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UNIVERSALITY OF THE OFF-EQUILIBRIUM CRITICAL FRAGMENTATION

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Abstract

Universality of the off-equilibrium fragmentation processes is discussed using the Fragmentation-Inactivation-Binary process. The signature of criticality associated with the scaling laws of the order parameter fluctuations are established and importance of these studies in various fields of physics including nuclear and condensed matter physics is pointed out.

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[1]Invited talk given by R. Botet at the Workshop on *Nuclear Matter in Different Phases and Transitions*, March 31 - April 10, 1998, Centre de Physique des Houches, Les Houches, France

I. INTRODUCTION

Fragmentation is the universal process which can be found at all scales in the nature, going from the parton cascading in ultrarelativistic collisions of leptons or hadrons, the multifragmentation in nuclear heavy-ion (HI) collisions or collisions of atomic clusters, up to the formation of large-scale structures in the Universe. Similarly as for the critical off-equilibrium aggregation processes [1], one expects only few universality classes among various off-equilibrium fragmentation processes. For that reason, we have investigated the general sequential binary and conservative fragmentation processes, called the Fragmentation-Inactivation Binary (FIB) process, with scale-invariant fragmentation and with either scale-invariant or scale-dependent inactivation [2–4] . The phase diagramme of these off-equilibrium processes has been established and the universal aspects of both the fragment size distribution and the total number of fragments distribution (*i.e.*, the multiplicity distribution) are known in the case of scale-invariant inactivation functions [3,5] . The FIB model has been successfully applied for the description of nuclear multifragmentation data for fragment-size distribution and fragment-fragment correlations at intermediate energy HI collisions [6] and to the multiplicity data in e^+e^- [7] .

One of the most important open problems in physics of fragmentation is the signature of possible criticality in small systems in terms of quantities which do not refer to the thermodynamic equilibrium and as such can be used both in the equilibrium as well as in the off-equilibrium fragmentation processes. In this context, we show that probability distribution of the order parameter, which has distinctly different features at the critical point and outside of it, may be extremely useful in phenomenological applications both in energetic collisions of hadrons, leptons or nuclei as well as in physics of atomic clusters and polymers.

II. THE MODEL

In the FIB model [2] , one deals with fragments characterized by some conserved scalar quantity that will be called in the following the *cluster mass* . The anscestor cluster of mass N is fragmenting via an ordered and irreversible sequence of steps. The first step is either a binary fragmentation, $(N) \to (j) + (N-j)$, or an inactivation $(N) \to (N)^*$. Once inactive, the cluster cannot be reactivated anymore. The fragmentation leads to two clusters, with the mass partition probability $\sim F_{j,N-j}$. In the following steps, the process continues independently for each active descendant cluster until either the low mass cutoff for further indivisible particles (monomers) is reached or all clusters are inactive. For any event, the fragmentation and inactivation occur with the probabilities per unit of time $\sim F_{j,k-j}$ and $\sim I_k$ respectively. The fragmenting system and its evolution is completely specified by these rate functions and by the initial state. It is also useful to consider the fragmentation probability p_F without specifying masses of descendants : $p_F(k) = \sum_{j=1}^{k-1} F_{j,k-j} \; (I_k + \sum_{j=1}^{k-1} F_{j,k-j})^{-1}$. If the instability of smaller clusters is smaller than instability of larger clusters, $p_F(k)$ is an increasing function of cluster mass and the total mass is converted into finite size clusters. This is the *shattered phase.* Average cluster multiplicity in this phase is : $< m > \sim N$. The cluster mass independence of $p_F(k)$ at any stage of the process until the cutoff-scale for monomers characterizes the *transition region.* In the domain $p_F > 1/2$, the average multiplicity is : $< m > \sim N^{\tau-1}$ $(1 \leq \tau \leq 2$, *i.e.*, $< m > \nearrow N$) , where τ is the exponent of the power-law cluster-size distribution : $n(k) \sim k^{-\tau}$. It is convenient to define the multiplicity anomalous dimension : $\gamma = d(\ln < m >)/d(\ln N)$, which changes from 0 to 1 for for p_F changing from 1/2 to 1. For $p_F < 1/2$ at the transition line and in the *infinite-cluster phase* [2] , the average multiplicity is always finite, *i.e.*, $\gamma = 0$, whereas $\gamma = 1$ in the shattered phase. Shattering is the second order phase transition and the order parameter in this transition is the cluster multiplicity or monomer multiplicity, both of them closely interrelated.

