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TOMOGRAPHIC RECONSTRUCTION OF TRANSVERSE PHASE SPACE FROM TURN-BY-TURN PROFILE DATA

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Abstract

Tomographic methods have the potential for useful application in beam diagnostics. The tomographic reconstruction of transverse phase space density from turn-by-turn profile data has been studied with particular attention to the effects of dispersion and chromaticity. It is shown that the modified Algebraic Reconstruction Technique (ART) that deals successfully with the problem of non-linear motion in the longitudinal plane cannot, in general, be extended to cover the transverse case. Instead, an approach is proposed in which the effect of dispersion is deconvoluted from the measured profiles before the phase space picture is reconstructed using either the modified ART algorithm or the inverse Radon Transform. This requires an accurate knowledge of the momentum distribution of the beam and the modified ART reconstruction of longitudinal phase space density yields just such information. The method has been tested extensively with simulated data.

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1 Tomography Algorithms

1.1 The Inverse Radon Transform

The idea of tomography is to reconstruct a distribution from a large number of projections taken at different angles. There are many algorithms for tomographic reconstruction.

The (2D) Radon transform [1] $r(s, \theta)$ of a distribution $\rho(x, y)$ is defined as

$$r(s,\theta) = \int_{-\infty}^{\infty} \rho(s\cos\theta - u\sin\theta, s\sin\theta + u\cos\theta)\, du, \tag{1}$$

where $0 < \theta < \pi$. For a given θ it gives the projection of $\rho(x, y)$ onto a line through the origin at an angle θ. The inverse of the Radon transform is

$$\rho(x,y) = \int_{0}^{\pi} \tilde{r}(x\cos\theta + y\sin\theta, \theta)\, d\theta, \tag{2}$$

where $\tilde{r}(s, \theta)$ can be written in terms of the Fourier transform $R(\xi, \theta)$ of $r(s, \theta)$ as

$$\tilde{r}(s,\theta) = \int_{-\infty}^{\infty} |\xi|\ R(\xi,\theta)\, e^{i2\pi\xi s}\, d\xi \tag{3}$$

which is just r filtered with a filter whose frequency response is $|\xi|$. If a number of projections $r(s_i, \theta_j)$ are known from measurements, an approximation of $\rho(x, y)$ can be made using a discrete version of (2). However, since $|\xi| \to \infty$ in (3), high frequency noise is strongly amplified and thus, in practice, an additional low-pass filter is needed.

1.2 The Algebraic Reconstruction Technique

Algebraic Reconstruction Technique (ART) [2] is an iterative method. It exploits the fact that each point in a projection corresponds to a line in the reconstructed picture. The projections are thus "back-projected" in such a way that the value at each point in a projection is distributed along the corresponding line in the picture. This yields a crude approximation. The approximation is then projected down again, and each projection of the approximation is compared to the original projection. The difference between the two is back-projected again. In this way, the approximation is improved until it converges, and the iteration is terminated.

2 Tomography of Transverse Phase Space

2.1 The Problem of Dispersion

If higher order effects are neglected, transverse phase space density performs a rigid rotation in phase space for each turn in a circular machine. This is manifest in normalised phase space, where all particles follow circular trajectories, rotating at the betatron frequency. Thus, performing a tomographic reconstruction from a number of transverse profiles where the dispersion is zero is trivial. With a non-zero dispersion, it is not as obvious. Two possible solutions have been tested and the results are summarised below.

2.2 Modified ART with Dispersion

Recently, a modified ART algorithm has been used with great success to reconstruct particle density in longitudinal phase space [3], even when the motion is strongly non-linear. The algorithm is based on the tracking of test particles. By changing the tracking routine, the code was adapted to tackle the transverse case with dispersion. Dispersion was included by giving the test particles an extra degree of freedom (momentum) with a statistical distribution given by the beam momentum spread, which was assumed to be Gaussian. It was found, however, that this approach does not work. An ART reconstruction that includes dispersion in the tracking code cannot resolve details blurred by dispersion.

