with

$$\frac{m_X^2}{E_X} = \text{const.} \tag{4}$$

The divergence of m_X^2, even though slower than the one of E_X^2, implies that the final hadronic state can be replaced with a partonic one, i.e. that the use of perturbation theory is fully justified. Limit (4) implies also the limit of infinite mass for the heavy flavour Q:

$$m_Q \rightarrow \infty, \tag{5}$$

since $m_Q \geq E_X$. Another consequence of (4) is that

$$\frac{m_X^2}{E_X^2} \rightarrow 0, \tag{6}$$

i.e. we are in the so-called threshold region[1].

The study of these processes has both a theoretical and a phenomenological interest. On the theoretical side, in heavy decays the infrared perturbative structure of gauge theories - the Sudakov form factor [1, 2] - enters in a rather "pure" form, owing to the absence of initial state mass singularities. On the phenomenological side, the computation of many relevant distributions requires a good theoretical control over the region (1). As examples, let us quote the electron spectrum $d\Gamma/dE_e$ close to the endpoint $E_e \lesssim m_B/2$ [3] and the hadron mass distribution $d\Gamma/dm_X$ at small m_X [4] in semileptonic $b \to u$ decays, such as

$$B \to X_u + e + \nu, \tag{7}$$

or the photon energy distribution $d\Gamma/dE_\gamma$ close to the endpoint $E_\gamma \lesssim m_B/2$ in $b \to s\gamma$ decays. For the electron or photon spectrum, the region (1) is involved because the requirement of a large energy of the lepton or of the photon pushes down to zero the mass of the recoiling hadronic system. As is well known, the above mentioned distributions in (7) allow an inclusive determination of the CKM-matrix element $|V_{ub}|$ [5], while a large photon energy in the rare decay, $E_\gamma \gtrsim 2.1$ GeV is required to cut experimental backgrounds.

In general, the dynamics in region (1) is rather intricate as it involves an interplay of non-perturbative and perturbative contributions. These are related to the Fermi motion of the heavy quark inside the hadron and to the Sudakov suppression in the threshold region (6), respectively. Even though these two effects are physically distinguishable and are treated as independent in various models [3], they are ultimately both described by the same quantum field theory, QCD. Therefore the problem arises of describing them consistently, i.e. without double countings, inconsistencies, etc. Our idea is to subtract from the hadronic tensor encoding *all* QCD dynamics,

$$W_{\mu\nu} \equiv \sum_X \langle\, H_Q | J_\nu^+ | X \rangle \, \langle X | J_\mu | H_Q \rangle \, \delta^4(p_B - q - p_X), \tag{8}$$

each of the perturbative components - associated with the Sudakov form factor and with other short-distance corrections - to end up with an explicit

[1]The converse is not true: limit (6) does not imply limit (4) (we thank G. Veneziano for pointing this out to us).

representation of the non-perturbative component. In eq. (8), we have defined

$$J_\mu(x) \equiv \overline{q}(x)\Gamma_\mu Q(x), \tag{9}$$

where Γ_μ is a matrix in Dirac algebra[2], p_X is the momentum of the final hadronic jet, and H_Q is a hadron containing the heavy quark Q. The non-perturbative component is identified with an ultraviolet (UV) regularized expression for the shape function in the effective theory. The shape function $f(k_+)$ has been introduced using the Operator Product Expansion (OPE) and can be defined as [6]

$$f(k_+) \equiv \langle H_Q \mid h_v^\dagger \, \delta(k_+ - iD_+) \, h_v \mid H_Q \rangle, \tag{10}$$

where h_v is a field in the Heavy Quark Effective Theory (HQET) with 4-velocity v; D_+ is the plus component of the covariant derivative, i.e. $D_+ \equiv D^0 + D^3$. The shape function represents the probability that the heavy quark has a momentum $m_B v + k'$ with a given plus component $k'_+ = k_+$. The shape function can also be interpreted (see section 4.4) as the probability that Q has an effective mass

$$m_* = m_B + k_+ \tag{11}$$

at disintegration time. The renormalization properties of the shape function have also been analysed [7, 8, 9, 10]. Because of UV divergences affecting its matrix elements, the shape function needs to be renormalized and it consequently acquires a dependence on the renormalization point μ. The non-perturbative information about Fermi motion enters in this framework as the initial value for the μ-evolution. The shape function can be extracted from a reference process and used to predict other processes, analogously to the parton distribution functions in usual hard processes such as Deep Inelastic Scattering (DIS) or Drell–Yan [11]. In principle, it can also be computed with a non-perturbative technique, for example lattice QCD [12].

Our approach aims at a deeper understanding of perturbative and non-perturbative effects with respect to the standard OPE in dimensional regularization. We compare different regularization schemes and find that the factorization procedure in the effective theory is scheme-dependent. The shape function, in contrast to naive expectations, is not a physical distribution, but it is affected by regularization scheme effects, even at the leading,

[2] For the left-handed currents of the Standard Model, $\Gamma_\mu = \gamma_\mu \, (1 - \gamma_5)$.

double-logarithmic level. We show, however, that it factorizes most of the perturbative effects in a class of regularization schemes.

This paper is devoted to a wide audience, i.e. not only to perturbative QCD experts, but also to phenomenologists who are interested in the field theoretic aspects of this area of B physics, as well as to lattice-QCD physicists who may wonder about the possibility of simulating the shape function. We have therefore tried to give a plain presentation of our method, together with a self-consistent description of the known results to be found in the literature.

In section 2 we give a simple introduction to the physics of semi-inclusive heavy flavour decays. In section 3 we present our strategy, based on factorization, in order to consistently combine perturbative and non-perturbative contributions and to arrive at a formal definition of the shape function in field theory; we outline the main steps and the relevant issues. In section 4 we review the standard OPE derivation of the shape function in the effective theory; this section can be skipped by the experienced reader. In section 5 we return to the strategy outlined in section 4 and apply our factorization procedure in the quantum theory to a specific class of loop corrections. Our technique is completely general, but we believe that it is better illustrated by treating in detail a simple computation, which illustrates most of the general features. In section 6 we describe the properties of the shape function in the effective theory in various regularizations and its evolution with the UV cutoff or renormalization scale. We also discuss our results on factorization and clarify a controversial factor of 2 in the evolution kernel of the shape function. Section 7 contains the conclusions.

2 Physics of semi-inclusive heavy flavour decays

Let us begin by discussing Fermi motion. This phenomenon, originally discovered in nuclear physics, is classically described as a small oscillatory motion of the heavy quark inside the hadron, due to the interaction with the valence quark; in the quantum theory it is also the *virtuality* of the heavy flavour that matters. Generally, as the mass of the heavy flavour becomes large, i.e. as we take the limit (5), we expect that the heavy particle decouples from the light degrees of freedom and becomes "frozen" with respect to strong interactions. That is indeed true in the "bulk" of the phase space of the decay products, but it is untrue close to kinematical boundaries, as in region (1). This is because a kinematical amplification effect occurs, according to which a small virtuality of the heavy flavour in the initial state produces relatively large variations of the fragmentation mass in the final state. To see how this works in detail, let us begin with a picture of the initial bound state. We assume that the momentum exchanges r_μ between the heavy flavour and the light degrees of freedom are of the order of the hadronic scale,

$$|r_\mu| \sim O\left(\Lambda_{QCD}\right), \tag{12}$$

as we take the infinite mass limit (5). In other words, we assume that the momentum transfer does not scale with the heavy mass but remains essentially constant[3]. This assumption, which is rather reasonable from a physical viewpoint, is at the basis of the application of the HQET [14]. Let us discuss for example the decay (7). The initial meson has momentum

$$p_B = m_B v, \tag{13}$$

where v is the 4-velocity, which we can take at rest without any loss of generality: $v^\mu = (1; 0, 0, 0)$. The final hadronic state X has a momentum

$$Q = m_B v - q \tag{14}$$

and invariant mass

$$m_X^2 \equiv Q^2. \tag{15}$$

In eq. (14) q_μ is the momentum of the virtual W or, equivalently, of the leptonic pair. We isolate in the decay a hard subprocess consisting in the fragmentation of the heavy quark. If the valence quark - in general the light degrees of freedom in the hadron - have momentum $-k'$, the heavy quark has a momentum[4]

$$p_Q = m_B v + k' \tag{16}$$

and a virtuality

$$p_Q^2 - m_B^2 = 2m_B v \cdot k' + k'^2 \neq 0 \quad \text{(in general)}. \tag{17}$$

[3]We must specify that we consider an initial hadron containing a *single* heavy quark: hadrons containing *more* than one heavy quark, such as for example quarkonium states, need a different theoretical treatment [13].

[4]For the appearance of m_B instead of m_b, see footnote in section 4.1.

The final invariant mass of the hard subprocess, i.e. the fragmentation mass, is

$$\widehat{m}_X^2 \equiv (p_Q - q)^2 = (Q + k')^2 = m_X^2 + 2Q \cdot k' + k'^2 \simeq m_X^2 + 2Q \cdot k' \tag{18}$$

and this is the mass that controls the dynamics of the hard subprocess, i.e. the Sudakov form factor (the difference between m_X^2 and $\widehat{m}_X^2$ is that we do not include in the latter the momentum of the valence quark). The term k'^2 has been neglected in the last member of eq. (18) because it is small, as gluon exchanges are soft according to the assumption (12). We take the motion of the final *up* quark in the $-z$ direction, so that the vector Q has large zero and third components, both of order E_X, and a small square; we have therefore for the average in the meson state:

$$\langle Q \cdot k' \rangle = Q \cdot \langle k' \rangle \sim O\left(E_X \Lambda_{QCD}\right). \tag{19}$$

A fluctuation in the heavy quark momentum of order Λ_{QCD} in the initial state produces a variation of the final invariant mass of the hard subprocess of order

$$\delta \widehat{m}_X^2 \sim O\left(\Lambda_{QCD}\, E_X\right). \tag{20}$$

An amplification by a factor E_X has occurred, as anticipated. The fluctuation (20) is of the order of (2) and so it must be taken into account.

We will discuss the shape function at length in sections 4 and 6, but let us introduce now some of its more important properties. If we consider a heavy quark with the given off-shell momentum (16), we find for the shape function[5]

$$f(k_+)^{part} = \delta\left(k_+ - k'_+\right) + O\left(\alpha_S\right), \tag{21}$$

where

$$k_+ \equiv -\frac{m_X^2}{2E_X}. \tag{22}$$

Selecting the hadronic final state, i.e. k_+, we select the light-cone virtuality $k'_+ = k_+$ of the heavy flavour which participates in the decay. After inclusion of the radiative corrections, we find that in general $k'_+ \geq k_+$ [6]. Equation (21)

[5] The final state consists of a massless on-shell quark at the tree level.

[6] To obtain the hadronic shape function, the "elementary" or "partonic" shape function in eq. (21) has to be convoluted with the distribution $\varphi_0\left(k'_+\right)$ of the primordial light-cone virtuality k'_+ of the heavy quark inside the hadron.

is analogous to the relation between the Bjorken variable $x_B \equiv -q^2/(2p \cdot q)$ (p is the momentum of the hadron and q that of the space-like photon) and the momentum fraction x of partons in the naive parton model, where we have

$$q(x)^{part} = \delta(x - x_B) + O(\alpha_S). \tag{23}$$

In this case, as is well known, by selecting final state kinematics, i.e. x_B, one selects the momentum fraction $x = x_B$ of the partons that participate in the hard scattering. Just as in the heavy flavour decay, radiative corrections lead to a softening of the above condition in $x \geq x_B$, due to the emission of collinear partons.

We note that even with the amplification effect (20), Fermi motion effects are irrelevant in most of the phase space, where typical values for the final hadron mass are

$$\widehat{m}_X^2 \sim O(E_X^2). \tag{24}$$

This is in agreement with physical intuition.

As will be proved in section 4.4, the shape function can be interpreted as the distribution of a *variable* mass. The virtuality of the heavy flavour can be represented by a shift of its mass, $m_b \to m_*$. In other words, an off-shell particle with a given mass, i.e. with the momentum (16), can be replaced by an on-shell particle with a variable mass, i.e. with a momentum $m_* v$. The physical distribution is obtained by convoluting the distribution of an isolated quark of mass m_* with the probability distribution for such a mass (see eq. (75)): this is the basis of the factorization theory for the semi-inclusive heavy flavour decays.

Fermi motion is a non-perturbative effect in QCD because it involves low momentum transfers to the heavy flavour (cf. eq. (12)), at which the coupling is large; it does however occur also in QED bound states, where it can be treated with perturbation theory [7].

The second phenomenon relevant in region (1) is related to soft gluon emission and it is of a perturbative nature - it is a case of the Sudakov form factor in QCD [15]. The *up* quark emitted by the fragmentation of the heavy flavour with a large virtuality - of the order of the final hadronic energy E_X - evolves in the final state, emitting soft and collinear partons, either real or virtual. Since the final state is selected to have a small invariant mass

[7] Consider for instance an atom composed of a μ and an e, decaying by μ fragmentation.

