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EXACT AND PERTURBATIVE SOLUTIONS FOR THE VACUUM POLARIZATION IN A SOLUBLE MODEL IN SCALAR FIELD THEORY

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A B S T R A C T

The relation between exact and perturbative solutions in a renormalizable field theory is studied in the case of all non-overlapping rainbow diagrams contributing to the self-energy of a scalar field in a $\lambda\varphi^2\Phi^2$ theory with one $g\Phi\varphi^2$ insertion. The graphs are summed using Bethe-Salpeter techniques. The exact solution exhibits a singularity on the light cone of the kind not usually considered in the Wilson expansion. In renormalized perturbation theory, the self-energy is determined up to an over-all additive constant and agrees, to order $O(g^2\lambda^2)$, with the exact solution for small values of the invariant mass q^2. At large q^2 perturbation theory is shown to fail. Because of the $\Phi\varphi^2$ insertion, the deep euclidean behaviour of the vacuum polarization is found to correspond to that of a four-point function in $\Phi^2\varphi^2$ theory, at exceptional momenta. It is argued that a particular discontinuity $\rho(q^2)$ of this class of graphs may be relevant to e^+e^- annihilation. $\rho(q^2)$ is found to rise with q^2 like $(q^2)^{\gamma(g)-1}$ where $\gamma(g)$ is tentatively associated with the anomalous dimension of the operator Φ^2.

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1. INTRODUCTION

The Bethe-Salpeter [1] equation in the ladder approximation has proven useful in the past for studying bound state problems [2] and high energy behaviour of physical scattering amplitudes [3]. In the scattering region the range of applicability of such approximations has been limited, however, to the special kinematical regions corresponding to production processes of the multiperipheral type [4].

On the other hand, renormalization group techniques [5] have provided a powerful tool for discussing the behaviour of vertex functions in the limit where some or all of the kinematical invariants tend to infinity. In particular there has been a recent attempt to combine these techniques with Bethe-Salpeter equations, in order to obtain Wilson type expansions for e^+e^- annihilation [6].

Renormalization group techniques are, however, intimately connected with multiplicative renormalizability of the theory, and hence can be applied only to classes of graphs where the divergences may be absorbed completely into a mass, coupling constant and wave function renormalization constant. This has limited non-perturbative calculations consistent with renormalization group transformations to fairly simple subclasses of graphs [7].

We shall take here the point of view that some interesting aspects about renormalizable theories may be learned from the study of non-perturbative solutions, even if the subclass of graphs considered is not invariant under a renormalization group transformation. Thus one may inquire about the relation existing between the solution to singular integral equations and the corresponding iterations in powers of the coupling constant ; the validity of asymptotic expansions in renormalized perturbation theory ; the singularity structure in the coupling constant plane ; the behaviour on the light cone and the related question concerning anomalous dimensions in non-asymptotically free theories, wherever such a concept is applicable. These are the main questions we shall address ourselves to. The class of graphs we consider are the self-energy diagrams shown in Fig. 1, corresponding to an interaction Lagrangian $\mathcal{L}_I = -\lambda_0 \Phi^2\varphi^2/(2!)^2 - g\,\Phi\varphi^2/2!$.

The motivation for considering this class of diagrams is essentially two-fold. One is a necessary condition : they are summable by standard Bethe-Salpeter techniques in the case where the rainbows (denoted by dotted lines) correspond to zero mass fields. The other motivation is a physical one : the discontinuity corresponding to cutting all dotted lines may be viewed as the cross-section for the cascade decay of a heavy virtual meson by the continuous emission of pion pairs. Cascade processes of this type, where the initial (vector) meson is coupled to a virtual photon, have been suggested by N.S. Craigie and the author [8)] as a possible mechanism for yielding an enhanced e^+e^- cross-section as the result of the continuous opening of new channels. A topology of this kind is also naturally generated in non-abelian gauge theories including electromagnetic interactions. Note that the topology of Fig. 1 corresponds to that of a $\lambda\Phi^2\varphi^2$ theory with a single $g\Phi\varphi^2$ insertion. Although this insertion has been motivated by the above physical picture, it has at the same time the effect of reducing the quadratic divergence of the self-energy graphs in a $\Phi^2\varphi^2$ theory, to a logarithmic one.

The material of the paper is arranged as follows : in Section 2 we formulate the integral and differential equation corresponding to summing over the infinite set of graphs shown in Fig. 1, and establish the boundary conditions to be satisfied by the solution. In Section 3 we then construct the solution satisfying these boundary conditions using Bethe-Salpeter techniques. In Section 4 we discuss the analytic properties of the exact solution and its behaviour as $q^2 \to 0$ and $q^2 \to \infty$. We then compare this behaviour with that found up to $O(g^2\lambda^2)$ in renormalized perturbation theory, with emphasis on the ambiguities involved when performing subtractions. We then rephrase our results in Section 5, in terms of the Callan-Symanzik equation for the discontinuity $\pi(s)$. A connection between the exponent in the asymptotic behaviour of $\pi(s)$ and the anomalous dimension of the operator Φ^2 is suggested. We conclude in Sections 6 and 7 with some remarks concerning e^+e^- annihilation and a summary of our results. Some of the mathematical details are relegated to the appendices.

2. EQUATION AND BOUNDARY CONDITIONS

We consider here the graphs shown in Fig. 1, corresponding to the self-energy of a scalar field of mass m coupled to a massless pseudoscalar field φ via the interaction Lagrangian

$$\mathcal{L}_I = -\frac{1}{2!} g\phi\varphi^2 - \frac{1}{(2!)^2}\lambda_0\phi^2\varphi^2$$

The sum over such diagrams is summarized by the integral equation shown in Fig. 2. Specifically one has for the self-energy $\Pi(q^2)$,

$$\Pi(q^2) = ig^2\tilde{I}_u(q^2) + (i\lambda_0)^2\int\frac{d^4q'}{(2\pi)^4}\,\frac{\tilde{I}_u((q-q')^2)}{(q'^2-m^2+i\varepsilon)^2}\,\Pi(q'^2) \quad , \tag{2.1}$$

where $\tilde{I}_u(q^2)$ is the (unrenormalized) potential corresponding to the exchange of a pair of zero-mass fields, and is formally given in terms of the integral,

$$\tilde{I}_u(q^2) = i^2\int\frac{d^4k}{(2\pi)^4}\,\frac{1}{(k^2+i\varepsilon)[(k+q)^2+i\varepsilon]} \quad . \tag{2.2}$$

Since the integral diverges logarithmically, we need to renormalize the potential by performing a subtraction at some (arbitrary) euclidean point $q^2 = -\varkappa^2$, $\varkappa^2 > 0$. The result is that $\lambda_0^2\,\tilde{I}_u$ in Eq. (2.1) is to be replaced by the renormalized potential (see appendix A)

$$\lambda^2\tilde{I}(q^2) = -\frac{i\lambda^2}{(4\pi)^2}\,\ln\left(\frac{-q^2-i0}{\varkappa^2}\right) \tag{2.3a}$$

with the Fourier transform

$$\lambda^2 I(x^2) = -\frac{\lambda^2}{(2\pi)^4}\,\frac{1}{(x^2-i0)^2} \tag{2.3b}$$

up to an additive term $b\delta^4(x)$, where b is arbitrary. For the subclass of graphs which we are considering, there is no need for renormalization of the vertex $g\varphi^2\chi$.