FIGURES

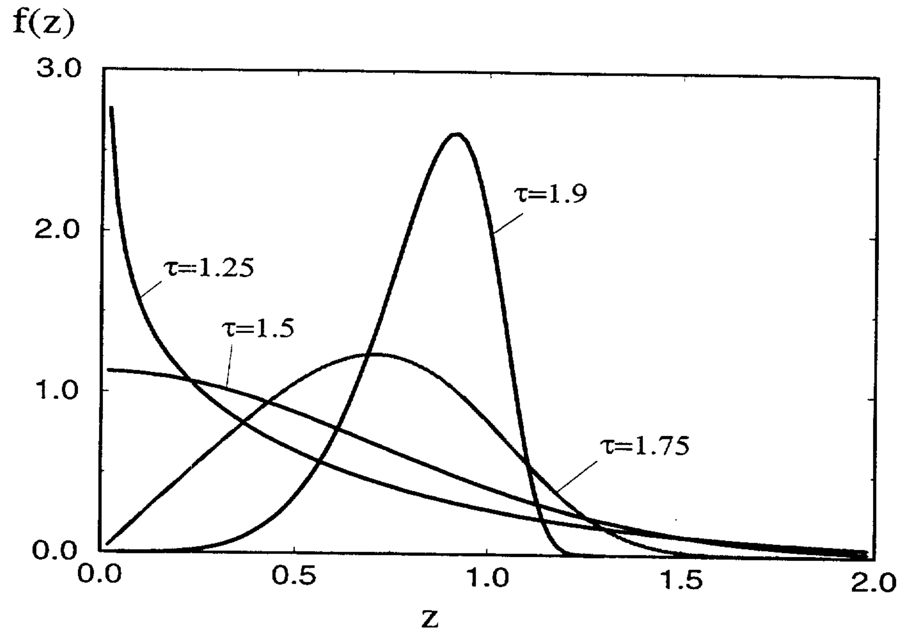

FIG. 1. **Typical multiplicity distributions in the critical fragmentation regime : $\alpha \geq -1$,$1/2 < p_F < 1$, are plotted in the first scaling variables ($\delta = 1$) (see Eq. (5)) for various values of τ .**

Most of interesting physical applications correspond to the homogeneous fragmentation functions : $F_{\lambda j,\lambda(N-j)} = \lambda^a F_{j,N-j}$, *e.g.*, $F_{j,N-j} \sim [j(N-j)]^\alpha$. These include the singular kernel $\alpha = -1$ in the perturbative quantum chromodynamics (PQCD) for gluons [8] , $\alpha = -2/3$ for the spinodal volume instabilities in three dimensions [2] , $\alpha = +1$ in the scalar $\lambda\phi_6^3$ field theory in six dimensions [9] , and many others [2] .

For the scale-invariant inactivation rate-function : $I_k \sim k^\beta$, the transition line corresponds to : $\beta = 2\alpha + 1$, and : $\beta < 2\alpha + 1$, in the shattered phase. In the latter case, the cluster-size distribution is a powerlaw with the exponent $\tau > 2$, though system is not critical.

A. Evolution equations of the multiplicity (the order parameter) distributions in FIB process

Below we shall discuss equations for the time-evolution of the multiplicity distribution in FIB model. They are important for several reasons. Firstly, the cluster multiplicity is the order parameter in FIB process and hence important informations about universal features of the order parameter fluctuations can be extracted from these equations. Secondly, after a suitable change of variables, these equations allow to study the relation between the FIB process and the PQCD, which provides a foundation of our understanding of the ultrarelativistic collisions.

Let us call $P_N[m;t]$ the probability to get a cluster multiplicity m at time t, starting from initial cluster of size N at $t = 0$. The time evolution equation for the multiplicity is given by the following non-linear rate equations :

$$\frac{\partial P_N[m;t]}{\partial t} + \left(I_N + \sum_{j=1}^{N-1} F_{j,N-j} \right) P_N[m;t] =$$

$$= \sum_{j=1}^{N-1} F_{j,N-j} \sum_{m'=1}^{m-1} P_j[m';t] \; P_{N-j}[m-m';t]$$

$$+ I_N \delta(m-1) \quad . \tag{1}$$

In terms of the generating function : $Z_N(u,t) = \sum_{m=1}^{\infty} P_N[m,t](1+u)^m$, one obtains :

