UPR-804T, PUPT-1793, CERN-TH/98-178, Imperial/TP/97-98/52
hep-th/9806051

Heterotic M–theory in Five Dimensions

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Abstract

We derive the five–dimensional effective action of strongly coupled heterotic string theory for the complete $(1,1)$ sector of the theory by performing a reduction, on a Calabi–Yau three–fold, of M–theory on S^1/Z_2. A crucial ingredient for a consistent truncation is a non–zero mode of the antisymmetric tensor field strength which arises due to magnetic sources on the orbifold planes. The correct effective theory is a *gauged* version of five–dimensional $N = 1$ supergravity coupled to Abelian vector multiplets, the universal hypermultiplet and four–dimensional boundary theories with gauge and gauge matter fields. The gauging is such that the dual of the four–form field strength in the universal multiplet is charged under a particular linear combination of the Abelian vector fields. In addition, the theory has potential terms for the moduli in the bulk as well as on the boundary. Because of these potential terms, the supersymmetric ground state of the theory is a multi–charged BPS three–brane domain wall, which we construct in general. We show that the five–dimensional theory together with this solution provides the correct starting point for particle phenomenology as well as early universe cosmology. As an application, we compute the four–dimensional $N = 1$ supergravity theory for the complete $(1,1)$ sector to leading nontrivial order by a reduction on the domain wall background. We find a correction to the matter field Kähler potential and threshold corrections to the gauge kinetic functions.

*Supported in part by Deutsche Forschungsgemeinschaft (DFG).
§Supported in part by a Senior Alexander von Humboldt Award.

1 Introduction

One of the phenomenologically most promising corners of the M–theory moduli space, in addition to the weakly coupled heterotic string, is the point described at low-energy by eleven-dimensional supergravity on the orbifold S^1/Z_2 due to Hořava and Witten [1, 2]. This theory gives the strongly coupled limit of the heterotic string with, in addition to the bulk supergravity, two sets of E_8 gauge fields residing one on each of the two ten–dimensional fixed hyperplanes of the orbifold. It has been shown [3] that this theory has phenomenologically interesting compactifications on deformed Calabi–Yau three–folds times the orbifold to four dimensions. Matching the 11–dimensional Newton constant κ, the Calabi–Yau volume and the orbifold radius to the known values of the Newton constant and the grand unification coupling and scale leads to an orbifold radius which is about an order of magnitude or so larger than the two other scales [3, 4]. This suggests that, near this "physical" point in moduli space, the theory appears effectively five–dimensional in some intermediate energy regime.

In a previous paper [5] we have derived this five–dimensional effective theory for the universal fields for the first time by directly reducing Hořava–Witten theory on a Calabi–Yau three–fold. We showed that a non–zero mode of the antisymmetric tensor field strength has to be included for a consistent reduction from eleven to five dimensions and that the correct five–dimensional effective theory of strongly coupled heterotic string is given by a gauged version of five–dimensional supergravity. A reduction of pure eleven–dimensional supergravity on a Calabi–Yau three–fold [6], on the other hand, leads to a non–gauged version of five–dimensional supergravity. Therefore, while this provides a consistent low–energy description of M–theory on a smooth manifold it is not the correct effective theory for M–theory on S^1/Z_2. The necessary additions are chiral four–dimensional boundary theories with potential terms for the bulk moduli and, most importantly, the aforementioned non–zero mode, living solely in the Calabi-Yau, which leads to the gauging of the bulk supergravity. As pointed out in ref. [5] this theory is the correct starting point for strongly coupled heterotic particle phenomenology as well as early universe cosmology if the theory indeed undergoes a five–dimensional phase as suggested by the data. Moreover, we have shown that contact with four–dimensional physics should not be made using flat space–time but rather a domain–wall solution as the background configuration. This domain wall arises as a BPS state of the five–dimensional theory [5] and its existence is intimately tied to the gauging of the theory. A reduction to four dimensions on this domain wall has been performed in [7] to lowest non–trivial order. The result agrees with ref. [8] where the complete four–dimensional effective action to that order has been derived directly from eleven dimensions. This is not surprising given that all reductions have been carried out consistently.

Various other aspects of the Hořava–Witten description of strongly coupled heterotic string theory have been addressed in the literature such as the structure of the four–dimensional effective action, its relation to 10–dimensional weakly coupled heterotic string, gaugino condensation, and anomaly cancelation [8 − 33]. Aspects of five–dimensional physics motivated by Hořava–Witten theory and related to particle phenomenology have been discussed in ref. [4, 27, 34, 35]. In refs. [36, 37, 38] five–dimensional early universe M–theory cosmology have been investigated. Recently, aspects of five–dimensional physics have also been discussed in ref. [39].

The main purpose of the present paper is to generalize the result of ref. [5] to include the full $(1, 1)$ sector of the theory. Our central result is to obtain the five–dimensional effective theory of strongly coupled heterotic string for all $(1, 1)$ fields and construct its fundamental BPS domain wall three–brane solutions. We show that, in the bulk, this theory is indeed a form of gauged supergravity. We argue that this effective theory together with the three–brane solutions is the proper starting point for particle phenomenology and early universe cosmology. As an explicit demonstration we derive the Kähler–potential, the superpotential and the gauge–kinetic functions of the four–dimensional theory by reducing on the five–dimensional theory on this three–brane solution. We also comment on how gaugino condensation appears in this formalism.

At first sight, it seems puzzling how a sensible five–dimensional theory can be derived by reducing Hořava–Witten theory. Generically, the antisymmetric tensor field of the 11–dimensional theory has magnetic sources provided by the E_8 gauge fields (and gravity) on the orbifold hyperplanes. In particular, for compactification on a Calabi–Yau three–fold using the standard embedding of the spin–connection into one of the E_8 gauge groups one finds that the magnetic sources are nonzero and of opposite strength. Hence, unlike in the weakly coupled heterotic string, one is forced to consider backgrounds with a nonvanishing internal antisymmetric tensor field strength. To satisfy the Killing spinor equations in the presence of this field one is led to "deform" the Calabi–Yau space and introduce dependence on the orbifold coordinates into the background metric. This procedure gives the solutions of ref. [3], which were explicitly constructed in detail in ref. [8]. Although those solutions are perfectly appropriate for a reduction to four dimensions [8], their explicit orbifold dependence makes them unsuitable to derive a five–dimensional effective action with the orbifold among those five dimensions.

