EMITTANCE GROWTH OF INTENSE HEAVY ION BEAMS *

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ABSTRACT
We investigate growth of the emittance in intense, strongly coupled heavy ion beams. A set of coupled differential equations is developed which describe the dynamics of the beam's envelopes and the emittances. The emerging estimates of emittance growth are compared with numerical results from molecular dynamics computer simulations.

1. INTRODUCTION

In most applications of high-current beam transport, a good quality of the beam and thus low emittances are desired. So, in these cases one has to deal with both the strong coupling between the ions and space charge effects. The usual treatment describes the dynamics of the beam's spatial extension through envelope equations which require the emittances as input (usually treated as constant). We aim to develop an extension of the envelope equations which allows to describe the coupled dynamics of the envelope and of the emittance. To this end, it is necessary to include collisional effects which play an important role in beam transport at high phase-space densities. The derivation of the extended envelope equations is presented in the following steps:
We first recall the well-known envelope equations in their special form when the beam's cross section is elliptic. We then give a schematic outline of our model and derive a set of temperature-relaxation equations which extend the envelope equations. Finally, we derive formulas for the averaged emittance growth rates and compare the predictions of our model with results derived from molecular dynamics (MD) simulations.

2. COLLISIONLESS BEAM TRANSPORT

We investigate a beam in a periodic quadrupole channel. Here, the external transversal forces F_x and F_y are proportional to the displacements x and y from an ideal trajectory. We will derive all our results in the rest-frame of the beam. We parametrize all quantities by means of the elapsed time $t = \frac{s}{v_z}$, where s is the distance along the accelerator and v_z is the mean beam velocity. Usually, the beam's transversal dimensions vary rather slowly in z-direction of beam propagation. Thus we can assume that the external forces act uniformly on the beam over its whole length and are switched on and off periodically. In space-charge-dominated beams one describes the beam dimensions in terms of the second moments $\langle x^2 \rangle$ and $\langle y^2 \rangle$ where the brackets denote avaraging over the whole phase-space:

$$\langle x^2 \rangle = \frac{\int d^3r \int d^3v \, n(\mathbf{r},\mathbf{v}) x^2}{\int d^3r \int d^3v \, n(\mathbf{r},\mathbf{v})} \quad . \tag{1}$$

We further assume, that the beam's cross section is elliptical and that the density is homogeneous. We derive the equations of motion for $x_{\text{rms}} = \sqrt{\langle x^2 \rangle}$ and $y_{\text{rms}} = \sqrt{\langle y^2 \rangle}$ by the second time-derivative of Eq. (1) and inserting the equations of motion for the phase-space density $n(\mathbf{r},\mathbf{v})$. This yields the envelope equation [1]

$$\ddot{x}_{\text{rms}} = k_{\text{ext},x} x_{\text{rms}} + \frac{n_0 Z^2 e^2 x_{\text{rms},0} y_{\text{rms},0}}{\varepsilon_0 m} \frac{1}{x_{\text{rms}} + y_{\text{rms}}} + \frac{\varepsilon^2_{\text{rms},x}}{x^3_{\text{rms}}} \tag{2}$$

where $k_{\text{ext},x}$ describes the external forces, Ze and m are the ions charge and mass, n_0 is the beam density, and $x_{\text{rms},0}$ and $y_{\text{rms},0}$ are the x- and y-envelopes at $t = 0$. The quantity

$$\varepsilon_{\text{rms},x} = \sqrt{\langle x^2 \rangle \langle \dot{x}^2 \rangle - \langle x\dot{x} \rangle^2} \tag{3}$$

is the root-mean-square emittance of the beam in x-direction. An analogous definition exists for the y-direction. The emittances $\varepsilon_{\mathrm{rms},x}$ and $\varepsilon_{\mathrm{rms},y}$ of the beam remain constant if the total forces F_x and F_y acting on an ion are proportional to its displacements x and y. In that case the equations for the envelopes are complete and one can solve them numerically where one looks in particular for solutions which are 'matched', which means that they have the same periodicity as the focussing structure.

3. TEMPERATURE OSCILLATIONS

We consider a FODO-section, where the beam is focussed and defocussed alternatingly.

