Successive Screw Method in Three-dimensional Ising Lattice Gauge Theory

Hiroshi KATSUMORI and Shingo OHYA

Department of Applied Physics, Chubu University, Kasugai, Aichi 487, Japan

Kramers and Wannier's successive screw method is applied to the three-dimensional Ising lattice gauge system ($l \times m \times \infty$). A sparse and large dimension $2^{3lm-1} \times 2^{3lm-1}$ transfer matrix is to be dealt with. The partition function as the largest eigenvalue of the transfer matrix is directly calculated by iteration on a supercomputer with several GFLOPS of speed and several Gbytes of storage. This approach is entirely different from a computer simulation.

Anomalous behavior of the specific heat thus obtaind even for rather small values l and m implies the existence of a phase transition of the second order.

Considering the high performance of presently available supercomputers, we are tempted to solve the eigenvalue problem in the lattice gauge theory directly by a huge computer, in a quite different way from computer experiments such as Monte Carlo simulations.

By means of Kramers and Wannier's successive screw technique [1], the eigenvalue problems of the two- and three-dimensional Ising spin systems were previously solved on general purpose big computers [2][3]. In fact, a similar calculation for the three-dimensional Ising lattice gauge system has required a computer with faster speed and more storage [4].

Assuming a hypercubical lattice in d-dimensions with unit spacing, we compute a partition function in the case of a pure gauge feild,

$$Z = 2^{-Nd} \sum_{\{\mu_{ij}=\pm 1\}} \exp(K \sum_{\text{plaquettes}} \mu_{ij}\mu_{jk}\mu_{kl}\mu_{li}), \tag{1}$$

where N is the total number of sites, a variable $\mu_{ij}(=\pm 1)$ is assigned to each link (ij) of neighboring sites, a set of four neighboring links is a plaquette $(ijkl)$, and K is the coupling parameter which is inversely proportional to the temperature.

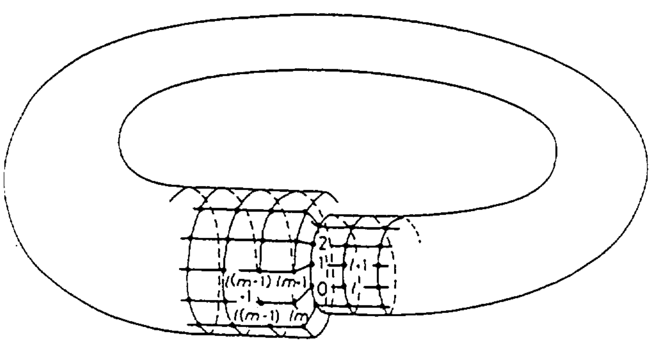

Fig.1. A part of the simple cubic lattice shows how each torus-like layer is continuously connected with the next layer.

Table I. The duality relation between pseudocritical parameters K_0 and K_0^* approximately holds.

$l \times m$	K_0	K_0^*	$\sinh 2K_0 \cdot \sinh 2K_0^*$
2×2	0.2886	0.6582	1.0555
2×3	0.2694	0.6681	1.0011
2×4	0.2626	0.6731	0.9847
3×2	0.2756	0.6675	1.0248
3×3	0.2577	(0.6774)	0.9740
3×4	0.2512	(0.6824)	0.9587
4×2	0.2698	0.6731	1.0142
4×3	0.2514	(0.6830)	0.9604
4×4	0.2451	(0.6880)	0.9453

In the three-dimensional Ising lattice gauge system, suppose that a multilayer torus consists of n layers. (Fig.1) Over the surface of each torus layer, lm sites are distributed along a continuous line twisting its way in a screw-like fashion. Each torus layer is continuously connected with the next torus layer, so that $lmn = N$ sites in all are distributed along a continuous line throughout a simple cubic lattice. As a periodic boundary condition, the $(n+1)$th layer is regarded as equivalent to the first layer. We eventually let $n \to \infty$.

Taking into account the Boltzmann exponentials, we obtain the eigenvalue equation from the figure,

$$M(K)A(K) = \lambda(K)A(K) \tag{2}$$

where $M(K)$ is the $2^{3lm} \times 2^{3lm}$ transfer matrix, which can be reduced to two $2^{3lm-1} \times 2^{3lm-1}$ irreducible matrices $V_{\pm}(K)$. Insofar as we discuss the thermal properties, we have only to find the largest eigenvalue of $V_+(K)$:

$$V_+(K) = \begin{pmatrix} f_1 & g_1 & f_2 & g_2 & g_1 & f_1 & g_2 & f_2 & \cdots \\ f_1' & g_1' & f_2' & g_2' & g_1' & f_1' & g_2' & f_2' & \cdots \\ g_2 & f_2 & g_1 & f_1 & f_2 & g_2 & f_1 & g_1 & \cdots \\ g_2' & f_2' & g_1' & f_1' & f_2' & g_2' & f_1' & g_1' & \cdots \\ g_4 & f_4 & g_3 & f_3 & f_4 & g_4 & f_3 & g_3 & \cdots \\ g_4' & f_4' & g_3' & f_3' & f_4' & g_4' & f_3' & g_3' & \cdots \\ f_3 & g_3 & f_4 & g_4 & g_3 & f_3 & g_4 & f_4 & \cdots \\ f_3' & g_3' & f_4' & g_4' & g_3' & f_3' & g_4' & f_4' & \cdots \end{pmatrix} \tag{3}$$

where f_i, g_i, f_i', g_i' stand for the submatrices, each row of which has only eight nonzero elements consisting of $e^{\pm K}$ and $e^{\pm 3K}$.