Which backgrounds should then be used to derive the five–dimensional action? An answer to this is provided by a change of perspective. Rather than viewing the nonzero antisymmetric tensor field, which is the cause of the metric deformation, as part of the pure background we will keep this field as an ingredient of the five–dimensional theory. As we will see, this internal antisymmetric tensor field is characterized by a topologically nontrivial solution for the four–form field strength on the Calabi–Yau space X and can hence be identified with the cohomology group $H^4(X)$. Such a configuration is called a nonzero mode of the antisymmetric tensor field. Alternatively, the nonzero mode can be described by a set of charges α_i, $i = 1, \cdots, h^{1,1}$ associated with the integrals of the boundary gauge–field and gravity sources over the Calabi–Yau four–cycles. As those boundary fields form magnetic sources for the antisymmetric tensor field, the charges α_i can be interpreted as five–brane charges with the associated five–branes confined to the orbifold planes. In conclusion, the correct, consistent reduction to five dimensions should then be performed on a conventional, undeformed Calabi–Yau three–fold but keeping the nonzero mode for the antisymmetric tensor field as dictated by the boundary sources.

The bulk action derived in such a way is a minimal $N = 1$ supergravity theory in five dimensions coupled to $h^{1,1} - 1$ vector multiplets and the universal hypermultiplet with sigma model coset $SU(2,1)/U(2)$. This action is not identical with the one obtained by a "conventional" reduction of pure 11–dimensional supergravity on a Calabi–Yau three–fold, as, for example, carried out in ref. [6]. The difference is due to the non–zero mode which leads to the gauging of a $U(1)$ subgroup in the hypermultiplet isotropy group with the corresponding gauge field being a certain linear combination of the vector multiplet gauge fields and the gauge field in the gravity multiplet. In particular it is the dual of the four-form antisymmetric field strength in five-dimensions which is gauged. This can be understood as a consequence of the Chern-Simons term in eleven dimensions. As a result of this gauging, the theory contains a potential for the Calabi–Yau breathing mode in the hypermultiplet and the Calabi–Yau "shape" moduli in the vector multiplets. The steepness of this potential is set by the five–brane charges α_i. Although five–dimensional supergravity theories coupled to matter have been studied in the literature [42, 44, 50, 51] the general case with gauging which we obtain from this reduction with non–zero mode has not been worked out previously. We will close this gap in an appendix thereby establishing that our result indeed represents a five–dimensional supergravity theory. One notes that the appearance of gauged supergravity when non-zero modes are included has been observed before in the context of Calabi-Yau compactification of type II theories [54, 55]. In further contrast to the pure 11–dimensional supergravity case, the bulk theory couples to four dimensional $N = 1$ theories with gauge and chiral multiplets confined to the now four–dimensional orbifold hyperplanes. In addition, we find "boundary potentials" for the projection of the bulk scalars onto the hyperplanes again specified by the five–brane charges α_i. Following ref. [9, 26], we will also explain how to properly incorporate gaugino condensates into the five–dimensional theory.

The existence of the potentials has important consequences for the "vacuum structure" of the five–dimensional theory. For a compact Calabi–Yau space, the potential terms are nonvanishing and, hence, flat space is not a solution of the theory. Instead, we find the fundamental solution of the theory is a new type of three–brane domain wall which couples to the bulk potential. This is in analogy with the eight–brane in massive type IIA supergravity which also couples to a scalar potential. The charges α_i play the role of the mass in the eight–brane theory. More precisely, this five–dimensional solution is a multi–charged double domain wall since it carries all the charges α_i and has two sources located one on each orbifold hyperplanes. Thus, each domain wall lies on one of the two orbifold planes and carries the four–dimensional physical fields. In this sense, upon reduction to four dimensions, 3 + 1–dimensional space–time is identified with the three–brane worldvolume. We find that, although the equations of motion can be solved, the three–brane can generally only be written implicitly in terms of the solution of a system of quadratic equations depending on the Calabi–Yau intersection numbers. In this form, however, the solution is quite general and applies to any Calabi–Yau space (that is any number of moduli and any set of intersection numbers). We also present various cases where the solution can be made more explicit, one being a linearized approximation in the charges α_i. This linearized version of the three–brane coincides with the zero mode part of the original deformed 11–dimensional solution [3] which was also determined to leading order in the charges. Hence, our three–brane represents a generalization of the 11–dimensional solution at all orders in the charges or, correspondingly, to all orders in $\kappa^{2/3}$ in the five–dimensional action.

From the eleven–dimensional viewpoint the three–brane originates from five–branes located on the 10–dimensional orbifold planes, with two dimensions wrapped over a Calabi–Yau two–cycle. It should be emphasized that this interpretation is not a result of any approximation, but is an *exact* statement in the context of the second-order effective theory that we consider here, since the five-dimensional theory that we shall derive is obtained *via* a consistent truncation of the eleven–dimensional theory. Consistency of ordinary Kaluza-Klein reduction on Calabi-Yau manifolds to gravity plus moduli fields was established in ref. [56]. In the present work and in [5], such a consistent reduction is generalized to include non-zero modes. The picture of wrapped 5-branes in $D = 11$ also provides an interpretation of the potential for the moduli which appears in the five dimensional theory. Unlike the case for a conventional Calabi–Yau compactification where all topologically equivalent Calabi–Yau spaces are on the same footing (resulting in a flat moduli space), in our case an expansion of the Calabi–Yau manifold reduces the energy stored in the non–trivial antisymmetric tensor field. This fact is reflected by the presence of the five–dimensional potential.