With the potential (2.3), the kernel of the integral equation is now well defined. In order to solve this equation, we proceed in the standard way[3] by considering instead the integral equation

$$\psi(x) = g^2 - \lambda^2\int d^4x'\,G(x-x')\,I(x')\,\psi(x') \quad , \tag{2.4}$$

where $G(x)$ is the Fourier transform of $(q^2-m^2+i\epsilon)^{-2}$. In terms of $\psi(x)$, the self-energy is given by

$$\Pi(q^2) = i\int d^4x\, e^{iqx} I(x)\psi(x) \, . \tag{2.5}$$

It is convenient to perform a Wick rotation [9] in Eq. (2.4). The Wick-rotated solution $\Phi(x) = \psi(-it,\vec{x})$, then satisfies the equation

$$\phi(x) = g^2 - \lambda^2\int d^4x' \, H(x-x')V(x')\phi(x') \tag{2.6}$$

where

$$V(x) = -\frac{1}{(2\pi)^4}\frac{1}{R^4} \quad , \quad R = \left(\sum_{i=1}^{4} x_i^2\right)^{\frac{1}{2}} \tag{2.7}$$

is the Wick rotated potential (2.3b) and

$$H(x-x') = \frac{1}{8\pi^2} K_0(m|x-x'|) \, . \tag{2.8}$$

In the following we set $m = 1$. Making use of the expansion

$$K_0(|x-x'|) = \sum_{n=0}^{\infty}\left[\epsilon_n I_n(R_<)K_n(R_>) - I_{n+2}(R_<)K_{n+2}(R_>)\right]C_n^1(\cos\theta)$$

$$\epsilon_0 = \frac{1}{2} \; ; \quad \epsilon_n = 1 , \quad n \neq 0$$

where θ is the angle between x and x', and of the potential (2.7), Eq. (2.6) reduces, on account of O_4 invariance, to

$$\phi(R) = g^2 + \frac{a}{8}\int_0^{\infty}\frac{dR'}{R'}\left[I_0(R_<)K_0(R_>) - 2\, I_2(R_<)K_2(R_>)\right]\phi(R') \tag{2.9}$$

where

$$a = \frac{\lambda^2}{(2\pi)^4} .$$

Equation (2.9) is equivalent to the differential equation

$$\left\{ \left(\frac{d^2}{dR^2} + \frac{3}{R}\frac{d}{dR} - 1 \right)^2 - \frac{a}{R^4} \right\} \phi(R) = g^2 \tag{2.10}$$

plus suitable boundary conditions. Following Ref. 3), we may factorize the left-hand side of Eq. (2.10),

$$\frac{1}{R^2}\left(O_{\nu_1} - 1\right) R^2 \left(O_{\nu_2} - 1\right)\phi(R) = g^2 \tag{2.11}$$

where

$$O_\nu \equiv \frac{d^2}{dR^2} + \frac{1}{R}\frac{d}{dR} - 1 - \frac{\nu^2}{R^2} , \tag{2.12}$$

$$\nu_1 = i\sqrt{2}\left(\sqrt{1 + \frac{a}{4}} - 1\right)^{\frac{1}{2}} ,$$

$$\nu_2 = \sqrt{2}\left(\sqrt{1 + \frac{a}{4}} + 1\right)^{\frac{1}{2}} . \tag{2.13}$$

Performing a Wick rotation in Eq. (2.5) and making use of the expansion [10]

$$e^{iQR\cos\theta} = (2\pi)^2 \sum_{n=0}^{\infty} i^n \frac{n+1}{2\pi^2} \frac{J_{n+1}(QR)}{QR} C_n^1(\cos\theta)$$

we have

$$(2\pi)^4 \Pi(-q^2) = \frac{(2\pi)^2}{(-q^2-i0)^{\frac{1}{2}}} \int_0^\infty dR\, R^2 J_1\left(R(-q^2-i0)^{\frac{1}{2}}\right) V(R)\,\phi(R)\,. \tag{2.14}$$

We next need to examine the boundary conditions which $\Phi(R)$ should satisfy.

- Boundary conditions :

a) At the origin

From the differential equation (2.11) we immediately deduce that

$$\phi(R) \underset{R\to 0}{\sim} R^{\pm\nu_1}\,,\; R^{\pm\nu_2}\,.$$

Since $\nu_2 \geq 2$ for $a \geq 0$, a solution behaving like $R^{-\nu_2}$ is incompatible with the integral equation (2.9) and must be discarded. ν_1, on the other hand is purely imaginary. Hence R^{ν_1} and $R^{-\nu_1}$ oscillate infinitely fast as $R \to 0$, damping completely the $R = 0$ singularity of the integrand in Eq. (2.9). We therefore require in accordance with the desired reality conditions for $\Phi(R)$,

$$\phi(R) \underset{R\to 0}{\sim} \left[a(\nu_1)R^{\nu_1} + a(-\nu_1)R^{-\nu_1}\right] + b(\nu_2)\,R^{\nu_2} \tag{2.15}$$

b) At infinity

The required behaviour at infinity is most easily deduced from the form (2.9) of the integral equation, from where we find

$$\phi(R) \xrightarrow[R\to\infty]{} g^2\left(1 - \frac{a}{8}\frac{1}{R^2} + \cdots\right)\,. \tag{2.16}$$

3. THE SELF-ENERGY

We are now in a position of constructing the appropriate Green's functions for the problem. Noting that the Bessel functions of imaginary argument $I_{\pm\nu_1}(R)$, $I_{\pm\nu_2}(R)$, $K_{\nu_1}(R)$ and $K_{\nu_2}(R)$ all are solutions to the homogeneous part of Eq. (2.11) and taking due account of the boundary conditions (2.15), (2.16), we have

$$G_a(R,R') = \gamma(a)\left[\frac{1}{2}\left(I_{\nu_1}(R_<) + I_{-\nu_1}(R_<)\right)K_{\nu_1}(R_>) - I_{\nu_2}(R_<)K_{\nu_2}(R_>)\right] \tag{3.1}$$

where

$$\gamma(a) = \frac{1}{\nu_2^2 - \nu_1^2} = \frac{1}{4\sqrt{1+\frac{a}{4}}} \; .$$

With the aid of the Wronskian

$$K_\nu'(R)I_\nu(R) - I_\nu'(R)K_\nu(R) = \frac{1}{R}$$

it is a straightforward matter to show that (3.1) is indeed a Green's function for the differential operator in Eq. (2.11). The solution to the inhomogeneous equation (2.11) is thus given by

$$\phi(R) = g^2\int_0^\infty dR' \, R'^3 G_a(R,R') \; . \tag{3.2}$$

From Eq. (3.2) we find [11]

$$\phi(R) \xrightarrow[R\to 0]{} 4g^2\gamma(a)\left\{\frac{1}{2}\frac{\Gamma(2+\frac{\nu_1}{2})\Gamma(2-\frac{\nu_1}{2})}{\Gamma(1+\nu_1)\Gamma(1-\nu_1)}\left[\Gamma(1-\nu_1)\left(\frac{R}{2}\right)^{\nu_1} + \Gamma(1+\nu_1)\left(\frac{R}{2}\right)^{-\nu_1}\right]\right.$$
$$\left. - \frac{\Gamma(2+\frac{\nu_2}{2})\Gamma(2-\frac{\nu_2}{2})}{\Gamma(1+\nu_2)}\left(\frac{R}{2}\right)^{\nu_2} + \frac{R^4}{(\nu_1^2-16)(\nu_2^2-16)}\right\}\left(1+O\left(\frac{1}{R}\right)\right)$$

in accordance with the required behaviour (2.15). Moreover, a little algebra shows that $\Phi(R)$ in Eq. (3.2) also behaves asymptotically as in Eq. (2.16). Hence $\Phi(R)$, as given by Eq. (3.2), is in fact the complete solution.

