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QCD_4 From A Five-Dimensional Point of View

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We propose a 5-dimensional definition for the physical $4D$-Yang–Mills theory. The fifth dimension corresponds to the Monte–Carlo time of numerical simulations of QCD_4. The 5-dimensional theory is a well-defined topological quantum field theory that can be renormalized at any given finite order of perturbation theory. The relation to non-perturbative physics is obtained by expressing the theory on a lattice, a la Wilson. The new fields that must be introduced in the context of a topological Yang–Mills theory have a simple lattice expression. We present a 5-dimensional critical limit for physical correlation functions and for dynamical auto-correlations, which allows new Monte–Carlo algorithm based on the time-step in lattice units given by $\epsilon = g_0^{-13/11}$ in pure gluodynamics. The gauge-fixing in five dimensions is such that no Gribov ambiguity occurs. The weight is strictly positive, because all ghost fields have parabolic propagators and yield trivial determinants. We indicate how our 5-dimensional description of the Yang–Mills theory may be extended to fermions.

1. Introduction

The 4-dimensional Yang–Mills theory seems to suffer from logical contradictions. In the continuum formulation, one has a gauge-fixed BRST invariant path integral, but one has the famous Gribov ambiguity for large gauge field configurations [1][2]. One often discards this problem, since the idea of defining the theory as a path integral of a gauge field can only be seriously advocated in perturbation theory or for semi-classical approximations. In the lattice formulation, which is by construction valid non-pertubatively, one chooses as variables the gauge group elements, but one has yet another contradiction. The way the continuum theory is approached is unclear, and if one tries to do a local gauge-fixing, the partition function vanishes [3]. This question is also often discarded, since for computing gauge-invariant quantities, one can factorize the volume of the gauge group that is finite on the lattice.

It is however frustrating not to have continuum and lattice formulations which would separately define both gauge-invariant and gauge non-invariant sectors, with a BRST symmetry controlling the correspondence of the theory to a physical sector, and with a natural limit from the lattice to the continuum formulation.

In this article, we will show that, in order to reconcile the continuum and lattice approaches, it is useful to define the theory in a 5-dimensional space, such that the 4-dimensional physical theory lives in a slice of this extended space. The theory that we will consider passes the tests that are obviously needed: in the continuous formulation it is perturbatively renormalizable by power counting without loosing its physical character, (due to Ward identities), and the gauge-fixing is no longer subject to the Gribov ambiguity; in the lattice formulation, all fields that one considers in the continuum formulation take their place, and one now obtains a formulation with a consistent gauge fixing.

The fifth-dimension will be the stochastic time that Parisi and Wu proposed long time ago for stochastically quantizing the Yang–Mills theory through a Langevin equation [4][5]. For reviews of stochastic quantization, see [6][7]. The 5-dimensional theory will be a supersymmetric theory of the topological type, that we can express as a path integral over 5-dimensional gauge fields for the continuum perturbative formulation, or as a lattice gauge theory which now depends on fields of a topological field theory. In one of the lattice formulations that we will present, the stochastic time is discretized. In the 5-dimensional critical limit controlled by $g_0 \to 0$, both the physical Euclidean correlation lengths and the dynamical Monte Carlo auto-correlation times diverge simultaneously. This allows new algorithms.

The link between topological field theory and stochastic quantization was first observed in [8]. The general idea is that the $D+1$-dimensional supersymmetric formulation of a D-dimensional quantum field theory that is obtained by expressing the field quantization by a Langevin equation, involves the fields of a topological field theory in $D+1$ dimensions. Then, supersymmetric cancellations wash away the detail of the theory in the bulk, while the relevant aspects of the physical theory are retained in some boundary of the $D+1$-dimensional space.

Moreover an extra gauge field component is needed to enforce the 5-dimensional gauge symmetry in the stochastic framework [9]. The latter can be identified as a potential for a drift force along the gauge orbits of the original D-dimensional gauge theory, as independently observed in [10][9][11]. A gauge-fixing drift force for stochastic quantization was actually introduced originally in [12], as a function of the 4-dimensional gauge fields A_μ. But the point is that, by promoting the potential for this force to a 5th field component, and by functionally integrating over all its possible values, [9], not only does one not alter the physical physical sector of the theory, but one softens the gauge condition of the D-dimensional quantum field theory. This actually gives the desired result that the Gribov ambiguity losses is relevance as an obstacle for the gauge fixing of the theory.

By postulating that QCD_4 should be considered from a 5-dimensional point of view, we will emphasize its relationship to a topological field theory which establishes a pleasant geometrical framework, both in the continuum and lattice descriptions.

Renormalizability of stochastically quantized scalar ϕ^4 theory was demonstrated in [13] using the BRST operator which encodes the supersymmetry of the stochastic process. Renormalizability of the 5-dimensional formulation of gauge theory was demonstrated in [14] using the BRST operator for gauge invariance supplemented by graphical arguments. In the present article we demonstrate renormalizability using the complete BRST operator that encodes both the supersymmetry of the stochastic process and the gauge symmetry of the theory, consistent with parabolic propagation of all ghost fields.

The disappearance of the Gribov ambiguity comes as an immediate consequence of being in five dimensions, in the context of renormalizable gauges (adapted to the power counting of this dimension). We will check this result by the verification that the ghost propagators cannot have zero modes. Alternatively, the mathematically oriented reader will notice that the argument of Singer [2] for having a Gribov copy phenomenon just disapears because the stochastic processes is defined on an infinite interval, even if we have a compact 4-dimensional physical space.

The consistency of our approach is ensured by the topological invariance, defined modulo ordinary gauge symmetry. We will also indicate that instantons in four dimensions are replaced by 5-dimensional solitons, but we leave open the question of anomalies, which is presumably connected to interesting 5-dimensional topological questions.

To make contact to non-perturbative physics, we will formulate the fields and the symmetry in a 5-dimensional lattice formulation. Not only this offers a new point of view for the topological symmetries, but this allows us to obtain a discrete lattice formulation with a BRST invariant gauge-fixing, offering thereby a new understanding of the continuous limit. It also gives a concrete definition of non-perturbative physics.

In secs. 2-5 we present the continuum 5-dimensional topological formulation. In sec. 6 we describe the lattice stochastic formulation including the lattice Fokker-Planck and Langevin equations, in which the 5th time $t = x_5$ is identified with the number of Monte Carlo sweeps. We describe a new Monte Carlo algorithm based on the 5-dimensional critical limit $g_0 \to 0$ which controls both physical correlation lengths and dynamical auto-correlation times, with time-step in lattice units given by $\epsilon \sim {g_0}^{-13/11}$ in gluodynamics. We conjecture that the physical spectrum may be given by the eigenvalues of the Fokker-Planck hamiltonian. In sec. 7 we present a 5-dimensional topological lattice formulation. In sec. 8 we indicate how our approach extends to fermions, and we conclude with some speculation on future developments. Readers who are primarily interested in Monte Carlo calculations may read sec. 6 independently.

2. The 5-dimensional continuum action

2.1. The Langevin equation

The Langevin equation proposed by Parisi and Wu was [4]:

$$\partial_5 A_\mu = D_\lambda F_{\lambda\mu} + \eta_\mu, \tag{2.1}$$

where η_μ is a Gaussian white noise. In the stochastic approach, the correlation functions are functionals of $A_\mu(x^\mu, x^5)$ that are the solutions of this equation. The 4-dimensional Green functions are obtained in the limit where all arguments in the stochastic time are equal, and $x^5 \to \infty$. Since we consider a stochastic process, we can as well take initial conditions at $x^5 \sim -\infty$, and define the Green functions at any given fixed time, for instance $x^5 = 0$. The convergence toward an equilibrium distribution relies on ergodicity theorems that accord with physical intuition for which the relaxation of a gas to the state of maximal entropy is obvious, whatever of the initial distribution of its constituents.

A difficulty with this Langevin equation is that it provides no restoring force along the gauge orbits, because the Euclidean action $S(A)$ is gauge-invariant. Consequently the probability escapes to infinity along the gauge orbits, and there is no normalizable equilibrium probability distribution. This may be remedied by modifying (2.1) by the introduction of a gauge-fixing term that is tangent to the gauge orbit [12] ,

$$\partial_5 A_\mu = D_\lambda F_{\lambda\mu} + D_\mu v[A] + \eta_\mu . \tag{2.2}$$

The gauge-fixing term $D_\mu v[A]$ has the form of an infinitesimal gauge transformation, and consequently it has no effect on the expectation-value of any gauge-invariant observable. The infinitesimal generator $v[A](x,t)$ may, in principle, be completely arbitrary, and the issue of a "correct" gauge fixing, without Gribov copies does not arise.

The last equation may be written in a way which respects gauge invariance in five dimensions. The idea is to identify $v[A]$ with an independent 5th field component

$$v = A_5(x,t) \tag{2.3}$$

which eventually be fixed in the functional integral by a gauge-fixing term in the action. Then the gauge-fixed Langevin equation (2.2) may be written as:

$$F_{5\mu} - D_\lambda F_{\lambda\mu} = \eta_\mu \tag{2.4}$$

where $F_{5\mu} = \partial_5 A_\mu - D_\mu A_5$. The relation

$$A_5 = a^{-1} \partial_\lambda A_\lambda \ . \tag{2.5}$$

will be imposed in the context of topological field theory by a 5-dimensional gauge condition.

We would like to obtain a functional integral representation for the Langevin equation (2.4). This is a non-trivial problem because there is a gauge invariance, and the determinant of the map which connects the A's and the noise η's has longitudinal zero modes. Instead of attempting to solve these questions by manipulations on the Langevin equation, we will pass directly to a 5-dimensional topological quantum field theory. Later with lattice regularization we shall verify the consistency of the 5-dimensional formulation, including the absence of Gribov copies, and we shall shall show that after elimination of auxiliary fields present in this theory, one recovers the Langevin equation and the equivalent Fokker-Planck equation.

2.2. The 5-dimensional BRST symmetry

The expression of the complete symmetry of the theory boils down to the knowledge of a BRST operator which encodes at once the supersymmetry of the stochastic process and the gauge symmetry of the theory. This operator is that of the topological symmetry of a gauge field in five dimensions, defined modulo gauge transformations. There is not much choice for exhibiting such a BRST operator, and the solution was given in [9]. It involves a fifth dimensional component for the gauge field, that we denote as A_5. Unavoidably, the vector fermion that is needed to express the Jacobian of the constraint (2.1) must be enlarged into a five dimensional vector (Ψ_μ, Ψ_5). If we denote by c the ordinary Faddeev–Popov ghost and Φ its ghost of ghost, the combined BRST operator for the stochastic supersymmetry and the gauge symmetry follows therefore from the following geometrical equation [9]:

$$\begin{aligned}(s+d)(A+c)+\frac{1}{2}[A+c,A+c]&=F+\Psi_\mu dx^\mu+\Psi_5 dx^5+\Phi\\(s+d)(F+\Psi_\mu dx^\mu+\Psi_5 dx^5+\Phi)&=-[A+c,F+\Psi_\mu dx^\mu+\Psi_5 dx^5+\Phi]\end{aligned}\tag{2.6}$$

The second equation is the Bianchi identity of the first, thus one has $s^2=0$ on all fields which ensures the consistency of this symmetry which mixes the Yang–Mills symmetry with the supersymmetry of the stochastic process. Here c, Ψ_μ and Ψ_5 are fermi ghost fields with ghost number $N_g=1$ and Φ is a bose ghost field with ghost number $N_g=2$. This equation gives, after expansion in ghost number:

$$\begin{aligned}sA_\mu&=\Psi_\mu+D_\mu c\\sA_5&=\Psi_5+D_5 c\\s\Psi_\mu&=-D_\mu\Phi-[c,\Psi_\mu]\\s\Psi_5&=-D_5\Phi-[c,\Psi_5]\\sc&=\Phi-\frac{1}{2}[c,c]\\s\Phi&=-[c,\Phi]\ .\end{aligned}\tag{2.7}$$

It is identical to the topological BRST symmetry for a Yang–Mills field in five dimensions.

We need anti-ghosts and lagrange multipliers, and their s-transformations. They are:

$$\begin{aligned}
s\bar{\Psi}_\mu &= b_\mu - [c, \bar{\Psi}_\mu] \\
s\bar{\Psi}_5 &= b_5 - [c, \bar{\Psi}_5] \\
sb_\mu &= -[c, b_\mu] + [\Phi, \bar{\Psi}_\mu] \\
sb_5 &= -[c, b_5] + [\Phi, \bar{\Psi}_5] \\
s\bar{\Phi} &= \bar{\eta} - [c, \bar{\Phi}] \\
s\bar{\eta} &= -[c, \bar{\eta}] + [\Phi, \bar{\Phi}] \ .
\end{aligned} \tag{2.8}$$

All fields in (2.7) and (2.8) are valued in the same Lie algebra representation as A. In the lattice formulation of this symmetry, it will be advantageous to redefine the anti-ghost and lagrange multiplier fields to eliminate the dependance on c in (2.8), that is, $s\bar{\Psi} = b',\ sb' = 0$, $s\bar{\Psi} = b',\ sb' = 0$, $s\bar{\Phi} = \bar{\eta}'$, and $s\bar{\eta}' = 0$.

There are five degrees of freedom for the choice of a dynamics, one for each component of A, with Lagrange multipliers fields b_μ and b_5. One of them will serve to gauge-fix the ordinary 5-dimensional gauge invariance, with the condition (2.5); the other four will enforce the Langevin equation (2.4), following the now standard methods of TQFTs. The gauge fixing of the longitudinal modes in Ψ will use $\bar{\eta}$ as a Lagrange multiplier.

2.3. The action

The complete action must be s-invariant. We will gauge fix the 5-dimensional invariant:

$$\int dx^\mu dx^5 \ \mathrm{Tr}(F_{5\mu} D_\nu F^{\mu\nu}) \tag{2.9}$$

This term is invariant under any local shift in A_μ and A_5, since the Lagrangian can locally be written as $\mathrm{Tr}\partial_\mu(F_5^\nu F_\nu^\mu) + \frac{1}{2}\partial_5(F_\mu^\nu F_\nu^\mu)$. Of course this means that we use special boundary conditions for the variations, and eventually on the Ψ's. The choice of this term may appear as quite intuitive: it enforces the idea that the stochastic time is unobservable, since it is independent of the metric component g_{55}, and it is compatible with Yang–Mills invariance. We will use an $s-$exact term for fixing all the invariances and introducing all relevant drift forces of stochastic quantization. Remarkably, the topological BRST symmetry just described is precisely what is needed to do so, that is, to represent a gauge theory in stochastic quantization by a functional integral with a high degree of symmetry [9].