The five–dimensional theory together with its three–brane ground state is the correct starting point for low energy particle phenomenology as well as early universe cosmology. In general, the low–energy four–dimensional $N = 1$ supergravity theory is obtained as a reduction of this five–dimensional theory on the domain wall background. At the same time, cosmologically interesting solutions are those which evolve into the domain wall solution. Correspondingly, one should be interested in cosmological solutions that depend on the orbifold coordinate as well as on time [37, 38]. As an application, we will derive the four–dimensional Kähler potential, the superpotential and the gauge kinetic functions to linear order in the charges α_i for the full $(1,1)$ sector of the theory. We find an α_i dependent correction to the matter part of the Kähler potential and threshold corrections to the gauge kinetic functions. These results are new in that they cover all $(1,1)$ modes of the theory and they generalize the universal expressions obtained in ref. [19, 8].

Let us now summarize our conventions. A more detailed account of the conventions for Calabi–Yau spaces and five–dimensional supergravity is given in the appendices. We will consider eleven-dimensional spacetime compactified on a Calabi-Yau space X, with the subsequent reduction down to four dimensions effectively provided by a double-domain-wall background, corresponding to an S^1/Z_2 orbifold. We use coordinates x^I with indices $I, J, K, \ldots = 0, \ldots, 9, 11$ to parameterize the full 11–dimensional space M_{11}. Throughout this paper, when we refer to orbifolds, we will work in the "upstairs" picture with the orbifold S^1/Z_2 in the x^{11}–direction. We choose the range $x^{11} \in [-\pi\rho, \pi\rho]$ with the endpoints being identified. The Z_2 orbifold symmetry acts as $x^{11} \to -x^{11}$. There then exist two ten–dimensional hyperplanes fixed under the Z_2 symmetry which we denote by $M_{10}^{(n)}$, $n = 1, 2$. Locally, they are specified by the conditions $x^{11} = 0, \pi\rho$. Upon reduction on a Calabi–Yau space to five dimensions they lead to four–dimensional fixed hyperplanes $M_4^{(n)}$. Barred indices $\bar{I}, \bar{J}, \bar{K}, \ldots = 0, \ldots, 9$ are used for the ten–dimensional space orthogonal to the orbifold. Upon reduction on the Calabi-Yau space we have a five-dimensional spacetime M_5 labeled by indices $\alpha, \beta, \gamma, \ldots = 0, \ldots, 3, 11$. The orbifold fixed planes become four-dimensional with indices $\mu, \nu, \rho, \ldots = 0, \ldots, 3$. We use indices $A, B, C, \ldots = 4, \ldots 9$ for the Calabi–Yau space. Holomorphic and anti–holomorphic indices on the Calabi–Yau space are denoted by $a, b, c, \ldots$ and $\bar{a}, \bar{b}, \bar{c}, \ldots$, respectively. The harmonic $(1,1)$–forms of the Calabi–Yau space on which we will concentrate throughout this paper are indexed by $i, j, k, \ldots = 1, \ldots, h^{1,1}$.

The 11-dimensional Dirac–matrices Γ^I with $\{\Gamma^I, \Gamma^J\} = 2g^{IJ}$ are decomposed as $\Gamma^I = \{\gamma^\alpha \otimes \lambda, \mathbf{1} \otimes \lambda^A\}$ where γ^α and λ^A are the five– and six–dimensional Dirac matrices, respectively. Here, λ is the chiral projection matrix in six dimensions with $\lambda^2 = 1$. Spinors in eleven dimensions are Majorana with 32 real components throughout the paper. In five dimensions we use symplectic–real spinors [43]. These are defined in appendix B. Fields will be required to have a definite behavior under the Z_2 orbifold symmetry in $D = 11$. We demand a bosonic field Φ to be even or odd; that is, $\Phi(x^{11}) = \pm\Phi(-x^{11})$. For a spinor Ψ the condition is $\Gamma_{11}\Psi(-x^{11}) = \pm\Psi(x^{11})$ and, depending on the sign, we also call the spinor even or odd. The projection to one of the orbifold planes leads then to a ten–dimensional Majorana–Weyl spinor with definite chirality. Similarly, in five dimensions, bosonic fields will be either even or odd, and there is a corresponding orbifold condition on spinors.

2 Eleven–dimensional Supergravity on an Orbifold

In this section we briefly review the formulation of the low–energy effective action of strongly coupled heterotic string theory as eleven–dimensional supergravity on the orbifold S^1/Z_2 due to Hořava and Witten [1, 2].

The bosonic part of the action is given by

$$S = S_{\rm SG} + S_{\rm YM} \tag{2.1}$$

where $S_{\rm SG}$ is the familiar 11–dimensional supergravity action

$$S_{\rm SG} = -\frac{1}{2\kappa^2}\int_{M^{11}} \sqrt{-g}\left[R + \frac{1}{24}G_{IJKL}G^{IJKL} + \frac{\sqrt{2}}{1728}\epsilon^{I_1\dots I_{11}}C_{I_1I_2I_3}G_{I_4\dots I_7}G_{I_8\dots I_{11}}\right] \tag{2.2}$$

and $S_{\rm YM}$ describes the two E_8 Yang–Mills theories on the orbifold planes, explicitly given by [1]

$$\begin{aligned} S_{\rm YM} = &-\frac{1}{8\pi\kappa^2}\left(\frac{\kappa}{4\pi}\right)^{2/3}\int_{M_{10}^{(1)}}\sqrt{-g}\;\left[{\rm tr}(F^{(1)})^2 - \frac{1}{2}{\rm tr}R^2\right] \\ &- \frac{1}{8\pi\kappa^2}\left(\frac{\kappa}{4\pi}\right)^{2/3}\int_{M_{10}^{(2)}}\sqrt{-g}\;\left[{\rm tr}(F^{(2)})^2 - \frac{1}{2}{\rm tr}R^2\right]\; . \end{aligned} \tag{2.3}$$

Here $F^{(n)}_{\bar{I}\bar{J}}$ are the two E_8 gauge field strengths and C_{IJK} is the 3–form with field strength $G_{IJKL} = 24\,\partial_{[I}C_{JKL]}$. The above action has to be supplemented by the Bianchi identity

$$(dG)_{11\bar{I}\bar{J}\bar{K}\bar{L}} = -\frac{1}{2\sqrt{2}\pi}\left(\frac{\kappa}{4\pi}\right)^{2/3}\left\{J^{(1)}\delta(x^{11}) + J^{(2)}\delta(x^{11}-\pi\rho)\right\}_{\bar{I}\bar{J}\bar{K}\bar{L}} \tag{2.4}$$

where the sources are defined by

$$J^{(n)} = {\rm tr}F^{(n)}\wedge F^{(n)} - \frac{1}{2}{\rm tr}R\wedge R\; . \tag{2.5}$$