$$\frac{\partial Z_N}{\partial t}(u,t) = \sum_{j=1}^{N-1} F_{j,N-j}[Z_j(u,t) \; Z_{N-j}(u,t) \; - \; Z_N(u,t)] +$$

$$+ I_N[1+u-Z_N(u,t)] \quad , \tag{2}$$

with the initial condition (monomer cannot break up) : $Z_1(u,t) = 1+u$ and the normalization condition : $Z_N(u=0,t) = 1$. Note that the partial derivative is taken at a fixed size N. The sum on the right hand side of eq. (2) represents binary fragmentation of the primary cluster N into the daughter clusters of mass j and $N-j$ respectively. The second term on the right hand side is responsible for the inactivation and it is in the essence the dissipative term.

1. Example : The gluodynamics with nonperturbative dissipation in QCD jets

We can transform the discrete variable j in (2) into a continuous one : $z = j/N$, which varies from 0 to 1. The time t appearing in (2), arises within the fragmentation and inactivation kernels , which themselves are probabilities per unit of t. We define then the time as : $t = T \ln Y$, where T is a constant, $Y = \ln(N\Theta/Q_0)$, $Q_0 = const$ and Θ plays the role of time, ordering the sequence of events.

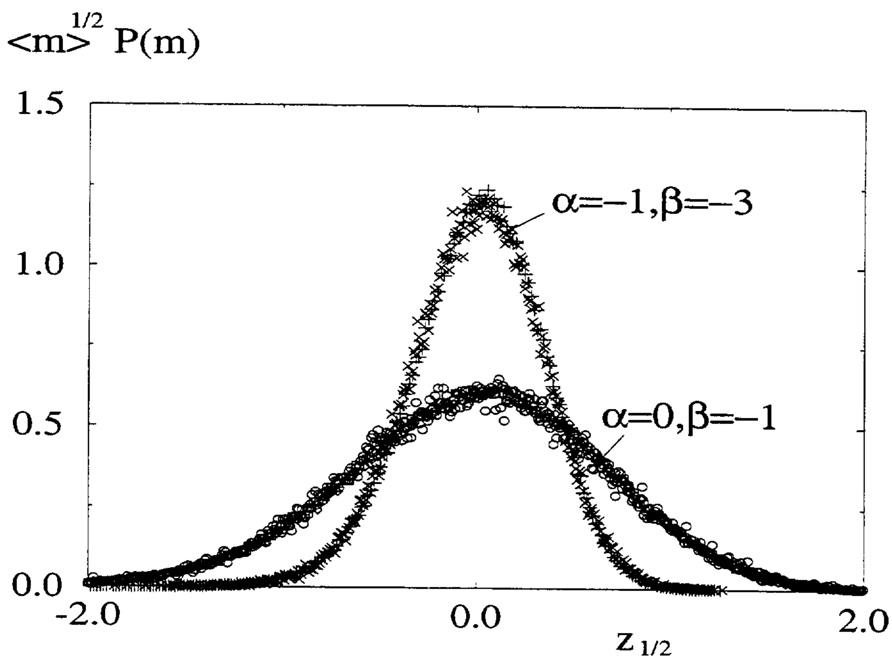

FIG. 2. Multiplicity distribution in the shattered phase are plotted in the variables of the second scaling ($\delta = 1/2$) (see Eqs. (4), (5)) for systems of different sizes $N = 2^{10}, 2^{12}, 2^{14}$. Two different sets of parameters are used, corresponding both to the same value $\tau = 4$: ($\alpha = -1, \beta = -3$ and $\alpha = 0, \beta = -1$) .

Assuming now that all physical quantities depend only on the variable Y and not on N and Θ separately, we transform (2) into :

$$\frac{\partial Z}{\partial Y}(Y,u) = \int_0^1 {\gamma_0}^2(Y)\Phi_{z,1-z}[Z(Y+\log z, u)Z(Y+\log(1-z), u) -$$

$$- Z(Y,u)]dz + R(Y,u) \quad , \tag{3}$$

where $R(Y,u) = \mathcal{I}(Y)[1+u-Z(Y,u)]$, and $\mathcal{I}(Y)$ is the inactivation function. The initial and normalization conditions are : $Z(0,u) = 1+u$, $Z(Y,0) = 1$, and : ${\gamma_0}^2(Y) \equiv (2N_C\pi)/\alpha_s(Y)$, where $\alpha_s(Y)$ is the QCD running coupling constant. Eq. (3) is analogous to the PQCD gluodynamics rate equations in the modified leading-log approximation if N is the initial momentum, Θ is the angular width of the gluon jet considered and T is chosen such that $T = 12N_C/(11N_C - 2N_f)$, where N_C and N_f are the number of colours and flavours respectively. Standard gluodynamics equations correspond to neglecting the *dissipation term* $R(Y,u)$ in (3). Notice, that the precise form of this term follows from the identification of the dissipation mechanism in QCD jets. Presently available data on multiplicity distributions in e^+e^- reactions are perfectly described using the finite-scale Gaussian inactivation [7].