To impose all relevant gauge conditions, we take the BRST-exact action:

$$
\begin{aligned}
I = \int dx^\mu dx^5 \; s \; \mathrm{Tr}\big(\bar\Psi^\mu(F_{5\mu} - D_\lambda F_{\lambda\mu} - \frac{1}{2} b_\mu) \\
+ \bar\Phi(a'\Psi_5 - D_\lambda \Psi_\lambda + \frac{\beta}{2}[\bar\eta, \Phi]) \\
+ \bar\Psi^5(a A_5 - \partial_\lambda A_\lambda)\big).
\end{aligned}
\tag{2.10}
$$

The consistency of this choice of this gauge function will be further justified by showing that it gives a perturbatively renormalizable theory. Here a, a' and β are arbitrary parameters. When β is non-zero, quartic interactions are introduced. Of course $\beta = 0$ gives simpler expressions, but nothing forbids $\beta \neq 0$, ($\beta = 0$ might be a stable fixed point under renormalization). [Alternatively one may make a linear gauge choice by setting $\beta = 0$ and taking $\partial_\lambda \Psi_\lambda$ instead of $D_\lambda \Psi_\lambda$.] The first two terms are invariant under 5-dimensional gauge transformations. The first term concentrates the path integral around the solutions of (2.4), modulo ordinary gauge transformations, while the second term fixes in a gauge-covariant way the internal gauge invariance of Ψ_μ that one detects in the BRST variation $s\Psi_\mu = D_\mu \Phi + ...$ The third term fixes the gauge invariance for A_μ, A_5. As we will see in section 3, all these terms are essentially determined by symmetry and power counting requirements.

We may introduce a background Yang–Mills symmetry, which transform all fields, but A, in the adjoint representation, while A transform as a gauge field. The shortest way to represent such a symmetry, is to define it through a background BRST symmetry, with generator σ and background ghost ω (which do not appear in the action). Since s and σ must anticommute, and $\sigma^2 = 0$, we easily find that we obtain both the action of s and σ by extending (2.6) into:

$$
\begin{aligned}
(s + \sigma + d)(A + c + \omega) + \frac{1}{2}[A + c + \omega, A + c + \omega] = F + \Psi_\mu dx^\mu + \Psi_5 dx^5 + \Phi \\
(s + \sigma + d)(F + \Psi_\mu dx^\mu + \Psi_5 dx^5 + \Phi) = -[A + c + \omega, F + \Psi_\mu dx^\mu + \Psi_5 dx^5 + \Phi]
\end{aligned}
\tag{2.11}
$$

When one expands the latter equation one must assign a new ghost number to σ and ω, which is independent of that carried by s and the propagating fields. This determines all transformations of s and σ on all fields, with $s^2 = \sigma^2 = s\sigma + \sigma s = 0$. One then observes that the first two terms in (2.10) are not only s-invariant as s-exact terms, but they are also σ-invariant, without being σ-exact. These terms involve gauge-covariant gauge functions for A and Ψ in five dimensions. The aim of the renormalization proof, using power counting, will be to show that these terms, up to multiplicative rescalings, are the only ones which satisfy both s and σ-invariances. The third term is not σ-invariant, and it will be necessary to prove that when the parameter a changes, this does not affect the sector of the theory defined by s and σ invariances.

To investigate the properties of our action, we must expand (2.10), which is a simple exercise. One gets:

$$
\begin{aligned}
I = \int dx^\mu dx^5 \ \mathrm{Tr}\Big(& -\frac{1}{2} b_\mu b_\mu + b_\mu (F_{5\mu} - D_\lambda F_{\lambda\mu}) \\
& + \bar{\Phi}(-a' D_5 \Phi + D_\mu D_\mu \Phi - [\Psi_\mu, \Psi_\mu]) - \frac{1}{2}\Phi[\bar{\Psi}_\mu, \bar{\Psi}_\mu] + \frac{\beta}{2}[\Phi, \bar{\Phi}][\Phi, \bar{\Phi}] \\
& + (b_5 - [c, \bar{\Psi}_5])(a A_5 - \partial_\mu A_\mu) \\
& - \bar{\Psi}_\mu (D_{[5}\Psi_{\mu]} - D_\lambda D_{[\lambda}\Psi_{\mu]} - [\Psi_\lambda, F_{\lambda\mu}]) \\
& + \bar{\eta}(a' \Psi_5 - D_\mu \Psi_\mu + \frac{\beta}{2}[\bar{\eta}, \Phi]) \\
& - \bar{\Psi}^5 (a(\Psi_5 + D_5 c) - \partial_\mu \Psi_\mu - \partial_\mu D_\mu c)\Big).
\end{aligned}
\tag{2.12}
$$

In order to see the dynamical content of this action, we identify and eliminate the auxiliary fields. The equation of motion of b_5 gives back the gauge-fixing condition that we wish to impose:

$$aA_5 = \partial_\lambda A_\lambda. \tag{2.13}$$

Moreover the equation of motion of b_μ has the same form as the Langevin equation,

$$F_{5\mu} - D_\lambda F_{\lambda\mu} = b_\mu. \tag{2.14}$$

We also eliminate Ψ_5 and $\bar{\eta}$ and we get,

$$a' \Psi_5 = D_\mu \Psi_\mu - \frac{\beta}{2}[\bar{\eta}, \Phi] \tag{2.15}$$

$$a' \bar{\eta} = a \bar{\Psi}_5 + D_\mu \bar{\Psi}_\mu. \tag{2.16}$$

Thus A_5 can be expressed as a function of A_μ, and Ψ_5 can be expressed as a function of Ψ_μ. This is the key of a gauge-fixing that does not suffer from the Gribov ambiguity, and the gauge condition $aA_5 = \partial_\lambda A_\lambda$ combines the virtues of axial and Laudau gauges. By integrating over all values of A_5, the Faddeev–Popov zero modes that one encounters in the genuine 4-dimensional theory just disappear. It is tempting to compare the integration over A_5 to the integration over moduli that also solves the problem of zero modes of reparametrization ghosts in string theory.

We will give much more details about the elimination of Gribov copies in sec. 7, although a Fourier transformation on the variable x^5 could help understanding directly how the multiple intersections of gauge orbit that occur in four dimensions could be split on different orbits in five dimensions.

Finally, after integrating out b_μ and b_5, we get:

$$\begin{aligned} I+\int dx^\mu dx^5\ \mathrm{Tr}(F_{5\mu}D_\lambda F^{\lambda\mu}) \sim \int dx^\mu dx^5\ \mathrm{Tr}\big(&-\frac{1}{2}F_{5_\mu}F^{5_\mu}-\frac{1}{2}D_\nu F^\nu_\mu D_\rho F^{\rho\mu} \\ &+\bar\Phi(a' D_5\Phi + D_\mu D^\mu\Phi) \\ &+\bar\Phi[\Psi_\mu,\Psi^\mu]) + \frac{1}{2}\Phi[\bar\Psi_\mu,\bar\Psi^\mu] + \frac{\beta}{2}[\Phi,\bar\Phi][\Phi,\bar\Phi] \\ &-\bar\Psi^\mu(D_5\Psi_\mu + D^\nu D_{[\mu}\Psi_{\nu]} + [\Psi^\nu, F_{\mu\nu}]) \\ &+a'^{-1}D_\mu\bar\Psi^\mu(D_\nu\Psi^\nu - \frac{\beta}{2}[\bar\Psi_5,\Phi]) \\ &-\bar\Psi^5(aD_5c + \partial^\mu D_\mu c + [A_\mu,\Psi_\mu])\ \big) \end{aligned} \tag{2.17}$$

One has $A_5 = a^{-1}\partial_\lambda A_\lambda$ in this action.

To derive Feynman rules, we examine the part of the resulting action that is quadratic in the fields. It is given by

$$I_0 = I_A + I_\Psi + I_\Phi + I_c \tag{2.18}$$

$$\begin{aligned} I_A &= \int dx^\mu dx^5 \mathrm{Tr}\big(\frac{1}{2}(\partial_5 A_\mu)^2 + \frac{1}{2}(\partial_\lambda\partial_\lambda A_\mu)^2 + \frac{1}{2}(a^{-2}-1)(\partial_\mu\partial_\lambda A_\lambda)^2\big) \\ I_\Psi &= \int dx^\mu dx^5 \mathrm{Tr}\big(\bar\Psi_\mu(-\partial_5\Psi_\mu + \partial_\lambda\partial_\lambda\Psi_\mu + (a'^{-1}-1)\partial_\mu\partial_\lambda\Psi_\lambda)\big) \\ I_\Phi &= \int dx^\mu dx^5 \mathrm{Tr}\big(\bar\Phi(-a'\partial_5\Phi + \partial_\mu\partial_\mu\Phi)\big) \\ I_c &= \int dx^\mu dx^5 \mathrm{Tr}\bar\Psi^5\big(-a\partial_5 c + \partial_\mu\partial_\mu c\big) \end{aligned} \tag{2.19}$$

where we have dropped exact derivatives in I_A.

An important observation is that all ghost-antighost pairs, $(\Psi_\mu, \bar\Psi_\mu)$, $(\Phi,\bar\Phi)$ and $(c,\bar\Psi_5)$ have a free action which is parabolic: it is first order in ∂_5, and the corresponding matrix of spatial derivatives is a negative operator. Consequently all ghost propagators are retarded, $D(x,t) = 0$ for $t = x_5 < 0$. For example for $a = 1$, all ghost-antighost pairs have the momentum-space propagator $(i\omega + k^2)^{-1}$, with fourier transform

$D(x,t) = \theta(t)(4\pi t)^{-2}\exp(-x^2/4t)$, where $\theta(t)$ is the step function. Moreover ghost number is conserved, and an arrow which represents the flow of ghost charge may be assigned to each ghost line: it points from the past to the future. On the other hand, the action of the A-field is second order in ∂_5, so A lines freely move forward and backward in time. (Its momentum-space propagator is $\Big(\omega^2 + (k^2)^2\Big)^{-1} = (2k^2)^{-1}\Big((i\omega + k^2)^{-1} + (-i\omega + k^2)^{-1}\Big)$.) Ghost number is also conserved at every vertex. Consequently in a diagrammatic expansion, every ghost line may be followed from the point where it enters a diagram from the past, as it moves monotonically from the past to the future, until it exits into the future. (The transformation of a Φ into two Ψ's and back is allowed.) It follows that in a diagram containing only external A-lines, there can be no closed ghost loops apart from tadples. With dimensional regularization the tadpole diagrams vanish, and in this representation of gauge theory, ghost diagrams provide accounting checks on the renormalization constants that are expressed in Ward identities, as usual, but they do not appear in the expansion of correlation functions the A-field. With lattice regularization, the tadpole diagrams yield the famous Ito term.

Another way to see that the ghost determinants have no Gribov problems is to note that the initial-value problem for a parabolic equation has a unique solution. In the section devoted to lattice regularization, we will indeed integrate out the ghosts exactly, and explicitly verify the absence of Gribov copies.

We now turn to the question of understanding the renormalization of this action.

3. Perturbative renormalization

The action (2.12) is renormalizable by power counting. Indeed, the form of the propagators implies that the canonical dimension of all fields with an index 5 is equal to 2, while other fields have dimensions 1. It follows that we start from an action where the coupling constant has dimension 0, and we can apply the general result that it can be renormalized order by order in perturbation theory by a finite number of counter-terms.

The main question is of course to investigate the structure of the counter-terms. We will only sketch the demonstration that this action is actually stable under renormalization, which means that the generating functional of Green functions can be defined, order by order in perturbation theory, while satisfying the same Ward identities as the action (2.10), corresponding to its invariance under s-symmetry, $SO(4)$-symmetry, ghost number conservation, and to the choice of a linear condition $aA_5-\partial_\mu A_\mu$ in the sector which violates the σ-invariance.

The action (2.12) that determines perturbation theory can indeed be be split in two parts:

$$
\begin{aligned}
I \sim &\hat{I}(A_\mu, A_5, \Psi_\mu, \bar{\Psi}_\mu, b_\mu, \Psi_5, \Phi, \bar{\Phi}, \bar{\eta}) \\
&+ \Big(b_5(aA_5 - \partial_\mu A_\mu) - \bar{\Psi}^5(a\partial_5 c - D^\mu\partial_\mu c + a\Psi_5 - \partial_\mu\Psi_\mu)\Big).
\end{aligned}
\tag{3.1}
$$

The first term, where the fields A_5 and Ψ_5 must be understood as independent fields, (before eliminating b_5), is s- and σ-invariant, while the last part is only s-invariant. In order to achieve the algebraic set-up which ensures perturbative renormalizability while maintaining the form of the action (up to so-called multiplicative renormalization), it is enough to remark that one can write the second term as:

$$
\begin{aligned}
&b_5(aA_5 - \partial_\mu A_\mu) - \bar{\Psi}^5(a\partial_5 c - D^\mu\partial_\mu c - a\Psi_5 - \partial_\mu\Psi_\mu) \\
&\qquad =b_5(aA_5 - \partial_\mu A_\mu) - \bar{\Psi}^5(aD_5 - \partial^\mu D_\mu)c'
\end{aligned}
\tag{3.2}
$$

where $c' = c+(a\partial_5-D^\mu\partial_\mu)^{-1}(a\Psi_5-\partial_\mu\Psi_\mu)$. The change of variable $c \to c'$ is perturbatively well defined, with a trivial Jacobian, because the operator $a\partial_5 - D^\mu\partial_\mu$ is parabolic. (This will be explained in detail in section 7, devoted to the lattice regularization.)