Note that, in analogy with the weakly coupled case, the boundary ${\rm tr}R^2$ terms in eq. (2.3) are required by supersymmetry as pointed out in ref. [8]. Under the Z_2 orbifold symmetry, the field components $g_{\bar{I}\bar{J}}$, $g_{11,11}$, $C_{\bar{I}\bar{J}11}$ are even, while $g_{\bar{I}11}$, $C_{\bar{I}\bar{J}\bar{K}}$ are odd. The above action is complete to order $\kappa^{2/3}$ relative to the bulk. Corrections, however, will appear as higher–dimension operators at order $\kappa^{4/3}$.

The fermionic fields of the theory are the 11–dimensional gravitino Ψ_I and the two 10–dimensional Majorana–Weyl spinors $\chi^{(n)}$, located on the boundaries, one for each E_8 gauge group. The components $\Psi_{\bar{I}}$ of the gravitino are even while Ψ_{11} is odd. The gravitino supersymmetry variation is given by

$$\delta\Psi_I = D_I\eta + \frac{\sqrt{2}}{288}\left(\Gamma_{IJKLM} - 8g_{IJ}\Gamma_{KLM}\right)G^{JKLM}\eta + \;\cdots\;, \tag{2.6}$$

where the dots indicate terms that involve fermion fields. The spinor η in this variation is Z_2 even.

[1] We note that there is a debate in the literature about the precise value of the Yang–Mills coupling constant in terms of κ. While we quote the original value [2, 40] the value found in ref. [14] is smaller. In the second case, the coefficients in the Yang-Mills action (2.3) and the Bianchi identity (2.4) should both be multiplied by $2^{-1/3}$. This potential factor will not be essential in the following discussion as it will simply lead to a redefinition of the five–dimensional coupling constants. In the following, we will give the necessary modifications where appropriate.

The appearance of the boundary source terms in the Bianchi identity has a simple interpretation by analogy with the theory of D-branes. It is well known that the $U(N)$ gauge fields describing the theory of N overlapping Dp-branes encode the charges for lower-dimensional D-branes embedded in the Dp-branes. For instance, the magnetic flux $\mathrm{tr}F$ couples to the $p-1$-form Ramond-Ramond potential, so describes $D(p-2)$-brane charge. Higher cohomology classes $\mathrm{tr}F \wedge \cdots \wedge F$ describe the embedding of lower-dimensional branes. Furthermore, if the Dp-brane is curved, then the cohomology classes of the tangent bundle also contribute. For instance $\mathrm{tr}R \wedge R$ induces $D(p-4)$-brane charge. We recall that in eleven dimensions it is M five-branes which are magnetic sources for G_{IJKL}. Thus we can interpret the magnetic sources in the Bianchi identity (2.4) as five-branes embedded in the orbifold fixed planes.

The relationship between the five-branes and the orbifold fixed planes requires some comment. Above, the theory is formulated with explicit delta-function sources describing the five-brane charges in the orbifold planes. In the following sections, we will reduce the theory on a Calabi-Yau, which means this five-brane charge is non-zero. In the reduced theory the five-branes appear as three-brane domain walls, localized in the fixed planes, which again provide the necessary delta-function sources to support them. We will see that the domain wall solution is the natural "vacuum" of the reduced theory. However, we could also change perspective and view the theory solely from the bulk eleven-dimensional, or reduced five-dimensional, supergravity theory, which certainly have an independent existence aside from the orbifold. Backgrounds with five-branes would then appear as solitons carrying a magnetic charge. Similarly, the domain wall solution is simply a soliton in the five-dimensional action. As with all solitonic solutions to supergravity theories, however, static vacuum solutions also have multiplets of zero-mode excitations associated to them, and owing to the presence of worldvolume anomalies, the spectrum of such zero-modes can be somewhat complex. It would be interesting to investigate the structure of the effective theory of the zero modes, analyzing the anomaly constraints along the lines of the case of superconducting strings [57] or M five-branes [58]. One might expect that this would reproduce at least some of the dynamics encoded in the orbifold plane Yang-Mills source action given above, and the orbifold description could, at least partially, be dispensed with.

This speculation takes us outside the immediate context of the present paper, however. For the meantime, we shall remain content to view the five-dimensional solitonic domain walls as simply being embedded into the orbifold fixed planes, whose curvature and instanton number provide the necessary delta-function sources and which naturally then lead to the five-brane charges associated to the domain walls in the $D = 11$ perspective. Physically, by analogy with the D-brane case, we can identify the instanton number with the number of physical five-branes living in the fixed-planes, while the curvature leads to an induced five-brane charge. This interpretation will prove a useful guide in understanding the structure of the reduced theory and its vacuum solution in the compactification to five dimensions.

3 The five–dimensional effective theory

As mentioned in the introduction, matching of scales suggests that strongly coupled heterotic string theory appears effectively five–dimensional in some intermediate energy range. In this section we derive the five–dimensional effective theory in this regime obtained by a compactification on a Calabi–Yau three–fold. We expect that this should lead to a theory with bulk $N = 1$ five-dimensional supersymmetry and four-dimensional $N = 1$ supersymmetry on the orbifold fixed planes. As we will see, doing this compactification consistently requires the inclusion of non–zero modes for the field strength of the anti–symmetric tensor field. These non–zero modes appear in the purely internal Calabi–Yau part of the anti–symmetric tensor field and correspond to harmonic $(1, 1)$ forms on the Calabi–Yau three–fold. Consequently, to capture the complete structure of these non–zero modes, we will have to consider the full $(1, 1)$ sector of the theory. We will not, however, explicitly include the $(2, 1)$ sector as it is largely unaffected by the specific structure of Hořava–Witten theory. Instead, we comment on the additions necessary to incorporate this sector along the way.