The largest eigenvalue λ_{max} of $V_+(K)$ is calculated by iteration and gives the partition function Z as

$$\lim_{N \to \infty} N^{-1} \ln Z = \ln \lambda_{max} \tag{4}$$

Then the specific heat is evaluated by

$$\frac{C_v}{R} = K^2 \frac{d^2 \ln \lambda_{max}}{dK^2} \tag{5}$$

Figure 2 shows the C_v/R versus K curves for the three-dimensional lattices $(l \times m \times \infty)$ for $lm \le 8$. The fact that a sharp peak in the specific heat has appeared even for rather small values of l, m, implies the existence of a phase transition in the three-dimensional Ising lattice gauge system. By means of an appropriate extrapolation, the maximum value of the specific heat for the lattices $(3 \times 3 \times \infty)$ and $(4 \times 4 \times \infty)$ are estimated as in Fig. 2.

The peak value of the specific heat linearly increases with the logarithms of l and m, which suggests the second order phase transition. (Fig. 3)

Combining the pseudocritical parameters K_0 and K_0^* calculated separately for the Ising spin system and for the Ising lattice gauge system, we see that the duality relation[5] approximately holds for rather small l and m. (Table I)

Then, conversely assuming the duality relation and using the data of the pseudocritical parameters for the Ising spin system $(l, m \le 12)$, we obtain Fig. 4. This figure shows how the pseudocritical parameter approaches to the bulk value, $K_c = 0.7613$, which is estimated from the Padé approximants value for the Ising spin system together with the duality relation.

As for the four-dimensional Ising lattice gauge system, the least lattice $(2 \times 2 \times 2 \times \infty)$ needs 32 Gbytes of storage. The calculation will be carried out soon.

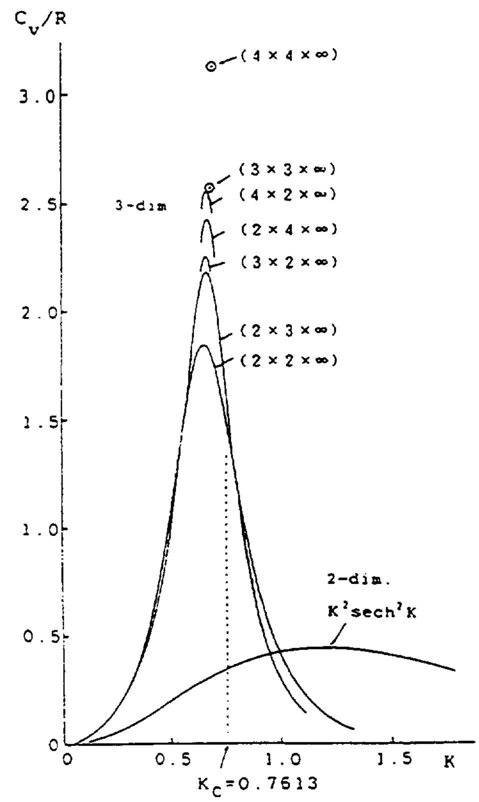

Fig.2. The specific heat versus inverse temperature curves are shown for the three-dimensional Ising lattice gauge system.

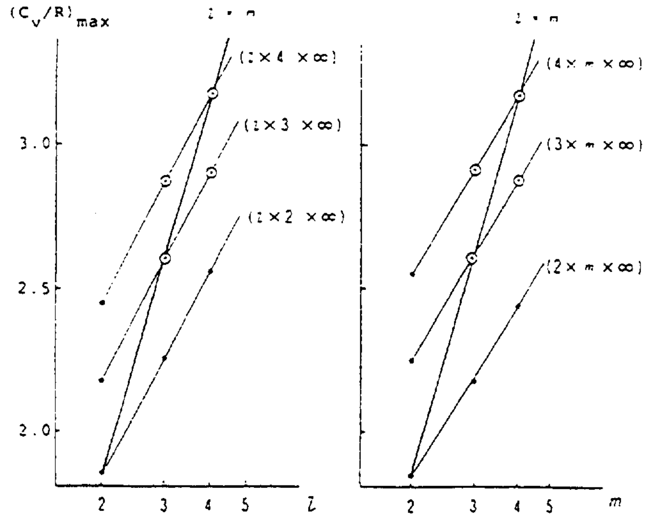

Fig.3. The peak value of the specific heat linearly increases with the logarithms of l and m.

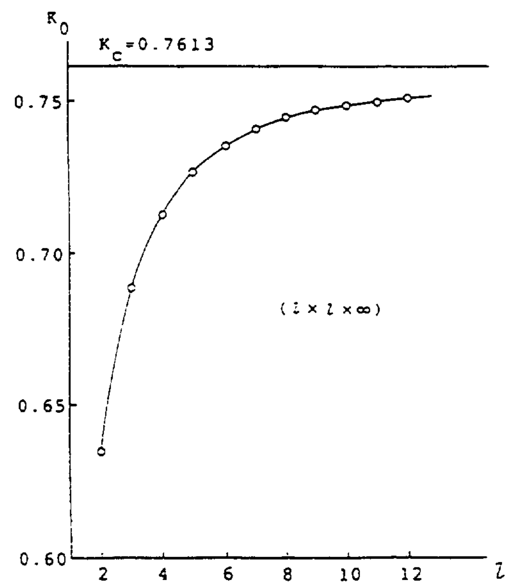

Fig.4. The pseudocritical parameter K_0 approaches to the bulk value K_c as $l = m$ increases.

References

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