(cf. eq. (6)), real radiation is inhibited with respect to the virtual one. That means that infrared (IR) singularities coming from real and virtual diagrams still cancel, but leave a large residual effect in the form of large logarithms[8]:

$$\alpha_S \left(\frac{\log m_X^2/E_X^2}{m_X^2/E_X^2} \right)_+ . \tag{26}$$

Schematically, the rate for final states with an invariant mass m_X^2 has double-logarithmic contributions at order α_S, of the form:

$$\text{real} = \alpha_S \int_0^{E_X} \int_0^1 \frac{d\epsilon}{\epsilon} \frac{d\theta^2}{\theta^2}\, \delta \left(\epsilon\theta^2 - \frac{m_X^2}{E_X} \right) \tag{27}$$

and

$$\text{virtual} = -\alpha_S\, \delta \left(\frac{m_X^2}{E_X} \right) \int_0^{E_X} \int_0^1 \frac{d\epsilon}{\epsilon} \frac{d\theta^2}{\theta^2}, \tag{28}$$

where ϵ is the gluon energy, θ is the angle between the *up* and the gluon, and $\Theta = \pi - \theta$ is the polar angle of the gluon 3-momentum. The perturbative corrections of the form (26) blow up at the Born kinematics $m_X = 0$, which is the threshold of the inelastic channels. For this reason, the above corrections are often called threshold logarithms and need a resummation to any order in α_S.

3 Overview of Factorization

The aim of this paper is a detailed study of factorization in semi-inclusive heavy flavour decays and of the properties of the shape function in field theory. In order to trace all the perturbative and non-perturbative contributions to the process, it is convenient to perform the factorization in two steps. In the first step the heavy flavour is replaced by a static quark. That is accomplished by taking the infinite mass limit (5), keeping E_X and m_X fixed.

[8]The plus-distribution is defined as

$$\left(\frac{\log x}{x} \right)_+ = \theta(x) \frac{\log x}{x} - \delta\,(x) \int_0^1 \frac{\log y}{y} dy. \tag{25}$$

With this, the hadronic tensor loses a kinematical scale, namely the heavy flavour mass m_Q:

$$W_{\mu\nu}\left(m_Q, E_X, m_X\right) \to \widetilde{W}_{\mu\nu}\left(E_X, m_X\right), \tag{29}$$

where the effective hadronic tensor is defined as

$$\widetilde{W}_{\mu\nu} \;\equiv \sum_X \langle H_Q|\widetilde{J}_\nu^\dagger|X\rangle\langle X|\widetilde{J}_\mu|H_Q\rangle\, \delta^4(p_B - q - p_X), \tag{30}$$

and it contains the static-to-light currents

$$\widetilde{J}_\mu(x) = \overline{q}(x)\Gamma_\mu \widetilde{Q}(x). \tag{31}$$

The difference between the two tensors in eq. (29) is incorporated into a first coefficient function or hard factor. While in full QCD the vector and axial currents are conserved, or partially conserved, so the renormalization constants are UV-finite and anomalous dimensions vanish, this property does not hold anymore in the HQET: the effective current with a static quark is not conserved and it acquires an anomalous dimension $\widetilde{\gamma}_J$ [9][16]:

$$\left(\frac{d}{d\log\mu} + \widetilde{\gamma}_J\right)\langle\widetilde{J}_\mu\rangle = 0. \tag{32}$$

As a consequence also the hadronic tensor acquires an anomalous dimension, which equals twice that of the vector or axial current:

$$\left(\frac{d}{d\log\mu} + 2\widetilde{\gamma}_J\right)\widetilde{W}_{\mu\nu} = 0. \tag{33}$$

All this is very easily understood by observing that the original QCD tensor $W_{\mu\nu}$ is UV-finite at one loop but it does contain $\alpha_S \log m_Q$ terms, and so it is divergent in the infinite-mass limit (5). If this limit is taken *ab initio,* i.e. before regularization, these terms manifest themselves as new ultraviolet divergences, an heritage of the log m_Q terms of the original tensor. We may

[9] In eq. (32) and (33), we are representing the evolution schematically, without details; f.i., we do not distinguish between the anomalous dimensions of the vector and axial currents.

say that the dependence on the heavy mass is promoted to UV divergence; in practice

$$\alpha_S \log \frac{m_Q}{E_X} \to \alpha_S \log \frac{\Lambda_1}{E_X}, \tag{34}$$

where Λ_1 is a UV cutoff if we deal with the bare theory, or a renormalization point if we deal with the renormalized theory; in principle $\Lambda_1 \ll m_Q$. At the end of the game, the effective hadronic tensor still depends on three scales, just like the original one,

$$\widetilde{W}_{\mu\nu} = \widetilde{W}_{\mu\nu} \left(\Lambda_1; E_X, m_X\right). \tag{35}$$

The original tensor $W_{\mu\nu}$ is parametrized in terms of five independent form factors [17]. For the HQET hadronic tensor (30) there are instead relations between the form factors originating from the spin-symmetry of the HQET. In particular, the structure in $\log m_Q/E_X$ of the original QCD tensor can be understood by looking at the UV divergences of $\widetilde{W}_{\mu\nu}$ [10].

After the first step $\widetilde{W}_{\mu\nu}$ still contains perturbative contributions. The latter are factorized with a second step, which corresponds to the limit (4). Additional UV divergences are introduced also with this second step, which must be regulated with a new cutoff Λ_2. In principle $\Lambda_2 \ll E_X$, since $E_X \to \infty$. As before with the heavy mass logarithms, soft and collinear logarithms are promoted to ultraviolet logarithms:

$$\alpha_S \left(\frac{\log m_X^2/E_X^2}{m_X^2/E_X^2} \right)_+ \to \alpha_S \left(\frac{\log\left(-2k_+/\Lambda_2\right)}{-2k_+/\Lambda_2} \right)_+ . \tag{36}$$

The second factorization step involves double-logarithmic effects of an infrared nature, in contrast with the single logarithms of the large mass of the first step. In practice, we separate perturbative contributions from non-perturbative ones starting with a cutoff

$$\Lambda_2 \sim E_X \tag{37}$$

and lowering it to a much smaller value[11]

$$\Lambda_2' \ll E_X. \tag{38}$$

[10] In Dimensional Regularization (DR), this means simple poles $1/\epsilon$.

[11] In order to avoid substantial finite cutoff effects, $\Lambda_2' \gg \Lambda_{QCD}$ must hold.

The contributions of the fluctuations with energy between Λ_2 and Λ_2' are put into a second coefficient function, while the contributions below Λ_2' are factorized inside the shape function. The latter is defined in the framework of a low-energy effective theory, with a cutoff given by

$$\Lambda_{ET} = \Lambda_2'. \tag{39}$$

Most of the non-perturbative effects in lattice-like regularizations are contained in the shape function, which uniquely determines the final, non-perturbative, hadronic tensor

$$\widetilde{\widetilde{\widetilde{W}}}_{\mu\nu} \equiv \sum_X \langle H_Q | \widetilde{\widetilde{J}}_\nu^\dagger | X \rangle \langle X | \widetilde{\widetilde{J}}_\mu | H_Q \rangle \, \delta^4 (p_B - q - p_X), \tag{40}$$

containing the effective-heavy-to-effective-light currents

$$\widetilde{\widetilde{J}}_\mu(x) = \overline{\widetilde{q}}(x) \Gamma_\mu \widetilde{Q}(x). \tag{41}$$

It is worth noting that the tensor (40) involves a single form factor, proportional to the shape function itself. That is again a consequence of the spin-symmetry of both HQET and Large Energy Effective Theory (LEET) [18], which is more efficient than that one of the HQET alone. The shape function is completely non-perturbative and perturbative factors can no longer be extracted.

The effect of lowering the UV cutoff (eqs. (37) and (38)) is incorporated inside a coefficient function, which, unlike more simple cases such as the light-cone expansion in DIS, is not completely short-distance dominated. Some long-distance effects are left in the coefficient function, but they are expected to be suppressed on physical grounds. Finally, the introduction of ultraviolet divergences with factorization implies scheme-dependence issues for the shape function, which are rather dramatic because of the double-logarithmic nature of the problem (cf. eqs.(27) and (28)).

In fig. 1, we give a pictorial description of the above procedure.

4 OPE in the effective theory

The amplitude for the decay (7), which we take as our example from now on, can be written at the lowest order in the weak coupling as

$$A = \frac{G_F}{\sqrt{2}} \langle l\nu \mid L_\mu \mid 0\rangle \, \langle X|J^\mu|B\rangle, \tag{42}$$

where L_μ is the leptonic current and J_μ is the hadronic one:

$$J_\mu(x) \equiv \overline{q}(x)\Gamma_\mu b(x) \tag{43}$$

with $\Gamma_\mu = \gamma_\mu(1-\gamma_5)$, $q(x)$ a light quark field and $b(x)$ the beauty quark field. Taking the square of (42) and summing over the final states, we arrive at the hadronic tensor defined in eq. (8). By the optical theorem, we can relate the hadronic tensor $W_{\mu\nu}$ to the imaginary part[12] of the Green function or forward hadronic tensor $T_{\mu\nu}$:

$$W_{\mu\nu} = -\frac{1}{\pi}\mathrm{Im}\ T_{\mu\nu}, \tag{44}$$

where

$$T_{\mu\nu} \ \equiv -i \int d^4x\, e^{-iqx} \langle B|T\left(J_\mu^\dagger(x)J_\nu(0)\right)|B\rangle. \tag{45}$$

4.1 The HQET

We are interested in the evaluation of $T_{\mu\nu}$ in the effective theory and we discuss in this section the first factorization step: replacing the beauty quark by a quark in the HQET. As is well known, we can decompose the heavy quark field $b(x)$ into two effective quark and antiquark fields h_v and H_v [13]

$$b(x) = e^{-im_B v\cdot x}\left[h_v(x) + H_v(x)\right], \tag{46}$$

satisfying

$$P_+\ h_v = h_v, \quad P_-\ h_v = 0, \quad P_-\ H_v = H_v, \quad P_+\ H_v = 0, \tag{47}$$

[12]Since Γ_μ is in general complex, we should say, more properly, the absorptive part.

[13]We prefer to refer to the physical B-meson mass rather than to the unphysical b-quark mass. Their difference is of order Λ_{QCD}, so it is a $1/m_B$ correction and can be neglected in our leading-order analysis. Furthermore, in perturbation theory there is no binding energy so that $m_b = m_B$.

where $P_\pm = (1 \pm \hat{v})/2$ are the projectors over the components with positive and negative energies, respectively. The field H_v is neglected (which amounts to neglecting heavy-pair creation), so that

$$b(x) \simeq e^{-im_B v\cdot x}\ h_v(x). \tag{48}$$

By using eq. (48) we obtain

$$\widetilde{T}_{\mu\nu} = -i \int d^4x\ e^{iQ\cdot x} \langle B(v)|T\ \bar{h}_v(x)\Gamma_\mu^\dagger\ q(x)\ \bar{q}(0)\Gamma_\nu\ h_v(0)|B(v)\rangle. \tag{49}$$

We now use the Wick theorem and we single out the only contraction that is relevant to B decay:

$$\widetilde{T}_{\mu\nu} = \int d^4x\ e^{iQx} \langle B|\bar{h}_v(x)\Gamma_\mu^\dagger\, S(x|0)\Gamma_\nu h_v(0)|B\rangle, \tag{50}$$

where $S(x|0)$ is the light quark propagator. Note that the operator entering the right-hand side of eq. (50) is already normal ordered, since h_v has only the component that annihilates heavy quarks, while $\bar{h}_v$ only the components that create those. We can express the Fourier transform of the light quark propagator as[14]

$$S(Q+iD) = \frac{1}{i\hat{D}+\hat{Q}+i0} = \frac{i\hat{D}+\hat{Q}}{Q^2+2iD\cdot Q - D^2 - g/2\ \sigma_{\mu\nu}G^{\mu\nu}+i0} \tag{51}$$

where $\sigma_{\mu\nu} \equiv i/2[\gamma_\mu, \gamma_\nu]$ is a generator of the Lorentz group, $G_{\mu\nu} \equiv -i/g[D_\mu, D_\nu]$ is the field strength and $D_\mu \equiv \partial_\mu - igA_\mu$ is the covariant derivative. In eq. (51), 0 denotes, as usual, an infinitesimal positive number and gives the prescription to deal with pole or branch-cut singularities. There are three different regions according to the value of the jet invariant mass, which are described by three different full or effective theories:

$$\begin{array}{rlcl} i) & m_X^2 & \sim & O(E_X^2), \\ ii) & m_X^2 & \sim & O(\Lambda_{QCD}^2), \\ iii) & m_X^2 & \sim & O(\Lambda_{QCD}\, E_X). \end{array} \tag{52}$$

[14] The notation is very compact. For more explicit representations of the propagator see ref. [19].

Since the derivative of the rescaled h_v field brings down the residual momentum k', and it is therefore an operator with matrix elements of order $O\left(\Lambda_{QCD}\right)$, the matrix elements of the operators entering the light quark propagator have a size of the order of

$$\begin{aligned} \langle\, i\hat{D}\,\rangle &\sim O(\Lambda_{QCD}), \\ \langle\, 2iD\cdot Q\,\rangle &\sim O(\Lambda_{QCD}\, E_X), \\ \langle\, D^2\,\rangle &\sim O(\Lambda_{QCD}^2), \\ \langle\, \sigma_{\mu\nu}G^{\mu\nu}\,\rangle &\sim O(\Lambda_{QCD}^2). \end{aligned} \tag{53}$$

Let us discuss these regions in turn in the next section.