To make contact with the compactifications to four-dimensions discussed by Witten [3], it is very natural to embed the spin-connection of the Calabi–Yau manifold in the gauge connection of one of the E_8 groups breaking it to E_6. In general, this implies that there is a non-zero instanton number on one of the orbifold planes. From the discussion of the previous section, this can be interpreted as including five-branes living in the orbifold plane in the compactification. It is this additional element to the compactification which introduces the non–zero mode and leads to much of the interesting structure of the five-dimensional theory. We note that the presence of five-brane charge is really unavoidable. Even without exciting instanton number, the curvature of the Calabi-Yau leads to an induced magnetic charge in the Bianchi identity (2.4), forcing us to include non-zero modes.

3.1 Zero modes

Let us now explain the structure of the zero mode fields used in the reduction to five dimensions. We begin with the bulk. The background space–time manifold is $M_{11} = X \times S_1/Z_2 \times M_4$, where X is a Calabi–Yau three–fold and M_4 is four–dimensional Minkowski space. Reduction on such a background leads to eight preserved supercharges and, hence, to minimal $N = 1$ supergravity in five dimensions. Due to the projection condition, this leads to four preserved supercharges on the orbifold planes implying four–dimensional $N = 1$ supersymmetry on those planes. Including the zero modes, the metric is given by

$$ds^2 = V^{-2/3} g_{\alpha\beta} dx^\alpha dx^\beta + g_{AB} dx^A dx^B \tag{3.1}$$

where g_{AB} is the metric of the Calabi–Yau space X. Its Kähler form is defined by [2] $\omega_{a\bar{b}} = i g_{a\bar{b}}$ and can be expanded in terms of the harmonic $(1,1)$–forms ω_{iAB}, $i = 1, \cdots, h^{1,1}$ as

$$\omega_{AB} = a^i \omega_{iAB} \; . \tag{3.2}$$

The coefficients $a^i = a^i(x^\alpha)$ are the $(1,1)$ moduli of the Calabi–Yau space. The Calabi–Yau volume modulus $V = V(x^\alpha)$ is defined by

$$V = \frac{1}{v} \int_X \sqrt{{}^6 g} \tag{3.3}$$

where ${}^6 g$ is the determinant of the Calabi–Yau metric g_{AB}. In order to make V dimensionless we have introduced a coordinate volume v in this definition which can be chosen for convenience. The modulus V then measures the Calabi–Yau volume in units of v. The factor $V^{-2/3}$ in eq. (3.1) has been chosen such that the metric $g_{\alpha\beta}$ is the five–dimensional Einstein frame metric. Clearly V is not independent of the $(1,1)$ moduli a^i but it can be expressed as

$$V = \frac{1}{6} \mathcal{K}(a) \; , \quad \mathcal{K}(a) = d_{ijk} a^i a^j a^k \tag{3.4}$$

where $\mathcal{K}(a)$ is the Kähler potential and d_{ijk} are the Calabi–Yau intersection numbers. Their definition, along with a more detailed account of Calabi–Yau geometry, can be found in appendix A.

Let us now turn to the zero modes of the antisymmetric tensor field. We have the potentials and field strengths,

$$\begin{array}{rcl} C_{\alpha\beta\gamma} & , & G_{\alpha\beta\gamma\delta} \\ C_{\alpha AB} = \frac{1}{6} \mathcal{A}^i_\alpha \omega_{iAB} & , & G_{\alpha\beta AB} = \mathcal{F}^i_{\alpha\beta} \omega_{iAB} \\ C_{abc} = \frac{1}{6} \xi \Omega_{abc} & , & G_{\alpha abc} = X_\alpha \Omega_{abc} \; . \end{array} \tag{3.5}$$

The five–dimensional fields are therefore an antisymmetric tensor field $C_{\alpha\beta\gamma}$ with field strength $G_{\alpha\beta\gamma\delta}$, $h^{1,1}$ vector fields $\mathcal{A}^i_\alpha$ with field strengths $\mathcal{F}^i_{\alpha\beta}$ and a complex scalar ξ with field strength X_α that arises from the harmonic $(3,0)$ form denoted by Ω_{abc}. In the bulk the relations between those fields and their field strengths are simply

$$\begin{array}{rcl} G_{\alpha\beta\gamma\delta} & = & 24\, \partial_{[\alpha} C_{\beta\gamma\delta]} \\ \mathcal{F}^i_{\alpha\beta} & = & \partial_\alpha \mathcal{A}^i_\beta - \partial_\beta \mathcal{A}^i_\alpha \\ X_\alpha & = & \partial_\alpha \xi \; . \end{array} \tag{3.6}$$

[2]Note here that we choose the opposite sign convention as in ref. [3, 8] to conform with the literature on Calabi–Yau reduction of 11–dimensional supergravity and type II theories [6, 67].

These relations, however, will receive corrections from the boundary controlled by the 11–dimensional Bianchi identity (2.4). We will derive the associated five–dimensional Bianchi identities later.

Next, we should set up the structure of the boundary fields. The starting point is the standard embedding of the spin connection in the first E_8 gauge group such that

$$\mathrm{tr}F^{(1)} \wedge F^{(1)} = \mathrm{tr}R \wedge R \,. \tag{3.7}$$

As a result, we have an E_6 gauge field $A_\alpha^{(1)}$ with field strength $F_{\mu\nu}^{(1)}$ on the first hyperplane and an E_8 gauge field $A_\mu^{(2)}$ with field strength $F_{\mu\nu}^{(2)}$ on the second hyperplane. In addition, there are $h^{1,1}$ gauge matter fields from the $(1,1)$ sector on the first plane. They are specified by

$$A_b^{(1)} = \bar{A}_b + \omega_{ib}{}^c T_{cp} C^{ip} \tag{3.8}$$

where $\bar{A}_b$ is the (embedded) spin connection. Furthermore, $p, q, r, \ldots = 1, \ldots, 27$ are indices in the fundamental **27** representation of E_6 and T_{ap} are the $(\mathbf{3}, \mathbf{27})$ generators of E_8 that arise in the decomposition under the subgroup $SU(3) \times E_6$. Their complex conjugate is denoted by T^{ap}. The C^{ip} are $h^{1,1}$ complex scalars in the **27** representation of E_6. Useful traces for these generators are $\mathrm{tr}(T_{ap}T^{bq}) = \delta_a^b \delta_p^q$ and $\mathrm{tr}(T_{ap}T_{bq}T_{cr}) = \Omega_{abc} f_{pqr}$ where f_{pqr} is the totally symmetric tensor that projects out the singlet in $\mathbf{27}^3$.