B. Scaling features of the order parameter distributions

Cluster multiplicity distribution : $P(m) = \sum_k P_k(m)$, where $P_k(m)$ is the probability distribution of the number of clusters of mass k is the basic observable in physics of fragmentation and aggregation phenomena. Of particular importance is the asymptotic scaling of multiplicity probability distributions [5,10] :

$$< m >^\delta P(m) = \Phi(z_{(\delta)}) \quad , \qquad z_{(\delta)} \equiv \frac{m- < m >}{< m >^\delta} \quad , \tag{4}$$

where the asymptotic behaviour is defined as $< m > \rightarrow \infty$, $m \rightarrow \infty$ for a fixed $(m/ < m >)$ – ratio. $< m >$ is the multiplicity of clusters averaged over an ensemble of events. The scaling law (4) means that, *e.g.*, data for differing energies (hence differing $< m >$) should fall on the same curve when $< m >^\delta P(m)$ is plotted against the scaled variable

$z_{(\delta)} \equiv (m- < m >)/ < m >^{\delta}$. Asymptotic scaling (4) with $\delta = 1$ has been suggested in the strong interaction physics [10] . $\delta = 1$ scaling, which corresponds to scale-independent fluctuations in $P(m)$, has been found in the *critical* transition region of scale-invariant FIB process for $1/2 < p_F < 1$ and $\alpha \geq -1$ [3] (see Fig. 1) . In the shattered phase, $P(m)$ obeys the second scaling limit with $\delta = 1/2$ [5] (see Fig. 2).

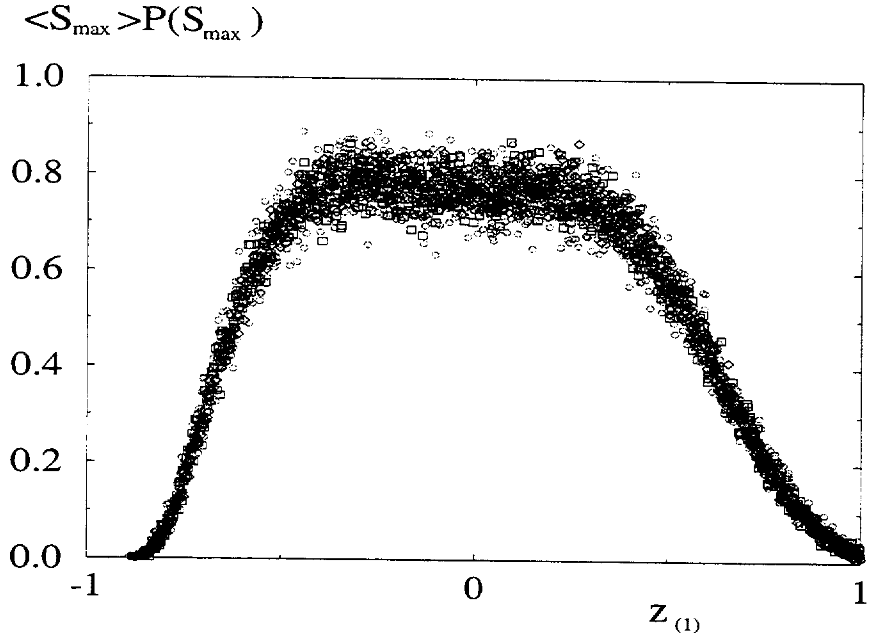

FIG. 3. The order parameter distribution in the 3D bond-percolation at the critical point is plotted in the variables of the first scaling ($\delta = 1$) (see Eq. (5)) for systems of different sizes $N = 10^3, 12^3, 14^3$ (respectively : diamonds, squares and circles) .