4.2 General kinematical regions

i) This region corresponds to a jet X with a large invariant mass, of the order of the energy:

$$m_X \sim E_X\,. \tag{54}$$

To a first approximation all the covariant derivative terms can be neglected, so that

$$S(Q+iD)\;\simeq\;\frac{\hat{Q}}{Q^2+i0}, \tag{55}$$

i.e. the light quark can be described as a free quark. A higher accurary is reached when expanding the propagator in powers of the covariant derivative operators up to the required order. We have here an application of the $1/m_B$ expansion up to a prescribed (finite) order [15]. In this region there are no large adimensional ratios of scales, the latter being all of the same order. This implies that in perturbation theory we do not hit large logarithmic corrections to be resummed to all orders in α_S. This region is not relevant to the endpoint electron spectrum because the hadronic jets takes away most of the available energy. This region will not be discussed further here.

[15] It is clear that a consistent inclusion of the $1/m_B$ corrections involves also the expansion of the heavy quark field $b(x)$ into the effective quark field $h_v(x)$ up to the required order.

ii) This region involves a recoiling hadronic system with a mass of the order of the hadronic one: it can be a single hadron or very few hadrons. The dynamics is dominated by the emission, with consequent decay, of resonances; it is a completely non-perturbative problem. According to the estimates (53), no term can be neglected in the light quark propagator. We are faced with full QCD dynamics as far as the final hadronic state is concerned. This region must be evaluated by an explicit sum over all the kinematically possible hadronic states, and the latter have to be computed with a non-perturbative technique such as a quark model or lattice QCD. This region will not be discussed here either.

iii) This region is intermediate between *i*) and *ii*) and as such it has both perturbative and non-perturbative components. Roughly speaking, we have to take into account non-perturbative effects for the initial state hadron, while we can neglect final state binding effects. This region is characterized by a small ratio of the jet invariant mass to the jet energy, and thus involves the large adimensional ratio in (3). As always is the case, perturbation theory generates logarithms of the above adimensional ratio, eq. (26). The term $2\,iD \cdot Q$ at the denominator cannot be brought at the numerator (with a truncated operatorial expansion) because it is of the same order as m_X^2. At lowest order, the other covariant derivative terms can be neglected, to give:

$$S(Q + iD) \simeq \frac{\hat{Q}}{m_X^2 + 2iD \cdot Q + i0}. \tag{56}$$

One can reach a higher level of accurary keeping these latter corrections up to a given order[16]. The rest of the paper deals with region *iii*) at the lowest order in $1/m_B$.

4.3 The LEET

In this section we discuss the second factorization step, which involves the description of the final *up* quark in the LEET, according to eq. (56). Let us

[16] We envisage a relation between the $1/E_X$ corrections to the shape function and the power-suppressed perturbative corrections of the form $\alpha_S/E_X \log^2(E_X/m_X)$.

define the adimensional vector n_μ as:

$$n_\mu = \frac{Q_\mu}{Q \cdot v} \ . \tag{57}$$

This n_μ has a normalized time component, $n_0 = 1$. In the "semi-inclusive" endpoint region *iii*):

$$\begin{aligned} n^2 &= \frac{m_X^2}{E_X^2} \\ &= O\left(\frac{\Lambda_{QCD}}{E_X}\right) \ll 1. \end{aligned} \tag{58}$$

We will show later that n can sometimes be replaced by a vector lying exactly on the light-cone, i.e.

$$n \to \overline{n}, \tag{59}$$

where $\overline{n}^\mu = (1; 0, 0, -1)$ $(\overline{n}^2 = 0)$, representing the direction of the hadronic jet, the $-z$ axis. We can write

$$S(Q + iD) = \frac{1}{2v \cdot Q} \ \frac{\hat{Q}}{iD_+ - k_+ + i0}, \tag{60}$$

where k_+ has been defined in eq. (22) and $D_+ \equiv \overline{n} \cdot D$. We can simplify the tensor structure of $T_{\mu\nu}$ by using the identity

$$\bar{h}_v \Gamma_\mu h_v = \frac{1}{2} \mathrm{Tr}(\Gamma_\mu P_+)\ \bar{h}_v h_v - \frac{1}{2} \mathrm{Tr}(\gamma_\mu \gamma_5 P_+ \Gamma_\mu P_+)\, \bar{h}_v \gamma^\mu \gamma_5 h_v, \tag{61}$$

which is valid for any Γ_μ. The matrix element of the axial vector current between the B-meson states vanishes by parity invariance, so that [17]:

$$\tilde{\tilde{T}}_{\mu\nu} = \ s_{\mu\nu} \ \frac{1}{2v \cdot Q} \ F(k_+), \tag{62}$$

where we have defined the "light-cone" function

$$F(k_+) \equiv \langle B(v) \mid h_v^\dagger \frac{1}{iD_+ - k_+ + i0} h_v \mid B(v) \rangle, \tag{63}$$

[17] A physical argument for the spin factorization is that, in the limit $m_B \to \infty$, the spin interaction of the b-quark in the B meson vanishes; therefore we can average over the helicity states of the b quark [20].

and

$$s_{\mu\nu} \equiv \frac{1}{2}\,\mathrm{Tr}\left[\Gamma_{\mu}^{\dagger}\hat{Q}\Gamma_{\nu}P_{+}\right] \tag{64}$$

is the “spin factor”, containg the leading spin effects. The factor $1/(2v \cdot Q)$ is a jacobian, which appears as we go from the full QCD variable Q^2 to the effective theory variable k_+.

Taking the imaginary part of $T_{\mu\nu}$, we obtain (see relation (44)):

$$\widetilde{\widetilde{W}}_{\mu\nu} = \; s_{\mu\nu}\,\frac{1}{2v \cdot Q}\; f(k_+), \tag{65}$$

where

$$f(k_+) \equiv -\frac{1}{\pi}\mathrm{Im}\,F(k_+) \tag{66}$$

is the shape function. By using the formula

$$\frac{1}{iD_+ - k_+ + i0} = \mathrm{P}\frac{1}{i\,D_+ - k_+} - i\pi\,\delta(iD_+ - k_+), \tag{67}$$

we recover the definition of the shape function given by eq. (10). Note that it involves the non-local operator $h_v^{\dagger}\,\delta(k_+ - iD_+)\,h_v$, which results from the resummation of the towers of operators of the form $(Q \cdot D)^n$.

4.4 The Variable Mass

The hadronic tensor can be written in the effective theory in terms of the shape function as:

$$\widetilde{\widetilde{W}}_{\mu\nu} = \; s_{\mu\nu}\int_{-\infty}^{0} dk_+\;\delta\left(Q^2 + 2k_+ v \cdot Q\right)\; f(k_+); \tag{68}$$

in the second member, k_+ is an integration, i.e. dummy, variable. In the free theory, with an on-shell b-quark,

$$f^0(k_+) = \delta(k_+ - 0), \tag{69}$$

so that

$$W^0_{\mu\nu} = s_{\mu\nu}\;\delta(Q^2 - 0). \tag{70}$$

The hadronic tensor can be written, up to terms of order $k_+^2 \sim O\left(\Lambda_{QCD}^2\right)$, as

$$\widetilde{\widetilde{W}}_{\mu\nu} = \; s_{\mu\nu}(Q) \int dm_* \; \delta\left(Q_*^2 - 0\right) \; f(m_* - m_B), \tag{71}$$

where we have defined

$$Q_* \equiv m_* v - q \tag{72}$$

and m_* is the "variable" or "fragmentation" mass, defined as

$$m_* = m_B + k_+. \tag{73}$$

Since m_* is just a shift of k_+, the range is

$$-\infty < m_* \le m_B. \tag{74}$$

Inside $s_{\mu\nu}$ we can replace Q with Q_*,because that amounts only to a correction of order $k_+ = O(\Lambda_{QCD})$, so that[18]

$$\widetilde{\widetilde{W}}_{\mu\nu}(v,Q) = \int_0^{m_B} dm_* \; \varphi(m_*) \;\; W^0_{\mu\nu}(v, Q_*), \tag{75}$$

where

$$W^0_{\mu\nu}(v, Q^*) = \; s_{\mu\nu}(Q_*) \; \delta\left(Q_*^2 - 0\right) \tag{76}$$

is the hadronic tensor in the free theory for a heavy quark of mass m_* and

$$\varphi(m_*) \equiv f(m_* - m_B) \tag{77}$$

is the distribution for the effective mass m_* of the b-quark inside the B-meson at disintegration time. Equation (75) is the fundamental result of factorization in semi-inclusive heavy flavour decays: it says that the hadronic tensor in the effective theory can be expressed as the convolution of the hadronic tensor in the free theory with a variable mass times a distribution probability for this mass. That offers also the physical interpretation to the shape function anticipated in the introduction: it represents the probability that the b quark has an effective mass m_* at the decay time. Since this tensor encodes all the hadron dynamics, *any* distribution can be expressed in a similar factorized form.

[18] We replace by 0 the lower limit of integration, because the relevant region is $m_* \sim m_B - O(\Lambda_{QCD})$.

5 Factorization in the quantum theory

In this section we discuss factorization in the quantum theory, i.e. the separation of short-distance and long-distance contributions, including loop effects.

A shape function $f(k_+)^{QCD}$ and a light-cone function $F(k_+)^{QCD}$ can also be defined in full QCD by means of the relations [10]:

$$T_{\mu\nu}^{QCD} \equiv (s_{\mu\nu} \; + \Delta s_{\mu\nu}) \, \frac{1}{2v \cdot Q} \; F(k_+)^{QCD}$$

and

$$W_{\mu\nu}^{QCD} \equiv \left(s_{\mu\nu} \; + \Delta s'_{\mu\nu}\right) \frac{1}{2v \cdot Q} \; f(k_+)^{QCD},$$

where $\Delta s_{\mu\nu}$ and $\Delta s'_{\mu\nu}$ are defined as the part of the spin structure not proportional to $s_{\mu\nu}$. The tensors $\Delta s_{\mu\nu}$ and $\Delta s'_{\mu\nu}$ represent residual spin effects not described by the effective theory (ET), which do not contribute to the Double-Logarithmic Approximation (DLA)[19]. In DLA the forward tensor can therefore be written as

$$T_{\mu\nu}^{QCD} = \; s_{\mu\nu} \, \frac{1}{2v \cdot Q} \; F(k_+)^{QCD}, \tag{78}$$

where the "light-cone function" is given by

$$F(k_+)^{QCD} \equiv \frac{1}{-k_+ + i0} \left[1 + a \, C\right], \tag{79}$$

when $a \equiv \alpha_S C_F / \pi$ and C is the scalar triangle diagram (see fig. 2):

$$C \equiv \; -i \, v \cdot Q \int \frac{d^4 l}{\pi^2} \, \frac{1}{(l+Q)^2 + i0} \;\; \frac{1}{v \cdot l + l^2/2m + i0} \;\; \frac{1}{l^2 + i0}. \tag{80}$$

We have set the light quark mass equal to zero [21].

The hadronic tensor relevant to the decay is obtained by taking the imaginary part according to eq. (44). This transforms the products in convolutions, which are converted again into ordinary products by the well-known Mellin transform [22].

[19] Note that $\Delta s_{\mu\nu}$ and $\Delta s'_{\mu\nu}$ are, in general, different; this was not noted in [10].

Infrared singularities (soft & collinear) are regulated by the virtuality $Q^2 \neq 0$ of the external *up* quark[20]. We may write

$$Q^\mu \cong E_X \left(1 + \frac{n^2}{4}; 0, 0, -1 + \frac{n^2}{4} \right) = E_X \left(v_- + \frac{n^2}{4} v_+ \right) \tag{81}$$

and

$$n^\mu \cong v_- + \frac{n^2}{4} v_+ \,, \tag{82}$$

where we defined the light-cone versors

$$v_+ \equiv (1; 0, 0, 1) \,, \qquad v_- \equiv (1; 0, 0, -1) \,. \tag{83}$$

Let us now consider the properties of the integral C. First, it is adimensional. Second, it is UV-finite for power counting: the integrand has three ordinary scalar propagators with a total of six powers of momentum at the denominator. This implies that C does not depend on an ultraviolet cutoff Λ_{UV} as long as it is larger than any physical scale of the process, namely

$$\Lambda_{UV} \gg m_B. \tag{84}$$

Third, as already discussed, C is also IR-finite as long as $Q^2 \neq 0$. For $Q^2 > 0$ there is an imaginary part, related to the propagation of the real *up* and gluon pair, while for $Q^2 < 0$ the integral is real. Therefore C does depend on adimensional ratios of three different scales: m_B, E_X and m_X. There are only two independent ratios, which we choose as m_B/E_X and m_X/E_X. We are going to decompose the integral C in a sum of various integrals; at the end, one of them will correspond to the double-logarithmic contribution to the shape function $f(k_+)$ in the low-energy ET. The other integrals represent additional contributions and they are mostly short-distance dominated in lattice-like regularizations. This decomposition consists of two separate steps, which will be described in the following sections.