3.2 The nonzero mode

So far, what we have considered is similar to a reduction of pure 11–dimensional supergravity on a Calabi–Yau space, as for example performed in ref. [6], with the addition of gauge and gauge matter fields on the boundaries. An important difference arises, however, because the standard embedding (3.7), unlike in the case of the weakly coupled heterotic string, no longer leads to vanishing sources in the Bianchi identity (2.4). Instead, there is a net five-brane charge, with opposite sources on each fixed plane, proportional to $\pm\mathrm{tr}R \wedge R$. The nontrivial components of the Bianchi identity (2.4) are given by

$$(dG)_{11ABCD} = -\frac{1}{4\sqrt{2}\pi}\left(\frac{\kappa}{4\pi}\right)^{2/3} \left\{\delta(x^{11}) - \delta(x^{11} - \pi\rho)\right\} (\mathrm{tr}R \wedge R)_{ABCD} \,. \tag{3.9}$$

As a result, the components G_{ABCD} and G_{ABC11} of the antisymmetric tensor field are nonvanishing. More precisely, the above equation has to be solved along with the equation of motion.

$$D_I G^{IJKL} = 0 \,. \tag{3.10}$$

(Note that the Chern–Simons contribution to the antisymmetric tensor field equation of motion vanishes if G_{ABCD} and G_{ABC11} are the only nonzero components of G_{IJKL}.) The general solution of these equations is quite complicated and has been given in ref. [8] as an expansion in Calabi–Yau harmonic functions. For the present purpose of deriving a five–dimensional effective action, we are only interested in the zero mode terms in this expansion because the heavy Calabi–Yau modes decouple as a result of the consistent Kaluza-Klein truncation to $D = 5$. To work out the zero mode part of the solution, we note that $\mathrm{tr}R \wedge R$ is a $(2, 2)$ form on the Calabi–Yau space (since the only nonvanishing components of a Calabi–Yau curvature tensor are $R_{a\bar{b}c\bar{d}}$). Let us, therefore, introduce a basis ν^i, $i = 1, \cdots, h^{2,2} = h^{1,1}$ of harmonic $(2, 2)$ forms and corresponding four–cycles C_i such that

$$\frac{1}{v}\int_X \omega_i \wedge \nu^j = \delta_i^j \ , \quad \frac{1}{v^{2/3}}\int_{C_i} \nu^j = \delta_i^j \ . \tag{3.11}$$

The zero mode part $\mathrm{tr}R \wedge R|_0$ of the source can then be expanded as

$$\mathrm{tr}R \wedge R|_0 = -8\sqrt{2}\pi \left(\frac{4\pi}{\kappa}\right)^{2/3} \alpha_i \, \nu^i \tag{3.12}$$

where the numerical factor has been included for convenience. The expansion coefficients α_i are

$$\alpha_i = \frac{\pi}{\sqrt{2}} \left(\frac{\kappa}{4\pi}\right)^{2/3} \frac{1}{v^{2/3}} \beta_i \ , \qquad \beta_i = -\frac{1}{8\pi^2}\int_{C_i} \mathrm{tr}R \wedge R \ . \tag{3.13}$$

Note that β_i are integers, characterizing the first Pontrjagin class of the Calabi-Yau. It is then straightforward to see that the zero mode part of the Bianchi identity (3.9) and the equation of motion (3.10) are solved by

$$G_{ABCD}|_0 = \alpha_i \, \nu^i_{ABCD} \, \epsilon(x^{11}) = \frac{1}{4V} \alpha^i \, \epsilon_{ABCD}{}^{EF} \, \omega_{iEF} \, \epsilon(x^{11}) \tag{3.14}$$

$$G_{ABC11}|_0 = 0 \ . \tag{3.15}$$

Here $\epsilon(x^{11})$ is the step function which is $+1$ for positive x^{11} and -1 otherwise. The index of the coefficient α^i in the second part of the first equation has been raised using the metric

$$G_{ij}(a) = \frac{1}{2V}\int_X \omega_i \wedge (*\omega_j) \tag{3.16}$$

on the $(1, 1)$ moduli space. Note that, while the coefficients α_i with lowered index are truly constants, as is apparent from eq. (3.13), the coefficients α^i depend on the $(1, 1)$ moduli a^i since the metric (3.16) does. From the expansion (3.12) we can derive an expression for the boundary $\mathrm{tr}F^2$ and $\mathrm{tr}R^2$ terms in the action (2.3) which will be essential for the reduction of the boundary theories. We have

$$\mathrm{tr}R_{AB}R^{AB}\big|_0 = \mathrm{tr}F^{(1)}_{AB}F^{(1)AB}\Big|_0 = -4\sqrt{2}\pi \left(\frac{4\pi}{\kappa}\right)^{2/3} V^{-1}\alpha^i \omega^{AB}\omega_{iAB} \tag{3.17}$$

while, of course

$$\mathrm{tr}F^{(2)}_{AB}F^{(2)AB} = 0 \ . \tag{3.18}$$

The expression (3.14) for G_{ABCD} with α_i as defined in (3.13) is the new and somewhat unconventional ingredient in our reduction. Using the terminology of ref. [47] we call this configuration for the antisymmetric tensor field strength a nonzero mode. Generally, a nonzero mode is defined as a nonzero internal antisymmetric tensor field strength G that solves the equation of motion. In contrast, conventional zero modes of an antisymmetric tensor field, like those in eq. (3.6), have vanishing field strength once the moduli fields are set to constants. Since the kinetic term G^2 is positive for a nonzero mode it corresponds to a nonzero energy configuration. Given that nonzero modes, for a p–form field strength, satisfy