Substituting the result (3.2) into Eq. (2.14), we have

$$\Pi(-Q^2) = \frac{1}{2}\left(\Pi_{\nu_1}(-Q^2) + \Pi_{-\nu_1}(-Q^2)\right) - \Pi_{\nu_2}(-Q^2) \tag{3.3}$$

where

$$(2\pi)^4 \Pi_\nu(-Q^2) = -\frac{(2\pi)^2}{Q} g^2 \gamma(a) \left\{ \int_0^1 dx\, x^3 \int_0^\infty dR\, R^2 J_1(QR) K_\nu(R) I_\nu(xR) + \int_1^\infty dx\, x^3 \int_0^\infty dR\, R^2 J_1(QR) I_\nu(R) K_\nu(xR) \right\}.$$

The integral over R at fixed x defines an analytic function of ν for $\mathrm{Re}\nu > -2$. Performing the integration we find

$$(2\pi)^4 \Pi_\nu(-Q^2) = -(2\pi)^{\frac{3}{2}} e^{-\frac{3}{2}i\pi} g^2 \gamma(a) \int_0^\infty dx\, x \frac{Q^{\frac{3}{2}}_{\nu-\frac{1}{2}}(u)}{(u^2-1)^{3/4}}, \quad \mathrm{Re}\,\nu > -2.$$

where Q^u_ν is the associated Legendre function of the second kind, and

$$u = \frac{1 + x^2 + Q^2}{2x} .$$

With $\nu = \pm\nu_1$ or ν_2 the remaining x integration converges for $a > 0$. Performing a change of variable we find

$$\begin{aligned}(2\pi)^4 \Pi_\nu(-Q^2) &= -2e^{-\frac{3}{2}i\pi} \gamma(a) g^2 (1+Q^2) \int_1^\infty dv \frac{2v^2-1}{\sqrt{v^2-1}} \frac{Q^{\frac{3}{2}}_{\nu-\frac{1}{2}}(v\sqrt{1+Q^2})}{[v^2(1+Q^2)-1]^{3/4}} \\ &= (2\pi)^2 g^2 \gamma(a) (1+Q^2)^{-\frac{\nu}{2}} \frac{\Gamma(2+\frac{\nu}{2})\Gamma(\frac{\nu}{2})}{\Gamma(1+\nu)} {}_2F_1\left(2+\frac{\nu}{2}, \frac{\nu}{2}; 1+\nu; \frac{1}{1+Q^2}\right).\end{aligned} \tag{3.4}$$

Continuing this result in Q^2 to the physical region, we may write the complete solution in the form [11]

$$(2\pi)^4 \Pi(q^2) = -\frac{g^2(2\pi)^2}{4\sqrt{1+\frac{a}{4}}} \Big\{ \frac{\Gamma(2+\frac{\nu_1}{2})\Gamma(\frac{\nu_1}{2})}{2\Gamma(1+\nu_1)} (-q^2)^{-\frac{\nu_1}{2}} {}_2F_1\left(\tfrac{\nu_1}{2}, \tfrac{\nu_1}{2}-1; 1+\nu_1; \tfrac{1}{q^2}\right)$$

$$+ \frac{\Gamma(2-\frac{\nu_1}{2})\Gamma(-\frac{\nu_1}{2})}{2\Gamma(1-\nu_1)} (-q^2)^{\frac{\nu_1}{2}} {}_2F_1\left(-\tfrac{\nu_1}{2}, -\tfrac{\nu_1}{2}-1; 1-\nu_1; \tfrac{1}{q^2}\right) \tag{3.5}$$

$$- \frac{\Gamma(2+\frac{\nu_2}{2})\Gamma(\frac{\nu_2}{2})}{\Gamma(1+\nu_2)} (-q^2)^{-\frac{\nu_2}{2}} {}_2F_1\left(\tfrac{\nu_2}{2}, \tfrac{\nu_2}{2}-1; 1+\nu_2; \tfrac{1}{q^2}\right) \Big\}$$

where $\nu_{1,2}$ have been defined in (2.13).

4. THE EXACT AND PERTURBATIVE SOLUTIONS

A. Properties of the solution :

The following properties may be read off the solution (3.5) :

a) It is an even function of ν_1. Hence the square root branch point at zero coupling constant associated with ν_1 is actually absent. The same applies to the pole at $a = 0$ associated with $\Gamma(\nu_1/2)$. This pole is an expression of the logarithmic divergence of the integrals in Eq. (3.4) at vanishing coupling constant, corresponding to the logarithmic divergence of the self-energy graphs in perturbation theory remaining after the coupling constant renormalization. The fact that this singularity actually cancels shows that our choice of boundary conditions has already accomplished the infinite mass renormalization.

b) $\Pi(q^2)$ has logarithmic branch points at $q^2 = 0$ and $q^2 = 1$ corresponding to the production of an arbitrary number of soft zero mass mesons φ, and the additional production of a massive quantum Φ, respectively. The discontinuity across the logarithmic cut in the region $0 \leq q^2 < 1$ is easily computed to be

$$\pi(s) = 2\,\mathrm{Im}\,\Pi(q^2)$$

$$= \frac{1}{\nu_2^2-\nu_1^2}\,\frac{g^2}{2\pi}\Big\{\left(\tfrac{\nu_2^2}{4}-1\right){}_2F_1\left(-\tfrac{\nu_2}{2}, \tfrac{\nu_2}{2}; 2; q^2\right) - \left(\tfrac{\nu_1^2}{4}-1\right){}_2F_1\left(-\tfrac{\nu_1}{2}, \tfrac{\nu_1}{2}; 2; q^2\right)\Big\}, \tag{4.1}$$

$$0 < q^2 < 1$$

Except for the appropriate flux factor, the result (4.1) gives the total cross-section for the production process shown in Fig. 3 in the energy range $0 \leq q^2 < 1$. It is a simple matter to check that $\mathrm{Im}\Pi(q^2)$ is indeed a positive definite quantity for all $a > 0$ in the range $q^2 = (0,1)$, corresponding to the absolute square of the production amplitudes of Fig. 3 summed over the infinite number of final states. Hence for $0 \leq q^2 < 1$, our $\Pi(q^2)$ satisfies a unitarity relation. Near threshold we have

$$\pi(q^2) \xrightarrow[q^2\to 0]{} \frac{g^2}{8\pi}\left\{1 + \frac{a}{192}(q^2)^2 + O((q^2)^3)\right\} \quad . \tag{4.2}$$

c) The renormalized propagator

The renormalized propagator of the φ field is given by

$$i\Delta_R(q^2) = \frac{i}{q^2 - m^2 - \Pi(q^2)} \quad .$$

In the approximation we have considered, this representation makes sense only in the range $0 \leq q^2 < 1$ where $\Pi(q^2)$ satisfies a kind of unitarity relation. $\Delta_R(q^2)$ is complex for all $q^2 > 0$, corresponding to the fact that the φ field is unstable and can decay. m^2 is not to be identified with the mass M_R of the "Φ resonance". We define

$$M_R^2 = m^2 + \mathrm{Re}\,\Pi(M_R^2)$$

$$\Gamma = \mathrm{Im}\,\Pi(M_R^2)$$

$$\Pi_R(q^2) = \Pi(q^2) - \Pi(M_R^2)$$

so that we may write

$$i\Delta_R(q^2) = \frac{i}{q^2 - M_R^2 - i\Gamma - \Pi_R(q^2)} \quad , \qquad 0 \leq q^2 < 1$$

provided that $M_R^2 < M^2$. In the small coupling limit, $g, a \to 0$, we find

$$M_R^2 = M^2 - \frac{g^2}{(4\pi)^2} \quad ,$$

$$\Gamma = \frac{\pi g^2}{(4\pi)^2} \quad ,$$

the width and mass of the Φ resonance being determined by the "2π decay". Note that the real part for this decay (as given in terms of the subtraction parameter $\varkappa^2$) is <u>uniquely</u> determined in our solution, and is negative, so that $M_R^2 < M^2$.