5.1 From QCD to HQET

In the first step we isolate a hard factor by simply subtracting and adding back the integral with the full beauty quark propagator replaced by a static

[20]This is consistent because a virtual massless quark is not degenerate with a quark and a soft and/or collinear gluon.

one (see fig. 3)

$$C = C_s + C_h, \tag{85}$$

where

$$C_s \equiv -iv \cdot Q \int \frac{d^4l}{\pi^2} \frac{1}{(l+Q)^2 + i0} \; \frac{1}{v \cdot l + i0} \; \frac{1}{l^2 + i0}, \tag{86}$$

and

$$C_h \equiv i\frac{v \cdot Q}{2m} \int \frac{d^4l}{\pi^2} \frac{1}{(l+Q)^2 + i0} \frac{1}{v \cdot l + i0} \frac{1}{v \cdot l + l^2/2m + i0}. \tag{87}$$

The above decomposition parallels that performed in section 4 in operatorial language. The light-cone function factorizes at order α_S according to

$$F(k_+)^{QCD} = \frac{1}{-k_+ + i0}\left[1 + a\, C\right] = \frac{1}{-k_+ + i0}\left[1 + a\, C_h\right]\left[1 + a\, C_s\right]. \tag{88}$$

We expect that C_s has the same infrared behaviour as C; it will be subjected to a further decomposition in the next sections; C_h is the "hard factor", i.e. the difference between QCD and the static limit for the b quark. The latter integral is both UV- and IR-finite. The ultraviolet finiteness stems from power counting: the integrand has two scalar propagators and a static propagator with a total of five powers of momentum in the denominator [21]. In the infrared region, all the components of the loop momentum are small

$$\mathrm{IR}: \quad l_\mu \to \rho l_\mu, \quad \rho \to 0, \tag{89}$$

so we can neglect the terms that are quadratic in l^μ in the propagator denominators:

$$C_{h,IR} \sim \int d^4l \frac{1}{2l \cdot Q + Q^2 + i0} \quad \frac{1}{(v \cdot l + i0)^2}. \tag{90}$$

Integrating over l_0, and closing the contour in the upper half of the l_0-plane, we see that there are no enclosed poles and the integral vanishes (QED). Inside C_h we can therefore make the replacement [22]

$$Q^\mu \quad \to \quad \overline{Q}^\mu \equiv E_X\, v_- \qquad\qquad (\overline{Q}^2 = 0). \tag{91}$$

[21] It is known that ultraviolet power counting may fail in effective theory integrals when there are LEET propagators because of the occurrence of an ultraviolet collinear region [10]. The integral C_h, however, contains only an HQET propagator.

[22] With this substitution, terms related to higher twist contributions of the forms $(m_X/m_B)^{n_1}$ and $(m_X/E_X)^{n_2}$ are neglected, but this is in agreeement with our leading-twist ideology (the indices n_1 and n_2 are integers).

It follows that C_h depends only on m_B and E_X: $C_h = C_h(m_B, E_X)$. Since it is adimensional, it may depend only on the adimensional hadronic energy

$$z \equiv \frac{2E_X}{m_B}, \tag{92}$$

i.e. $C_h = C_h(z)$. An explicit computation gives

$$C_h = \log(1-z)\log z + Li_2(z) \simeq -z\log z \quad (z \ll 1), \tag{93}$$

C_h does not contain large contributions in the limit $z \to 0$, i.e. $\log z$ terms. This is related to the fact that C_h and C_s are UV-convergent. In general, single logarithms of the hadronic energy, $\log z$, do appear, representing the difference between the interaction of a full propagating heavy quark, of mass m_B, and the one of a static quark. The relevant interaction energies are between the beauty mass m_B and the hadronic energy E_X,

$$\alpha_S \int_{z^2 m_B^2}^{m_B^2} \frac{dk^2}{k^2} = -2\,\alpha_S \log z. \tag{94}$$

The large logarithms (94) are resummed, as usual, by replacing the bare coupling with the running coupling and exponentiating, so that the above formula is corrected into:

$$\begin{aligned} 1 + \gamma_0 \alpha_S \int_{z^2 m_B^2}^{m_B^2} \frac{dk^2}{k^2} \quad &\rightarrow \quad \exp\left[\gamma_0 \int_{z^2 m_B^2}^{m_B^2} \frac{dk^2}{k^2}\,\alpha_S(k^2)\right] \\ &= \quad \exp\left[-2\,\alpha_S \gamma_0 \log z + 2\gamma_0 \beta_0\,\alpha_S^2 \log^2 z + \cdots\right], \end{aligned} \tag{95}$$

where $\beta_0 \equiv 1/(4\pi)(11/3\,N_c - 2/3\,n_f)$ and $\alpha_S \equiv \alpha_S(m_B)$.

Let us summarize the above discussion. A first coefficient function is introduced, which takes into account the fluctuations with energy ε in the range

$$E_X < \varepsilon < m_B. \tag{96}$$

In the language of Wilson's renormalization group, we are lowering the UV cutoff of the effective hamiltonian from m_B to E_X.

5.2 Infrared factorization

The second factorization step involves the separation of the various infrared contributions to the process, one of which will ultimately lead to the shape function. This step forces us to introduce explicitly an ultraviolet cutoff Λ from which the various factors depend separately. In other words, the decomposition of C_s introduces fictitious UV divergences, which cancel in the sum. We will see that there are substantial regularization effects, even for the leading DLA terms.

C_s is UV-convergent, as is clear again from power counting, and it does not depend on the beauty mass m_B, which has been sent to infinity, so that $C_s = C_s(E_X, m_X)$. The only adimensional variable that can be constructed out of E_X and m_X is their ratio or, equivalently, n^2 (see eq. (58)). Since C_s is adimensional it may depend only on n^2: $C_s = C_s(n^2)$. The explicit computation in DLA gives

$$C_s = -\frac{1}{2}\log^2(-n^2 - i0). \tag{97}$$

The infrared factorization is performed by integrating C_s over the energy l_0 using the Cauchy theorem. There are three poles in the lower half of the l_0-plane related to the propagation of a *real* static quark, a *real* gluon and a *real up* quark, located respectively at

$$l_0 = -i0, \qquad l_0 = +|\overrightarrow{l}| - i0, \qquad l_0 = -Q_0 + \sqrt{(Q_3 + l_3)^2 + l_\perp^2} - i0. \tag{98}$$

In the upper half-plane, instead, there are only two poles, related to the gluon and the up propagator:

$$l_0 = -|\overrightarrow{l}| + i0, \qquad l_0 = -Q_0 - \sqrt{(Q_3 + l_3)^2 + l_\perp^2} + i0 \tag{99}$$

The poles in (99) are conventionally related to a propagation that goes backward in time; they will therefore be called the antiparticle poles. We close for simplicity the integration contour in the upper half-plane and we have two residue contributions related to the antigluon pole ad the anti-*up* pole respectively (see fig.4):

$$C_s = C_g + C_q. \tag{100}$$

The antigluon and the anti-*up* contributions are given respectively by

$$C_g = \frac{2}{\pi} v \cdot Q \int d^3 l \quad \frac{1}{l_0 - |\overrightarrow{l}| + i0} \frac{1}{v \cdot l + i0} \frac{1}{Q^2 + 2l \cdot Q + i0}\Bigg|_{l_0 = -|\overrightarrow{l}| + i0} \tag{101}$$

$$C_q = \frac{2}{\pi} v \cdot Q \int d^3 l \quad \frac{1}{l_0 + Q_0 - |\overrightarrow{l} + \overrightarrow{Q}| + i0} \frac{1}{v \cdot l + i0} \frac{1}{l^2 + i0}\Bigg|_{l_0 = -Q_0 - \sqrt{(Q_3 + l_3)^2 + l_\perp^2} + i0}$$

The above terms are usually called “soft” and “jet” (or “collinear”) factor respectively, even though we believe that this terminology can be rather misleading, as the redistribution of soft and collinear contributions in C_g and C_q is substantially dependent on the regularization. Indeed, as we see by power counting, C_g and C_q are UV-divergent and it is therefore necessary to introduce an ultraviolet regularization to treat them separately. We will show later that it is possible, within a specific class of regularization schemes, to confine all the soft contributions in C_g.

The light-cone function factorizes after this second step as

$$F(k_+)^{QCD} = \frac{1}{-k_+ + i0} \left[1 + a\, C_h\right] \left[1 + a\, C_q\right] \left[1 + a\, C_g\right], \tag{102}$$

i.e. as a product of three factors.

5.2.1 Wilson line representation

Before explicitly computing these 3-dimensional integrals, let us represent them as 4-dimensional ones, i.e. as one-loop integrals of a properly chosen field theory:

$$C_g \equiv -i v \cdot Q \int \frac{d^4 l}{\pi^2} \frac{1}{Q^2 + 2l \cdot Q + i0} \quad \frac{1}{v \cdot l + i0} \quad \frac{1}{l^2 + i0} \tag{103}$$

and

$$C_q \equiv i v \cdot Q \int \frac{d^4 l}{\pi^2} \frac{1}{(l + Q)^2 + i0} \quad \frac{1}{v \cdot l + i0} \quad \frac{1}{Q^2 + 2l \cdot Q + i0} \tag{104}$$

The proof of the above equations is just by integration over l_0: closing the integration contour in the upper half-plane of l_0, we enclose a single pole, whose residue gives the 3-dimensional integrals in eqs. (101); C_g and C_q involve one full - i.e. quadratic - propagator and two eikonal - i.e. linear - propagators. It is easy to check that the algebraic sum of C_q and C_s in the above expressions reproduces the integral C_s defined in eq. (86). Introducing the variable k_+, the integral C_g can be written in the "familiar" form

$$C_g(k_+) \equiv \frac{-i}{2} \int \frac{d^4 l}{\pi^2} \, \frac{1}{-k_+ + l \cdot n + i0} \, \frac{1}{v \cdot l + i0} \, \frac{1}{l^2 + i0}. \tag{105}$$

The geometrical interpretation is the following: $C_g(k_+)$ is the one-loop correction to a vertex composed of an on-shell Wilson line along the time axis, and a Wilson line along the direction n off-shell by k_+. That is exactly the vertex correction in Feynman gauge to the light-cone function $F(k_+)$, which was computed in ref. [10] in the limit $n^2 \to 0$ (see section 5.3.2).

We can give a similar description for C_q. The 4-dimensional representation for C_q involves an on-shell Wilson line along the time direction, a Wilson line along the direction n off-shell by k_+, and a light quark propagator with momentum $l + Q$

$$C_q \equiv \frac{i}{2} \int \frac{d^4 l}{\pi^2} \, \frac{1}{-k_+ + l \cdot n + i0} \, \frac{1}{v \cdot l + i0} \, \frac{1}{(l + Q)^2 + i0}. \tag{106}$$

Note that the expressions for C_g and C_q are very similar: they differ by an overall sign and by the replacement in the quadratic propagator of $l \to l + Q$. For future reference, let us note that the latter shift involves only the zero and the third components of l, not the transverse ones.

In fig.5 the decomposition of C_s into C_g and C_q is represented.

5.3 Regularization Effects

The decomposition of C_s into C_g and C_q is strongly dependent on the regularization scheme adopted, as a consequence of the fact that double-infrared logarithms are promoted to ultraviolet ones with the splitting. We study two different classes of regularizations. To the first class belongs the regularization considered in ref. [10]: a sharp cutoff on the spatial loop momenta

$$|\vec{l}\,| < \Lambda_S,$$

and a loop energy on the entire real axis,

$$-\infty < l_0 < \infty. \tag{107}$$

That means, roughly speaking, a discrete space and a continuous time. We believe that this regularization gives the same double-logarithm as the ordinary lattice regularization - the Wilson action [23]. In the latter case all the components of the loop 4-momentum are cutoff, not only the spatial ones

$$|l^\mu| < \Lambda_4 \equiv \frac{\pi}{a}, \tag{108}$$

where a is the lattice spacing. The physical reason for the equality of the double-logarithmic coefficients in the regularizations (107) and (108) is the following. Soft and collinear logarithms are both related to quasi-real gluon configurations, for which

$$l_0 \sim |\overrightarrow{l}|. \tag{109}$$

Cutting off the spatial momenta therefore should cut off also the relevant energies as far as soft and collinear singularities are concerned.

As a representative of the second class of UV regularizations, consider a sharp cutoff on the transverse momenta (the x–y plane):

$$|\overrightarrow{l}_\perp| < \Lambda_\perp, \quad \text{while} \quad -\infty < l_+,\ l_- < \infty. \tag{110}$$

This regularization is "effective", i.e. it is sufficient to cut on the transverse momenta to render the integrals finite. To this class of regularizations belongs the Dimensional Regularization (DR), in which most of the effective field theory computations have been performed. Let us treat the two cases in turn.