If we now introduce an ordinary BRST symmetry s', with c' as a Faddeev–Popov ghost, we are almost in the ordinary situation of the renormalization, except that we have more fields. We have that s' acts on $A_\mu, A_5, \Psi_\mu, \bar{\Psi}_\mu, b_\mu, \Psi_5, \Phi, \bar{\Phi}, \bar{\eta}$ as ordinary gauge transformations with the parameter equal to c', $s'c' = -c'c'$, $s'\bar{\Psi}_5 = b_5$ and $s'b_5 = 0$. The action (3.2) can be written as:

$$
\begin{aligned}
I \sim &\hat{I}(A_\mu, A_5, \Psi_\mu, \bar{\Psi}_\mu, b_\mu, \Psi_5, \Phi, \bar{\Phi}, \bar{\eta}) \\
&+ s'\big(\bar{\Psi}^5(aA_5 - \partial_\mu A_\mu)\big)
\end{aligned}
\tag{3.3}
$$

The first term is s'-invariant as a consequence of its σ-invariance: it corresponds to the cohomology with ghost number zero of s', restricted to the terms with the correct power counting (6 with our power assignmant), while the second term is s'-exact, and will remain stable order by order in perturbation theory because of the linear gauge condition $aA_5 - \partial_\mu A_\mu$. Notice that $\hat{I}$ depends on a', but A_μ-dependent observables are independent of this parameter, as will be shown in sec. 7.

The complete proof that the theory can be renormalized multiplicatively is just a slight amplification of the standard proof of the renormalizability of the 4-dimensional theory in a linear gauge. It involves introducting sources for all s' variations, as well as using the equation of motion of the antighost $\bar{\Psi}_5$ as a Ward identity. When the Ward identity of the s'-invariance is combined with that of the topological s-invariance, using locality, one can prove from purely algebraic considerations that the counter-terms can be built order by order in perturbation theory such that, when they are added to the starting action (2.10), they provide an identical action, up to mere multiplicative renormalization constants for the fields and parameters. Essential is in this proof is the use of a linear gauge function $aA_5 - \partial_\mu A_\mu$.

A notable feature of the renormalization is that besides the multiplicative renormalization of the fields and coupling constants there is also a renormalization of the stochastic time

$$t = Z_t t_r \ . \tag{3.4}$$

4. Observables and gauge invariance

It is now clear how observables should be defined: they are the cohomology with ghost number zero of s'; moreover, they must be computed at a same value of x^5. Using the translation invariance under x^5, and provided that the stochastic process starts at $x^5 = -\infty$, the physical theory can be defined in any given slice of the 5-dimensional manifold. This points out the relevance of the boundary term (2.9), which is independent of the metric component g_{55}. Perturbatively, the equal-time gauge-invariant correlation functions are guaranteed to agree with those calculated in 4-dimensional theory by the usual Faddeev-Popov method. On the other hand, gauge non-invariant correlation functions are not expected to agree within the framework of local renormalizable 5-dimensional gauges that we consider in this paper [1].

There are also Green functions that can be computed with fields at different values of x^5, as well as the Green functions that involves the topological ghosts. We leave their interpretation as an interesting open question, although they are not directly relevant to physics.

Once the renormalization has been done properly, one can integrate out all the ghost fields of the type Ψ, Φ and c, provided that one considers observables which depend on

[1] To recover the local 4-dimensional Faddeev–Popov distribution, one needs a non-local 5-dimensional gauge-fixing on A_5 [15].

A_μ only. This follows from the parabolic or retarded behavior of all ghost propagators in the 5-dimensional theory. It will be demonstrated by explicit calculation with lattice regularization in section 7 that, when the ghost fields are integrated out, only the tadpole diagram survives so the ghost determinent is trivial and contributes only a local term to the effective action which is in fact the famous Ito term. This result, which explains the disappearance of the problem of Gribov copies, has an immediate application: mean values of observables are independent of the parameter a'. It should be clear however that the ghosts and ghosts of ghosts are nevertheless necessary to unveil the topological and gauge properties of the theory and to control its Ward identities.

The other question is obviously that of the independence of physical expectation values of the gauge parameter a which appears in the gauge function $aA_5 - \partial_\mu A_\mu$. To prove this, one observes that the observables are s'-invariant without being s'-exact, while the a-dependence is through an s'-exact term, so the standard BRST method based on Ward identities applies. One can also prove directly the non-renormalization of the s'-exact term in (3.3).

We mention the possibility of introducing the interpolating gauges used in [16] for defining the "physical" Coulomb gauge as the limit of a renormalizable gauge.

Finally, one must of course verify that no anomaly in the s- and σ-symmetries can occur. The absence of s-anomalies is quite obvious if one examines the consistency conditions for a topological symmetry; for the background Yang–Mills symmetry, no anomaly is expected, since we are in the case of a pure Yang–Mills theory, without 4-dimensional chiral fermion. The inclusion of spinors is a most interesting question, on which we will comment in the last section.

5. Non-perturbative aspects

In the previous section, we have verified that perturbation theory is as well defined in the 5-dimensional approach as it is in the 4-dimensional one.

5.1. Global properties of gauge-fixing

The gauge choice made here has the global property of a restoring force because it is derivable from a "minimizing" functional. Consider the functional

$$\mathcal{F}[A] = (2a)^{-1}(A, A) = (2a)^{-1} \int d^4x A^2, \tag{5.1}$$

which is proportional to the Hilbert square norm. Here $a > 0$ is a gauge parameter. (More generally, one can take $\mathcal{F}[A] = \int d^4x A_\lambda \alpha_{\lambda\mu} A_\mu$, where α is a strictly positive symmetric matrix that characterizes a class of interpolating gauges [16].) The gauge condition used here, $aA_5 = \partial_\mu A_\mu$, may be expressed as

$$A_5(x) = -G(x)\mathcal{F}, \tag{5.2}$$

where

$$G(x) \equiv -D_\mu \frac{\delta}{\delta A_\mu(x)} \tag{5.3}$$

is the generator of local gauge transformations. It satisfies the Lie algebra commutation relations

$$[G^a(x), G^b(y)] = \delta(x-y) f^{abc} G^c(x) \ . \tag{5.4}$$

Indeed, one can verify that

$$G(x)\mathcal{F} = -a^{-1}\partial_\mu A_\mu(x). \tag{5.5}$$

To see that the gauge-fixing force is globally restoring, consider the flow defined by the gauge-fixing force alone

$$\partial_5 A_\mu = D_\mu v = -D_\mu G\mathcal{F}, \tag{5.6}$$

in the Langevin equation (2.2). Under this flow the minimizing functional $\mathcal{F}$ decreases monotonically, since

$$\begin{aligned} \partial_5 \mathcal{F}[A] &= (\frac{\delta \mathcal{F}}{\delta A_\mu}, \partial_5 A_\mu) = -(\frac{\delta \mathcal{F}}{\delta A_\mu}, D_\mu G\mathcal{F}) \\ &= -(G\mathcal{F}, G\mathcal{F}) \leq 0. \end{aligned} \tag{5.7}$$

A consequence of our gauge choice is that the Langevin equation (2.4) is parabolic. Indeed, let $A_5 = a^{-1}\partial_\mu A_\mu$ be substituted into this equation, and consider the highest derivatives on the right-hand side,

$$\frac{\partial A_\mu}{\partial t} = \partial_\lambda \partial_\lambda A_\mu + (a^{-1} - 1)\partial_\mu \partial_\lambda A_\lambda + \dots \ , \tag{5.8}$$

where the dots represent lower order derivatives. One sees that for $a > 0$, the operator that appears on the right is negative, which assures convergence at large t.

5.2. How the Gribov problem is solved

Let us now discuss the Gribov question in more detail, and explain how our use of the gauge function $aA_5 - \partial_\mu A_\mu$ escapes the ambiguity.

The origin of this ambiguity in the 4-dimensional formulation, as given by Singer [2], was the compactness of the (Euclidian) space. It does not apply anymore, since the 5-dimensional formulation is necessarily non-compact: the system requires an infinite amount of time to relax to equilibrium. In our formulation the stochastic time runs from $-\infty$ to ∞, and one evaluates the Green function at an arbitrary intermediate time, say $x^5 = 0$.

Then, there is the explicit argument, which we already mentioned, of considering all ghost propagators, whether they are the Ψ and Φ topological ghosts or the Faddeev–Popov ghost: by taking the suitable boundary conditions at $x_5 = -\infty$, no zero modes can occur since we have parabolic propagation. It is a convincing argument, and we will check it in great detail in section 7, in the framework of the lattice regularization of our 5-dimensional formulation. We will prove that the integration on all ghosts does not lead to zero modes that would make ambiguous their elimination. In our opinion, this a concrete verification of the assertion that the Gribov problem is solved.

To see intuitively where the gauge-fixing force concentrates the probability, observe from (5.7) that the gauge fixing force is in equilibrium only where $G(x)\mathcal{F} = 0$. This equilibrium may be stable or unstable, but the gauge fixing force drives the system toward stable equilibrium only, namely where the second variation of $\mathcal{F}[A]$ under infinitesimal gauge transformation is a positive matrix. This is equivalent to the condition that the Faddeev-Popov matrix

$$M^{ab}(x, y) = G^a(x)\ G^b(y)\mathcal{F} \tag{5.9}$$

has positive eigenvalues. With $G\mathcal{F} = -a^{-1}\partial_\mu A_\mu$, the matrix M and the condition for stable equilibrium are given by

$$M = -a^{-1}\partial_\mu D_\mu \geq 0. \tag{5.10}$$

Thus the conditions for stable equilibrium are (a) transversality of the vector potential, $\partial_\mu A_\mu = 0$ and (b) positivity of the Faddeev–Popov operator, $-\partial_\mu D_\mu \geq 0$. These two conditions define the Gribov region, a region which Gribov suggested (not quite correctly) was free of copies. As long as the gauge-parameter $a > 0$ is finite, the gauge-fixing is soft in the sense that the weight on each gauge orbit has a spread that is centered on the Gribov region. As a approaches 0, the equilibrium probability gets concentrated close to the Gribov region. The gauge-fixing force drives the system by steepest descent along a gauge orbit toward a relative or absolute minimum of the functional $\mathcal{I}_{[A]}[g] \equiv \mathcal{F}[{}^gA]$, so Gribov copies that are unstable equilibria (saddle points of $\mathcal{I}_{[A]}[g]$) are avoided in the limit $a \to 0$. However the relative minima are in fact Gribov copies of the absolute minimum, so in the limit $a \to 0$ this gauge-fixing distributes the probability in some way among these relative minimum, possibly even entirely at the absolute minimum. However the validity of the present approach in no way depends on taking the limit $a \to 0$.

It is interesting to ask how instantons play a role in the 5-dimensional presentation, for instance in a semi-classical approximation. Their interpretation is actually quite simple. The bosonic part of our action is a sum of squares, namely $|F_{5\mu}|^2 + |D_\lambda F_{\lambda\mu}|^2$. It obviously gives and an absolute minimum when it vanishes. This occurs for x^5-independent solutions, with $A_5 = \partial_\mu A_\mu = 0$ and satisfying $D_\lambda F_{\lambda\mu} = 0$. This means that the classical 4-dimensional instanton solutions also minimize the 5-dimensional path integral: they are the solitons of the 5-dimensional theory.

We are therefore in a situation where, from various points of view, the path integral seems well defined. We can now seriously consider its evaluation for "large" field configurations, since the question of duplicating orbits does not occur. It is however clear that the space of connections is not appropriate for defining a meaningful measure, and we will shortly look at the lattice formulation of this extended version of the Yang–Mill theory, keeping in mind that the use of the fifth time allows a more complete presentation of the theory.

6. Stochastic lattice gauge theory and new Monte Carlo algorithm

6.1. Lattice gauge theory with discrete time

Numerical calculations in lattice gauge theory are effected by Monte Carlo methods which rely on the simulation of a stochastic process. The stochastic process in question is specified by a matrix $\mathcal{T}(U_{t+1}, U_t) > 0$ of transition probabilities $U_t \to U_{t+1}$, on configurations of "horizontal" link variables, $U_t = \{U_{x,t,\mu}\}$ and $U_{t+1} = \{U_{x,t+1,\mu}\}$, where $\mu = 1, ...4$. Consider a transition probability of the form

$$\mathcal{T}(U_{t+1}, U_t) = \mathcal{N} \exp\{-\beta \sum_{x,\mu} \mathrm{ReTr}\Big(I - (U^{-1}_{x,t+1,\mu} V_{x,t,\mu})\Big) \} \; , \tag{6.1}$$

where β is a positive parameter. Here $V_{x,t,\mu} = V_{x,\mu}(U_t)$ is a group element that depends on the horizontal configuration at time t that will be specified shortly. For appropriately chosen normalization constant $\mathcal{N}$, this expression satisfies the requirement that the sum of transition probabilities out of any state is unity

$$\int \prod_{x,\mu} dU_{x,t+1,\mu} \mathcal{T}(U_{t+1}, U_t) = 1 \ . \tag{6.2}$$

This follows from invariance of the Haar measure under translation on the group, $dU_{x,t+1,\mu} = dU'_{x,t+1,\mu}$, where $U'_{x,t+1,\mu} = V^{-1}_{x,t,\mu} U_{x,t+1,\mu}$.

We set

$$V_{x,t,\mu} = U^{-1}_{x,t,5} \exp(\epsilon f_{x,t,\mu}) U_{x,t,\mu} U_{x+\hat{\mu},t,5}, \tag{6.3}$$

where ϵ is another positive parameter, and $f_{x,t,\mu}$ is a lattice analog of the continuum drift force $(D_\lambda F_{\lambda\mu})^{\mathrm{cont}}$ that is specified below. This gives

$$\mathcal{T}(U_{t+1}, U_t) = \mathcal{N} \exp\{-\beta \sum_{x,\mu} \mathrm{ReTr}\Big(I - \exp(\epsilon f_{x,t,\mu}) U_{x,t,\mu} U_{x+\hat{\mu},t,5} U^{-1}_{x,t+1,\mu} U^{-1}_{x,t,5}\Big) \ \} \ . \tag{6.4}$$

where the product of 4 U's is the transporter around the plaquette in the (μ-5) plane, starting and ending at the site (x,t). This expression is manifestly invariant under 5-dimensional local gauge transformations provided that $f_{x,t,\mu}$ transforms like a site variable at x in the adjoint representation,

$$f_{x,t,\mu} \to {}^g f_{x,t,\mu} = (g_{x,t})^{-1} f_{x,t,\mu} \ g_{x,t} \ , \tag{6.5}$$

which will be the case.

In the last expression for $\mathcal{T}$, the variables $U_{x,t,5}$ associated to time-like links serve merely to effect a gauge transformation $g_{x,t} = U_{x,t,5}$ on the variables $U_{x,t+1,\mu}$ associated to the space-like links. Consequently the $U_{x,t,5}$ may be assigned arbitrarily without affecting expectation values of the observables which are the gauge-invariant functions of the space-like link variables at a fixed time. We shall shortly gauge-fix $U_{x,t,5}$ by a lattice analog of continuum gauge fixing implemented above, where by $U_{x,t,5}$ will be expressed in terms of the $U_{x,t,\mu}$ so that eq. (6.2) is satisfied.