2.3 Deconvolution of Dispersive Effects

The dispersive blurring effect can be removed directly from the individual profiles, knowing that the physical beam profile is given by the convolution

$$\rho_\sigma(x) = \int \rho_\beta(z) \cdot \rho_D(x-z)\, dz, \tag{4}$$

where $\rho_\beta(x)$ is the pure betatronic profile, $\rho_D(x)$ is the dispersive spread and $\rho_\sigma(x)$ the measured profile. Since the profiles are measured in discrete points, the discrete equation

$$\rho_\sigma(x_i) = \sum_j \delta_D(x_{i-j}) \cdot \rho_\beta(x_j) \tag{5}$$

applies, which can be written in matrix form as

$$\overline{\rho}_\sigma = H \cdot \overline{\rho}_\beta \tag{6}$$

where H is a band matrix constructed from $\rho_D(x)$. Thus, the betatronic profile $\rho_\beta(x)$ can be recovered by inverting the matrix H. However, band matrices are known to be numerically ill-conditioned. Therefore, measurement errors and noise will be strongly amplified. In order to achieve a useful result, (6) has to be regularized in some way to make it numerically stable. Several schemes exist. The scheme which has been mainly used in this work is the so-called Hunt regularization method [4]. It is an extension of the well known least squares fit, where an extra term is added in the minimization function to control the second derivative of the result. If $\tilde{\rho}_\sigma$ denotes the noisy measured profile, the deconvolution result is the vector $\tilde{\rho}_\beta$ that minimizes the functional

$$J_\alpha(\rho_\beta) = (\tilde{\rho}_\sigma - \tilde{H} \cdot \rho_\beta)^T (\tilde{\rho}_\sigma - \tilde{H} \cdot \rho_\beta) + \alpha (C \cdot \rho_\beta)^T (C \cdot \rho_\beta) \tag{7}$$

for $\alpha > 0$. The parameter α is the regularization constant and the regularization function C is the matrix of the numeric second derivative

$$C = \begin{pmatrix} -2 & 1 & 0 & \cdots \\ 1 & -2 & 1 & \\ 0 & 1 & -2 & \\ \vdots & & & \ddots \end{pmatrix}. \tag{8}$$

The solution to the minimization problem can be written as

$$\tilde{\rho}_\beta = (\tilde{H}^T \cdot \tilde{H} + \alpha C^T \cdot C)^{-1} \cdot H^T \cdot \tilde{\rho}_\sigma. \tag{9}$$

The result is dependent of the value of α. As a rule of thumb, its value should be chosen so that

$$(\tilde{\rho}_\sigma - \tilde{H} \cdot \tilde{\rho}_\beta)^T (\tilde{\rho}_\sigma - \tilde{H} \cdot \tilde{\rho}_\beta) = (\tilde{\rho}_\sigma - \rho_\sigma)^T (\tilde{\rho}_\sigma - \rho_\sigma), \tag{10}$$

where the right hand side of the equation is just the rms error of the measured profiles, which can be estimated.

The Hunt regularization scheme is not the only possible one. An approximate solution of (6) can also be obtained for example by using singular value decomposition (SVD) techniques. The idea is to eliminate very small singular values from the matrix H before solving the equation by means of the least squares fit.

When the profiles have been deconvoluted, transverse phase space can be reconstructed using either ART or the inverse Radon transform.

2.4 How to Measure the Dispersive Spread?

In order to deconvolute the dispersive effect from the measured beam profiles, a very accurate measurement of the momentum distribution is needed. Attempts to use an assumed distribution (ie Gaussian or parabolic) with the right measured rms width have been made with little success due to the strong error amplification. However, using tomographic methods in the longitudinal plane an accurate picture of the longitudinal phase plane can be obtained. Projecting onto the energy axis gives the momentum spread. The expected accuracy, estimated from simulations, is better than 1%. The momentum distribution then has to be scaled by the dispersion at the transverse profile monitor to obtain the dispersive spread. This has to be done for each transverse profile, since in particular the dispersion can vary significantly between the first few turns after injection into a circular machine.

3 Simulation Results

3.1 Simulation and Reconstruction Codes

Several pieces of code have been written to test the method. A simple 4D tracking code has been implemented, which produces both longitudinal and transverse mountain range data for the reconstructions.

For the longitudinal reconstruction, the code was readily available and tested [5].

The Hunt deconvolution has been implemented in *Mathematica*, as well as the inverse Radon transform for transverse reconstruction. A modified version of the longitudinal reconstruction code has been produced, which can handle the ART reconstruction in the transverse plane.

It was found that the inverse Radon transform and the ART code give similar results. However, the results from the inverse Radon transform tend to be noisier. Therefore, the ART code has been used mainly, with occasional cross-checks using the inverse Radon transform. ART also has the additional advantage that nonlinear transverse beam dynamics can be treated, although this has not been done here.

3.2 Reconstructions

For the simulations presented here, a parabolic energy-phase distribution was used. The momentum spread (2σ) was roughly 1.2×10^{-3}, and the dispersion at the profile measurement device 3.0m.

In the transverse plane, a test distribution with a doughnut shape was used to check the resolving power of the method. The distribution is shown in Fig. 1. To simulate noise and cut the simulation time, an insufficient number of test particles was used. Ideally, a larger number of particles should be tracked and noise added afterwards, but for a proof-of-principle this was considered sufficient.