5.3.1 Space Momenta Cutoff

Integrating C_g over l_0 by closing the integration contour upward and integrating over the azimuthal angle, we obtain

$$C_g = -\int_0^{\Lambda_S} dl \int_{-1}^{1} d\cos\theta \frac{1}{k_+ + l\,(1-\cos\theta) + n^2 l/4\,(1+\cos\theta)}, \tag{111}$$

where we have used the definition of Q^μ in eq. (81) and $l \equiv |\vec{l}|$. Integrating over the polar angle, we obtain

$$C_g = -\int_0^{2\Lambda_S} \frac{dl}{l} \log\left[\frac{k_+ + l}{k_+ + n^2\, l/4}\right]. \tag{112}$$

Assuming a cutoff much larger than any physical scale in the process, i.e.[23]

$$\Lambda_S \gg E_X, \tag{113}$$

we obtain in DLA [24]

$$C_g = -\frac{1}{2}\log^2 \frac{E_X}{k_+ - i0} - \log\frac{\Lambda_S}{E_X}\log\frac{E_X}{-k_+ + i0}. \tag{114}$$

Three scales enter in C_g: k_+, Λ_S and E_X. The appearance of k_+ and the cutoff Λ_S was expected, because these two quantities represent the infrared and the ultraviolet scale, respectively. The noticeable fact is that also the hadronic energy E_X makes its appearance. C_g contains a double-logarithm of the infrared kinematical scale k_+ (related to the overlap of the soft and the collinear region, which extends up to Λ_S); but it also contains a single logarithm of the cutoff. The appearance of the hadronic energy E_X comes from the necessity of a third mass scale for the function C_g, which behaves like $\log^2 k_+$ in k_+ and like $\log \Lambda_S$ in Λ_S.

When $l \gg E_X$ the integrand behaves as

$$-\frac{1}{l}\log\left[\frac{k_+ + l}{k_+ + n^2\, l/4}\right] \sim \frac{1}{l}\log\frac{n^2}{4} \tag{115}$$

and produces a single-logarithmic ultraviolet divergence. As eq. (115) clearly indicates, $n^2 \neq 0$ regulates the collinear or light-cone singularity. It is interesting to note that if we take a cutoff much smaller than the hadronic energy (as we will do in the "final" low-energy effective theory),

$$\Lambda'_S \ll E_X, \tag{116}$$

[23] This is done consistently with the relation (84), in which we have taken a large cutoff for the computation of C_s.

[24] The last member in eq. (114) is an artificial absorptive part that cancels against an opposite one in C_q (see eq. (120).

we have

$$n^2\, l \leq \frac{m_X^2}{E_X^2}\, \Lambda_S' \ll k_+, \tag{117}$$

and C_g simplifies in

$$C_g\left(\Lambda_S'\right) \simeq -\int_0^{2\Lambda_S'} \frac{dl}{l} \log\left(\frac{k_+ + l}{k_+}\right). \tag{118}$$

The quantity n^2 does not enter anymore and the integrand is the same as that with $n^2 = 0$, i.e. with n replaced by $\overline{n}$. The physical explanation is that soft gluons are not able to distinguish between the two slightly different directions n and $\overline{n}$. In other words, with the small cutoff (116) we can take the limit (6) inside the integral. For $l \gg k_+$, the integrand in eq. (118) has the asymptotic behaviour

$$\frac{1}{l} \log \frac{l}{k_+}, \tag{119}$$

implying a double-logarithmic behaviour with respect to Λ_S' upon integration over l, in contrast with the single logarithmic behaviour of the integrand in (115).

For the computation of C_q, it is convenient to first make the shift $l \to l - Q$ in the expression of C_s, eq. (86), and then to compute the residue of the light quark pole at $l_0 = -|\vec{l}\,| + i0$: this is legitimate if condition (113) holds. We find

$$C_q = \log \frac{\Lambda_S}{E_X} \log \frac{E_X}{-k_+ + i0} \qquad (\Lambda_S \gg E_X). \tag{120}$$

The three scales appearing in C_g do appear also in C_q. We note that C_q contains a single logarithm of k_+, i.e. it is subleading by one logarithm in the infrared counting with respect to C_g. It has a single-logarithmic UV divergence.

If we compute the integral in eq. (120) with a small cutoff (116), we do not find any infrared logarithm, in contrast with what happens instead with C_g. Thus the logarithmic contributions to C_q come from high-energy gluons and it is therefore an indication that C_q, unlike C_g, is short-distance dominated.

One can check that the correct value of C_s is reproduced by summing C_g and C_q; in particular UV divergences cancel.

At the level of logarithmic accuracy, we can replace the strong inequality (113) with a weaker one:

$$\Lambda_S \geq E_X. \tag{121}$$

Setting in particular

$$\Lambda_S = E_X, \tag{122}$$

the expressions for the gluon and quark-pole residue read

$$\begin{aligned} C_g\left(\Lambda_S = E_X\right) &= -\frac{1}{2}\log^2\frac{E_X}{k_+ - i\epsilon} = C_s, \\ C_q\left(\Lambda_S = E_X\right) &= 0, \end{aligned} \tag{123}$$

i.e. the gluon-pole term gives the whole contribution while the quark-pole factor vanishes. The term C_q therefore has the role of correcting C_g when $\Lambda_S \neq E_X$.

5.3.2 The effective theory on the light-cone, the LEET

In full QCD infrared singularities are regulated by the unique quantity $m_X^2 \neq 0$. In the effective theory, the light quark propagator is replaced by an eikonal one

$$\frac{1}{\left(l+Q\right)^2 + i0} = \frac{1}{Q^2 + 2l\cdot Q + l^2 + i0} \;\rightarrow\; \frac{1}{Q^2 + 2l\cdot Q + i0}. \tag{124}$$

In the expression on the right-hand side l^2 has been neglected and as a consequence Q_μ enters in two distinct and independent ways: its square Q^2 represents the virtuality of the eikonal line at $l_\mu = 0$, while its components Q^μ are the coefficients of the linear combination of the loop-momentum components l_μ in the term $2\,l_\mu\,Q^\mu$. Unlike full QCD, Q^2 and Q^μ can be considered as independent quantities in the effective theory. We can ask ourselves what happens if we take the limit $Q^2 \rightarrow 0$ inside the term $2l\cdot Q \rightarrow 2l\cdot\overline{Q}$ while keeping at the same time $Q^2 \rightarrow \text{const} \neq 0$, i.e. if we make the replacement

$$\frac{1}{Q^2 + 2l\cdot Q + i0} \;\rightarrow\; \frac{1}{Q^2 + 2l\cdot \overline{Q} + i0}. \tag{125}$$

In the usual notation, the above replacement reads

$$\frac{1}{-k_+ + n\cdot l + i0} \;\rightarrow\; \frac{1}{-k_+ + \overline{n}\cdot l + i0}, \tag{126}$$

corresponding to the limit

$$n^2 \rightarrow 0, \quad k_+ \rightarrow \text{const} \neq 0. \tag{127}$$

This means that we are considering an eikonal propagator that lies exactly on the light-cone, with collinear singularities regulated now by $k_+ \neq 0$ only, instead of by $n^2 \neq 0$ [19, 24, 25]. As we saw before, the limit $n^2 \to 0$ is "invisible" with a small cutoff, simply because the integral does not depend on n^2 in this case. We now want to see what happens in the limit (127) with a *large* cutoff. The explicit computation gives

$$\begin{aligned} C_{g,\, n^2=0} &= -\frac{1}{2} \log^2 \frac{\Lambda_S}{k_+ - i0} \\ C_{q,\, n^2=0} &= \frac{1}{2} \log^2 \frac{\Lambda_S}{k_+ - i0} - \frac{1}{2} \log^2 \frac{E_X}{k_+ - i0}. \end{aligned} \tag{128}$$

The behaviour with respect to k_+ is the same as in the case $n^2 \neq 0$. Ultraviolet divergences are now more severe than in the previous case, being of the double-logarithmic kind. However, the sum is again the correct one that vanish at $\Lambda_S = E_X$

$$C_{g,\, n^2=0} + C_{q,\, n^2=0} = C_s. \tag{129}$$

In other words, the transition to the light-cone theory implies a rearrangement of the ultraviolet structure, but the physical observable, C_s, is unchanged. Note that, from an algebraic point of view, the equality (129) only holds when C_s is calculated in the limit (127) too. We have verified that the DLA value of C_s of eq. (97) is reproduced in this limit, which corresponds to the substitution

$$\frac{1}{(l+Q)^2 + i0} \quad \to \quad \frac{1}{Q^2 + 2l \cdot \overline{Q} + l^2 + i0} \tag{130}$$

in the full QCD propagator.

5.3.3 Transverse momenta cutoff

The factor C_g is better computed in this case by introducing light-cone coordinates:

$$l_+ = l_0 + l_3, \quad l_- = l_0 - l_3. \tag{131}$$

Integrating over l_- by closing the integration contour upward and over the transverse momentum, we obtain

$$C_g = -2 \int_0^\infty \frac{dy\, y}{1+y^2} \frac{1}{1+n^2y^2/4} \log \left[1 + \frac{\Lambda_\perp}{k_+ y} \left(1 + \frac{n^2 y^2}{4} \right) \right]. \tag{132}$$

Performing the final integration in the simpler case $n^2 = 0$, we obtain

$$C_{g,\, n^2=0} = -\log^2 \frac{\Lambda_\perp}{k_+ - i0}. \tag{133}$$

For the quark-pole factor C_q, the integration over l_- gives

$$C_q = \int_0^\infty dx\, x \int_0^{\Lambda_\perp^2} dl_\perp^2 \, \frac{1}{x^2 l_\perp^2 + 2E_X\, x + 1 - i0} \; \frac{1}{k_+\, x + 1 - i0}. \tag{134}$$

Integrating over $l_\perp$ we obtain, in the light-cone limit:

$$C_q = \int_0^\infty \frac{dx}{x} \, \frac{1}{1 - n^2\, x/4 - i0} \, \log \frac{(\overline{\Lambda}_\perp^2 - n^2/4)\, x^2 + x + 1 - i0}{1 + x}, \tag{135}$$

with[25]

$$\overline{\Lambda}_\perp \equiv \frac{\Lambda_\perp}{2E_X}. \tag{136}$$

The above integral has two double-logarithmic regions for $\Lambda_\perp \gg E_X$,

$$(1):\; 1 \ll x \ll \frac{4}{n^2}, \quad (2):\; \frac{1}{\overline{\Lambda}_\perp} \ll x \ll 1. \tag{137}$$

Performing the integration in the two regions, we find

$$C_{q,\, n^2=0} = \frac{1}{2} \log^2 \frac{\Lambda_\perp^2}{k_+\, E_X - i0} - \log^2 \frac{\Lambda_\perp}{E_X} \qquad (\Lambda_S \gg E_X). \tag{138}$$

The first double-logarithm on the right-hand side is related to region (1), the second one to region (2).

For a smaller UV cutoff, we obtain instead:

$$C_{q,\, n^2=0} = \frac{1}{2} \log^2 \frac{\Lambda_\perp^2}{k_+\, E_X - i0}, \qquad E_X\, |k_+| \ll \Lambda_\perp^2 \ll E_X^2. \tag{139}$$

Finally, for $\Lambda_{QCD}^2 \ll \Lambda_\perp^2 \ll E_X\, |k_+|$, C_q vanishes in DLA. Let us comment on the results (133) and (138). As with the 3-momentum regularization, C_g and C_q have double-logarithmic UV divergences, again a consequence of $n^2 = 0$.

[25] n^2 in the above formula has to be interpreted as $-2k_+/E_X$.

The most important point, however, is that C_g has an additional factor of 2 with respect to the spatial cutoff case in the coefficient of the double-logarithm of the infrared scale, $\log^2 k_+$ (cf. eqs. (114) and (133)). With the Λ_S regularization, C_q has no $\log^2 k_+$ term, while with the $\Lambda_\perp$ regularization it does. The same double logarithm is obtained in the sum C_s in both regularizations. In general, the appearance of $\log^2(-k_+)$ in $C_q(\Lambda_\perp)$ implies that, with the $\Lambda_\perp$ regularization, C_q does not describe only collinear contributions but also soft ones [26]. We interpret this fact by saying that C_q is non-perturbative within this regularization and that the shape function, in general, does not have any physical meaning, but it just represents the gluon-pole contribution to a physical process: that result is also, as far as we know, new. One generally attaches to the shape function a physical meaning - related to the Fermi motion; thus, to understand what is happening, we have to start again from the beginning. The shape function is obtained from the original QCD tensor $W_{\mu\nu}$ considering the infrared limit of small momenta compared with the hadronic energy:

$$|l_\mu| \ll E_X. \tag{140}$$

The tree-level rate in the ET equals the QCD one by construction. However, in loops, the condition (140) is not guaranteed: its validity depends on the regularization scheme adopted. If we cut all the loop-momentum components with a hard cutoff much smaller than the hard scale,

$$|l_\mu| \leq \Lambda_{UV} \ll E_X, \tag{141}$$

then the condition (140) is still valid at the loop level. As a consequence, we expect that the leading, double-logarithmic term of the ET shape function will match the QCD one. That is indeed what happens with the spatial momentum regularization, as we have seen explicitly. On the other hand, when one uses a regularization such as DR or $\Lambda_\perp$, the equality of the double-logs is no longer guaranteed, and indeed it does not occur in $\Lambda_\perp$-regularization, as we have seen explicitly. This is because the longitudinal momentum of the gluon l_z, or equivalently its energy ϵ, can become arbitrarily large. For the latter regularizations, even for the double-logarithm, one has to come back to the original QCD loop diagram and perform factorization into a factor C_g

[26] The double logarithm necessarily comes from the overlap.

and a factor C_q, as we have shown in detail. In ref. [10] it was shown that the factor of 2 in the $\log^2 k_+$ term in DR is a regularization effect, i.e. it can be removed by going to a non-minimal dimensional scheme. We explicitly see, with the similar $\Lambda_\perp$ regularization, that by including C_q the scheme-dependence automatically disappears. The origin of the additional factor of 2 in the transverse-momentum regularization is related to the occurrence of a second double-logarithmic region for $l_z \to \infty$ (infinite rapidity).