d) Threshold behaviour

For $q^2 \to 0$ we find

$$(2\pi)^4 \Pi(q^2) \xrightarrow[q^2\to 0]{} \pi^2 g^2 \left[\ln(-q^2) + \gamma - 1 \right] \tag{4.3}$$

$$- \frac{\pi^2 g^2}{\sqrt{1+\frac{a}{4}}} \left\{ \left(\frac{\nu_1^2}{4} - 1\right)\left(\psi\left(\frac{\nu_1}{2}\right) + \psi\left(-\frac{\nu_1}{2}\right)\right) - \left(\frac{\nu_2^2}{4} - 1\right)\left(\psi\left(\frac{\nu_2}{2}\right) + \psi\left(\frac{\nu_2}{2}+1\right)\right) \right.$$

$$\left. - \frac{a}{32} q^2 \left[\psi\left(\frac{\nu_1}{2}+1\right) + \psi\left(-\frac{\nu_1}{2}+1\right) - \psi\left(\frac{\nu_2}{2}+1\right) - \psi\left(\frac{\nu_2}{2}\right) \right] \right\} + O(q^2)$$

or expanding in powers of the coupling constant

$$(2\pi)^4 \Pi(q^2) \xrightarrow[q^2\to 0]{} \pi^2 g^2 \left\{ \left(\ln(-q^2) - 1\right) + \frac{a}{16}\left(1 + 2\zeta(3) - \frac{1}{2} q^2\right) \right\} + O(a^2 q^4) \tag{4.4}$$

where γ denotes Euler's constant and $\zeta(z)$ is the Riemann Zeta function.

e) Asymptotic behaviour

For $q^2 \to \infty$,

$$(2\pi)^4 \Pi(q^2) \xrightarrow[q^2\to\infty]{} - \frac{\pi^2 g^2}{\sqrt{1+\frac{a}{4}}} \left\{ \frac{\Gamma\left(2+\frac{\nu_1}{2}\right)\Gamma\left(\frac{\nu_1}{2}\right)}{2\Gamma(1+\nu_1)} (-q^2)^{-\frac{\nu_1}{2}} \right. \tag{4.5}$$

$$\left. + \frac{\Gamma\left(2-\frac{\nu_1}{2}\right)\Gamma\left(-\frac{\nu_1}{2}\right)}{2\Gamma(1-\nu_1)} (-q^2)^{\frac{\nu_1}{2}} + O\left(\frac{1}{q^2}\right) \right\} .$$

Noting that

$$\frac{\Gamma\left(1+\frac{\nu_1}{2}\right)^2}{\Gamma(1+\nu_1)} = 1 + \sum_{n=1}^{\infty} \left[\left(\frac{1}{2}\right)^{n-1} - 1 \right] \frac{(-1)^n}{n} \zeta(n) \left(\frac{\nu_1}{2}\right)^n$$

we have to order $O(a^2)$,

$$(2\pi)^4 \Pi(q^2) \xrightarrow[q^2\to\infty]{} \pi^2 q^2 \Big\{ \big(\ln(-q^2) - 1\big) - \frac{a}{8}\Big[\big(-1 + \zeta(2) - 6\zeta(3)\big) + \big(1 - \zeta(2)\big)\ln(-q^2) - \frac{1}{4}\ln^2(-q^2) + \frac{1}{12}\ln^3(-q^2)\Big] + O(a^2)\Big\}. \tag{4.6}$$

Note that $\log^{2n+1}(-q^2)$ is the maximum power of $\log(-q^2)$ occuring in a given order a^n, which is reminiscent of analogous relations required by the renormalization group constants in QED [12].

B. Connection with renormalized perturbation theory

The non-perturbative result Eqs. (3.5) and (4.5) show that summation to all orders in the coupling constant λ leads to strong oscillations in the limit $q^2 \to \infty$. The asymptotic behaviour, Eq. (4.5), corresponds to a light cone singularity of the type $(x^2)^{-2}\cos(\bar{\nu}_1/2 \ln x^2/4)$ and $(x^2)^{-2}\sin(\bar{\nu}_1/2 \ln x^2/4)$, $(\bar{\nu}_1 = -i\nu_1)$, which are non-perturbative in character, and are usually not considered in arguments relating to the Wilson expansion. It is unclear whether this feature is a result of restricting ourselves only to a subclass of graphs, or whether it may in fact be a feature of the full set of graphs in a non-asymptotically free theory without an ultraviolet stable fixed point. Nevertheless, the expansions (4.4) and (4.6) in powers of the coupling constant exhibit the features expected from renormalized perturbation theory, and it is instructive to investigate in some detail this connection.

It is clear from the outset, that due to the required renormalization in perturbation theory, agreement with our unique solution (3.5) can at best be achieved up to an arbitrary constant. The usual normalization conditions required to define uniquely the Green's functions in renormalized perturbation theory would then be sufficient to remove the remaining ambiguity to each order in perturbation theory.

The self-energy graphs of Fig. 1 require two types of subtractions : a subtraction for each of the $n-1$ potentials occuring in a $2n$'th order graph, corresponding to a coupling constant renormalization (see appendix A) ; an over-all subtraction at each stage of the iteration, corresponding to a mass renormalization.

The arbitrary constant associated with the coupling constant renormalization is essentially the same as the arbitrary scaling parameter b occurring in the definition of the Fourier transform of the potential $I(x)$, Eq. (A.6). As pointed out in Ref. 13), it is related to the dilatation group, and its arbitrariness is a reflection of the fact that the definition of the finite part of a singular integral is not invariant under scale transformations. In perturbation theory this arbitrariness is removed by a coupling constant renormalization ; in the general case it is removed if we choose a specific test function. In particular, regarding our solution $\psi(x)$ as a test function in Eq. (2.5), and comparing the leading term in the expansion (4.4) with that given by renormalized perturbation theory, Eq. (2.3a), we see that our non-perturbative treatment has uniquely fixed the value of the subtraction constant $\varkappa^2$ to be $\varkappa^2 = e$. Similarly, the arbitrary constant associated with the mass renormalization has also been uniquely fixed in our non-perturbative approach.

Since we have restricted ourselves to a subclass of graphs, the divergences remaining after the coupling constant renormalization cannot be absorbed unambiguously into a mass renormalization. For extracting the finite parts we shall adopt the method of "complex extension" [13], replacing everywhere the Φ propagator by $(M^2-k^2-i\epsilon)^{-\lambda}$, and extracting the poles in λ at $\lambda = 1$. Setting $\varkappa = e$ we then have to second order in perturbation theory,

$$(2\pi)^4 \Pi(q^2) = \pi^2 g \left[\left(\ln(-q^2) - 1 \right) - \frac{a}{16} \, \mathrm{Pf} \; T_1(q^2) \right] + O(a^2) \tag{4.7}$$

where "Pf" stands for "part finie" in the above sense, and

$$T_\lambda(q^2) = -\frac{i}{\pi^2} \int d^4k \, \ln\left(-\frac{(q-k)^2}{e}\right) \frac{1}{(1-k^2-i\varepsilon)^{2\lambda}} \ln\left(-\frac{k^2}{e}\right) .$$

Performing a Wick rotation, and introducing four-dimensional spherical coordinates, we may carry out the angular integration by noting that

$$\int_0^\pi d\theta \sin^2\theta \ln\left(Q^2+K^2-2QK\cos\theta\right) = \frac{\pi}{2} \begin{cases} \ln Q^2 + \frac{1}{2}\frac{K^2}{Q^2} , & K^2 < Q^2 \\ \ln K^2 + \frac{1}{2}\frac{Q^2}{K^2} , & K^2 > Q^2 \end{cases}$$

We find, after some calculation,

$$T_\lambda(-Q^2) = \frac{1}{2}\left[\frac{1}{(\lambda-1)^3} - \frac{1}{(\lambda-1)^2} + \frac{1}{(\lambda-1)}\right] - 2$$

$$- Q^4\left[C_1^2(Q^2) + C_1^1(Q^2)(\ln Q^2 - 1) - \frac{1}{2}C_2^1(Q^2)\right] \tag{4.8}$$

$$- \frac{1}{2}Q^2\left[\frac{Q^2\ln Q^2}{1+Q^2} - \ln(1+Q^2)\right]$$

$$+ \frac{1}{2}\ln\frac{Q^2}{e}\left[1 + \frac{1}{1+Q^2} - 2\frac{\ln(1+Q^2)}{Q^2}\right]$$

$$- \frac{1}{2}\frac{Q^2}{1+Q^2} + O(\lambda-1)$$

where

$$C_\alpha^\beta(Q^2) = \int_0^1 dy\, \frac{y^\alpha(\ln y)^\beta}{(1+Q^2y)^2}\ .$$