To show the importance of handling the dispersion problem, a reconstruction of transverse phase space without preceding deconvolution of the measured profiles is shown in Fig. 2.

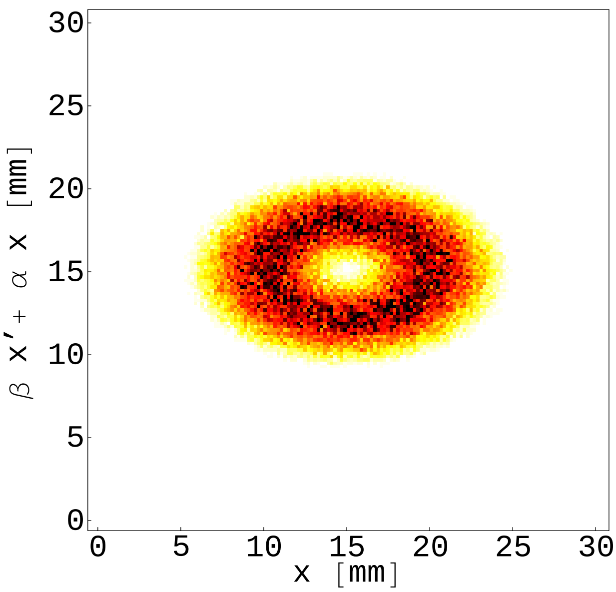

Figure 1: The test distribution in transverse phase space.

With deconvolution of the dispersion effect, the picture is much more like the original (Fig. 3). Deconvolution using SVD also works and yields a very similar picture for the tested distribution.

4 Concluding Remarks

It has been shown that the transverse phase space distribution can be reconstructed using tomographic methods, even in the presence of dispersion. The deconvolution of the dispersive effect requires very accurate input data, since any noise is strongly amplified. The final result depends on the momentum spread, the dispersion, and the accuracy of the measurement.

Tests of the method with measured data will be done in 1999, using a SEM-grid in the CERN PS and an OTR (Optical Transition Radiation) screen in the SPS for the acquisition of transverse profiles.

References

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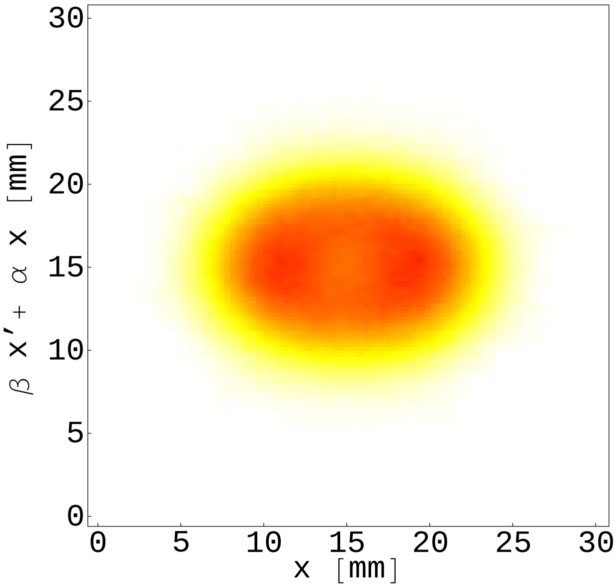

Figure 2: Reconstruction with ART, using no dispersion correction.

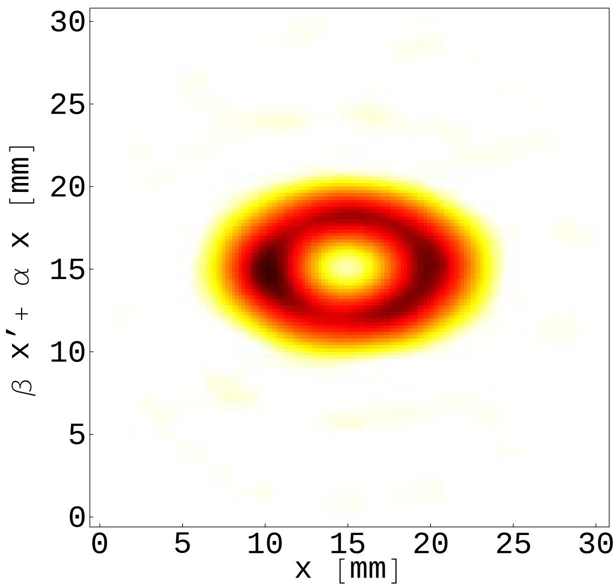

Figure 3: Reconstruction with ART, using dispersion correction.