Recall that the 5-dimensional continuum theory contains no dimensionful constant, apart from the cut-off, when engineering dimensions are assigned according to $[t] = [x^2]$. Consequently the discretized action should depend on the lattice spacings only through the ratio

$$\epsilon \equiv \frac{a_t}{a_s^2} \ , \tag{6.6}$$

which defines the parameter ϵ. Here a_s and a_t are the lattice spacings in the space and time directions respectively, with a_t being the "time" for a single sweep of the lattice in a Monte Carlo updating. We write

$$\begin{aligned} U_{x,t,\mu} &\sim \exp(g_0 a_s A^{\rm cont}_{x,t,\mu}) \\ U_{x,t,5} &\sim \exp(g_0 a_t A^{\rm cont}_{x,t,5}) \ , \end{aligned} \tag{6.7}$$

where g_0 is the unrenormalized coupling constant, and "cont" designates continuum perturbative variables.

To verify this point, we estimate the quantities that appear in eqs. (6.1) and (6.3). The lattice drift force, specified below, is of order

$$f_{x,t,\mu} \sim g_0 a_s^3 (D_\lambda F_{\lambda\mu})^{\rm cont} \ . \tag{6.8}$$

This gives

$$\begin{aligned} \exp(\epsilon f_{x,t,\mu}) &\sim \exp\Big(g_0 a_s a_t \ (D_\lambda F_{\lambda\mu})^{\rm cont}\Big) \\ U_{x,t,\mu} U_{x+\hat\mu,t,5} U^{-1}_{x,t+1,\mu} U^{-1}_{x,t,5} &\sim \exp\Big(\ - \ g_0 a_s a_t \ (F_{5,\mu})^{\rm cont}\Big) \ , \end{aligned} \tag{6.9}$$

and finally

$$\begin{aligned} {\rm ReTr}\Big(1 - \exp(\epsilon f_{x,t,\mu}) \ U_{x,t,\mu} U_{x+\hat\mu,t,5} U^{-1}_{x,t+1,\mu} U^{-1}_{x,t,5}\Big) \\ \sim \Big(\frac{g_0 a_s a_t}{2}\Big)^2 \Big((F^a_{5\mu} - D_\lambda F^a_{\lambda\mu})^{\rm cont}\Big)^2 \ , \end{aligned} \tag{6.10}$$

where a sum on μ and a is understood. We have

$$\prod_t \mathcal{T}(U_{t+1}, U_t) = \exp(-\beta I) \ , \tag{6.11}$$

where the five-dimensional discretized action is given by

$$I \equiv \sum_{x,t,\mu} {\rm Re \ Tr}\Big(1 - \exp(\epsilon f_{x,t,\mu}) U_{x,t,\mu} U_{x+\hat\mu,t,5} U^{-1}_{x,t+1,\mu} U^{-1}_{x,t,5}\Big) \ . \tag{6.12}$$

We define

$$\beta \equiv \frac{1}{g_0^2 \epsilon} = \frac{a_s^2}{g_0^2 a_t}, \tag{6.13}$$

and we obtain

$$\beta I \sim \frac{1}{4} a_t a_s^4 \sum_{x,t,\mu} \Big((F^a_{5\mu} - D_\lambda F^a_{\lambda\mu})^{\rm cont}\Big)^2 \ , \tag{6.14}$$

which is the correct volume element, $a_t a_s^4$, and the correct normalization of the 5-dimensional action.

There remains to specify the lattice drift force $f_{x,\mu}$ and the gauge-fixing of $U_{x,t,5}$. For this purpose we shall establish a correspondence between lattice and continuum quantities. Let $S = S(U)$ be the gauge-invariant 4-dimensional Euclidean lattice action normalized to

$$S(U) \Longleftrightarrow g_0^2 S^{\rm cont} = \frac{g_0^2}{4} \int d^4x (F_{\mu\nu}^{a\ {\rm cont}})^2 \ , \tag{6.15}$$

which depends only on the horizontal link variables $U = \{U_{x,\mu}\}$ at a fixed time. It may be the Wilson plaquette action

$$S_{\rm W} = 2 \sum_p {\rm Re\ Tr}(I - U_p) \tag{6.16}$$

for pure gluodynamics, where the sum extends over all plaquettes p, or the effective action that results from integrating out the quark degrees of freedom,

$$S(U) = S_{\rm W} - \ g_0^2 {\rm Tr}\ \ln(\gamma_\mu D_\mu + m) \ , \tag{6.17}$$

where $\gamma_\mu D_\mu + m$ represents the lattice Dirac operator of choice. Let the lattice color-"electric" field operator $E_{x,\mu,a}$ be defined by

$$E_{x,\mu,a} \equiv (t_a U_{x,\mu})_{\alpha\beta} \frac{\partial}{\partial (U_{x,\mu})_{\alpha\beta}} \ , \tag{6.18}$$

where $[t_a, t_b] = f^{abc} t_c$, and ${\rm tr}(t_a t_b) = -(1/2)\delta_{ab}$. It has 4 components, $\mu = 1, ...4$, because our perspective is 5-dimensional. With $(U_{x,\mu})_{\alpha\beta} \sim \delta_{\alpha\beta} + (t_a)_{\alpha\beta}\ g_0 a_s A_{x,\mu}^{a\ {\rm cont}}$, we have the correspondence between lattice and continuum quantities,

$$E_{x,\mu,a} S(U) \Longleftrightarrow g_0 a_s^3\ \frac{\delta S^{\rm cont}}{\delta A_{x,\mu}^{a\ {\rm cont}}} = -g_0 a_s^3 (D_\lambda F_{\lambda\mu})^{\rm cont} \ , \tag{6.19}$$

and we take for the lattice drift force

$$f_{x,\mu,a} = -E_{x,\mu,a} S(U) \ . \tag{6.20}$$

The lattice color-electric field operator defined by the remarkably simple expression (6.18) represents the color-electric flux carried by the link (x, μ). It satisfies the Lie algebra commutation relations

$$[E_{x,\mu,a}, E_{y,\nu,b}] = -\delta_{x,y} \delta_{\mu,\nu}\ f^{abc} E_{x,\mu,c} \ , \tag{6.21}$$

as well as

$$
\begin{aligned}
[E_{x,\mu,a}, U_{y,\nu}] &= \delta_{x,y}\delta_{\mu,\nu}\ t_a U_{y,\nu} \\
[E_{x,\mu,a}, U^{-1}_{y,\nu}] &= -\delta_{x,y}\delta_{\mu,\nu}\ U^{-1}_{y,\nu} t_a\ .
\end{aligned}
\tag{6.22}
$$

We may visualize the color-electric field operator $E_{x,\mu,a}$ acting on $S(U)$ as inserting t_a into the plaquettes that contain the link (x,μ) on the left side of the matrix $U_{x,\mu}$.

The color-electric field operator that operates by right multiplication of t_a,

$$
E'_{x,\mu,a} \equiv (U_{x,\mu} t_a)_{\alpha\beta} \frac{\partial}{\partial (U_{x,\mu})_{\alpha\beta}}
\tag{6.23}
$$

satisfies the Lie algebra commutation relations with opposite sign

$$
[E'_{x,\mu,a}, E'_{y,\nu,b}] = \delta_{x,y}\delta_{\mu,\nu}\ f^{abc} E'_{x,\mu,c}\ .
\tag{6.24}
$$

Since left and right multiplication commute, we have

$$
[E_{x,\mu,a}, E'_{y,\nu,b}] = 0\ .
\tag{6.25}
$$

With

$$
U_{x,\mu} t_a = (U_{x,\mu} t_a U^{-1}_{x,\mu})\ U_{x,\mu} = O_{ba}(U_{x,\mu})\ t_b U_{x,\mu}\ ,
\tag{6.26}
$$

where the real orthogonal matrices $O_{ba}(U_{x,\mu}) = O_{ab}(U^{-1}_{x,\mu})$ form the adjoint representation of the group, the two are related by

$$
E'_{x,\mu,a} = O_{ba}(U_{x,\mu}) E_{x,\mu,b}\ ,
\tag{6.27}
$$

and satisfy

$$
\sum_a E'^2_{x,\mu,a} = \sum_a E^2_{x,\mu,a}\ .
\tag{6.28}
$$

The generator of infinitesimal gauge transformations $G_{x,a}$ is easily expressed in terms of these operators. A generic infinitesimal gauge transformation $U_{x,\mu} \to g_x^{-1} U_{x,\mu} g_{x+\hat\mu}$, with $g_x = 1 + \omega_x$, is given by

$$
\begin{aligned}
\delta U_{x,\mu} &= -\omega_x U_{x,\mu} + U_{x,\mu}\omega_{x+\hat\mu} \\
&= (D_\mu \omega)_x U_{x,m}\ ,
\end{aligned}
\tag{6.29}
$$

where

$$
(D_\mu\omega)_x \equiv U_{x,\mu}\omega_{x+\hat\mu} U^{-1}_{x,\mu} - \omega_x = t_a (D_\mu\omega)_{x,a}\ .
\tag{6.30}
$$

This defines the lattice gauge-covariant difference D_μ that corresponds to the continuum gauge-covariant derivative. It has a simple geometric meaning, being the difference between ω at $x + \hat{\mu}$ transported back to x, and ω at x. This infinitesimal gauge transformation induces the following change in a generic function $\mathcal{F}(U)$:

$$\begin{aligned} \delta\mathcal{F} &= \mathcal{F}(U + \delta U) - \mathcal{F}(U) \\ &= \sum_{x,\mu} (D_\mu \omega)_{x,a} \; E_{x,\mu,a} \mathcal{F} \\ &= \sum_x \omega_{x,a} \; G_{x,a} \mathcal{F} \; . \end{aligned} \tag{6.31}$$

This defines the generator of local gauge transformations $G_{x,a}$. From

$$(D_\mu \omega)_{x,a} = O_{ab}(U_{x,\mu}) \; \omega_{x+\hat{\mu},b} - \omega_{x,a} \tag{6.32}$$

we obtain

$$G^a_x \equiv \sum_\mu D^\dagger_\mu E_\mu = \sum_\mu (E'^a_{x-\hat{\mu},\mu} - E^a_{x,\mu}), \tag{6.33}$$

which is the total flux of color-electric field leaving the site x. As in the continuum theory, it is the left-hand side of Gauss's law. One easily verifies the commutation relations

$$\begin{aligned} [G^a_x, E^b_{y,\mu}] &= \delta_{x,y} \; f^{abc} E^c_{y,\mu} \\ [G^a_x, E'^b_{y,\mu}] &= \delta_{x-\hat{\mu},y} \; f^{abc} E'^c_{y,\mu} \\ [G^a_x, G^b_y] &= \delta_{x,y} \; f^{abc} G^c_y \; . \end{aligned} \tag{6.34}$$

Under local gauge transformation, $E^b_{x,\mu}$ transforms like a site variable at x in the adjoint representation, as required for the gauge-invariance of the transition matrix $\mathcal{T}(U_{t+1}, U_t)$. Moreover $E'^b_{x,\mu}$ transforms like a site variable in the adjoint representation at $x + \hat{\mu}$.

We shall implement a lattice gauge fixing that corresponds to the continuum gauge fixing described above. We introduce a minimizing function, $\mathcal{F}(U_t)$, and we impose the gauge condition

$$U_{x,t,5} = \exp(A_{x,t,5}) = \exp\Big(- \epsilon G_{x,t} \mathcal{F}(U_t) \Big) \; . \tag{6.35}$$

With this definition, eq. (6.2) is satisfied, because both $U_{x,t,5}$ and the gauge-fixing force $f_{x,t,\mu}$ are expressed in terms of variables that live only on the hyperplane t. For the minimizing function we take a lattice analog of $a^{-1} \int d^4x (A^{\rm cont})^2$, for example,

$$\begin{aligned} \mathcal{F}(U) &= 2a^{-1} \sum_{x,\mu} {\rm ReTr}(1 - U_{x,\mu}) \\ \mathcal{F}(U) &\sim g_0^2 a^{-1} \int d^4x (A^{\rm cont})^2 \; . \end{aligned} \tag{6.36}$$

where $a > 0$ is a gauge parameter (not to be confused with the lattice units a_s and a_t). This gives the gauge condition

$$\begin{aligned} A_{x,t,5} &= -\epsilon G_x^b \mathcal{F}(U) \\ &= -\epsilon a^{-1} \sum_\mu \mathrm{Tr}\Big(\ t^b \ (U_{x,\mu} - U_{x,\mu}^{-1} - U_{x-\hat{\mu},\mu} + U_{x-\hat{\mu},\mu}^{-1})\Big) \ . \end{aligned} \tag{6.37}$$

6.2. Five-dimensional critical limit and new algorithm

The renormalizability of the 5-dimensional continuum theory demonstrated above controls the critical limit of the 5-dimensional lattice theory just described. In particular it fixes the dependence of the parameter $\epsilon = \epsilon(g_0)$ on g_0, and thereby provides a new accelerated Langevin algorithm, as we shall see.

The probability for transition from configuration U_t into the volume element $\prod_{x,\mu} dU_{x,t+1,\mu}$, where $dU_{x,t+1,\mu}$ is Haar measure, is given by

$$\prod_{x,\mu} dU_{x,t+1,\mu} \mathcal{T}(U_{t+1}, U_t) \ . \tag{6.38}$$

From eq. (6.4), and invariance of Haar measure under translation on the group, it follows that the probability for transition from the configuration U_t to the configuration

$$U_{x,t+1,\mu} = U_{x,t,5}^{-1} \exp(\epsilon f_{x,t,\mu}) U_{x,t,\mu} U_{x+\hat{\mu},t,5} W_{x,t,\mu} \tag{6.39}$$

depends only on the group element $W_{x,t,\mu}$,

$$\prod_{x,\mu} dU_{x,t+1,\mu} \mathcal{T}(U_{t+1}, U_t) = \mathcal{N} \prod_{x,\mu} \{ dW_{x,t,\mu} \exp\Big(-\beta \mathrm{Re}\ \mathrm{Tr}(I - W_{x,t,\mu})\Big) \ \}. \tag{6.40}$$

Thus to generate a new configuration it is sufficient to generate the last probability distribution independently for each group element $W_{x,t,\mu}$, and also independently of the configuration U_t, and then calculate $U_{x,t+1,\mu}$ from eq. (6.39). (If one is not interested in gauge-fixing, one may set $U_{x,t,5} = 1$. However the gauge-fixing defined above smooths out configurations, which may help to accelerate the inversion of the Dirac operator.) One recognizes the familiar Langevin algorithm for the SU(N) group [17] and [18], with $W_{x,t,\mu}$ being white noise on the group. The new element is that we shall determine $\epsilon = \epsilon(g_0)$ from the perturbative renormalization group.