Taking the finite part and combining these results in Eq. (4.7), we find

$$i(2\pi)^4\Pi(q^2) \xrightarrow[q^2\to 0]{} \pi^2 g^2\left\{(\ln(-q^2) - 1) + \frac{a}{16}\left(2 - \frac{1}{2}q^2\right)\right\}, \tag{4.9}$$

$$i(2\pi)^4\Pi(q^2) \xrightarrow[q^2\to\infty]{} \pi^2 g^2\Big\{(\ln(-q^2) - 1)$$

$$- \frac{a}{8}\left[\frac{1}{12}\ln^3(-q^2) - \frac{1}{4}\ln^2(-q^2) + \frac{1}{2}\ln(-q^2) - \frac{3}{2}\right] \tag{4.10}$$

$$+ O(a^2)\Big\}\ .$$

Upon comparing the perturbation theory results (4.9) and (4.10) with the corresponding expansions (4.4), (4.6) of the exact solution, we see that the two results only differ by an over-all additive constant for $q^2 \to 0$, where the perturbation series is expected to converge. This discrepancy is removed by imposing the proper normalization condition. For $q^2 \to \infty$, on the other hand, the two results only formally agree as regards the powers of $\log(-q^2)$ appearing in the expansion, but differ in the coefficients. This illustrates the failure of perturbation theory when large momenta are involved, a fact which constitutes the basic motivation for the renormalization group approach to the deep euclidean region.

5. CALLAN-SYMANZIK EQUATION

The graphs we have considered require regularization. However, since we have restricted ourselves to the particular subclass of graphs shown in Fig. 1, the infinities cannot be eliminated unambiguously to each order in perturbation theory, since they cannot be entirely absorbed into a mass and coupling constant renormalization. Hence this set of graphs does not satisfy a renormalization group equation [5], nor a Callan-Symanzik equation [14].

If, however, we replace the potential in Eq. (2.1) by its discontinuity as shown in Fig. 4, this leads to a Volterra integral equation. The perturbation series generated by iteration of this equation is thus finite and requires no subtractions. The exact solution $\pi(s)$ has already been given in Eq. (4.1) and will be identical with the corresponding power series expansion in λ wherever it converges.

Since $\pi(s)$ requires no subtractions in any order of perturbation theory, it satisfies the (trivial) Callan-Symanzik equation

$$-s\frac{\partial}{\partial s}\pi\left(\frac{s}{m^2},g^2,a\right) = m^2\pi_{\phi^2}\left(0;\frac{s}{m^2},g^2,a\right) \tag{5.1}$$

corresponding to a vanishing anomalous dimension and β function. Here $m^2\pi_{\Phi^2} \equiv m^2(\partial/\partial m^2)\pi$ is just $\pi(s)$ with a mass insertion at zero momentum.

Now, from the solution (4.1) we obtain the asymptotic behaviour [11]

$$\pi(s) \xrightarrow[s\to\infty]{} \frac{g^2}{8\pi}\frac{1}{\sqrt{1+\frac{a}{4}}}\left\{\frac{\Gamma(\nu_2)}{\Gamma(\frac{\nu_2}{2}-1)\Gamma(\frac{\nu_2}{2}+1)}\left(-\frac{s}{m^2}-i0\right)^{\frac{\nu_2}{2}} + \frac{\Gamma(-\nu_2)}{\Gamma(-\frac{\nu_2}{2}-1)\Gamma(-\frac{\nu_2}{2}+1)}\left(-\frac{s}{m^2}-i0\right)^{-\frac{\nu_2}{2}}\right\} - (\nu_2\to\nu_1) \tag{5.2}$$

Expression (5.2) exhibits two interesting features. It shows that the solution associated with the potential of Fig. 4 rises faster than $\rho \sim s$, ν_2 rather than ν_1 determining now the leading behaviour. Furthermore,

$$m^2 \frac{\partial}{\partial m^2} \ln \pi(s) \xrightarrow[s \to \infty]{} - \nu_2(a) \neq 0 \tag{5.3}$$

so that differentiation with respect to the mass m does not lower the asymptotic behaviour. The right-hand side of the Callan-Symanzik equation (5.1) thus does not vanish in the deep euclidean region, contrary to what one might have expected on the basis of a corollary to Weinberg's theorem [15]. One may in fact easily verify that this is not only a property of the non-perturbative solution, but also of each diagram in perturbation theory, the reason being essentially that the flow of large amount of momentum through the external loop in Fig. 1 does represents one of the dominant momentum configurations for $q^2 \to -\infty$. This property of the diagrams in question is due to the $g\varphi^2\Phi$ insertion occurring once to every order in perturbation theory : if viewed within the framework of a pure $\lambda\Phi^2\varphi^2$ theory, it has the effect of replacing our vacuum polarization by a four-point function evaluated at the two exceptional momenta $p_1 = p_2 = 0$.

Making use of the asymptotic expansion (5.2), Eq. (5.1) reads for $s \to \infty$,

$$\left(s\frac{\partial}{\partial s} - \gamma(a) - 1 \right) \pi^{\infty}(s) = 0$$

where

$$\gamma(a) = \frac{\nu_2}{2} - 1 \underset{a \to 0}{\sim} \frac{a}{32} .$$

Noting that $\pi(s)$ carries the dimensions of g (mass), and keeping in mind the above remarks concerning the exceptional momenta, it is tempting to associate $\gamma(a)$ with the anomalous dimension γ_{Φ^2} of the operator Φ^2 in a Wilson expansion.

Since the Callan-Symanzik β function is identically zero for the $\pi(s)$ under consideration, the asymptotic behaviour of $\pi(s)$ is given by $\pi(s) \sim s^{\gamma(a)-1}$ with γ a function of the coupling strength a, rather than a fixed point of $\beta(a)$. For this same reason the theory does not become asymptotically free if we replace a by $-a$.

6. CONNECTION WITH e^+e^- ANNIHILATION

As it was pointed out in the introduction, the topology of the graphs we have considered was motivated in part by imagining multipion production in e^+e^- annihilation to proceed via the "decay" of a virtual, heavy meson (coupled to a timelike photon) cascading down in energy as it emits pairs of pions at each stage of the cascade (see Fig. 3). The corresponding cross-section would be given in terms of $\rho(s) = \sum_n |A_n|^2$, where A_n are the production amplitudes shown in Fig. 3. $\rho(s)$ satisfies the Volterra integral equation (see appendix B)

$$\rho(s) = \frac{g^2}{8\pi} + \frac{\pi}{8}\,\frac{a}{s}\int_0^s ds' \,\frac{[(s^2-s'^2)+2s's\ln(s'/s)]}{(s'-m^2-i\varepsilon)(s'-m^2+i\varepsilon)}\,\rho(s') \tag{6.1}$$

obtained from Eq. (2.1) by replacing the potential (2.2) by the appropriate discontinuity, Eq. (B.1) (Fig. 4). Using Mellin transform techniques (see appendix B) one can show that (we set again $m = 1$)

$$\rho(s) \xrightarrow[s\to\infty]{} \left[c(\nu_2)s^{\frac{\nu_2}{2}} + c(-\tfrac{\nu_2}{2})s^{-\frac{\nu_2}{2}} + c(\nu_1)s^{\frac{\nu_1}{2}} + c(-\nu_1)s^{-\frac{\nu_1}{2}}\right]\left(1+O(\tfrac{1}{s})\right) \tag{6.2}$$

where $\nu_{1,2}$ are the solutions to an "eigenvalue equation",

$$\nu^2(\nu^2-1) = \frac{a}{16}$$

and are identical with Eqs. (2.13). The asymptotic behaviour of $\rho(s)$ is thus similar to that of $\pi(s)$ as given by Eq. (5.2), although it is to be kept in mind that $\rho(s)$ is a real, positive quantity for all $s > 0$, whereas $\pi(s)$ is not.