The fluctuations of order parameter are scale-invariant ($\delta = 1$) *only* in the critical sector of the FIB model ($\alpha \geq -1$,$1/2 < p_F < 1$) and correspond to the second scaling ($\delta = 1/2$) in all non-critical sectors whenever $< m > \nearrow N$. This property is more general and , actually, for *all* second-order phase transitions, both equilibrium and non-equilibrium ones, fluctuations of the order parameter M obey the first scaling ($\delta = 1$) at the critical point and the second scaling ($\delta = 1/2$) outside of it, *i.e.* :

$$< M >^{\delta} P(M) = f(z_{(\delta)}) \ , \qquad z_{(\delta)} \equiv \frac{M- < M >}{< M >^{\delta}} \quad . \tag{5}$$

In percolation, for example, the order parameter M is the reduced size of the largest cluster : $M \equiv S_{max}$. Figs. 3 and 4 show the scaling features of S_{max} in the 3D-percolation at the critical point (the first scaling) and outside of it (the second scaling) . M-fluctuations for the critical aggregation process have been discussed in [11] .

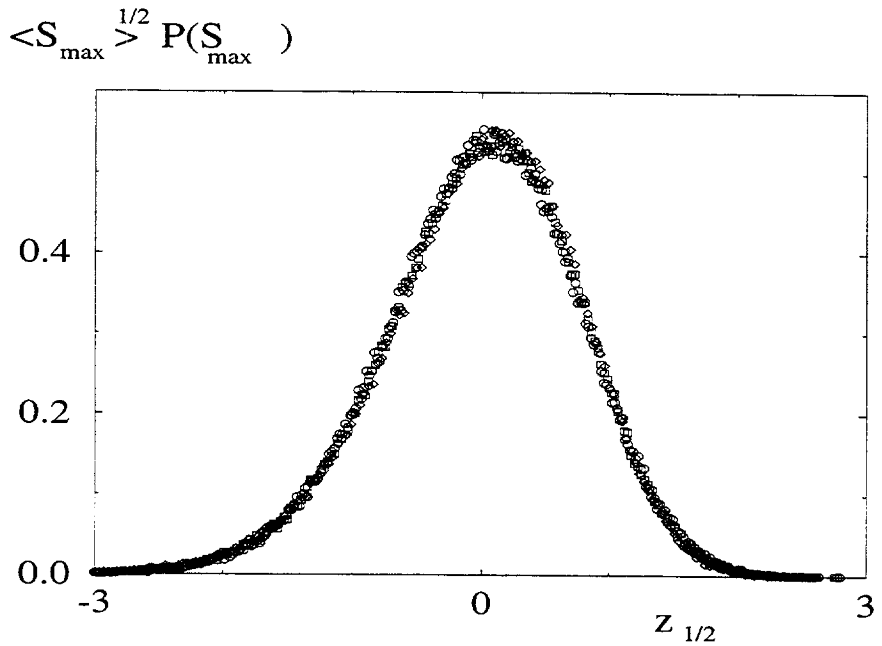

FIG. 4. The order parameter distribution in the 3D-percolation outside of the critical point ($p = 0.35 \neq p_{cr}$) is plotted in the variables of the second scaling ($\delta = 1/2$) (see Eq. (5)) for systems of different sizes $N = 10^3, 12^3, 14^3$ (respectively : diamonds, squares and circles) .

The essential information about the phase transition, the order parameter and its fluctuations is contained in the scaling function $f(z_{(\delta)})$, which is the benchmark of the phase transition. We have found that the scaling features are well seen even in relatively small systems what is important in the studies of transitional behaviour in small finite systems.

C. FIB model with the finite-scale dissipative effects

The dissipation which slows down and finally stops the off-equilibrium fragmentation process, is often characterized by a finite and usually small length scale. It is then an open question to which extent this process,