We noted in our discussion of the continuum theory that the time is renormalized according to $t = Z_t t_r$. The parameter $\epsilon = a_t/a_s^2$ rescales in the same way as the time t,

$$\epsilon = Z_t \epsilon_r \ , \tag{6.41}$$

where ϵ_r is independent of the ultraviolet cut-off $\Lambda = a_s^{-1}$. Consequently it satisfies the renormalization-group equation

$$\begin{aligned} \Lambda \frac{d \ln \epsilon}{d\Lambda} &= \Lambda \frac{d \ln Z_t}{d\Lambda} \\ &\equiv \beta_t(g_0) \\ &= -b_{t,0} g_0^2 + O(g_0^4) \ , \end{aligned} \tag{6.42}$$

where $\beta_t(g_0)$ is a β-function for the renormalization of the Monte Carlo time. The coefficient $b_{t,0}$ has been calculated for a theory without quarks, [19]and [20], and is independent of the gauge parameter a,

$$b_{t,0} = -\frac{13N}{3} \frac{1}{16\pi^2} \ . \tag{6.43}$$

It is convenient to change dependent variable from Λ to g_0, using

$$\Lambda \frac{d \ln \epsilon}{d\Lambda} = \beta(g_0) \frac{d \ln \epsilon}{d g_0} \tag{6.44}$$

where

$$\beta(g_0) \equiv \Lambda \frac{d g_0}{d\Lambda} \tag{6.45}$$

is the usual β-function,

$$\beta(g_0) = -b_0 g_0^3 - b_1 g_0^5 + O(g_0^7) \ . \tag{6.46}$$

It has been verified to one-loop level for the 5-dimensional theory without quarks that $\beta(g_0)$ is the same as in the Faddeev-Popov theory, [19]and [20]. For the theory without quarks we have

$$b_0 = \frac{11N}{3} \frac{1}{16\pi^2} \ , \tag{6.47}$$

which gives

$$\begin{aligned} \frac{d \ln \epsilon}{d g_0} &= \frac{\beta_t(g_0)}{\beta(g_0)} \\ &= \frac{b_{t,0}}{b_0} g_0^{-1} + O(g_0) \ , \end{aligned} \tag{6.48}$$

so

$$\epsilon = (b_0 g_0^2)^{b_{t_0}/(2b_0)} \exp\Big(O(g_0^2)\Big) \ C \ , \tag{6.49}$$

where C is a constant of integration, and

$$\frac{b_{t,0}}{b_0} = -\frac{13}{11} \tag{6.50}$$

for the theory without quarks. The higher loop corrections are negligible in the critical limit $g_0 \to 0$. This allows us to take for the purposes of numerical simulation

$$\epsilon = (b_0 g_0^2)^{b_{t_0}/(2b_0)} \; C \; , \tag{6.51}$$

and

$$\beta = (g_0^2 \epsilon)^{-1} = \Big((g_0^2)^{1+b_{t_0}/(2b_0)} \; b_0^{b_{t_0}/(2b_0)} \; C \Big)^{-1} \; . \tag{6.52}$$

In a theory without quarks, this gives

$$\begin{aligned} \epsilon &= (b_0 g_0^2)^{-13/22} \; C \\ \beta &= (g_0^2 \epsilon)^{-1} = \Big((g_0^2)^{9/22} \; b_0^{-13/22} \; C \Big)^{-1} \; . \end{aligned} \tag{6.53}$$

We note that β diverges in the critical limit $g_0 \to 0$, but more slowly than $1/g_0^2$. Moreover the time-step in lattice units diverges, $\epsilon = a_t/s_s^2 \to \infty$ because b_0 and $b_{0,t}$ have opposite sign in a gauge theory. This provides a highly accelerated algorithm. However one may doubt whether this algorithm converges because high wave-number components, $k \sim a_s^{-1}$ will overshoot and thus may appear to fluctuate wildly. However they are not of physical interest in the critical limit. On the other hand the modes that are of physical interest $k \sim \Lambda_{QCD}$ will not overshoot. Indeed the real problem is likely to be that they evolve too slowly as usual, in which case the divergence of the step-size $\epsilon(g_0) \to \infty$ as $g_0 \to 0$ is a great asset. Obviously the algorithm must be studied in practice before conclusions can be drawn. In order to apply the algorithm proposed here to a theory with quarks, the coefficient $b_{t,0}$ must also be calculated with quarks.

If one takes $C \to \infty$, one obtains a different, or non-existent, critical theory from the continuum theory discussed above. If one takes $C \to 0$, one obtains the continuous-time stochastic theory discussed in the next section with a Euclidean spatial cut-off.

The 5-dimensional critical limit is approached in the limit $g_0 \to 0$. The value of g_0 determines the physical correlation length as measured in Euclidean lattice units a_s in the same way that it does in the Faddeev-Popov theory,

$$a_s = \Lambda_{QCD}^{-1} (b_0 g_0^2)^{-b_1/(2b_0^2)} \exp\Big(-\frac{1}{2b_0 g_0^2} \Big) \; , \tag{6.54}$$

where Λ_{QCD} is the standard QCD mass scale, although this has been verified to one-loop level in the theory without quarks [19] and [20]. This formula and the above expression for $\epsilon(g_0)$ fix the "time", a_t, for a single Monte Carlo sweep in terms of Λ_{QCD},

$$\begin{aligned} a_t &= \epsilon a_s^2 \\ a_t &= \Lambda_{QCD}^{-2} C (b_0 g_0^2)^{(b_{t,0} b_0 - 2b_1)/(2b_0^2)} \exp\Big(- \frac{1}{b_0 g_0^2} \Big). \end{aligned} \tag{6.55}$$

In the asymptotic scaling region, auto-correlation times of physical observables are finite and independent of g_0 and C when measured in units of Λ_{QCD}^{-2}. The criteria for choosing the value of C are similar to those for g_0. The auto-correlation time τ for a physical observable should be large compared to one sweep-time, $\tau >> a_t$, but small compared the total running time T, $\tau << T$, so that the statistical uncertainty, of order $N^{-1/2}$, is small, where $N \equiv T/\tau$.

If it could be established that Monte Carlo auto-correlation times τ of physical observables are related to physical Euclidean correlation lengths L, by a universal relation $\tau = KL^2$, where K is the same for all physical observables, then physical information such mass ratios, could be extracted directly from Monte-Carlo auto-correlation times. In this case the mass spectrum is given by the eigenvalues of the Fokker-Planck hamiltonian, as explained in sec. 6.3.

We briefly mention some attractive features of the algorithm proposed here, which are those of the Langevin algorithm with a time-step in lattice units that diverges in the critical limit $g_0 \to \infty$. There is no accept/reject criterion. Every updating $W_{x,t,\mu}$ is accepted, even when the (effective) action is non-local as occurs with dynamical quarks. Only the derivative of the Euclidean action S appears in

$$f_{x,t,\mu} = -E_{x,\mu} S = -E_{x,\mu} S_W + \ g_0^2 \mathrm{Tr}\Big((\gamma_\lambda D_\lambda + m)^{-1} E_{x,\mu} (\gamma_\nu D_\nu) \Big) \ , \tag{6.56}$$

so one does not have to calculate the determinent of the lattice Dirac operator, but only its inverse. Moreover it is sufficient to calculate this inverse only once per sweep, because a whole sweep of the Euclidean lattice, described by eqs. (6.39) and (6.40), corresponds to a single hyperplane of the 5-dimensional lattice with action (6.12).

Of course other algorithms may be invented that are based on other discretizations in which the 5-dimensional critical limit is also achived in the limit $g_0 \to 0$.

6.3. Lattice Fokker–Planck and Langevin equations

One would be tempted to formulate a BRST version of the lattice gauge theory descrobed in the previous section, which has discrete stochastic time t and group-valued link variables $U_{x,t,5}$ asociated to the vertical (time-like) links. However Neuberger [21] has shown that lattice BRST gauge-fixing fails if the gauge-fixing function is continuous on the group manifold. Moreover Testa [22] has provided a counter-example which shows that if one tries to get around Neuberger's theorem by choosing a gauge-fixing function that is not continuous on the group manifold, then inconsistencies in the BRST procedure appear in the form of BRST-exact quantities sX with non-zero expectation-value $< sX >\neq 0$. Therefore in order to comply with Neuberger's theorem we shall make $t = x_5$ continuous while keeping x_μ discrete, so $A_{x,5}$ is in the Lie algebra whereas $U_{x,\mu}$ is in the Lie group, where $\mu = 1, ...4$. Gauge-fixing will be of the form $A_{x,5} = v_x(U)$, so that a quantity in the algebra is fixed instead of a quantity in the group. An alternative approach to BRST gauge-fixing in lattice gauge theory has recently been provided by Baulieu and Schaden [23].

Before developing the BRST formulation of this 5-dimensional lattice theory, we first consider the stochastic process described by the lattice Fokker-Planck and Langevin equations with random variable which is a 4-dimensional Euclidean lattice configuration $U = \{U_{x,\mu}\}$, where $U_{x,\mu} \in SU(N)$. Note that a Euclidean lattice configuration U is a point on a finite product of Lie-group manifolds and that a continuous stochastic process on Lie-group manifolds is the probabilistic analog of quantum mechanics on Lie-group manifolds. Both allow a well-defined path-integral formulation.

The continuous-time stochastic process is described by the Fokker-Planck equation

$$\frac{\partial P}{\partial t} = -H_{\rm FP} P \ , \tag{6.57}$$

where $P = P(U, t)$ is a probability distribution, and the Fokker-Planck hamiltonian is given by

$$\begin{aligned} H_{\rm FP} &= -\sum_{x,\mu} E^a_{x,\mu}\Big(E^a_{x,\mu} - g_0^{-2}(D_\mu A_5)^a_x + g_0^{-2} E^a_{x,\mu} S\Big) \\ H^\dagger_{\rm FP} &= -\sum_x [\Big(E^a_{x,\mu} - g_0^{-2}(E^a_{x,\mu} S)\Big) E^a_{x,\mu} - g_0^{-2} A^a_{x,5} G^a_x] \ , \end{aligned} \tag{6.58}$$

where

$$A_{x,5} = -G_x \mathcal{F}(U) \ , \tag{6.59}$$

and D_μ is the lattice gauge-covariant difference, eq. (6.30). In Appendix A we show that this equation describes the limit of the discrete-time stochastic process defined previous paragraph in which the time-step approaches zero, $\epsilon = a_t/a_s^2 \to 0$.

In the previous paragraph it was conjectured that the auto-correlation times τ of physical observables might be related to Euclidean auto-correlation lengths L by the universal relation $\tau = KL^2$, where K is the same for all observables. If this is true, then the mass spectrum is given by the eigenvalue equation

$$\begin{aligned} H_{\mathrm{FP}}\Psi &= m^2\Psi \\ H_{\mathrm{FP}}^{\dagger}\Phi &= m^2\Phi\ . \end{aligned} \tag{6.60}$$

But for the gauge-fixing "force", $(D_\mu A_5)_x$ in H_{FP}, the equilibrium solution, satisfying $H_{\mathrm{FP}}P_{\mathrm{eq}} = 0$ would be given by $P_{\mathrm{eq}} = \exp(-g_0^{-2}S)$. We cannot give the exact equilibrium solution for non-zero gauge-fixing force because it is not conservative. However with lattice regularization gauge-fixing may be dispensed with. Then a similarity transformation by $\exp[-(g_0^{-2}/2)S]$ is sufficient to make the the Fokker-Planck hamiltonian hermitian

$$H'_{\mathrm{FP}} = -\sum_{x,\mu}[\ \Big(E^a_{x,\mu} - (g_0^{-2}/2)(E^a_{x,\mu}S)\Big)\ \Big(E^a_{x,\mu} + (g_0^{-2}/2)E^a_{x,\mu}S\Big)\ ]\ , \tag{6.61}$$

and $H'_{\mathrm{FP}} = H'^{\dagger}_{\mathrm{FP}}$. Moreover the physically relevant eigenfunctions are gauge-invariant, $G_x\Phi' = 0$. The eigenvalue equation

$$H'_{\mathrm{FP}}\Psi' = m^2\Psi', \tag{6.62}$$

is interesting. It is both Euclidean- and gauge-invariant. Moreover the ground-state or vacuum wave-function is known exactly, $\Psi'_0 = \exp[-(g_0^{-2}/2)S]$, so variational calculations of excited-state eigenvalues are possible.

The corresponding Langevin equation is obtained from the $\epsilon \to 0$ limit of the stochastic process defined in eqs. (6.39) and (6.40). It is given by

$$\dot{U}_{x,\mu}U^{-1}_{x,\mu} = g_0^{-2}(D_\mu A_5)_x - g_0^{-2}E_{x,\mu}S + \eta_{x,\mu}, \tag{6.63}$$

where $\eta_{x,\mu}$ is Gaussian white noise with two-point correlator

$$< \eta_{x,\mu}(s)\eta_{y,\nu}(t) >= 2\delta_{x,y}\delta_{\mu,\nu}\delta(s-t). \tag{6.64}$$

The gauge choice $A_{x,5} = -G_x \mathcal{F}(U)$ has the global property that for the flow determined by the gauge-fixing force alone,

$$\partial_5 U_{x,\mu} U_{x,\mu}^{-1} = g_0^{-2} (D_\mu A_5)_x, \tag{6.65}$$

the minimizing function $\mathcal{F}$ decreases monotonically

$$\partial_5 \mathcal{F} \leq 0 \ , \tag{6.66}$$

for we have

$$\partial_5 \mathcal{F} = g_0^{-2} \sum_{x,\mu} E_{x,\mu} \mathcal{F} \ D_\mu A_5 = g_0^{-2} \sum_x G_x \mathcal{F} \ A_{x,5} = -g_0^{-2} \sum_x G_x \mathcal{F} \ G_x \mathcal{F} \ . \tag{6.67}$$

For the minimizing function (6.36) , this drives all the $U_{x,\mu}$ toward unity.