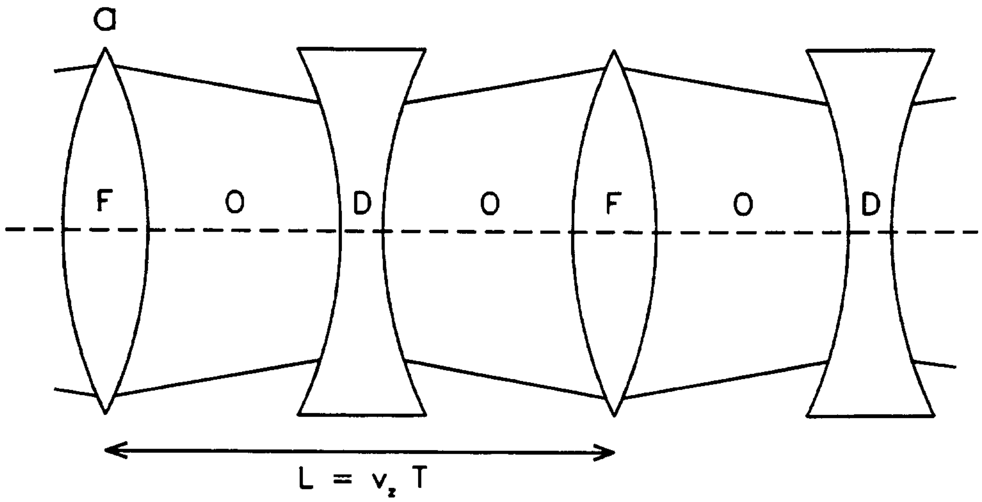

Figure 1: *Lens arrangement of a FODO-section in the x-z-plane. L is the length of one focussing period.*

Fig. 1 shows the arrangement of the lenses in the x-z-plane. A corresponding arrangement exists for the y-z-plane such that the beam is defocussed in y-direction when it is focussed in x-direction and vice versa. Now consider point a) in Fig. 1. There, the x-envelope is large and thus the y-envelope must be small. The phase-space volume in x-$\dot{x}$- and y-$\dot{y}$-space is approximately conserved, and consequently, the width of the phase-space density must be small in $\dot{x}$-direction and large in $\dot{y}$-direction, delivering a small kinetic energy in x-direction and a large one in y-direction. Relating a temperature to the kinetic energy, one can interpret this as a temperature anisotropy in x- and y-direction driven by beam guiding fields. Collisions between the ions drive towards thermal equilibrium, diminishing the anisotropy, increasing the entropy, and also increasing the emittance. The repeated deformation of the beam will then lead to a steady growth of the emittance.

4. TEMPERATURE-RELAXATION EQUATIONS

We define the local beam temperature in x-direction from the kinetic energy as

$$T_x(\mathbf{r}) = \frac{m}{k_B}\overline{(\dot{x} - \bar{\dot{x}}(\mathbf{r}))^2} \tag{4}$$

where the bar denotes averaging over the velocity space

$$\bar{\dot{x}}(\mathbf{r}) = \frac{\int d^3v\, n(\mathbf{r},\mathbf{v})\dot{x}}{\int d^3v\, n(\mathbf{r},\mathbf{v})} \quad . \tag{5}$$

In the case of linear external forces the correlation between $\bar{\dot{x}}(\mathbf{r})$ and x is linear and we can write

$$\bar{\dot{x}}(\mathbf{r}) = \frac{\langle x\dot{x}\rangle}{\langle x^2\rangle} x \quad . \tag{6}$$

In addition, we assume that the beam temperature varies little over $\mathbf{r}$ such that we can approximate the local temperature by its global value

$$T_x = \frac{\int d^3r\, n(\mathbf{r})T_x(\mathbf{r})}{\int d^3r\, n(\mathbf{r})} = \frac{m}{k_B}\frac{\langle x^2\rangle\langle \dot{x}^2\rangle - \langle x\dot{x}\rangle^2}{\langle x^2\rangle} = \frac{m}{k_B}\frac{\varepsilon^2_{\mathrm{rms},x}}{\langle x^2\rangle} \tag{7}$$

This establishes a relationship between beam temperature, emittance and envelope. As discussed above, there are two mechanisms which change the temperature:

1. envelope oscillations cause the temperature to oscillate with the squared inverse envelope as

$$\dot{T}_{x,\mathrm{osc}} = \frac{m}{k_B}\varepsilon^2_{\mathrm{rms},x}\frac{d}{dt}\frac{1}{\langle x^2\rangle} = -\frac{m}{k_B}\varepsilon^2_{\mathrm{rms},x}\frac{\langle x\dot{x}\rangle}{\langle x^2\rangle^2} \tag{8}$$

2. collisions diminish the temperature anisotropy. We describe this with a relaxation ansatz:

$$\dot{T}_{x,\mathrm{rel}} = -D\left[(T_x - T_y) + (T_x - T_z)\right] \tag{9}$$

where D is an inverse relaxation time.

The total time derivative of the temperature is thus $\dot{T}_x = \dot{T}_{x,\mathrm{osc}} + \dot{T}_{x,\mathrm{rel}}$. This is inserted into Eq. (7) and yields the time evolution for the emittance

$$\frac{d}{dt}\varepsilon^2_{\mathrm{rms},x} = -D\left[2\frac{\varepsilon^2_{\mathrm{rms},x}}{\langle x^2\rangle} - \frac{\varepsilon^2_{\mathrm{rms},y}}{\langle y^2\rangle} - \langle\dot{z}^2\rangle\right]\langle x^2\rangle \quad . \tag{10}$$

Analogous results are obtained for $\varepsilon^2_{\mathrm{rms},y}$ and $\langle\dot{z}\rangle^2$. The envelope equation (2) together with the emittance equation (10) and with the analgous equations in y- and z-directions form a set of coupled differential equations taking care of space-charge as well as collisional effects. The inverse relaxation time D is a parameter of the model. The dielectric theory predicts for weak coupling [2]

$$D \propto \omega_p \Gamma^{3/2}\ln\Lambda \tag{11}$$

where $\omega_p = \sqrt{\frac{n_0 Z^2 e^2}{\varepsilon_0 m}}$ is the plasma frequency and $\ln\Lambda \propto \ln 3 - \ln\Gamma$ is the Coulomb-logarithm. In practice, we adjust D to data from MD simulations.

5. AVERAGE EMITTANCE GROWTH

The total emittance $\varepsilon^6_{\mathrm{rms}} = \varepsilon^2_{\mathrm{rms},x}\varepsilon^2_{\mathrm{rms},y}\langle\dot{z}^2\rangle$, serves as a measure of total phase-space volume. Its time derivative is $\frac{d}{dt}\varepsilon^6_{\mathrm{rms}}$ as:

$$\begin{aligned}\frac{d}{dt}\varepsilon^6_{\mathrm{rms}} &= -D\varepsilon^6_{\mathrm{rms}}\left[\left(2 - \frac{\varepsilon^2_{\mathrm{rms},y}\langle x^2\rangle}{\varepsilon^2_{\mathrm{rms},x}\langle y^2\rangle} - \frac{\varepsilon^2_{\mathrm{rms},x}\langle y^2\rangle}{\varepsilon^2_{\mathrm{rms},y}\langle x^2\rangle}\right) + \left(2 - \frac{\langle\dot{z}^2\rangle\langle x^2\rangle}{\varepsilon^2_{\mathrm{rms},x}} - \frac{\varepsilon^2_{\mathrm{rms},x}}{\langle\dot{z}^2\rangle\langle x^2\rangle}\right)\right.\\ &\quad\left. + \left(2 - \frac{\langle\dot{z}^2\rangle\langle y^2\rangle}{\varepsilon^2_{\mathrm{rms},y}} - \frac{\varepsilon^2_{\mathrm{rms},y}}{\langle\dot{z}^2\rangle\langle y^2\rangle}\right)\right]\end{aligned} \tag{12}$$

The terms in the round brackets are each of the form $2 - \frac{1}{c} - c \leq 0$ and thus the total emittance is monotonously increasing,

$$\frac{d}{dt}\varepsilon^6_{\mathrm{rms}} \geq 0 \quad , \tag{13}$$

similar as the entropy. In order to estimate the average growth of the emittance, we approximate the beam's envelopes as

$$\langle x^2\rangle(t) = r^2\left(1 + \Delta\sin\frac{2\pi t}{T}\right) \tag{14}$$

$$\langle y^2\rangle(t) = r^2\left(1 - \Delta\sin\frac{2\pi t}{T}\right) \quad . \tag{15}$$