$$dG = d^*G = 0 \tag{3.19}$$

they correspond to harmonic forms of degree p. Hence, they can be identified with the pth cohomology group $H^p(X)$ of the internal manifold X. In the present case, we are dealing with a four–form field strength on a Calabi–Yau three–fold X so that the relevant cohomology group is $H^4(X)$. The expression (3.14) is just an expansion of the nonzero mode in terms of the basis $\{\nu^i\}$ of $H^4(X)$. The appearance of all harmonic $(2,2)$ forms shows that it is necessary to include the complete $(1,1)$ sector into the low energy effective action in order to fully describe the nonzero mode, as argued in the beginning of this section. On the other hand, harmonic $(2,1)$ forms do not appear here and are hence less important in our context. We stress that the nonzero mode (3.14), for a given Calabi–Yau space, specifies a fixed element in $H^4(X)$ since the coefficients α_i are fixed in terms of Calabi–Yau properties. In fact, they are related to the integers β_i characterizing the first Pontrjagin class of the tangent bundle. Thus we see that, correctly normalized, G is in the integer cohomology of the Calabi-Yau. This quantization condition has been described in [48]

In a dimensional reduction of pure 11–dimensional supergravity, non–zero modes can be considered as well but are usually dismissed as non–zero energy configuration. Compactifications of 11–dimensional supergravity on various manifolds including Calabi–Yau three-folds with non–zero modes have been considered in the literature [49]. The difference in our case is that we are not free to turn off the non–zero mode. Its presence is simply dictated by the nonvanishing boundary sources.

3.3 The five–dimensional effective action

Let us now summarize the field content which we have obtained above and discuss how it fits into the multiplets of five–dimensional $N = 1$ supergravity. The form of these multiplets and in particular the conditions on the fermions is discussed in more detail in appendix B. We know that the gravitational multiplet should contain one vector field, the graviphoton. Thus since the reduction leads to $h^{1,1}$ vectors, we must have $h^{1,1} - 1$ vector multiplets. This leaves us with the $h^{1,1}$ scalars a^i, the complex scalar ξ and the three-form $C_{\alpha\beta\gamma}$. Since there is one scalar in each vector multiplet, we are left with three unaccounted for real scalars (one from the set of a^i, and ξ) and the three-form. Together, these fields form the "universal hypermultiplet;" universal because it is present independently of the particular form of the Calabi-Yau manifold. From this, it is clear that it must be the overall volume breathing mode $V = \frac{1}{6}d_{ijk}a^i a^j a^k$ that is the additional scalar from the set of the a^i which enters the universal multiplet. The three-form may appear a little unusual, but one should recall that in five dimensions a three-form is dual to a scalar σ. Thus, the bosonic sector of the universal hypermultiplet consists of the four scalars $(V, \sigma, \xi, \bar{\xi})$.

The $h^{1,1} - 1$ vector multiplet scalars are the remaining a^i. More properly, since the breathing mode V is already part of a hypermultiplet it should be first scaled out when defining the shape moduli

$$b^i = V^{-1/3} a^i \ . \tag{3.20}$$

Note that the $h^{1,1}$ moduli b^i represent only $h^{1,1} - 1$ independent degrees of freedom as they satisfy the constraint

$$\mathcal{K}(b) \equiv d_{ijk} b^i b^j b^k = 6 \ . \tag{3.21}$$

Alternatively, as described in appendix B, we can introduce $h^{1,1} - 1$ independent fields ϕ^x with $b^i = b^i(\phi^x)$. The bosonic fields in the vector multiplets are then given by $(\phi^x, b_i^x \mathcal{A}_\alpha^i)$ (b_i^x represents a projection onto the ϕ^x subspace). Meanwhile the graviton and graviphoton of the gravity multiplet are given by $(g_{\alpha\beta}, \frac{2}{3} b_i \mathcal{A}_\alpha^i)$. Again, the details of this decomposition are described in appendix B.

Therefore, in total, the five dimensional bulk theory contains a gravity multiplet, the universal hypermultiplet and $h^{1,1} - 1$ vector multiplets. The inclusion of the $(2,1)$ sector of the Calabi–Yau space would lead to an additional $h^{2,1}$ set of hypermultiplets in the theory. Since they will not play a prominent rôle in our context they will not be explicitly included in the following.

On the boundary $M_4^{(1)}$ we have an E_6 gauge multiplet $(A_\mu^{(1)}, \chi^{(1)})$ and $h^{1,1}$ chiral multiplets (C^{ip}, η^{ip}) in the fundamental $\mathbf{27}$ representation of E_6. Here C^{ip} denote the complex scalars and η^{ip} the chiral fermions. The other boundary, $M_4^{(2)}$, carries an E_8 gauge multiplet $(A_\mu^{(2)}, \chi^{(2)})$ only. Inclusion of the $(2,1)$ sector would add $h^{2,1}$ chiral multiplets in the $\overline{\mathbf{27}}$ representation of E_6 to the field content of the boundary $M_4^{(1)}$. Any even bulk field will also survive on the boundary. Thus, in addition to the four–dimensional part of the metric, the scalars b^i together with $\mathcal{A}_{11}^i$, and V and σ survive on the boundaries. These pair into $h^{1,1}$ chiral muliplets.