7. Five-dimensional topological lattice gauge theory

7.1. Lattice BRST operator

We determine the lattice BRST operator s which corresponds to the continuum BRST operator defined previously. It possesses the symmetries of the hypercubic lattice. In order to exhibit these symmetries, we shall, in this subsection only, use a symmetric 5-dimensional notation, whereby $x = \{x_\mu\}$ for $\mu = 1, ...5$ represents lattice sites (previously denoted (x,t)), and we shall also denote link variables by $U_{xy} \in$ SU(N), with $U_{yx} = (U_{xy})^{-1}$, where (xy) is a pair of nearest neighbors.

Consider a generic infinitesimal transformation U_{xy},

$$U_{xy} + \delta U_{xy} = (1 + \omega_{xy}) U_{xy} = U_{xy} (1 + \omega'_{xy}) \ , \tag{7.1}$$

where ω_{xy} and $\omega'_{xy} = (U_{xy})^{-1} \omega_{xy} U_{xy}$ are elements of the Lie algebra, $\omega_{xy} = t_a \omega^a_{xy}$. We write $\delta = \epsilon s$, $\omega_{xy} = \epsilon \Omega_{xy}$ and $\omega'_{xy} = \epsilon \Omega'_{xy}$, where ϵ is an infinitesimal Grassmann variable, and Ω_{xy} and $\Omega'_{xy}$are Lie algebra-valued Grassmann variables, with

$$\Omega'_{xy} = (U_{xy})^{-1} \Omega_{xy} U_{xy} \ . \tag{7.2}$$

This determines the action of s on U_{xy},

$$s U_{xy} = \Omega_{xy} U_{xy} = U_{xy} \Omega'_{xy} \ . \tag{7.3}$$

From the condition $s^2 = 0$, we obtain

$$
\begin{aligned}
s\Omega_{xy} &= \Omega_{xy}^2 \\
s\Omega'_{xy} &= -\Omega'^2_{xy} \ .
\end{aligned}
\tag{7.4}
$$

We will not write out explicitly further relations for Ω'_{xy} since they may be obtained for those of Ω_{xy}.

As in the continuum theory, we wish to distinguish the infinitesimal gauge transformations from among all possible transformations of U_{xy}. We pose $\omega_x = \epsilon c_x$, where ϵ is an infinitesimal Grassmann variable, and c_x is a Lie algebra-valued Grassmann variable, so the infinitesimal gauge transformation defined in the previous section reads

$$
\begin{aligned}
\delta_g U_{xy} &= \epsilon(-c_x U_{xy} + U_{xy} c_y) \\
&= \epsilon(-c_x + U_{xy} c_y U_{yx}) U_{xy} \ ,
\end{aligned}
\tag{7.5}
$$

where we have written $U_{yx} \equiv (U_{xy})^{-1}$. We separate this infinitesimal gauge transformation out of $\Omega_{x,\mu}$, and write

$$
\Omega_{xy} \equiv \Psi_{xy} - c_x + U_{xy} c_y U_{yx} \ , \tag{7.6}
$$

which defines Ψ_{xy}. The action of s on U_{xy} now reads

$$
sU_{xy} = (\Psi_{xy} - c_x) U_{xy} + U_{xy} c_y \ . \tag{7.7}
$$

We identify the lattice site variable c_x with the corresponding continuum scalar ghost $c(x)$, and similarly for Φ_x, and assign to them the same transformation law as in the continuum case namely

$$
\begin{aligned}
sc_x &= \Phi_x - c_x^2 \\
s\Phi_x &= -c_x \Phi_x + \Phi_x c_x \ .
\end{aligned}
\tag{7.8}
$$

The action of s on Ψ_{xy} is obtained from

$$
s\Psi_{xy} = s\Omega_{xy} + sc_x - s(U_{xy} c_y U_{yx}) \ . \tag{7.9}
$$

This determines the action of s on all the lattice fields and ghosts.

We write it in the notation in which it will be used, namely with lattice sites designated by (x, t), where x, is a 4-vector $x = \{x_\mu\}$ for $\mu = 1, ...4$, and $t = x_5$. "Horizontal" links, corresponding to links in 4-dimensional Euclidean space-time, are designated by (x, t, μ), and "vertical" links by $(x, t, 5)$. In this notation s acts according to

$$
\begin{aligned}
sU_{x,t,\mu} &= (\Psi_{x,t,\mu} + (D_\mu c)_{x,t})\ U_{x,t,\mu} \\
sU_{x,t,5} &= (\Psi_{x,t,5} + (D_\mu c)_{x,t})\ U_{x,t,5} \\
s\Psi_{x,t,\mu} &= -(D_\mu \Phi)_{x,t} - [c_{x,t}, \Psi_{x,t,\mu}] + (\Psi_{x,t,\mu})^2 \\
s\Psi_{x,t,5} &= -(D_5 \Phi)_{x,t} - [c_{x,t}, \Psi_{x,t,5}] + (\Psi_{x,t,5})^2 \\
sc_{x,t} &= \Phi_{x,t} - (c_{x,t})^2 \\
s\Phi_{x,t} &= -[c_{x,t}, \Phi_{x,t}]\ ,
\end{aligned}
\tag{7.10}
$$

which satisfies $s^2 = 0$, and where the lattice gauge-covariant difference operator D_μ is defined in (6.30). Under gauge transformation $\Psi_{x,t,\mu}$ and $\Psi_{x,t,5}$ transform like site variables located at (x, t).

We shall shortly take $t = x_5$ to be a continuous variable, while x remains the discrete label of a 4-dimensional lattice site. In this case, instead of the link variable $U_{x,t,5}$ in the Lie group, we require a variable $A_{x,t,5}$ in the Lie algebra. The BRST operator acts on $A_{x,t,5}$ and $\Psi_{x,t,5}$ as in the continuum theory,

$$
\begin{aligned}
sA_{x,t,5} &= \Psi_{x,t,5} + D_5 c_{x,t} \\
s\Psi_{x,t,5} &= -D_5 \Phi_{x,t} - [c_{x,t}, \Psi_{x,t,5}]\ ,
\end{aligned}
\tag{7.11}
$$

where D_5 is the continuum covariant derivative and s acts on the remaining variables as above.

In terms of the lattice color-electric field operator $E_{x,\mu}$ and generator of local gauge transformations G_x obtained above, the explicit form of the lattice BRST operator acting on functions $f = f(U, \Psi, c, \Phi)$ is given by $sf = Qf$, where Q is the differential operator

$$
\begin{aligned}
Q \equiv \sum_x \Big(\Psi_{x,\mu} E_{x,\mu} + c_x G_x + \{ -(D_\mu \Phi)_x - [c_x, \Psi_{x,\mu}] + \Psi_{x,\mu}^2 \} \frac{\partial}{\partial \Psi_{x,\mu}} \\
+ (\Phi_x - c_x^2) \frac{\partial}{\partial c_x} - [c_x, \Phi_x] \frac{\partial}{\partial \Phi_x} \Big).
\end{aligned}
\tag{7.12}
$$

The action of s on the anti-ghosts and Lagrange multipliers is defined simply by

$$
\begin{aligned}
s\bar{\Psi}_{x,t,\mu} &= b'_{x,t,\mu} \\
s\bar{\Psi}_{x,t,5} &= b'_{x,t,5} \\
sb'_{x,t,\mu} &= 0 \\
sb'_{x,t,5} &= 0 \\
s\bar{\Phi}_{x,t} &= \bar{\eta}'_{x,t} \\
s\bar{\eta}'_{x,t} &= 0\ .
\end{aligned}
\tag{7.13}
$$

This is the same as in the continuum theory after the change of variable

$$
\begin{aligned}
b'_{x,t,\mu} &\equiv b_{x,t,\mu} - [c_{x,t}, \bar{\Psi}_{x,t,\mu}] \\
b'_{x,t,5} &\equiv b_{x,t,5} - [c_{x,t}, \bar{\Psi}_{x,t,5}] \\
\bar{\eta}'_{x,t} &\equiv \bar{\eta}_{x,t} - [c_{x,t}, \bar{\Phi}_{x,t}] \ .
\end{aligned}
\tag{7.14}
$$

Note that $\Phi_{x,t}, \bar{\Phi}_{x,t}, b_{x,t,\mu}$ and $b_{x,t,5}$ are real variables.

It is remarkable that the lattice BRST transformation requires only minimal change from the continuum BRST transformation, even though there is no local lattice curvature and no lattice Bianchi identity, whereas our starting point for the continuum transformation was the curvature equation and the Bianchi identity (2.6).

7.2. Five-dimensional topological lattice action

The most powerful representation of the continuous stochastic process just defined is by path integral. We start with the 5-dimensional topological lattice action which we write by analogy with the 5-dimensional continuum action presented above,

$$
\begin{aligned}
I = g_0^{-2} \int dt \sum_x s\mathrm{Tr}\{ \ & \bar{\Psi}_{x,\mu}\Big((D_5 U)_{x,\mu} U^{-1}_{x,\mu} + E_{x,\mu} S(U) + \frac{1}{2} b'_{x,\mu}\Big) \\
& + \bar{\Psi}_{x,5}(A_{x,5} + G_x \mathcal{F}) \\
& + \bar{\Phi}_x(\Psi_{x,5} + \Psi_{y,\mu} \ E_{y,\mu} G_x \mathcal{F}') \ \} \ ,
\end{aligned}
\tag{7.15}
$$

where a sum on y and μ is understood. Here $\mathcal{F}(U)$ is the minimizing function introduced above, and $\mathcal{F}'(U)$ is the same or another minimizing function.

We expand the above action and obtain

$$
I = I_1 + I_{A_5} + I_{\Psi_5} + I_{\Psi_\mu} + I_c + I_\Phi \ , \tag{7.16}
$$

$$
\begin{aligned}
I_1 &= g_0^{-2} \int dt \sum_x \mathrm{Tr}\{ \ \bar{b}'_{x,\mu} \ \Big((D_5 U)_{x,\mu} U^{-1}_{x,\mu} + E_{x,\mu} S(U) + \frac{1}{2} b'_{x,\mu}\Big) \ \} \\
I_{A_5} &= g_0^{-2} \int dt \sum_x \mathrm{Tr}\Big(\ \bar{b}'_{x,5} \ (A_{x,5} + G_x \mathcal{F})\Big) \\
I_{\Psi_5} &= g_0^{-2} \int dt \sum_x \mathrm{Tr}\Big(\bar{\eta}'_x \ (\Psi_{x,5} + \Psi_{y,\mu} \ E_{y,\mu} G_x \mathcal{F}')\Big)
\end{aligned}
\tag{7.17}
$$

$$
\begin{aligned}
I_{\Psi_\mu} = -g_0^{-2} \int dt \sum_x \mathrm{Tr}\Big(\bar{\Psi}_{x,\mu} \ \{ \ & D_5\Big(\Psi_{x,\mu} + (D_\mu c)_x\Big) - D_\mu\Big(\Psi_{x,5} + (D_5 c)_x\Big) \\
& + \Big(\ \Psi_{x,\mu} + (D_\mu c)_x \ , \ \partial_5 U_{x,\mu} U^{-1}_{x,\mu} - (D_\mu A_5)_x\Big) \\
& + \Big(\ \Psi_{y,\nu} + (D_\nu c)_y \ \Big) E_{y,\nu} E_{x,\mu} S(U) \ \} \ \Big)
\end{aligned}
\tag{7.18}
$$

$$I_c = -g_0^{-2} \int dt \sum_x \mathrm{Tr}\Big(\bar{\Psi}_{x,5} \ \{ \ (D_5 c)_x + \Psi_{x,5} + \Big((D_\mu c)_y + \Psi_{y,\mu} \Big) E_{y,\mu} G_x \mathcal{F} \ \} \ \Big) \tag{7.19}$$

$$\begin{aligned} I_\Phi = g_0^{-2} \int dt \sum_x \mathrm{Tr}\Big(\ \bar{\Phi}_x \ \{ -(D_5\Phi)_x - [c_x, \Psi_{x,5}] \\ + \Big(-(D_\mu \Phi)_y - [c_y, \Psi_{y,\mu}] + \Psi_{y,\mu}^2 \Big) \ E_{y,\mu} G_x \mathcal{F}' \ \} \ \Big) \ . \end{aligned} \tag{7.20}$$

The terms I_{A_5} and I_{Ψ_5} serve to gauge fix A_5 and Ψ_5. Indeed upon integration with respect to $\bar{b}'_5$ and $\bar{\eta}'$ they impose the constraints,

$$\begin{aligned} A_{x,5}^a &= -G_x^a \mathcal{F} \\ \Psi_{x,5}^a &= -\Psi_{y,\mu}^b \ E_{y,\mu}^b G_x^a \mathcal{F}' \ . \end{aligned} \tag{7.21}$$

7.3. Proof of gauge invariance and absence of Gribov copies.

We must show that the expectation-values of physical observables, namely the Wilson loops, are independent of the gauge-fixing functions $\mathcal{F}$ and $\mathcal{F}'$. For this purpose we integrate out $\bar{\eta}'$ and $\bar{\Psi}_5$, which results in Ψ_5 being assigned its gauge-fixed value. We next integrate out the bose ghosts $\bar{\Phi}$ and Φ, which appear only in

$$Z_\Phi \equiv \int d\Phi d\bar{\Phi} \exp(-I_\Phi) \ . \tag{7.22}$$

The integral on $\bar{\Phi}$ yields

$$Z_\Phi = \int \prod_{t,x} d\Phi_{t,x} \prod_{t,x} \delta\Big(-\partial_5 \Phi_x + (L\Phi)_x + f_x \Big) \ , \tag{7.23}$$

where f_x is independent of Φ, and the finite matrix L is defined by

$$(L\Phi)_x^a \equiv -[A_{x,5}, \Phi_x]^a - \sum_y G_y^b G_x^a \mathcal{F}' \ \Phi_y^b \ , \tag{7.24}$$

and we have used $\sum_\mu (D_\mu^\dagger E)_{y,\mu} = G_y$. To evaluate Z_Φ, we observe that the stochastic process starts at some initial time $t = t_0$, which may be taken to be $t_0 = -\infty$. At the initial time, $\Phi_x(t_0)$, and the other fields U, Ψ, c, are assigned some definite initial value that is *not* integrated over, for example $\Phi_x(t_0) = 0$. With this initial value, the differential equation

$$\partial_5 \phi_x - (L\phi)_x = f_x \tag{7.25}$$

is equivalent to the integral equation

$$\phi_x^b(t) = \int_{t_0}^{t} du \Big(f_x^b(u) + L_{x,y}^{b,c}(u)\phi_x^c(u) \Big). \tag{7.26}$$

This equation possesses a *unique* solution $\phi_x(t)$. Thus unlike gauge-fixing on the group, or in a 4-dimensional covariant gauge, there are no Gribov copies.