It turns out, that the emittance growth is dominantly driven by the oscillations of the envelopes. Integration of Eq. (12) over one period yields

$$\varepsilon^6_{\mathrm{rms}}(t_0 + T) = \varepsilon^6_{\mathrm{rms}}(t_0)\left(1 + 6DT\frac{1 - \sqrt{1-\Delta^2}}{\sqrt{1-\Delta^2}}\right) \quad . \tag{16}$$

And thus the average growth of the emittance can be written approximately as

$$\Delta\varepsilon_{\mathrm{rms}} = \varepsilon_{\mathrm{rms}}(t_0 + T) - \varepsilon_{\mathrm{rms}}(t_0) \approx DT\frac{1 - \sqrt{1-\Delta^2}}{\sqrt{1-\Delta^2}}\varepsilon_{\mathrm{rms}}(t_0) \quad . \tag{17}$$

6. NUMERICAL RESULTS

The results from MD-simulations for the envelopes and emittances in x- and y-direction are shown in Figs. 2 and 3. The emittances oscillate as discussed in section 3. For example, if the x-envelope is larger

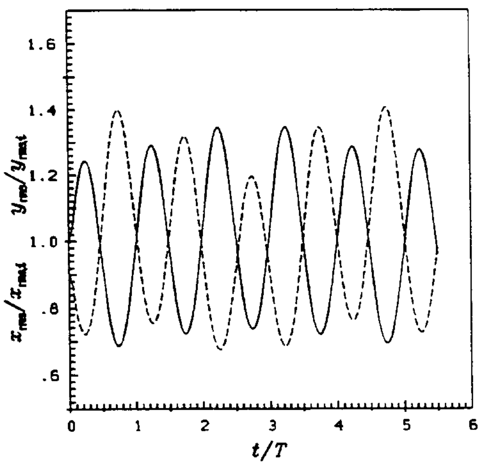

Figure 2: *Envelopes in x-direction (solid curve) and in y-direction (dashed curve).*

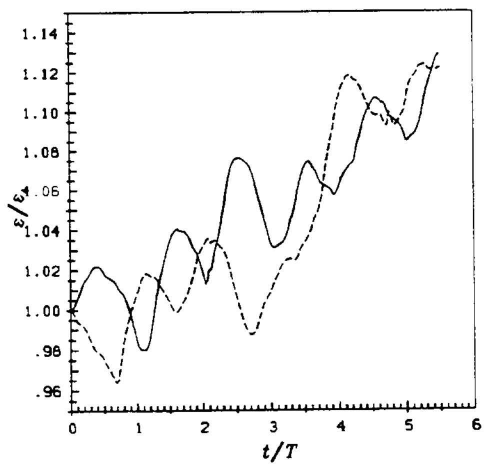

Figure 3: *Emittances in x-direction (solid curve) and in y-direction (dashed curve).*

than the y-envelope, then the emittance in x-direction increases because the temperature T_x is lower than the temperature T_y. Similar patterns are obtained by solving the extended envelope equations. Note that the emittance increases on the average because the relaxation process is irreversible. This is in accordance with the result of Eq. (17) from the extended envelope equations. The development of the emittance over a long distance of about 350 focussing structures is shown in Fig. 4, for the MD-simulation (solid curve) as well as for a solution of the extended envelope equations (dashed curve).

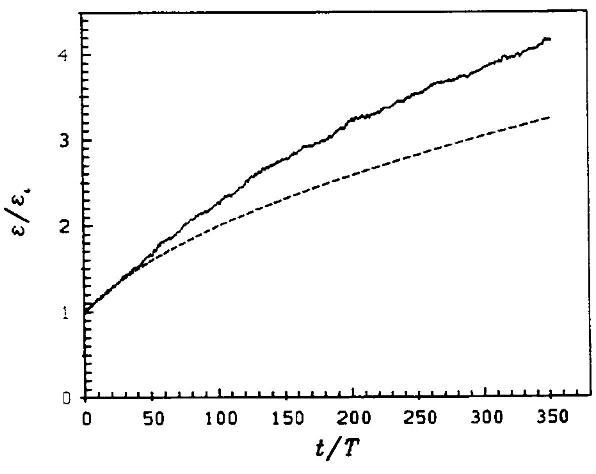

Figure 4: *Emittances drawn from MD-simulations (solid line) and from a solution of the extended envelope equations*