Finally, let us observe that in the case $n^2 \neq 0$ we expect the transverse momentum cutoff to give double-logarithmic results similar to those from the space momentum cutoff. That is because $n^2 \neq 0$ cuts the collinear emission at infinite rapidity.

6 The shape function in the low-energy effective theory

With the $\Lambda_\perp$ regularization, double logarithms, tracing long-distance effects, are contained in C_g as well as in C_q. Since we want to confine long-distance effects inside the shape function only, let us consider from now on the Λ_S regularization only. The factor C_q is short-distance dominated in the latter regularization, so it is computed once and for all in perturbation theory and "leaves the game".

Let us therefore return to formula (111) for C_g. Calling $\epsilon = |\overrightarrow{l}|$, and $t = \theta^2$, C_g can be written as

$$\begin{aligned} C_g\left(\Lambda_S, k_+\right) &\simeq -\int_0^{\Lambda_S} d\epsilon \int_0^1 dt \frac{1}{2k_+ + \epsilon\, t} \\ &\simeq -\int_0^{\Lambda_S} \frac{d\epsilon}{\epsilon} \int_0^1 \frac{dt}{t} \theta\left(\epsilon t - k_+\right) \\ &\simeq -\frac{1}{2} \log^2 \frac{\Lambda_S}{k_+}, \end{aligned} \tag{142}$$

where we have assumed $\Lambda_S \lesssim O\left(E_X\right)$ and we have used the approximation $1/\left(2k_+ + \epsilon\, t\right) \simeq \theta\left(\epsilon t - 2k_+\right)/\left(\epsilon\, t\right)$, which is valid in DLA. This form helps visualizing the origin of the double logarithm. We see that contributions come from soft regions, where $\epsilon \sim O\left(k_+\right)$, as well as from hard regions, where $\epsilon \sim O\left(\Lambda_S\right)$. In order to separate them, the simplest way is to introduce another UV cutoff Λ_{ET}, this time well below the hadronic energy E_X (the hard scale of the process), such as

$$k_+ \ll \Lambda_{ET} \ll \Lambda_S. \tag{143}$$

We can write

$$C_g\left(\Lambda_S, k_+\right) = \delta Z\left(\Lambda_S, \Lambda_{ET}, k_+\right) + \delta \bar{F}^{ET}\left(\Lambda_{ET}, k_+\right), \tag{144}$$

where

$$\begin{aligned} \delta Z\left(\Lambda_S, \Lambda_{ET}, k_+\right) &\equiv -\int_{\Lambda_{ET}}^{\Lambda_S} \frac{d\epsilon}{\epsilon} \int_0^1 \frac{dt}{t} \theta\left(\epsilon t - k_+\right) \\ &= -\frac{1}{2}\log^2\frac{\Lambda_S}{k_+} + \frac{1}{2}\log^2\frac{\Lambda_{ET}}{k_+} \end{aligned} \tag{145}$$

is a coefficient function and $\delta \bar{F}^{ET}\left(\Lambda_{ET}, k_+\right)$ is the one-loop contribution to the light-cone function δF^{ET}, multiplied by the propagator: $\delta F^{ET} = \delta \bar{F}^{ET}/(-k_+ + i0)$, as defined in eq. (63),

$$\begin{aligned} \delta \bar{F}^{ET}\left(\Lambda_{ET}, k_+\right) &\equiv -\int_0^{\Lambda_{ET}} \frac{d\epsilon}{\epsilon} \int_0^1 \frac{dt}{t} \theta\left(\epsilon t - k_+\right) \\ &= -\frac{1}{2}\log^2\left(\frac{\Lambda_{ET}}{k_+}\right). \end{aligned} \tag{146}$$

Note that $\delta \bar{F}^{ET}$ depends only on the two scales k_+ and Λ_{ET}. This is in line with the idea of a simple low-energy effective theory, which describes infrared phenomena characterized by the scale k_+, apart from the UV cutoff that enters through loop effects.

We assume that long-distance effects can be traced by the growth of the coupling constant in the proximity of the Landau pole, and that the coupling constant must be evaluated at the transverse momentum squared [26]:

$$\alpha_S \to \alpha_S\left(k_\perp^2\right), \tag{147}$$

where

$$k_\perp^2 \cong \epsilon^2 t. \tag{148}$$

From the expression of $\delta\, Z$ we see that transverse momenta have a lower bound given by

$$l_\perp^2 > l_{\perp\,\mathrm{min}}^2 = \Lambda_{ET}\, k_+. \tag{149}$$

According to our criteria, non-perturbative effects are absent from Z as long as

$$l_{\perp \min}^2 \gg \Lambda_{QCD}^2. \tag{150}$$

According to eq. (149), this occurs when k_+ is non-vanishing, as it is for example if

$$k_+ \sim O\left(\Lambda_{QCD}\right), \tag{151}$$

as expected from Fermi motion (since $\Lambda_{ET} \gg \Lambda_{QCD}$). However, by taking the imaginary part of $T_{\mu\nu}$ to obtain $W_{\mu\nu}$, i.e. the rate, the product of factors is converted into a convolution over k_+ and the point $k_+ = 0$ is included in the integration range. This implies that transverse momenta down to zero contribute to the coefficient function in $W_{\mu\nu}$, i.e. that factorization of short- and long-distance effects breaks down at this point. The breakdown is related to the fact that we are cutting the energies of the gluons, but not the emission angles, which can go down to zero, implying the vanishing of the transverse momenta. That is one of the most important outcomes of our analysis. However, we believe that these long-distance contributions are suppressed. Let us present a qualitative argument. As we can see from inequalities (149) and (150), transverse momenta of the order of the hadronic scale occur for a very small slice of values of k_+,

$$k_+ \lesssim \frac{\Lambda_{QCD}^2}{\Lambda_{ET}} \ll \Lambda_{QCD}. \tag{152}$$

If the integrand is not singular in this small slice, as it is natural to assume, it gives a reasonally small fraction of the total. Note that the usual factorization of mass singularities is instead "exact". If we consider for example the moments of DIS cross-section, factorization involves a splitting of the long- and short-distance contributions of the form

$$\begin{aligned} M_N\left(Q^2\right) &= \int_0^1 dx_B\, x_B^{N-1}\, \sigma_{DIS}\left(x_B, Q^2\right) \qquad (153) \\ &= 1 + \gamma_N \alpha_S \int_{m^2}^{Q^2} \frac{dl_\perp^2}{l_\perp^2} = \left(1 + \gamma_N\, \alpha_S \int_{\Lambda^2}^{Q^2} \frac{dl_\perp^2}{l_\perp^2}\right)\left(1 + \gamma_N\, \alpha_S \int_{m^2}^{\Lambda^2} \frac{dl_\perp^2}{l_\perp^2}\right), \end{aligned}$$

where m is the mass of a light quark.

After the last step (145), the forward hadronic tensor takes the final form

$$T_{\mu\nu}^{QCD} = \frac{s_{\mu\nu}}{2v\cdot Q}\, F(k_+)^{QCD} \tag{154}$$
$$= \frac{s_{\mu\nu}}{2v\cdot Q}\, \frac{1}{-k_+ + i0}\left[1 + a\, C_h\right]\left[1 + a\, C_q\right]\left[1 + a\,\delta Z\right]\left[1 + a\,\delta \bar{F}^{ET}\right],$$

where the various factors have been introduced in eqs. (78), (79), (88) and (102). Taking the imaginary (absorptive) part, according to the optical theorem (44), we have for $W_{\mu\nu}$ the multiple convolution

$$\begin{aligned} W_{\mu\nu} = \frac{s_{\mu\nu}}{2v\cdot Q}\int & dk_1\, dk_2\, dk_3\, dk_4\, \delta\left(k_+ - k_1 - k_2 - k_3 - k_4\right) \\ & \left[\delta\left(k_1\right) + a\, c_h\left(k_1\right)\right]\left[\delta\left(k_2\right) + a\, c_q\left(k_2\right)\right] \\ & \left[\delta\left(k_3\right) + a\,\delta z\left(k_3\right)\right]\left[\delta\left(k_4\right) + a\delta f^{ET}\left(k_4\right)\right], \end{aligned} \tag{155}$$

where

$$f^{ET}\left(k_+, \Lambda_{ET}\right) = \delta\left(k_+\right) + a\,\delta f^{ET}\left(k_+\right) + O(a^2) \tag{156}$$

is the shape function, defined in eq. (10), for an on-shell quark ($k'_+ = 0$); moreover, we have defined

$$\begin{aligned} c_h\left(k_+\right) &\equiv -\frac{1}{\pi}\mathrm{Im}\left[\frac{1}{-k_+ + i0} C_h\left(k_+ - i0\right)\right] \\ &= \delta\left(k_+\right) C_h\left(k_+\right) - \frac{1}{k_+}\left(-\frac{1}{\pi}\right)\mathrm{Im}\, C_h\left(k_+ - i0\right) \end{aligned} \tag{157}$$

and analogously for the other factors[27]. Typically, by taking the imaginary parts, for double-logarithmic contributions, we have

$$\begin{aligned} -\frac{1}{\pi}\,\frac{\log^2\left(k_+ - i0\right)}{-k_+ + i0} &\rightarrow \delta\left(k_+\right)\log^2(-k_+) + 2\theta\left(-k_+\right)\frac{\log\left(-k_+\right)}{-k_+} \\ &= \frac{d}{dk_+}\left(-\theta\left(-k_+\right)\log^2\left(-k_+\right)\right) \end{aligned} \tag{158}$$

[27] Note that, according to our actual notation, formula (9) in ref. [10] should be substituted by $f^{QCD}(k_+) = \int dk_1\, dk_2\, \delta(k_+ - k_1 - k_2)\,(\delta(k_1) + a\,\delta z(k_1))\, f^{ET}(k_2)$, where $Z = 1 + a\,\delta Z$.

and for single-logarithmic ones

$$
\begin{aligned}
-\frac{1}{\pi}\,\frac{\log\left(k_+ - i0\right)}{-k_+ + i0} \quad &\rightarrow \quad \delta\left(k_+\right)\log(-k_+) + \theta\left(-k_+\right)\frac{1}{-k_+} \\
&= \quad \frac{d}{dk_+}\left(-\theta\left(-k_+\right)\log\left(-k_+\right)\right). \qquad (159)
\end{aligned}
$$

The last members of the above equations have to be interpreted as distributions. In DLA, according to eq. (146), f^{ET} up to one loop reads

$$
\begin{aligned}
f^{ET}\left(k_+, \Lambda_{ET}\right) \quad &= \quad \delta\left(k_+\right) + a\;\theta\left(-k_+\right)\frac{\log\Lambda_{ET}/\left(-k_+\right)}{-k_+} - \frac{a}{2}\,\delta\left(k_+\right)\log^2\left(\frac{\Lambda_{ET}}{-k_+}\right) \\
&= \quad \delta\left(k_+\right) + \frac{a}{2}\frac{d}{dk_+}\left(\theta\left(-k_+\right)\log^2\left(\frac{\Lambda_{ET}}{-k_+}\right)\right). \qquad (160)
\end{aligned}
$$

6.1 Evolution

Taking a derivative with respect to the logarithm of the cutoff, we obtain

$$
\begin{aligned}
\frac{d\,f\left(k_+, \Lambda_{ET}\right)}{d\,\log\Lambda_{ET}} \quad &= \quad -a\,\delta\left(k_+\right)\log\left(\frac{\Lambda_{ET}}{-k_+}\right) + a\;\frac{\theta\left(-k_+\right)}{-k_+} \\
&= \quad a\frac{d}{dk_+}\left(\theta\left(-k_+\right)\log\left(\frac{\Lambda_{ET}}{-k_+}\right)\right). \qquad (161)
\end{aligned}
$$

Comparing the above equation with the evolution equation for the shape function

$$
\frac{d\,f\left(k_+, \Lambda_{ET}\right)}{d\,\log\Lambda_{ET}} = \int dk'_+\,\mathrm{K}_S\left(k_+ - k'_+; \Lambda_{ET}\right)\,f\left(k'_+, \Lambda_{ET}\right), \qquad (162)
$$

and taking into account that, at lowest order in α_S, $f\left(k'_+, \Lambda_{ET}\right) = \delta\left(k'_+\right)$ holds, we find for the evolution kernel at one loop

$$
\begin{aligned}
\mathrm{K}_S\left(k_+ - k'_+; \Lambda_{ET}\right) \quad &= \quad -a\,\delta\left(k'_+ - k_+\right)\log\left(\frac{\Lambda_{ET}}{k'_+ - k_+}\right) + a\;\frac{\theta\left(k'_+ - k_+\right)}{k'_+ - k_+} \\
&= \quad a\frac{d}{dk_+}\left(\theta\left(k'_+ - k_+\right)\log\left(\frac{\Lambda_{ET}}{k'_+ - k_+}\right)\right) \qquad (163) \\
&= \quad a\left[\frac{\theta\left(k'_+ - k_+\right)}{k'_+ - k_+} - \delta\left(k'_+ - k_+\right)\int_0^{\Lambda_{ET}}\frac{dk}{k}\right].
\end{aligned}
$$

If we consider the $\Lambda_\perp$-regularization, the evolution kernel for the shape function is instead given by (eq. (133)):

$$\begin{aligned} K_\perp\left(k_+ - k'_+;\Lambda_\perp\right) &= -2a\,\delta\left(k'_+ - k_+\right)\log\left(\frac{\Lambda_\perp}{k'_+ - k_+}\right) + 2a\,\frac{\theta\left(k'_+ - k_+\right)}{k'_+ - k_+} \\ &= 2a\frac{d}{dk_+}\left(\theta\left(k'_+ - k_+\right)\log\left(\frac{\Lambda_\perp}{k'_+ - k_+}\right)\right). \end{aligned} \tag{164}$$

We notice that there is a factor of 2 between the kernels (163) and (164) for the shape function in the two regularizations. The kernel in DR is the same as that in eq. (164), with $\Lambda_\perp \to \mu$.