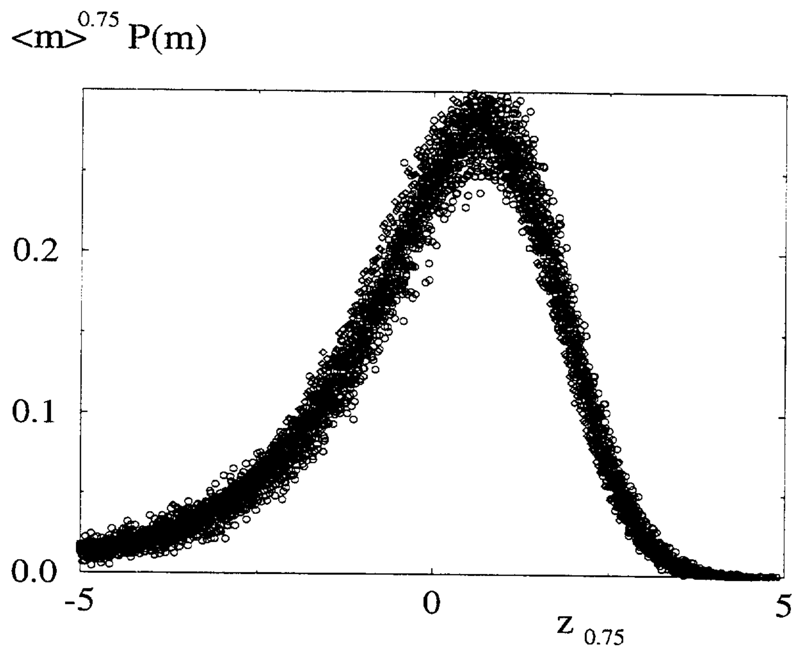

FIG. 5. The multiplicity distribution in FIB model for $\alpha = -1/3$ with the Gaussian inactivation (6) ($\beta = 0, c = 1, \sigma = 1$) in the scaling variables $\delta = 0.75$ are plotted for systems of different sizes : $N = 2^{10}, 2^{12}, 2^{14}$.

which on one side is driven by the homogeneous scale-invariant fragmentation rate function and on other side it is inactivated at a certain fixed scale by the random inactivation process, may develop scale-invariant and universal features in both the cluster mass distribution $n(k)$ and the cluster multiplicity distribution $P(m)$. This question is important in view of the widespread occurrence of scale-invariant cluster mass distributions and the lack of convincing arguments for using homogeneous dissipation functions in many processes including parton cascading in the PQCD [7] or the fragmentation of highly excited atomic nuclei, atomic clusters or polymers. This important question was investigated in [4] for certain fixed values of exponents : $\alpha = -1, +1$, of the fragmentation function $F_{j,k-j}$ and for the dissipation at small scales approximated by the Gaussian inactivation rate function :

$$I_k = ck^{\beta} \exp[-\frac{1}{2\sigma^2}\left(\frac{k-1}{N}\right)^2] \quad . \tag{6}$$

Limit $\sigma \to \infty$ yields ordinary scale-invariant inactivation.

As a generic case for $\alpha = -1$, one finds the scale-invariant region of power law fragment mass distributions with $\tau \le 2$ for σ above ~ 0.5 . The power law region is completely analogous to the critical sector of scale-invariant FIB model ($\alpha > -1$ and $1/2 < p_F < 1$) [2,3] . As in the latter case, the multiplicity anomalous dimension is : $0 \le \gamma \le 1$.

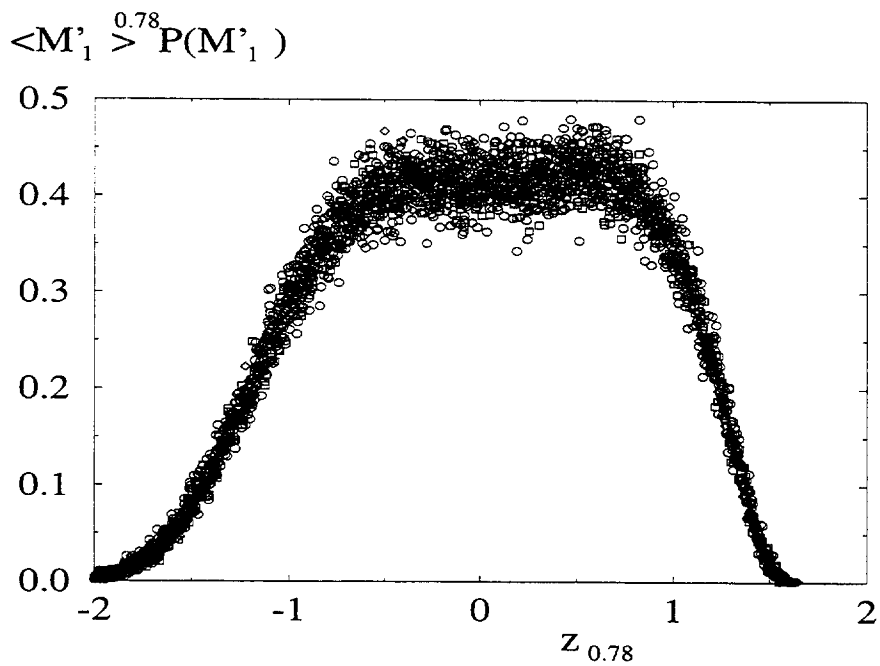

FIG. 6. 3D bond-percolation probability distribution of the variable $M_1' \equiv M_1 - S_{max}$, with M_1 the first moment of the size-distribution (*i.e.*, M_1' ≡mass of all clusters except for the largest one) at the percolation threshold is plotted for three different lattice sizes : $10^3, 12^3$ and 14^3 (respectively : diamonds, squares, circles) in the scaling variables (5) with $\delta = 0.78$.