The asymptotic behaviour (6.2) shows that the opening of new channels as $s \to \infty$ (actually these thresholds all lie at $q^2 = 0$ for the zero-mass case in question) enhances the high energy behaviour of $\rho(s)$ and could thus provide a dynamical mechanism for the observed enhancement in the e^+e^- cross-section, an idea that was already explored in Ref. 8). It is instructive to examine the origin of this enhancement.

Defining $\sigma(s)$ and $h(\lambda)$ as in Eq. (B.4b), the n'th order term $\sigma_n(s)$ in the iteration of Eq. (B.4a) for $s \to \infty$ is given by

$$\sigma_n(s) \approx \left(\frac{a}{32}\right)^n \int_0^1 d\lambda_1\, h(\lambda_1) \int_0^1 d\lambda_2\, h(\lambda_2) \ldots \int_0^1 d\lambda_n\, h(\lambda_n)\, \sigma_0(\lambda_1 \ldots \lambda_n s) \quad ,$$

or making the change of variable

$$\eta_i = \lambda_1 \lambda_2 \ldots \lambda_i$$

$$d\lambda_1 \ldots d\lambda_n = \frac{d\eta_1}{\eta_1} \ldots \frac{d\eta_n}{\eta_n}$$

we have

$$\sigma_n(s) \sim \left(\frac{a}{32}\right)^n \int_0^1 \frac{d\eta_1}{\eta_1} h(\eta_1) \int_0^{\eta_1} \frac{d\eta_2}{\eta_2} h\left(\frac{\eta_2}{\eta_1}\right) \ldots \int_0^{\eta_{n-2}} \frac{d\eta_{n-1}}{\eta_{n-1}} h\left(\frac{\eta_{n-1}}{\eta_{n-2}}\right) \int_0^{\eta_{n-1}} d\eta_n\, h\left(\frac{\eta_n}{\eta_{n-1}}\right) \frac{1}{(\eta_n s - 1)^2} \tag{6.3}$$

In the limit $s \to \infty$ the integral is dominated by the integration region $\eta_\ell / \eta_{\ell-1} \ll 1$, so that

$$\sigma_n(s) \sim \left(\frac{a}{32}\right)^n \int_0^1 \frac{d\eta_1}{\eta_1} \int_0^{\eta_1} \frac{d\eta_2}{\eta_2} \ldots \int_0^{\eta_{n-3}} \frac{d\eta_{n-2}}{\eta_{n-2}} \ln(1 - \eta_{n-2} s)$$

(6.4)

$$\approx \left(\frac{a}{32}\right)^n \frac{\ln^{n-1}(-s)}{(n-1)!} + O\left(\ln^{n-2}(-s)\right) \quad .$$

Two interesting pieces of information may be abstracted from Eq. (6.4) :

1) the dominant contribution to the integral in Eq. (6.3) arises from the integration regions

$$y_1 > y_2 > \cdots > y_n$$

in rapidity $y_i = \ln(s_i/m^2)$, where $s_i = \lambda_i s$ are the masses of the heavy meson in the successive decay stages ;

2) the leading log-summation fails to give the correct asymptotic behaviour, just as one would have expected.

Finally we would like to remark that we do not expect the behaviour (6.2) to correspond to the asymptotic behaviour in the complete theory, where the Callan-Symanzik β function is expected to play a crucial role. Moreover, unitarity corrections will necessarily have to come into play at sufficiently high energies. That is, the subset of graphs in Fig. 3 can at best dominate in some intermediate energy region.

7. SUMMARY AND CONCLUSION

It has been our primary aim to study the connection existing between the solution to integral equations occurring in renormalizable field theories, and renormalized perturbation theory. As the basis for our investigation we have chosen the subset of non-overlapping rainbow insertions contributing to the hadronic vacuum polarization in $\lambda\Phi^2\varphi^2$ theory (with one $g\Phi\varphi^2$ insertion) (see Fig. 1). The corresponding integral equation (2.1) was singular in the sense that a formal iteration leads to (logarithmically) divergent integrals to each order in λ.

The following interesting points have emerged from our study :

a) the full solution oscillates infinitely fast on the light cone, a feature which is usually not considered in connection with Wilson expansions in renormalizable theories. These strong oscillations are precisely the reason for the existence of the integral in Eq. (2.1) in the non-perturbative sense. As the coupling constant λ tends to zero, these oscillations are "turned off" ; the integral in Eq. (2.1) diverges in the limit $\lambda = 0$, which manifests itself as a singularity in $\Pi_{\nu_1}(q^2)$, Eq.(3.4), at vanishing coupling constant [however, no such singularity is present in the full solution (3.5) !]. This is at the same time the reason for the failure of the iterative solution, in the absence of some (cut-off independent) regularization procedure (peratrization) : a formal power-series expansion of

$(x^2)^{-2}\cos(\bar{\nu}_1/2 \ln x^2/4)$ in powers of λ eliminates the crucial convergence factor coming from these oscillations. It is thus tempting to conjecture that the divergences remaining in the graphs of Fig. 1 after performing the coupling constant renormalization [necessary to define the kernel of the integral equation (2.1)] are a result of choosing the wrong expansion parameter (λ), and are actually absent in the sum over all graphs [16];

b) for q^2 small we have found the expansion of the exact solution in powers of λ to agree, up to an additive constant, with that given by renormalized perturbation theory to order $g^2\lambda^2$. The additive constant is arbitrary in the iterative solution, but is uniquely fixed in the non-perturbative case. The discrepancy is removed, however, by imposing a normalization condition, or equivalently, by a suitable redefinition of the coupling constant. On the other hand, at large q^2, perturbation theory was shown to fail ;

c) because of the $g\Phi^2\varphi$ insertion, differentiation with respect to the bare mass did not lower the asymptotic behaviour of neither $\Pi(q^2)$ nor $\pi(q^2)$. We tentatively identified the exponent in $\pi(q^2)/q^2 \sim (q^2)^{\nu_2-1}$ with the anomalous dimension of the Φ^2 operator. Moreover, in the model under consideration, $m^2\partial/\partial m^2 \text{ disc } \Pi(q^2) \underset{m \to 0}{\to} \infty$, rather than tending to zero as in a pure Φ^4 theory [6].

Our study of a simple soluble model has indicated that the connection between the solution of singular integral equations occurring in renormalizable field theories and renormalized perturbation theory to arbitrary orders in the coupling constant may not be a simple one *). It would certainly seem of interest to extend our study to larger subclasses of graphs also satisfying the renormalization group equations.

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*) After completing this work the author became aware of work by W. Güttinger and E. Pfaffelhuber [Ref. 17)] in which similar questions have been raised in connection with unrenormalizable interactions.