There is a clear analogy of the evolution of the shape function with the Altarelli–Parisi evolution equation, but with an important difference: the evolution kernel in this case explicitly depends on the cutoff Λ_{ET} of the bare theory or on the renormalization point μ if we consider the renormalized theory[28]. All this is related to the fact that the Altarelli–Parisi evolution involves a single collinear logarithm for each loop, while our problem is double-logarithmic. Let us discuss this point with a simple analogy. The Altarelli–Parisi evolution, or in general the usual Callan–Symanzik evolution, is analogous to a first-order differential equation, which is autonomous (i.e. time-independent):

$$\frac{dx}{dt} = h(x), \tag{165}$$

or, in discrete form,

$$x_{n+1} = O\left(x_n\right), \tag{166}$$

where O is a generic operator, such that the formal solution reads

$$x_n = O^n\left(x_0\right). \tag{167}$$

The evolution in eq. (162) is instead analogous to an evolution equation of the form

$$\frac{dx}{dt} = h(x,t), \tag{168}$$

or, in discrete form

$$x_{n+1} = O_n\left(x_n\right). \tag{169}$$

[28] We thank S. Catani for a discussion on this point.

In the latter case there is a different evolution operator at each step[29].

We clarify at this point a discrepancy of a factor of 2 in the evolution kernel K of the shape function, computed at one loop in DR in both refs. [8] and [9]. We agree with ref. [8], where the kernel is computed from the Green function in the ET taking a μ derivative, as in eq. (164). We disagree with ref. [9], where the kernel is computed by taking the difference of the QCD Green function with the ET Green function and then differentiating with respect to μ; their kernel is two times smaller than the one in eq. (164). The latter authors give for the QCD amplitude, in our notation, the result

$$F\left(k_+\right)^{QCD} \stackrel{?}{=} \frac{1}{-k_+ + i0}\left(-\frac{1}{2}\right) a \log^2\left(\frac{\mu}{k_+ - i0}\right). \tag{171}$$

They find a dependence on the renormalization point μ, which we do not find as the QCD diagram is ultraviolet - as well as infrared - finite [10]. If we replace in their renormalization condition, which determines the kernel, our μ-independent result for the QCD Green function,

$$F\left(k_+\right)^{QCD} = \frac{1}{-k_+ + i0}\left(-\frac{1}{2}\right) a \log^2\left(\frac{m_b}{k_+ - i0}\right), \tag{172}$$

we find a vanishing kernel. Since the effective theory is UV-divergent and consequently μ-dependent, we believe that there may be a problem with the renormalization conditions. To summarize, the matrix element of a bare operator is schematically of the form

$$\langle p|O_B|p\rangle = 1 + c\frac{\alpha_B^{dim}}{\epsilon}\left(p^2\right)^{-\epsilon} + (\text{finite for } \epsilon \to 0), \tag{173}$$

where c is a numerical constant, p^2 refers to an overall momentum scale in the external state, and $\left(p^2\right)^{-\epsilon}$ comes from the one-loop integral in $D = 4 - 2\epsilon$ dimensions; α_B^{dim} is the bare coupling of the original D-dimensional theory:

[29] In double-logarithmic problems, one can obtain an autonomous differential equation at the price of having a second-order equation, i.e. of the form

$$\frac{d^2x}{dt^2} = h(x). \tag{170}$$

This, anyway, is not an evolution equation.

it has a positive mass dimension $4-D=2\epsilon$ and must be kept fixed as we vary μ, which is just an arbitrary mass scale:

$$\frac{d}{d\mu}\alpha_B^{dim}=0. \tag{174}$$

This implies the well-known condition

$$\frac{d}{d\mu}\langle p|O_B|p\rangle=0. \tag{175}$$

One usually introduces an adimensionalized bare coupling multiplying α_B^{dim} by $\mu^{2\epsilon}$, where μ is just an arbitrary mass scale as we said before,

$$\alpha_B^{\rm adim}\equiv\mu^{-2\epsilon}\alpha_B^{dim}, \tag{176}$$

so that the bare Green function reads

$$\begin{aligned}\langle p|O_B|p\rangle &= 1+c\,\frac{\alpha_B^{\rm adim}}{\epsilon}\left(\frac{\mu^2}{p^2}\right)^{\epsilon}+(\text{finite for }\epsilon\to 0)\\ &= 1+c\,\frac{\alpha_B^{\rm adim}}{\epsilon}+c\,\alpha_B^{\rm adim}\log\frac{\mu^2}{p^2}+\cdots\end{aligned} \tag{177}$$

In the minimal-dimensional scheme (MS), we include the pole term in the renormalization constant

$$Z_{MS}=1+c\,\frac{\alpha_B^{\rm adim}}{\epsilon}, \tag{178}$$

and the remaining terms in the matrix element of the renormalized operator,

$$\langle p|O_{MS}|p\rangle=1+c\,\alpha_B^{\rm adim}\log\frac{\mu^2}{p^2}+\cdots, \tag{179}$$

since $O_B=Z\,O_R$. It is only *after* this splitting that a dependence on μ is introduced separately in the renormalization constant and in the renormalized operator[30].

[30] In the notation of ref. [9], $\log\langle O_B\rangle=\partial\ln\widetilde{f_B}\,(\xi)\,/\partial\log\xi$, with $\xi\sim 1/\sqrt{p^2}$.

The anomalous dimension is computed from the renormalization constant keeping α_B^{dim} fixed:

$$\gamma \equiv \frac{d\log Z}{d\log\mu} = \frac{d}{d\log\mu}\left(c\,\frac{\alpha_B^{dim}\mu^{-2\epsilon}}{\epsilon}\right) = -2c\,\alpha_B^{\mathrm{adim}}. \tag{180}$$

It seems to us that a vanishing kernel or anomalous dimension in the effective theory is obtained in ref. [9] because the renormalization constant Z has been identified with the whole matrix element (177).

7 Conclusions

We have discussed the properties of decays of heavy flavour hadrons into an inclusive hadron state X with an invariant mass m_X that is small compared to its energy E_X, $m_X \ll E_X$. We have introduced a factorization procedure in $T_{\mu\nu}^{QCD}$; at one-loop order, in DLA, we have the final formula (see eq. (155)):

$$T_{\mu\nu}^{QCD} = \frac{s_{\mu\nu}}{2v\cdot Q}\,\frac{1}{-k_+ + i0}\left[1 + a\,C_h\right]\left[1 + a\,C_q\right]\left[1 + a\,\delta Z\right]\left[1 + a\,\delta\bar{F}^{ET}\right]. \tag{181}$$

The coefficient C_h is a hard factor that takes into account the fluctuations with energy ε in the range $E_X < \varepsilon < m_B$. The other two coefficients C_q and δZ are both short-distance dominated only within a specific class of lattice-like regularization schemes. The tensor $W_{\mu\nu}$, i.e. the rate, is obtained by taking the imaginary part of $T_{\mu\nu}$, according to the optical theorem. We have found that the shape function $f\,(k_+)$, contrary to naive physical expectations at DLA, has no direct physical meaning, as it represents a partial contribution to the complete physical process. Changing regularization, we have explicitly shown that the leading double-logarithmic contribution to $f\,(k_+)$ can be changed by a factor of 2, i.e. that the shape function is substantially regularization-scheme dependent. Only after summing the shape function with the other contributions, is a physical, scheme-independent result recovered. We have also shown that in lattice-like regularization the shape function factorizes a large part of the non-perturbative effects.

To summarize, we have presented an explicit separation of perturbative and non-perturbative effects.However, contrary to single logarithmic problems, factorization in this (double-logarithmic) problem turns out not to be exact, even in lattice-like regularization schemes. Some long-distance effects are present in the coefficient function even at leading twist, even though they are expected to be suppressed on physical grounds. Finally, we have clarified some discrepancy in the literature about the evolution kernel for the shape-function computed inside dimensional regularization.

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References

[1] V. Sudakov, Sov. Phys. JETP 3:65 (1956); R. Jackiw, Annals of Physics 48:292 (1968); E. Eichten and R. Jackiw, Phys. Rev. D 4:439 (1971); Y. Dokshitzer, D. Diakonov and S. Troian, Phys. Rep. 58:269 (1980); G. Parisi and R. Petronzio, Nucl. Phys. B 154:427 (1979); G. Curci, M. Greco and Y. Srivastava, Phys. Rev. Lett. 43:134 (1979).

[2] J. Kodaira and L. Trentadue, SLAC preprint SLAC-PUB-2934 (1982) and Phys. Lett. B 112:66 (1982), B 123:335 (1983); S. Catani and M. Ciafaloni, Nucl. Phys. B 249:301 (1985), B 264:588 (1986); S. Catani, E. D'Emilio and L. Trentadue, Phys. Lett. B 211:335(1988); U. Aglietti, G. Corbó and L. Trentadue, Int. J. Mod. Phys. A 14:1769 (1999).

[3] G. Altarelli, N. Cabibbo, G. Corbó, L. Maiani and G. Martinelli, Nucl. Phys. B 208:365 (1982).

[4] V. Barger, C. Kim and J. Phillips, Phys. Lett. B 251:629 (1990); R. Dikeman and N. Uraltsev, Nucl. Phys. B 509:378 (1998); A. Falk, Z. Ligeti and M. Wise, Phys. Lett. B 406:225 (1997).

[5] A. Liparteliani, V. Monich, Y. Nikitin and G. Volkov, Nucl. Phys. B 195:425 (1982); Erratum, ibid. B 206:496 (1982).

[6] I. Bigi, M. Shifman, N. Uraltsev and A. Vainshtein, Phys. Rev. Lett. 71:496 (1993); A. Manohar and M. Wise, Phys. Rev. D 49:1310 (1994); M. Neubert, Phys. Rev. D 49:3392 (1994), ibid. D 49:4623 (1994); I. Bigi, M. Shifman, N. Uraltsev and A. Vainshtein, Int. J. Mod. Phys. A 9:2467 (1994); T. Mannel and M. Neubert, Phys. Rev. D 50:2037 (1994).

[7] I. Korchemskaya and G. Korchemsky, Phys. Lett. B 287:169 (1992); G. Korchemsky and G. Sterman, Phys. Lett. B 340:96 (1994); R. Akhoury and I. Rothstein, Phys. Rev. D 54:2349 (1996); A. Leibovich and I. Rothstein, Phys. Rev. D 61:074006 (2000); A. Leibovich, I. Low and I. Rothstein, Phys. Rev. D 61:053006 (2000).

[8] A. Grozin and G. Korchemsky, Phys. Rev. D 53:1378 (1996).

[9] C. Balzereit, W. Kilian and T. Mannel, Phys. Rev. D 58:114029 (1998).

[10] U. Aglietti and G. Ricciardi, Phys. Lett. B 466:313 (1999).

[11] G. Altarelli, K. Ellis and G. Martinelli, Nucl. Phys. B 143:521 (1978), Erratum, ibid. B 146:544 (1978); J. Kubar-Andre and F. Paige, Phys. Rev. D 19:221 (1979).

[12] U. Aglietti, M. Ciuchini, G. Corbó, E. Franco, G. Martinelli and L. Silvestrini, Phys. Lett. B 432:411 (1998).

[13] M. Cacciari, hep-ph/9910412, to be published on Nucl. Phys. B.

[14] F. Bloch and A. Nordsieck, Phys. Rev. 52:54 (1937); L. Foldy and S. Wouthuysen, Phys. Rev. 78:29 (1950); D. Yennie, S. Frautschi and H. Suura, Ann. Phys. 13:379 (1961); E. Eichten and F. Feinberg, Phys. Rev. D 23:2724 (1981); N. Isgur and M. Wise, Phys. Lett. B 237:527 (1990); H. Georgi, Phys. Lett. B 240:447 (1990); F. Hussain, J. Korner, K. Schilcher, G. Thompson and Y. Wu, Phys. Lett. B 249:295 (1990).

[15] K. Ellis, G. Marchesini and B. Webber, QCD and Collider Physics (Cambridge University Press, 1996).

[16] D. Politzer and M. Wise, Phys. Lett. B 206:681 (1988), B 208:504 (1988).