The beam parameters were identical in both calculations. The relaxation coefficient D in the extended envelope equations was fitted to reproduce the initial emittance growth of the MD simulation. The Fig. 4 shows that the temperature model provides an appropriate description of the general trend and order of magnitude, but it underestimates the emittance growth somewhat in comparison to the MD-simulation.

Finally, we want to discuss the dependence of the emittance growth rates on the plasma frequency ω_p and the plasma parameter Γ of the beam. According to Eqs. (17) and (11) one expects $\frac{d}{dt}\varepsilon_{\rm rms}/\varepsilon_{\rm rms}$ to be proportional to $\omega_p \Gamma^{3/2} \ln \Lambda$. We test this hypothesis against the results of full MD simulations.

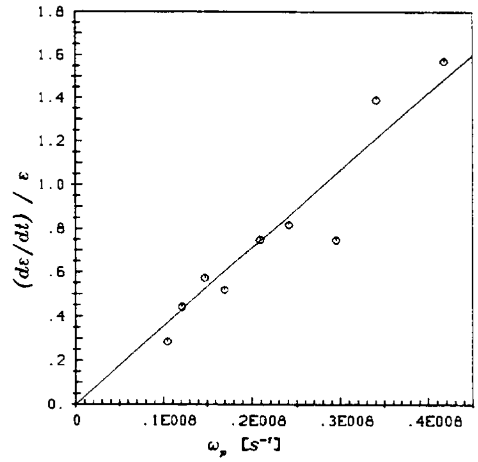

Figure 5: *Emittance growth rates versus plasma frequency (arbitrary units).*

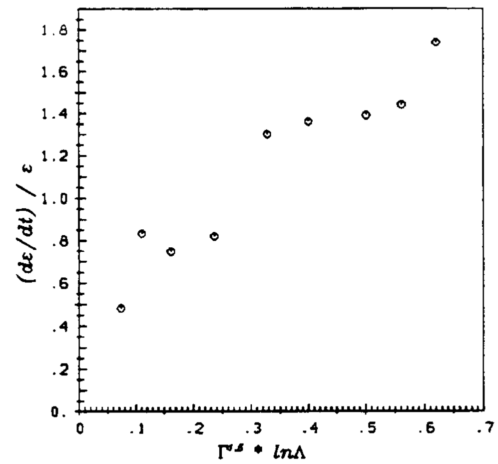

Figure 6: *Emittance growth rates versus plasma parameter (arbitrary units).*

The dependence of $\frac{d}{dt}\varepsilon_{\rm rms}/\varepsilon_{\rm rms}$ versus ω_p for fixed Γ is shown in Fig. 5. The linear dependence on ω_p as predicted by the dielectric theory is nicely visible. The dependence of $\frac{d}{dt}\varepsilon_{\rm rms}/\varepsilon_{\rm rms}$ on $\Gamma^{3/2}\ln\Lambda$ for fixed ω_p is shown in Fig. 6. The growth rate increases with increasing $\Gamma^{3/2}\ln\Lambda$, but there remains a non-zero intercept at $\Gamma^{3/2}\ln\Lambda = 0$ deviating from the dielectric prediction. The most probable reason is that the dielectric theory overestimates the relaxation coefficient D at high Γ. Moreover, our starting configuration for the MD-simulation (elliptical cylinder with sharp edge in ordinary space and Maxwellian in velocity space) may lead to a transient softening of the beam edge at high temperatures (i.e. low Γ) which causes an additional emittance growth.

7. CONCLUSION

We have shown that the envelope oscillations produce emittance growth in space-charge-dominated, strongly coupled heavy ion beams. The MD-simulation results confirm the prediction of our simple model based on a temperature anisotropy and equilibration.
The emittance growth rates increase with increasing Γ. Therefore, we think that this effect must be taken into account in attempts to produce crystalline ion beams with a more complicated structure than a linear chain.

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