For $\alpha = +1$ and σ above ~ 0.5 , the fragment size distributions is the power law with the exponent $\tau > 2$. The multiplicity anomalous dimension is : $\gamma = 1$, and the second scaling ($\delta = 1/2$) holds like in the shattered phase of scale-invariant FIB model [2] . These generic situations can be summarized as follows : if the fragment size distribution is a power law, the first scaling ($\delta = 1$) of multiplicity distributions is associated with $\tau \le 2$ and the second scaling ($\delta = 1/2$) of multiplicity distributions with $\tau > 2$ in both scale-invariant and scale-dependent regimes of dissipation. This clearly indicates a relation between the multiplicity scaling law and the fragment mass distribution scaling regimes in the FIB model. It also proves that in this range of parameters, the nature of the order parameter and its singularity is not modified by the finiteness of the dissipation scale in the Gaussian FIB model. Like in the scale-invariant FIB process, the scale-dependent fragmentation processes may also develop strong scale-invariant fluctuations (the first scaling) , though the region of their appearance is restricted to : $-1 < \alpha < -1/2$, of the homogeneous fragmentation function. For $\alpha > 0$, the fragment multiplicity distributions obey the second scaling ($\delta = 1/2$) , i.e. the small amplitude limit of scaling multiplicity fluctuations.

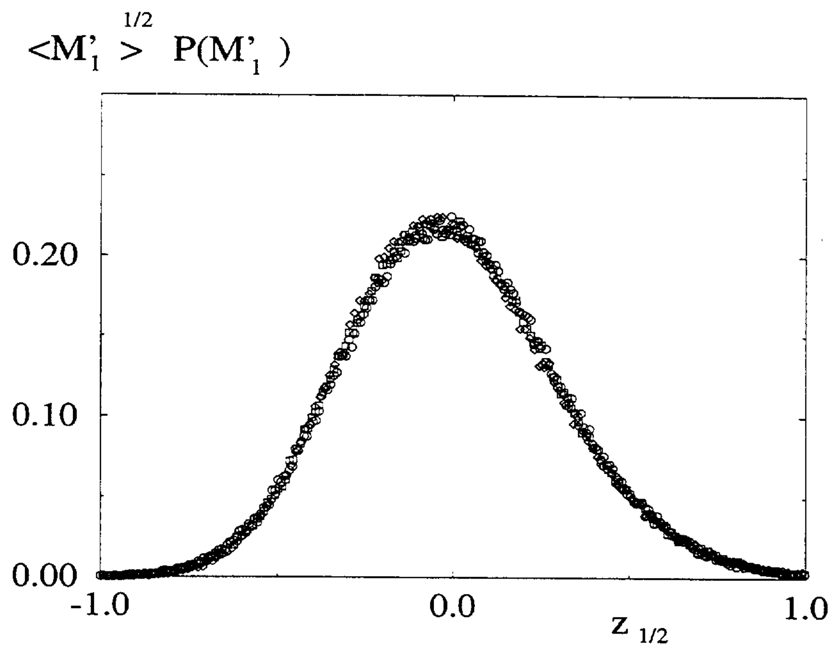

FIG. 7. 3D bond-percolation probability distribution of the variable $M_1' = M_1 - S_{max}$, with M_1 the first moment of the size-distribution (i.e. M_1'=mass of all clusters except for the largest one) away from the percolation threshold ($p = 0.35 \neq p_{cr}$) is plotted for different lattice sizes : $N = 10^3, 12^3, 14^3$, in the scaling variables (5) with $\delta = 1/2$.