APPENDIX A

a) The renormalized potential

With a Pauli-Villars regularization, we have

$$\tilde{I}_\Lambda(q^2) = -\frac{i}{(4\pi)^2}\int_0^1 d\beta \, \ln\left(\frac{\beta(\beta-1)(q^2+i0)}{\beta(\beta-1)(q^2+i0)+\Lambda^2}\right) .$$

Hence, to second order in the coupling constant the four-point vertex function for the corresponding kinematical configuration is given by (Fig. 5)

$$\Gamma^{(4)}\left(\frac{q^2}{\Lambda^2},\lambda_0\right) = -i\lambda_0 + \lambda_0^2\,\tilde{I}_\Lambda(q^2)$$

$$= -i\lambda_0 - \frac{i\lambda_0^2}{(4\pi)^2}\left[\ln\left(\frac{-q^2-i0}{\Lambda^2}\right) - 2\right]$$

The physical coupling constant λ is usually defined as the value of the four-point vertex function $\Gamma^{(4)}(k_i)$ evaluated at the symmetric point $k_ik_j = m^2/3(4\delta_{ij}-1)$. This definition is not suitable in our case since the potential involves zero mass particles. Instead we arbitrarily define the renormalized coupling constant λ by

$$\Gamma^{(4)}\left(-\frac{\varkappa^2}{\Lambda^2},\lambda_0\right) = -i\lambda$$

or to second order in λ

$$\lambda = Z\left(\lambda_0,\frac{\Lambda}{\varkappa}\right)\lambda_0 \quad ,$$

$$Z\left(\lambda_0,\frac{\Lambda}{\varkappa}\right) = 1 + \frac{2\lambda_0}{(4\pi)^2}\left(\ln\left(\frac{\varkappa}{\Lambda}\right) - 1\right) . \qquad (A.1)$$

Hence, to order λ^2, the renormalized potential is given by

$$\lambda^2\tilde{I}(q^2) = -\frac{i\lambda^2}{(4\pi)^2}\ln\left(\frac{-q^2-i0}{\varkappa^2}\right) . \qquad (A.2)$$

b) Fourier transform of the potential

The Fourier transform of $\tilde{I}(q^2)$ may be written in the form

$$i(4\pi)^2 I(x) = \int_{-\infty}^{\infty} \frac{ds}{2\pi} \Delta^{(1)}(x,s) \ln\left(-\frac{s-i0}{\varkappa}\right) \tag{A.3a}$$

where

$$\Delta^{(1)}(x,s) = \frac{is}{8\pi}\left(\frac{H^{(1)}(\sqrt{sx^2+i0})}{\sqrt{sx^2+i0}} + \frac{H^{(1)}(\sqrt{sx^2-i0})}{\sqrt{sx^2-i0}}\right)$$

$$= -\left(\theta(s)\theta(-x^2) + \theta(-s)\theta(x^2)\right)\frac{s}{2\pi^2}\frac{K_1(\sqrt{-sx^2})}{\sqrt{-sx^2}} \tag{A.3b}$$

$$-\left(\theta(s)\theta(x^2) + \theta(-s)\theta(-x^2)\right)\frac{s}{4\pi}\frac{N_1(\sqrt{sx^2})}{\sqrt{sx^2}}$$

$N_1(z)$ and $K_1(z)$ being the Neumann and modified Bessel functions, respectively. We define $i(4\pi)^2 I(x)$ in terms of the analytic continuation of the integral

$$\int_{-\infty}^{\infty} \frac{ds}{2\pi} |s|^{\nu-1} \Delta^{(1)}(x,s) \ln\left(-\frac{s+i0}{\varkappa^2}\right)$$

from $0 < \nu < \frac{1}{2}$ to $\nu = 1$; thus, using Eqs. (A.3), we have

$$i(4\pi)^2 I(x) = \lim_{\nu\to 1} \frac{2}{x^2}\left\{\theta(x^2)\mathcal{F}_\nu(|x^2|, \varkappa^2) + \theta(-x^2)\mathcal{F}_\nu(|x^2|, -\varkappa^2)\right\}$$

where

$$\mathcal{F}_\nu(|x^2|, \varkappa^2) = \frac{1}{4\pi^3}\frac{1}{|x^2|}\int_0^{\infty} dz\, z^\nu K_1(z) \ln\left(\frac{z}{\varkappa\sqrt{|x^2|}}\right)$$

$$-\frac{1}{8\pi^2}\frac{1}{|x^2|}\int_0^{\infty} dz\, z^\nu N_1(z) \ln\left(\frac{-z}{\varkappa\sqrt{|x^2|}}\right) .$$

$F_\nu(|x^2|,\varkappa^2)$ is readily evaluated to be

$$\mathcal{F}_\nu(|x^2|,\varkappa^2) = -\frac{2^\nu}{8\pi^3}\Gamma(1+\tfrac{\nu}{2})\Gamma(\tfrac{\nu}{2})\left(1-\cos(1-\tfrac{\nu}{2})\pi\right)\frac{\ln(\varkappa\sqrt{|x^2|})}{|x^2|}$$

$$-\frac{i}{2\pi^2}\Gamma(1+\tfrac{\nu}{2})\Gamma(\tfrac{\nu}{2})\frac{1}{|x^2|} .$$

Continuing analytically this result to $\nu = 1$, we find

$$I(x) = -\frac{1}{(2\pi)^4}\frac{1}{(x^2-i0)^2} . \tag{A.4}$$

This result is independent of the renormalization point, a property intimately related to the specific analytic regularization procedure chosen. It is therefore clear that the method of analytic continuation only defines the Fourier transform up to an arbitrary function whose support is the origin. Conversely, the Fourier transform of (A.4) is only determined up to an arbitrary constant, which is essentially our subtraction constant $\varkappa$.

In order to make this more explicit we observe that in the terminology of Güttinger [13], $(x^2-i0)^{-2}$ is to be regarded as a generalized function with an algebraic singularity on the light cone. Following Ref. 13), the Fourier transform of $I(x)$ is given by

$$\mathcal{F}I(x^2-i0) = -i4\pi^2 \operatorname*{Res}_{z=0}\left[\frac{1}{z}\frac{1}{(-q^2-i0)^{\frac{1}{2}}}\int_0^\infty d\tau\, \tau^2 J_1\!\left(\tau(-q^2-i0)^{\frac{1}{2}}\right) I(-\tau^2-i0)\left(-\frac{b^2}{\tau^2}\right)^{z}\right] \tag{A.5}$$

replacing our formula (A.3). The constant b is arbitrary, reflecting the fact that the definition of the finite part of a singular integral is not invariant under scale transformations [13]. In fact, substituting (A.4) into Eq. (A.5) one finds [13]

$$\mathcal{F}I(x^2-i0) = -\frac{i}{(4\pi)^2}\left[\ln\left(\frac{b^2}{4}(-q^2-i0)\right) - 2\gamma - 1\right] . \tag{A.6}$$

The Fourier transform of (A.4) is thus defined only up to an arbitrary constant b, which is essentially our subtraction constant $\varkappa$.

APPENDIX B

Integral equation for $\rho(s)$

Replacing the potential in Eq. (2.1) by its discontinuity (see Fig. 4) (to start with we consider the exchange of massive quanta)

$$i^2 \int \frac{d^4k}{(2\pi)^4}\, \delta_+(k^2-\mu^2)\,\delta_+((q-k)^2-\mu^2) \tag{B.1}$$

and defining the variables

$$s = q^2 \,, \qquad s' = (q-k_1-k_2)^2$$

we arrive at the Volterra integral equation

$$\rho(s) = \rho_0(s) + a\int_0^{(\sqrt{s}-\mu)^2} ds' \frac{\Delta(s,s')\,\rho(s')}{(s'-m^2-i\varepsilon)(s'-m^2+i\varepsilon)} \,, \tag{B.2}$$

$$\rho_0(s) = \frac{1}{8\pi}\left(1-\frac{4\mu^2}{s}\right)^{\frac{1}{2}} ,$$

where

$$(2\pi)^2\Delta(s,s') = \int d^4k_1\, d^4k_2\, d^4Q\; \delta_+(k_1^2-\mu^2)\,\delta_+(k_2^2-\mu^2)\,\delta_+(Q^2-s')\,\delta^4(q-k_1-k_2-Q)$$

$$= \frac{1}{4}\,\frac{2\pi}{8s}\int_{9\mu^2}^{(\sqrt{s}-\mu)^2} ds_1 \int_{9\mu^2}^{(\sqrt{s}-\mu)^2} ds_2\; \theta(s_1+s_2-2\mu^2)\int_{-1}^{1} dx_1\, dx_2\, \frac{\theta(K(\Phi,x_1,x_2))}{\sqrt{K(\Phi,x_1,x_2)}}\,\theta(1-|\Phi|)$$

with

$$\Phi \equiv \Phi(s,s_1,s_2)$$

$$= \frac{2s(s_1+s_2-s-s') + (s-s_1+\mu^2)(s-s_2+\mu^2)}{\sqrt{\lambda(s,s_1,\mu^2)\,\lambda(s,s_2,\mu^2)}} \,.$$