We shift Φ by ϕ which cancels f_x in the path integral for Z_Φ. We also multiply as usual by the formally infinite normalization constant $\det \partial_5$. This gives

$$Z_\Phi = \int \prod_{t,x} d\Phi_x(t) \prod_{t,x} \delta\Big(\Phi_x(t) - (\mathcal{L}\Phi)_x(t) \Big) \ , \tag{7.27}$$

where $\mathcal{L} = \partial_5^{-1} L$ is the integral operator

$$(\mathcal{L}\Phi)_x^a(t) = \int_{t_0}^{\infty} du \ \mathcal{L}_{x,y}^{ab}(t,u)\Phi_y^b(u), \tag{7.28}$$

with kernel

$$\mathcal{L}_{x,y}^{a,b}(t,u) = \theta(t-u) L_{x,y}^{a,b}(u) \ . \tag{7.29}$$

This gives

$$Z_\Phi = \det{}^{-1}(1-\mathcal{L}) = \exp\Big(- \mathrm{Tr}\ \ln(1-\mathcal{L}) \Big) = \exp\Big(\mathrm{Tr}\mathcal{L} + ... \Big) \ , \tag{7.30}$$

Because $\mathcal{L}_{x,y}(t,u)$ is retarded, $\mathcal{L}(t,u) = 0$ for $u > t$, only the tadpole term

$$\begin{aligned} \mathrm{Tr}\mathcal{L} &= \int_{t_0}^{\infty} dt \ \sum_{x,a} \mathcal{L}_{x,x}^{a,a}(t,t) \\ &= \theta(0) \int_{t_0}^{\infty} dt \sum_{x,a} L_{x,x}^{aa}(t) \end{aligned} \tag{7.31}$$

survives in the last expansion. It is natural to assign the ambiguous expression $\theta(0)$ its mean value 1/2, which is consistent with other determinations (see Appendix B), and we obtain

$$Z_\Phi = \exp\Big(\frac{1}{2} \int_{t_0}^{\infty} dt \sum_{x,a} L_{x,x}^{a,a}(t) \Big) \ , \tag{7.32}$$

and so, by eq. (7.24),

$$Z_\Phi = \exp\Big(- \frac{1}{2} \int_{t_0}^{\infty} dt \sum_{x,a} G_x^a G_x^a \mathcal{F}' \Big) \ . \tag{7.33}$$

We emphasize that we have made an exact evaluation of the ghost determinant, and obtained a purely local contribution to the effective action.

In the same way we integrate out the Fermi ghosts Ψ_μ and $\bar{\Psi}_\mu$, after setting Ψ_5 to its gauge-fixed value. They only appear in the factor

$$Z_{\Psi_\mu} \equiv \int d\Psi_\mu d\bar{\Psi}_\mu \exp(-I_{\Psi_\mu}) \ . \tag{7.34}$$

As before only the tadpole term survives and gives

$$Z_{\Psi_\mu} = \exp\Big(\frac{1}{2}\int_{t_0}^{\infty} dt \sum_{x,a}(\sum_{\mu} E^a_{x,\mu}E^a_{x,\mu}S + G^a_x G^a_x \mathcal{F}')\Big) \ . \tag{7.35}$$

The dependence on $\mathcal{F}'$ cancels in the product

$$Z_{\Psi_\mu} Z_\Phi = \exp(-I_2) \ , \tag{7.36}$$

where

$$I_2 \equiv -\frac{1}{2}\int_{t_0}^{\infty} dt \sum_{x,\mu,a} E^a_{x,\mu}E^a_{x,\mu}S \ . \tag{7.37}$$

Thus physical observables are independent of the gauge-fixing function $\mathcal{F}'$, as asserted.

We next show that they are also independent of $\mathcal{F}$. The integration on ${b_\mu}'$ is effected, which causes I_1 to be changed to ${I_1}'$, given below. The variable of integration c may be translated, just as Φ was translated above, to cancel the inhomogeneous term in I_c, which changes I_c to ${I_c}'$ given below. Putting these factors together we obtain

$$Z = \int \prod_{t,x} \Big(dU_{x,\mu}(t) dA_{x,5}(t) dc_x(t) d\bar{\Psi}_{x,5}(t) db_{x,5}(t)\Big) \ \exp(-I') \ , \tag{7.38}$$

where

$$\begin{aligned} I' &= {I_1}' + I_2 + I_{A_5} + {I_c}' \\ {I_1}' &= -\frac{1}{2}{g_0}^{-2} \int dt \sum_x \mathrm{Tr}\ \Big((D_5 U)_{x,\mu} U^{-1}_{x,\mu} + E_{x,\mu} S(U)\Big)^2 \\ {I_c}' &= -{g_0}^{-2} \int dt \sum_x \mathrm{Tr}\{\ \bar{\Psi}_{x,5}\ \Big((D_5 c)_x + (D_\mu c)_y E_{y,\mu} G_x \mathcal{F}\Big)\ \}, \end{aligned} \tag{7.39}$$

and I_2 and I_{A_5} are defined above.

We next use standard BRST arguments to show that the expectation-value of any gauge-invariant observable is independent of the gauge-fixing function $\mathcal{F}$. Observe first that $I_1{}'$ and I_2 are gauge invariant. They are therefore invariant under a "little" BRST operator s' that acts according to

$$\begin{aligned} s'U_{x,\mu} &= (D_\mu c)_x U_{x,\mu} \qquad s'A_{x,5} = (D_5 c)_x \\ s'c_x &= -c_x^2 \\ s'\bar{\Psi}_{x,5} &= b_{x,5} \qquad\qquad\quad s'b_{x,5} = 0 \ , \end{aligned} \tag{7.40}$$

which satisfies $s'^2 = 0$. Indeed, s' is the usual BRST operator with $\bar{c}_x \to \bar{\Psi}_{x,5}$ and $b_x \to b_{x,5}$, and it acts on $U_{x,\mu}$ and $A_{x,5}$ like an infinitesimal gauge transformation. Because $I_1{}'$ and I_2 are gauge-invariant they are also invariant under s',

$$s'I_1{}' = 0; \qquad\qquad s'I_2 = 0 \ . \tag{7.41}$$

Moreover we may express the total action I' as

$$I' = I_1{}' + I_2 + s'J \ , \tag{7.42}$$

where

$$J \equiv g_0^{-2} \int dt \sum_x \mathrm{Tr}\Big(\bar{\Psi}_{x,5} \ (\ A_{x,5} + G_x \mathcal{F} \ )\Big). \tag{7.43}$$

Thus I' is s'-invariant, $s'I' = 0$, and moreover the gauge-fixing function $\mathcal{F}$ appears only in the term $s'J$ that is s'-exact. By standard arguments it follows that the expectation-value of any gauge-invariant observable is independent of $\mathcal{F}$, as asserted, which completes the proof.

We next integrate out the remaining auxiliary variables $b_5, A_5, \bar{\Psi}$ and c to obtain a path-integral in terms of the actual random variables $U_{x,\mu}(t)$ only. In eq. (7.38) , we integrate out $b_{x,5}(t)$ and $A_{x,5}(t)$, which fixes $A_{x,5} = -G_x \mathcal{F}(U)$. The integral on the Fermi ghosts $c_x(t)$ and $\bar{\Psi}_{x,5}(t)$ namely

$$Z_c \equiv \int \prod_{t,x} [dc_x(t) d\bar{\Psi}_{x,5}(t)] \ \exp(-I_c{}') \ , \tag{7.44}$$

may be effected just like the integral on the bose ghosts $\Phi_x(t)$ and $\bar{\Phi}_x(t)$ that was done explicitly above. Again only the tadpole term survives and gives

$$\begin{aligned} Z_c &= \exp(-I_3) \\ I_3 &= -\frac{1}{2} \int_{t_0}^{\infty} dt \sum_{x,a} G_x^a G_x^a \mathcal{F} \ . \end{aligned} \tag{7.45}$$

This yields the desired partition function of the stochastic process in terms of the random variables $U_{x,\mu}(t)$ only,

$$Z = \int \prod_{t,x} dU_{x,\mu}(t) \ \exp(-I'') \ , \tag{7.46}$$

where

$$I'' = I_1{}' + I_2 + I_3 \tag{7.47}$$

is a local action given by

$$\begin{aligned} I'' = -\frac{1}{2} \int_{t_0}^{\infty} dt \sum_x \{ \ g_0^{-2} \mathrm{Tr} \Big((D_5 U)_{x,\mu} U_{x,\mu}^{-1} + E_{x,\mu} S(U) \Big)^2 \\ + E_{x,\mu}^a E_{x,\mu}^a S + G_x^a G_x^a \mathcal{F} \ \} \ . \end{aligned} \tag{7.48}$$

The last two terms are the famous Ito terms that are discussed in Appendix B, and we see that they are produced automatically by the 5-dimensional topological action when all auxiliary and ghost fields are integrated out. Thus the ghost determinant is well-defined, and moreover its explicit evaluation gives a local Ito term to the effective action which comes from the tadpole diagrams only. The Ito terms, which involve the second derivative of the action S and of the minimizing function $\mathcal{F}$ with an over-all minus sign, favor minima of S and $\mathcal{F}$ over maxima and saddle points. This is physically natural and transparent, so it is somewhat surprising that the tadpole diagrams, which give the Ito terms with lattice regularization, vanish with dimensional regularization [14].

8. Conclusion and Perspectives

We have seen that the description of the 4-dimensional Yang–Mills theory in the 5-dimensional framework allows one to obtain a field theory description that does not suffers from the contradictions of the four dimensional description. From a physical point of view, we have introduced in a gauge-invariant way the additional time of the Monte Carlo description of the quantum theory and unified it with the Euclidean space-time coordinates. It is of course quite remarkable that this approach permits one to get a fully consistent lattice formulation of the theory with a natural limit toward a continuum formulation, while the Gribov ambiguity is just absent. Our work shows that this ambiguity is merely an artifact of a purely 4-dimensional description. It is only in five dimensions that one has a globally correct gauge fixing which is purely local. It thus appears that we can consider the 4-dimensional physical space as a slice in a 5-dimensional manifold, in which one can express the Yang–Mills theory under the form of a particular topological field theory. A way to this this schematically is to observe that the 4-dimensional action can always be rewritten as the integral of a topological term in five dimensions:

$$\int d^4x\mathcal{L}(\phi) = \int d^4xdt\frac{\delta\mathcal{L}(\phi)}{\delta\phi}\frac{d\phi}{dt} \tag{8.1}$$

Thus, the definition of a quantum field theory amounts to the BRST invariant gauge-fixing of the action, a task that we have explained in detail above and which leads to the solutions of several problems of the purely 4-dimensional formalism.

We briefly indicate how our method extends to fermions. To obtain the 5-dimensionsal fermionic action of a 4-dimensional Dirac spinor q, which automatically extends to a 5-dimensional spinor, it is now quite natural to introduce a topological BRST operator, and to enforce the relevant Langevin equation by a BRST-exact action. Call Ψ_q and $\bar{\Psi}_q$ the commuting topological ghost and anti-ghost of the anti-commuting spinor q, and b_q the anti-commuting Lagrange multiplier field. Then one has $sq = \Psi_q - cq$, $s\Psi_q = -c\Psi_q + \Phi q$, and one may take $s\bar{\Psi}_q = b_q$, $sb_q = 0$ or $s\bar{\Psi}_q = b_q - c\bar{\Psi}_q$, $sb_q = -cb_q + \Phi\bar{\Psi}_q$. In the Euclidean formulation the anti-commuting anti-quark spinor field $q^\dagger$ is an independent field, and it has corresponding commuting topological ghost and anti-ghost $\Psi_q^\dagger$ and $\bar{\Psi}_q^\dagger$, and Lagrange multiplier field $b_q^\dagger$, with $sq^\dagger = \Psi_q^\dagger - q^\dagger c$, $s\Psi_q^\dagger = \Psi_q^\dagger c - q^\dagger\Phi$. For the anti-ghost one has again $s\bar{\Psi}_q^\dagger = b_q^\dagger$, $sb_q^\dagger = 0$ or $s\bar{\Psi}_q^\dagger = b_q^\dagger + \bar{\Psi}_q^\dagger c$, $sb_q^\dagger = -b_q^\dagger c - \bar{\Psi}_q^\dagger\Phi$. All these fields are functions of x and t. We take the topological action,

$$\begin{aligned} I_q &= \int d^4xdt\ s\big(\bar{\Psi}_q^\dagger\{D_5q - K[(\gamma^\mu D_\mu + m)q + ab_q]\ \}\big) \\ &= \int d^4xdt\ \big(b_q^\dagger\{D_5q - K[(\gamma^\mu D_\mu + m)q + ab_q]\ \} \\ &\qquad + \bar{\Psi}_q^\dagger[D_5\Psi_q - K(\gamma^\mu D_\mu + m)\Psi_q] + \ldots\big)\ , \end{aligned} \tag{8.2}$$

which must be added to the 5-dimensional action that we have introduced in the pure Yang–Mills case. Here $(\gamma^\mu D_\mu + m)q = 0$ is the Dirac equation of motion of the 4-dimensional theory and K is a kernel [24]. One can formally prove, by mere Gaussian integrations, that the equilibrium distribution is independent of the choice of the kernel K [5]. Since we are in the context of a renormalizable theory, it is in fact necessary to introduce a kernel with canonical dimensional equal to 1, that is $K = -\gamma^\mu D_\mu + M$. This is remarkable, since this coincides with the necessity of having a well-defined Langevin process for spinors, as discussed for instance in [5]. Quite interestingly, power counting would allows a term of the type $\Gamma^{\mu\nu}F_{\mu\nu}$ that is not proportional to the kernel: such term would contradict the stochastic interpretation of the 5-dimensional theory and could be the signal of an anomaly. For vector theories, such term should not be generated perturbatively.

Beyond solving the ambiguities which occur in a purely 4-dimensional approach, our work might offer new perspectives. We have already mentioned in sec. 6 the existence of a new algorithm and the conjecture that the mass spectrum is given by the eigenvalues of the Fokker-Planck hamiltonian.