The situation is different for $-1/2 < \alpha < 0$. Here, the multiplicity distributions scale as in (5) but with the exponent δ which takes an intermediate value between the first scaling ($\delta = 1$) and the second scaling ($\delta = 1/2$). In Fig. 5 we show the cluster multiplicity distributions in the scaled variables $z_{(\delta)}$ for $\delta = 0.75$ for the parameters $\alpha = -1/3$, $\beta = 0, c = 1, \sigma = 1$ of the fragmentation and inactivation rate functions. Of course, if $\sigma \rightarrow \infty$, the inactivation becomes scale-invariant again and, for this choice of parameters α and β, the process is in the shattering phase and, hence, the second scaling ($\delta = 1/2$) holds. What happens then for small values of σ in the range of parameters α from -1/2 to 0? To understand meaning of this unusual scaling, it is instructive to refer to the properties of percolation model. What happens at the percolation threshold ($p = p_{cr}$) if instead of $P(M)$, where $M \equiv S_{max}$, we plot the probability distribution of $M_1' = M_1 - S_{max}$, where $M_1 = \sum_k kn(k)$? M_1' is related in a non-trivial way to the order parameter S_{max} and, in particular, it conserves the singularity of S_{max} . Fig. 6 shows the probability distributions of the variable M_1' at the 3D bond percolation threshold. $P(M_1')$ for systems of different sizes obeys the scaling law (5) but with the non-trivial exponent : $\delta = 0.78$. This non-trivial value of δ , *i.e.*, different both from $\delta = 1/2$ and 1 , is here a signature of the phase transition and disappears, *i.e.*, δ becomes equal 1/2, for $p \neq p_{cr}$ (see Fig. 7). Of course, this sign of criticality is not the same as the first scaling ($\delta = 1$), but on the other hand M_1' is not exactly the order parameter : $M \equiv S_{max}/N = (N - M_1')/N$ for which the first scaling ($\delta = 1$) holds (see Fig. 4). The results we have obtained so far for the off-equilibrium fragmentation systems as well as for the percolation which is a static equilibrium model, lead us to formulate the following conjecture :

- The occurrence of first scaling ($\delta = 1$) in the probability distribution $P(M')$ of a certain macroscopic quantity M', is the sign of critical behavior and M' is the order parameter in this transition;

- The occurrence of generalized scaling with : $1/2 < \delta < 1$ in the probability distribution $P(M')$ is the sign of a critical behavior in the system but M' is not the order parameter M . M' is in this case closely related to the true order parameter M and, in particular, it is singular at the critical point as M is;

- The occurrence of a second scaling ($\delta = 1/2$) in the probability distribution $P(M')$ is the sign that the variable M' is not singular.

In a certain range of parameters of the Gaussian FIB model, one observes a change of the order parameter as compared to the scale-invariant FIB model. This change manifest itself in the modification of exponent δ of the scaling law (5) at the critical point. In general, it is possible to reconstruct the true order parameter though this procedure may turn out to be inaccurate in some cases due to the finite-size effects. This possibility could be important in various phenomenological applications where the order parameter may not be directly related to observed quantities. Nevertheless, one may deduce from observables the most essential properties of the order parameter and obtain the information about scaling exponents [12] .

III. CONCLUSIONS

The analysis of critical behaviour and, hence, the determination of critical exponents in finite systems may be strongly perturbed by the finite-size effects. Moreover, for many phenomena even the order parameter is not known. In these cases, investigation of fluctuations in the order parameter or in the related quantity may be very important. We have shown that both equilibrium and off-equilibrium processes exhibiting a second-order phase transition, obey the scaling law of order parameter fluctuations (5) for systems of different sizes. This scaling, which is well seen even in small systems, permits to tell whether the studied process is critical (*i.e.*, $1/2 < \delta \leq 1$ in Eq. (5)) and the selected macroscopic quantity which shows a singular behaviour in a critical sector is a true order parameter in this process (*i.e.*, $\delta = 1$ in this case). The case $\delta = 1/2$ in Eq. (5), is an interesting special case. This scaling appears outside of a critical region (*e.g.*, in the shattering phase of the off-equilibrium FIB process or away from the critical point in the equilibrium percolation process) where the order parameter is non-singular.

The universal features of the order parameter fluctuations do not depend on whether the studied process is an equilibrium or an off-equilibrium process. In the latter case, the arsenal of available tools to characterize statistical properties of the system is strongly limited, and the universality of the order parameter fluctuations may be a unique tool in many phenomenological applications. For example, in ultrarelativistic collisions of leptons or hadrons, the study of multiplicity fluctuations and their deviations from the $\delta = 1$ scaling as seen by UA5 Collaboration [13] may give a unique insight into the dynamics of parton cascading and subsequent hadronization, allowing for the determination of both the relevant observables and the nature of criticality.

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