[17] M. Jezabek and J. Kühn, Nucl. Phys. B 320:20 (1989); F. De Fazio, M. Neubert, JHEP 9906:017 (1999).

[18] J. Charles, A. Le Yaouanc, L. Oliver, O. Péne and J. Raynal, Phys. Rev. D 60:014001 (1999).

[19] U. Aglietti and G. Corbó, Phys. Lett. B 431:166 (1998); Roma1 1188/97 preprint, hep-ph/9712242, to appear in Int. J. Mod. Phys. A.

[20] U. Aglietti, Phys. Lett. B 281:341 (1992).

[21] M. Abud, G. Ricciardi and G. Sterman, Phys. Lett. B 437:169 (1998).

[22] S. Catani and L. Trentadue, Nucl. Phys. B 327:323 (1989).

[23] U. Aglietti, Nucl. Phys. B 421:191 (1994).

[24] M. Dugan and B. Grinstein, Phys. Lett. B 255:583 (1991).

[25] U. Aglietti, Phys. Lett. B 292:424 (1992).

[26] D. Amati, A. Bassetto, M. Ciafaloni, G. Marchesini and G. Veneziano, Nucl. Phys. B 173:429 (1980).

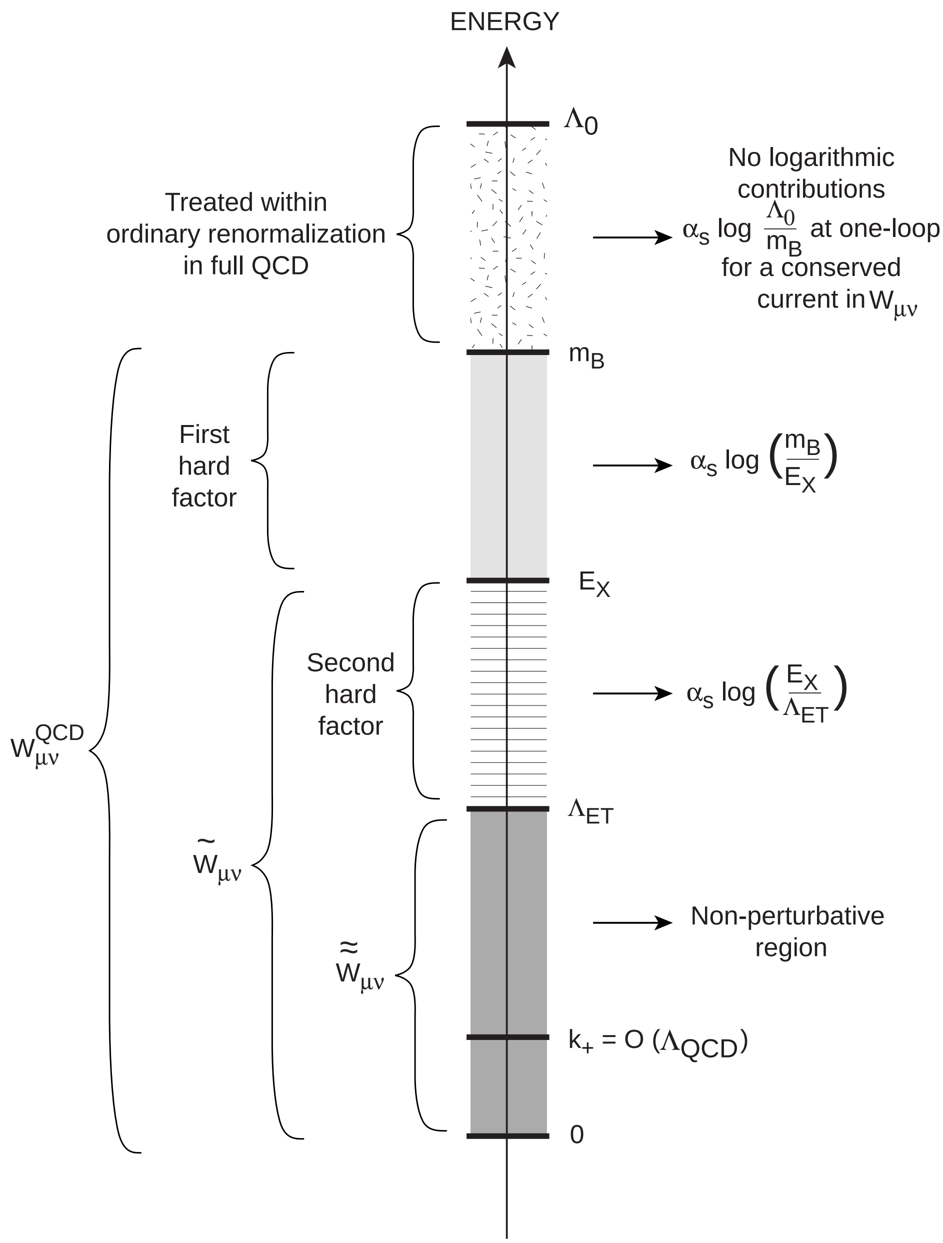

Figure 1: Pictorial description of factorization in the effective theory.

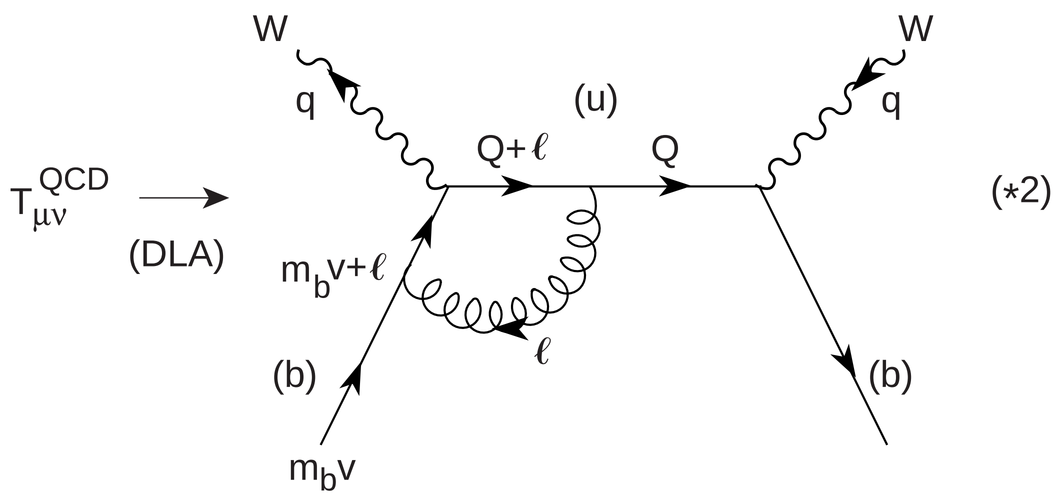

Figure 2: Vertex corrections to the light-cone function $F^{QCD}(k_+)$

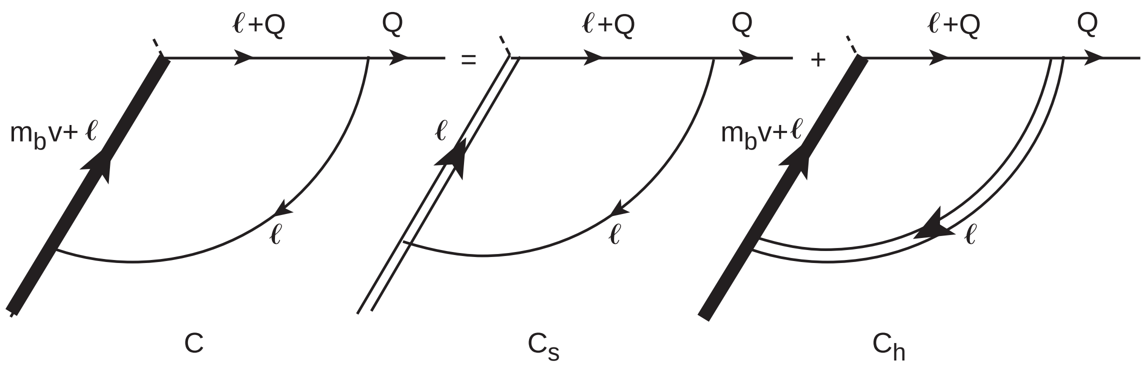

Figure 3: First decomposition in $T_{\mu\nu}$.
The thick, thin and double lines represent the massive quadratic, massless quadratic and time-like eikonal propagators, respectively.

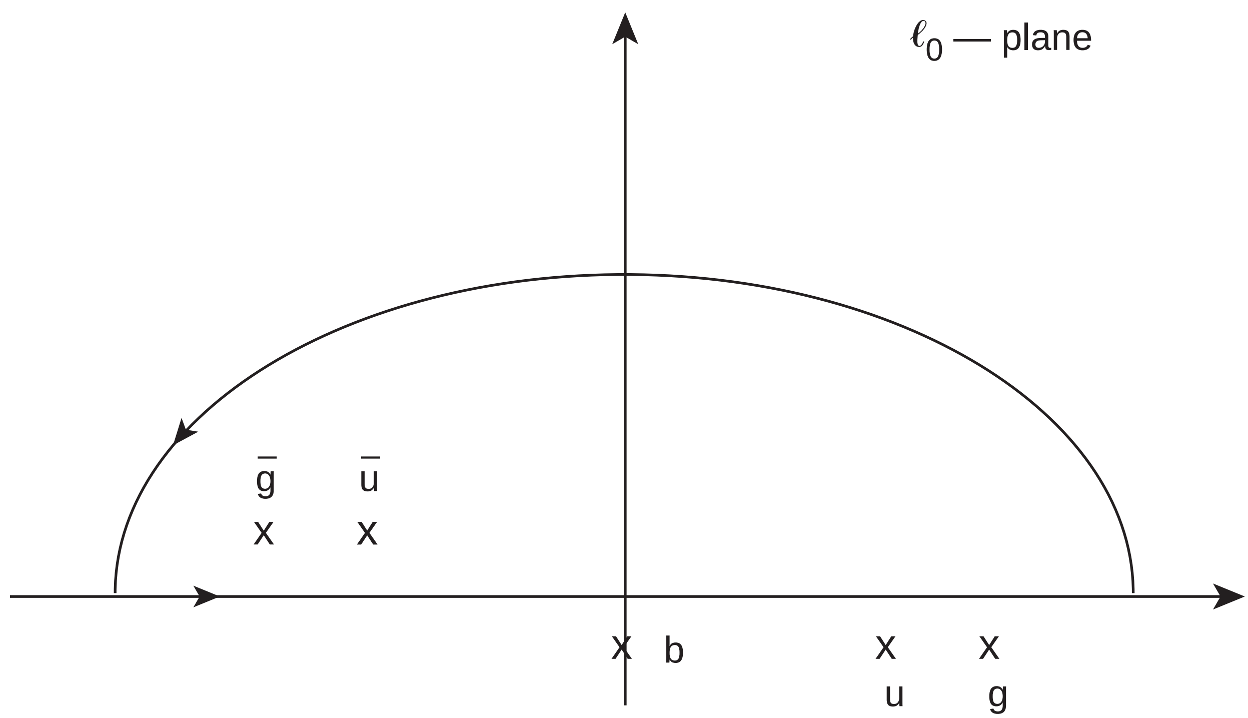

Figure 4: Poles of C_s in the l_0-plane.
The crosses labelled by b, u and g represent the particle poles in the static beauty, *up* and gluon propagators, respectively. The crosses labelled by $\bar{u}$ and $\bar{g}$, instead, represent the antiparticle poles in the *up*-quark and gluon propagators.

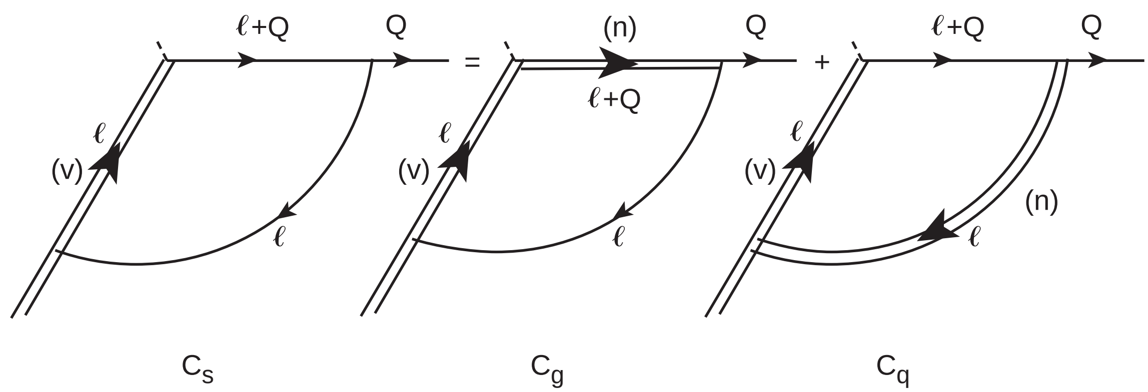

Figure 5: Second decomposition in $T_{\mu\nu}$.
The thick, thin and double lines represent the massive quadratic, massless quadratic and eikonal propagators, respectively. The symbols (v) and (n) indicate that the eikonal propagators must be taken along the directions v and n.