$\Delta(s,s')$ is just the three-body phase space associated with particles of mass μ^2, μ^2 and s', respectively. Furthermore, K is the Mandelstam K function

$$K(x,x_1,x_2) = 1 - x_1^2 - x_2^2 - x^2 + 2xx_1x_2$$

and λ is the usual triangle function

$$\lambda(x,y,z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz$$

For the purpose of studying the high energy behaviour of $\rho(s)$ and $m^2(\partial/\partial m^2)\rho(s)$ we may set $\mu = 0$. The integrals may then be carried out in closed form. Setting $s_i/s = \lambda_i$, $s'/s = \lambda$, we have

$$(2\pi)^2\Delta = \frac{\pi^2}{4} s \int_0^1 d\lambda_1 \int_0^1 d\lambda_2\, \theta(1-\lambda_1-\lambda_2+\lambda)\,\theta(\lambda_1\lambda_2-\lambda)$$
$$= \frac{\pi^2}{8} s \left\{ \left(1 - \frac{s'^2}{s^2}\right) + 2\frac{s'}{s} \ln \frac{s'}{s} \right\} \quad . \tag{B.3}$$

In order to deduce the asymptotic behaviour of $\rho(s)$, it is convenient to rewrite Eq. (B.2) in terms of $\sigma(s) = \rho(s)/(s-m^2)^2$. After a change of variable one has (we set $\mu = 0$)

$$\sigma(s) = \sigma_0(s) + \frac{a}{32} \frac{1}{(1-\frac{m^2}{s})^2} \int_0^1 d\lambda\, h(\lambda)\, \sigma(\lambda s) \tag{B.4a}$$

where

$$h(\lambda) = (1-\lambda^2) + 2\lambda \ln \lambda \tag{B.4b}$$

and

$$\sigma_0(s) = \frac{1}{8\pi} \frac{1}{(s-m^2)^2}$$

To leading order we may neglect the factor $(1-m^2/s)^{-2}$. Taking the Mellin transform we obtain

$$\sigma(j) = \frac{\sigma_0(j)}{1 - aH(j)} \quad , \quad \sigma(s) = \int_C \frac{dj}{2\pi i}\, s^j \sigma(j)$$

where

$$\begin{aligned} H(j) &= \frac{1}{32}\int_0^1 d\lambda\, h(\lambda)\, \lambda^j \\ &= \frac{1}{16}\,\frac{1}{(j+1)(j+2)^2(j+3)} \quad . \end{aligned}$$

$\sigma(j)$ may be rewritten in the form

$$\sigma(j) = \frac{(j+1)(j+2)^2(j+3)}{[(j-2)^2-\nu_1^2][(j-2)^2-\nu_2^2]}\,\sigma_0(j) \tag{B.5}$$

where $\nu_{1,2}$ have already been defined in Eq. (2.14) ; or taking the inverse Mellin transform we have

$$\rho(s) \sim \left[c(\nu_2)s^{\frac{\nu_2}{2}} + c(-\nu_2)s^{-\frac{\nu_2}{2}} + c(\nu_1)s^{\frac{\nu_1}{2}} + c(-\nu_1)s^{-\frac{\nu_1}{2}}\right]\left(1+O\left(\tfrac{1}{s}\right)\right). \tag{B.6}$$

Now, $\hat{\sigma}(s) \equiv m^2(\partial/\partial m^2)\sigma(s)$ satisfies a Volterra integral equation with the same kernel as in Eq. (B.4a). It is a straightforward matter to show, using the above Mellin transform technique that

$$\frac{1}{\rho(s)}\, m^2 \frac{\partial}{\partial m^2}\, \rho(s) \xrightarrow[s\to\infty]{} r$$

with

$$r = \left.\frac{\hat{\sigma}(j)}{\sigma(j)}\right|_{j=j_0} = -\frac{\nu_2}{2} \tag{B.7}$$

where j_0 is the location of the zero of $1-aH(j)$ farthest to the right in the j plane. The result (B.7) could also have been deduced, of course, from pure dimensional analysis.

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FIGURE CAPTIONS

Fig. 1 : Non-overlapping rainbow diagrams contributing to the self-energy of the Φ field. The dotted lines refer to the zero-mass field φ.

Fig. 2 : Integral equation for the self-energy of the Φ field.

Fig. 3 : Production of pion pairs in the cascade decay of a virtual Φ meson.

Fig. 4 : Discontinuity of the potential $\tilde{I}(q)$.

Fig. 5 : Four-point function to second order in $\lambda\Phi^2\varphi^2$ theory.

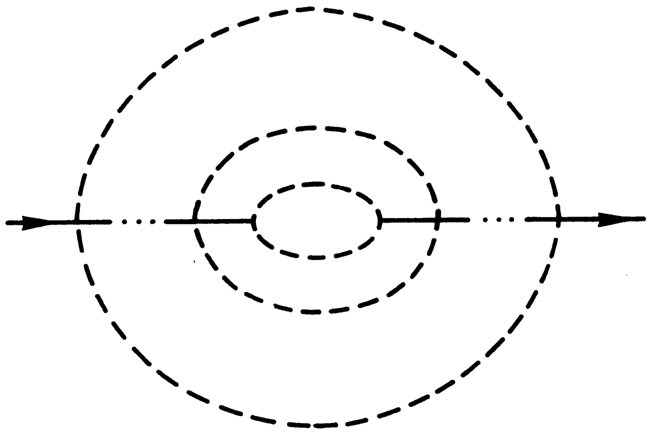

FIG. 1

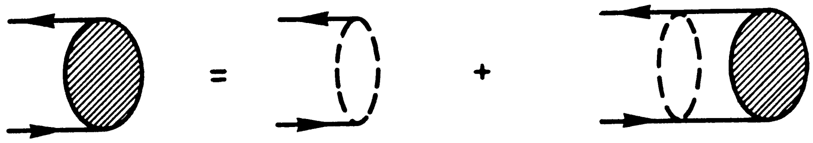

FIG. 2

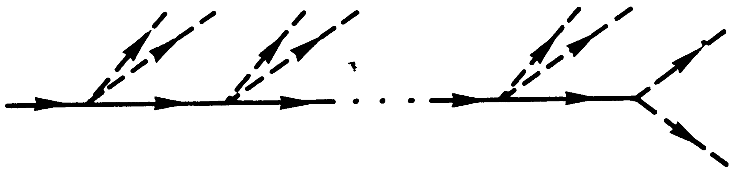

FIG. 3

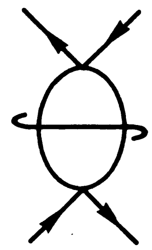

FIG. 4

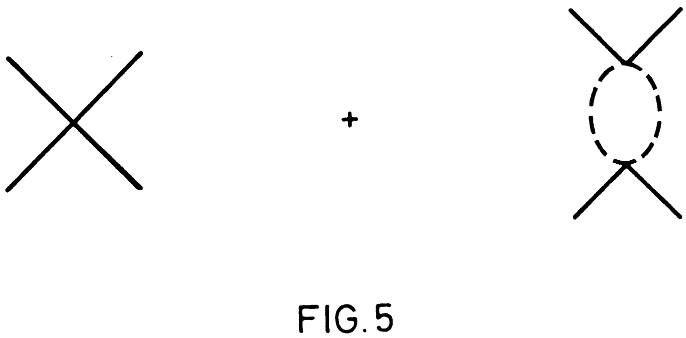

FIG. 5