One speculation is the question of understanding chiral fermions in a lattice formulation. The topological fermionic action (8.2) is interesting from this point of view. One may consider introducing a t-dependent mass term $\mu(t)$ in (8.2). This might lead to an action of the type used by Kaplan and Neuberger to solve the chiral fermion problem for the lattice [25][26][27]. In their picture, one introduces a tower of additional fermions, which can be combined by Fourier transform into a 5-dimensional fermion. One notices that the equation of motion of (8.2) reproduces their 5-dimensional Dirac equation when $\mu(t)$ is a step function. Moreover, if one gauges the fermion conservation number in the fifth dimension, which gives an additional $U(1)$ gauge field $A_{\mathrm{abel}}(x,t)$, then $\mu(t)$ could be understood as a special value for the gauge field component $A_{5\mathrm{abel}}(x,t)$, similar to a kink that cannot be set globally to zero by local gauge-fixing. Therefore the choice of $\mu(t)$ could be determined by topological considerations. This issue of seeing whether one can understand the 4-dimensional space in which the stochastic process stops as a domain wall in which everything that remains from the fields of the topological multiplet of q is a chiral fermion seems to us to be great interest.

The second perspective is that of a dual formulation which would now involve 2-form gauge fields. It is worth indicating the idea here. In the 5-dimensional framework, the dual of a one-form is a 2-form, since the sum of the degrees of the curvatures of A and its dual must equal 5.

If one restricts to the abelian part, one has various possibilities for expressing the theory by using arguments as in [28]. The dual action, up to gauge-fixing and supersymmetric terms, should take the form of a very simple 5-dimensional topological term:

$$\int_5 B_2 \wedge dB_2. \tag{8.3}$$

Such a Chern–Simons type action is interesting, since it can be gauge-fixed in the framework of the self-duality for 2-form gauge fields [29]. Thus, the question also arises whether a reliable link can be established between the TQFT generated by (8.3) and the physical Yang–Mills theory!

Let us finally mention the following observation. Suppose we start from an action with two abelian two-forms in six dimensions, $\int_6 dB_2 \wedge d\bar{B}_2$. The field content that one obtains by dimensional reduction in four dimensions will be made of 2 two-forms, 4 one-forms and 2 zero-forms. But the 2 two-forms are equivalent by duality to 2 zero-form in four dimensions. So, we obtain a field content equivalent to 4 one-forms and 4 zero-forms. It is striking that this is the bosonic content of the Weinberg-Salam model, with its four gauge fields and four scalars, which are the components of a complex $SU(2)$ doublet.

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Appendix A. Derivation of lattice Fokker-Planck equation

We shall demonstrate that the discrete Markov process defined in sec. 6 is described by the Fokker-Plack equation of sec. 6 in the limit $\epsilon \to 0$. We shall derive the lattice Fokker-Planck hamiltonian H_{FP} from the transition matrix $\mathcal{T}(U_{t+1}, U_t)$, eq. (6.1) , in the same way that the Kogut-Susskind hamiltonian is derived from the transfer matrix of lattice gauge theory. We follow here the discussion of Creutz [30]. We introduce a basis of states labelled by group elements $|U>$, where $U = \{U_{x,\mu}\}$, with operators

$$\hat{U}_{x,\mu}|U> = U_{x,\mu}|U> \quad . \tag{A.1}$$

We must find an operator T with the property

$$< U'|T|U> = \mathcal{T}(U', U). \tag{A.2}$$

To order ϵ it will have the form $T = 1 - \epsilon H_{FP}$, which will allow us to identify H_{FP}.

From eqs. (6.1), (6.3), (6.20) and (6.37), we have

$$\mathcal{T}(U',U) = \mathcal{N} \exp\Big(- (\epsilon g_0^2)^{-1} \sum_{x,\mu} \mathrm{Re}\ \mathrm{Tr}(I - U'^{-1}_{x,\mu} H_{x,\mu} U_{x,\mu})\Big) \ , \tag{A.3}$$

Here $H_{x,\mu} \in SU(N)$ is given by

$$H_{x,\mu} \equiv \exp(-\epsilon v_x) \exp(\epsilon f_{x,\mu}) \exp(\epsilon\ U_{x,\mu} f_{x+\hat{\mu},\mu} U^{-1}_{x,\mu}) \ , \tag{A.4}$$

and the gauge-fixed value of $A_{x,5}$ appears, namely $A_{x,5} = \epsilon v_x$ where

$$v_x \equiv -G_x \mathcal{F} \ . \tag{A.5}$$

We write

$$H_{x,\mu} = \exp(\epsilon k_{x,\mu}) \ , \tag{A.6}$$

where $k_{x,\mu}$ may be interpreted as the total drift force. To order ϵ, it is given by

$$k_{x,\mu} \ = f_{x,\mu} + (D_\mu v)_x \ , \tag{A.7}$$

where D_μ is the lattice gauger covariant difference, eq. (6.30). It is the sum of the drift force $f_{x,\mu} = -E_{x,\mu} S$ and a "gauge-fixing force" $(D_\mu v)$.

Let T_0 be the operator whose matrix elements give the transition probability with total drift force $k_{x,\mu} = 0$,

$$< U'|T_0|U >= \mathcal{N} \exp\Big(- (\epsilon g_0^2)^{-1} \sum_{x,\mu} \mathrm{Re}\ \mathrm{Tr}(I - U'^{-1}_{x,\mu} U_{x,\mu})\Big). \tag{A.8}$$

We have

$$< U'|T|U >=< U'|T_0| \exp(\epsilon k) U >, \tag{A.9}$$

where $\exp(\epsilon k) U \equiv \{\exp(\epsilon k_{x,\mu}) U_{x,\mu}\}$. We insert a complete set of states,

$$< U'|T|U >= \int dU'' < U'|T_0|U'' >< U''| \exp(\epsilon k) U > \ . \tag{A.10}$$

To order ϵ we have

$$\begin{aligned} < U''| \exp(\epsilon k) U > &= \delta(U'', \exp(\epsilon k) U) = \delta(\exp(-\epsilon k) U'', U) \\ &= \Big(1 - \epsilon \sum_{x,\mu} E''_{x,\mu} k_{x,\mu}(U)\Big) \delta(U'', U), \end{aligned} \tag{A.11}$$

where $E''_{x,\mu}$ is the color-electric field derivative operator defined in sec. 6, that acts on the variable U''. Let $\hat{E}_{x,\mu}$ be the corresponding quantum mechanical color-electric field operator. In terms of operators we have shown that to order ϵ

$$T = T_0 \Big(1 - \epsilon \sum_{x,\mu} \hat{E}^a_{x,\mu} k^a_{x,\mu}(\hat{U})\Big) \ . \tag{A.12}$$

The evaluation of T_0 may be found in [30], with the result

$$T = 1 + \epsilon \sum_{x,m} \{\hat{E}^a_{x,\mu}\Big(g_0^2 \hat{E}^a_{x,\mu} - k^a_{x,\mu}(\hat{U})\Big)\} \ . \tag{A.13}$$

Thus with $T = 1 - \epsilon H_{FP}$ and $k_{x,\mu} = f_{x,\mu} + (D_\mu v)_x$, and $f_{x,\mu} = -E_{x,\mu} S$, we obtain the Fokker-Planck hamiltonian

$$H_{FP} = -\sum_{x,m} E^a_{x,\mu}\Big(g_0^2 E^a_{x,\mu} - (D_\mu v)_x + E_{x,\mu} S\Big) \ , \tag{A.14}$$

where $v_x = -G_x \mathcal{F}$.

Appendix B. Equivalence of topological lattice action and lattice Fokker-Planck equation

We wish to show that the 5-dimensional partition function Z, eqs. (7.46) and (7.48), that was obtained from the topological lattice action by integrating out all the auxiliary fields, is a path integral representation of the solution to the Fokker-Plank equation, $\partial_5 P = -H_{FP} P$, with Fokker-Planck hamiltonian H_{FP} given in eq. (A.14).

For this purpose consider the integral defined for arbitrary $O(U')$ by

$$\begin{aligned} \bar{O}(U) &\equiv \int dU' O(U') \mathcal{K}(U', U) \\ &\equiv \int dU' \ O(U') \ \mathcal{N} \exp\Big(- (4\epsilon)^{-1} \sum_{x,\mu} (B_{x,\mu} - \epsilon k_{x,\mu})^2\Big) \ , \end{aligned} \tag{B.1}$$

where $k_{x,\mu} = k_{x,\mu}(U)$ is defined in eq. (A.7), and $B_{x,\mu}$ is defined by $\exp(t_a B^a_{x,\mu}) = U'_{x,\mu} U^{-1}_{x,\mu}$. It will be sufficient to show that in the limit $\epsilon \to 0$ this expression corresponds on the one hand to a discretization of the lattice action (7.48), and on the other hand yields

$$\bar{O}(U) = (1 - \epsilon H^\dagger_{\rm FP}) O(U) \ . \tag{B.2}$$

We change variable of integration by translation on the group $U' = VU$, with $dU' = dV$. With $V_{x,\mu}$ parametrized by $V_{x,\mu} = \exp(t_a B^a_{x,\mu})$, we have

$$\bar{O}(U) = \mathcal{N} \int dB \rho(B)\ O\Big(\exp(B)U\Big)\ \exp\Big(-(4\epsilon)^{-1} \sum_{x,\mu} (B_{x,\mu} - \epsilon k_{x,\mu})^2 \Big)\ , \tag{B.3}$$

where $\rho(B)$ is Haar measure. We again change variable of integration by $B_{x,\mu} = \epsilon^{\frac{1}{2}} C_{x,\mu} + \epsilon k_{x,\mu}$, and obtain

$$\bar{O}(U) = \int dC\ \rho(\epsilon^{\frac{1}{2}} C + \epsilon k)\ O(\exp(\epsilon^{\frac{1}{2}} C + \epsilon k)U)\ \mathcal{N} \exp\Big(-\frac{1}{4} \sum_{x,\mu} C^2_{x,\mu} \Big)\ . \tag{B.4}$$

We now expand to order ϵ,

$$\begin{aligned} \rho_{x,\mu}(\epsilon^{\frac{1}{2}} C + \epsilon k) &= 1 + \text{const} \times \epsilon C^2_{x,\mu} \\ O\Big(\exp(\epsilon^{\frac{1}{2}} C + \epsilon k)U\Big) &= \Big(1 + (\epsilon^{\frac{1}{2}} C^a_{x,\mu} + \epsilon k^a_{x,\mu}) E^a_{x,\mu} + \frac{1}{2} \epsilon C^a_{x,\mu} C^b_{y,\nu} E^a_{x,\mu} E^b_{y,\nu}\Big) O(U), \end{aligned} \tag{B.5}$$

where a sum over repeated indices is understood. When this is substituted into (B.4), the contribution from the expansion of ρ is cancelled by the normalization constant $\mathcal{N}$, and one obtains eq. (B.2) , with

$$H^\dagger_{\rm FP} = -\sum_{x,\mu} \Big(E_{x,\mu} + k_{x,\mu}(U)\Big) E_{x,\mu}\ , \tag{B.6}$$

as desired.

We next show that the kernel in (B.1) corresponds to a discretization of the path integral with action (7.48). In the preceding calculation the drift force $k = k(U)$ depends on U but not on U', whereas the path integral (7.46) and (7.48) implicitly uses the symmetrized drift force

$$k_s \equiv \frac{1}{2}\Big(k(U) + k(U')\Big)\ . \tag{B.7}$$

So to compare with the path integral, we express the integral (B.1) in terms of k_s plus a correction which will turn out to be the well-known Ito term. We have

$$k(U) = k_s + \frac{1}{2}\Big(k(U) - k(U')\Big)\ , \tag{B.8}$$

so, with $U' = \exp(B)U$, we may write to the order of interest

$$k(U) = k_s - \frac{1}{2} \sum_{y,\nu} B^a_{y,\nu} E^a_{y,\nu} k\ , \tag{B.9}$$

because the last term gives a contribution of order ϵ as we shall show. This gives, to the order of interest,

$$\exp\{-(4\epsilon)^{-1}\sum_{x,\mu}\Big(B_{x,\mu}-\epsilon k_{x,\mu}(U)\Big)^2\} = \Big(1-\frac{1}{4}\sum_{x,\mu,y,\nu}B^b_{x,\mu}B^a_{y,\nu}E^a_{y,\nu}k^b_{x,\mu}\Big) \times \exp\Big(-(4\epsilon)^{-1}\sum_{x,\mu}(B_{x,\mu}-\epsilon k_{s,x,\mu})^2\Big)\ . \tag{B.10}$$

Upon substituting this expression into (B.1), one obtains a Gaussian integral and Gaussian expectation value

$$< B^b_{x,\mu}B^a_{y,\nu} >= 2\epsilon\delta_{x,y}\delta_{\mu,\nu}\delta^{a,b}\ , \tag{B.11}$$

which is indeed of order ϵ, as asserted. Consequently in the exponent we may replace $B^b_{x,\mu}B^a_{y,\nu}$ by its expectation value, and to the order of interest we have

$$\exp\{-(4\epsilon)^{-1}\sum_{x,\mu}\Big(B_{x,\mu}-\epsilon k_{x,\mu}(U)\Big)^2\} \to \exp\{-\sum_{x,\mu}\Big((4\epsilon)^{-1}(B_{x,\mu}-\epsilon k_{s,x,\mu})^2 + \frac{1}{2}\epsilon E^a_{x,\mu}k^a_{x,\mu}\Big)\}\ , \tag{B.12}$$

the last term being the Ito term. Observe that with $U' \to U_{t+1}$ and $U \to U_t$, and $U'_{x,\mu}U^{-1}_{x,\mu} = \exp(B_{x,\mu})$, we have $B_{x,\mu} \sim \epsilon\dot{U}_{x,\mu}U^{-1}_{x,\mu}$, so the last expression may be written formally as

$$\exp\{-\epsilon\sum_{x,\mu}\Big(\frac{1}{4}(\dot{U}_{x,\mu}U^{-1}_{x,\mu}-\epsilon k_{s,x,\mu})^2+\frac{1}{2}E^a_{x,\mu}k^a_{x,\mu}\Big)\}\ . \tag{B.13}$$

We have

$$E^a_{x,\mu}k^a_{x,\mu} = -E^a_{x,\mu}E^a_{x,\mu}S - G^a_xG^a_x\mathcal{F}\ , \tag{B.14}$$

and the last expression corresponds to the action (7.48), as asserted.

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