After this survey we are ready to derive the bosonic part of the five–dimensional effective action for the $(1,1)$ sector. Inserting the expressions for the various fields from the previous subsection into the action (2.1), using the formulae given in appendix A and dropping higher derivative terms we find

$$S_5 = S_{\text{grav,vec}} + S_{\text{hyper}} + S_{\text{bound}} + S_{\text{matter}} \tag{3.22}$$

with

$$S_{\rm grav,vec} = -\frac{1}{2\kappa_5^2}\int_{M_5}\sqrt{-g}\,\Big[R + G_{ij}\partial_\alpha b^i\partial^\alpha b^j + G_{ij}\mathcal{F}^i_{\alpha\beta}\mathcal{F}^{j\alpha\beta} + \frac{\sqrt{2}}{12}\epsilon^{\alpha\beta\gamma\delta\epsilon}d_{ijk}\mathcal{A}^i_\alpha\mathcal{F}^j_{\beta\gamma}\mathcal{F}^k_{\delta\epsilon}\Big] \tag{3.23a}$$

$$S_{\rm hyper} = -\frac{1}{2\kappa_5^2}\int_{M_5}\sqrt{-g}\,\Big[\frac{1}{2}V^{-2}\partial_\alpha V\partial^\alpha V + 2V^{-1}X_\alpha\bar{X}^\alpha + \frac{1}{24}V^2G_{\alpha\beta\gamma\delta}G^{\alpha\beta\gamma\delta} + \frac{\sqrt{2}}{24}\epsilon^{\alpha\beta\gamma\delta\epsilon}G_{\alpha\beta\gamma\delta}\left(i(\xi\bar{X}_\epsilon - \bar{\xi}X_\epsilon) + 2\epsilon(x^{11})\alpha_i\mathcal{A}^i_\epsilon\right) + \frac{1}{2}V^{-2}G^{ij}\alpha_i\alpha_j\Big] \tag{3.23b}$$

$$S_{\rm bound} = \frac{\sqrt{2}}{\kappa_5^2}\int_{M_4^{(1)}}\sqrt{-g}\,V^{-1}\alpha_i b^i - \frac{\sqrt{2}}{\kappa_5^2}\int_{M_4^{(2)}}\sqrt{-g}\,V^{-1}\alpha_i b^i \tag{3.23c}$$

$$S_{\rm matter} = -\frac{1}{16\pi\alpha_{\rm GUT}}\sum_{n=1}^{2}\int_{M_4^{(n)}}\sqrt{-g}\,V{\rm tr}F^{(n)\,2}_{\mu\nu} - \frac{1}{2\pi\alpha_{\rm GUT}}\int_{M_4^{(1)}}\sqrt{-g}\,\Big[G_{ij}(D_\mu C)^i(D^\mu\bar{C})^j + V^{-1}G^{ij}\frac{\partial W}{\partial C^{ip}}\frac{\partial\bar{W}}{\partial\bar{C}^j_p} + D^{(u)}D^{(u)}\Big]\ . \tag{3.23d}$$

All fields in this action that originate from the 11–dimensional antisymmetric tensor field are subject to a nontrivial Bianchi identity. Specifically, from eq. (2.4) we have

$$(dG)_{11\mu\nu\rho\sigma} = -\frac{\kappa_5^2}{4\sqrt{2}\pi\alpha_{\rm GUT}}\left\{J^{(1)}\delta(x^{11}) + J^{(2)}\delta(x^{11}-\pi\rho)\right\}_{\mu\nu\rho\sigma} \tag{3.24a}$$

$$(d\mathcal{F}^i)_{11\mu\nu} = -\frac{\kappa_5^2}{4\sqrt{2}\pi\alpha_{\rm GUT}}J^i_{\mu\nu}\delta(x^{11}) \tag{3.24b}$$

$$(dX)_{11\mu} = -\frac{\kappa_5^2}{4\sqrt{2}\pi\alpha_{\rm GUT}}J_\mu\delta(x^{11}) \tag{3.24c}$$

with the currents defined by

$$J^{(n)}_{\mu\nu\rho\sigma} = \left({\rm tr}F^{(n)}\wedge F^{(n)} - \frac{1}{2}{\rm tr}R\wedge R\right)_{\mu\nu\rho\sigma} \tag{3.25a}$$

$$J^i_{\mu\nu} = -2iV^{-1}\Gamma^i_{jk}\left((D_\mu C)^{jp}(D_\nu\bar{C})^k_p - (D_\mu\bar{C})^k_p(D_\nu C)^{jp}\right) \tag{3.25b}$$

$$J_\mu = -\frac{i}{2}V^{-1}d_{ijk}f_{pqr}(D_\mu C)^{ip}C^{jq}C^{kr}\ . \tag{3.25c}$$

The five–dimensional Newton constant κ_5 and the Yang–Mills coupling $\alpha_{\rm GUT}$ are expressed in terms of 11–dimensional quantities as [3]

$$\kappa_5^2 = \frac{\kappa^2}{v} , \qquad \alpha_{\rm GUT} = \frac{\kappa^2}{2v}\left(\frac{4\pi}{\kappa}\right)^{2/3} . \tag{3.26}$$

We still need to define various quantities in the above action. The metric G_{ij} is given in terms of the Kähler potential $\mathcal{K}$ as

$$G_{ij} = -\frac{1}{2}\frac{\partial}{\partial b^i}\frac{\partial}{\partial b^j}\ln\mathcal{K} . \tag{3.27}$$

The corresponding connection Γ^i_{jk} is defined as

$$\Gamma^i_{jk} = \frac{1}{2}G^{il}\frac{\partial G_{jk}}{\partial b^l} . \tag{3.28}$$

We recall that

$$\mathcal{K} = d_{ijk}b^ib^jb^k , \tag{3.29}$$

where d_{ijk} are the Calabi–Yau intersection numbers. All indices $i, j, k, \cdots$ in the five–dimensional theory are raised and lowered with the metric G_{ij}. A more explicit form of this metric can be found in appendix A. We also recall that the fields b^i are subject to the constraint

$$\mathcal{K} = 6 \tag{3.30}$$

which should be taken into account when equations of motion are derived from the above action. Most conveniently, it can be implemented by adding a Lagrange multiplier term $\sqrt{-g}\lambda(\mathcal{K}(b) - 6)$ to the bulk action. Furthermore, we need to define the superpotential

$$W = \frac{1}{6}d_{ijk}f_{pqr}C^{ip}C^{jq}C^{kr} \tag{3.31}$$

and the D–term

$$D^{(u)} = G_{ij}\bar{C}^jT^{(u)}C^i \tag{3.32}$$

where $T^{(u)}$, $u = 1, \ldots , 78$ are the E_6 generators in the fundamental representation. The consistency of the above theory has been explicitly checked by a reduction of the 11–dimensional equations of motion.

The most notable features of this action, at first sight, are the bulk and boundary potentials for the $(1,1)$ moduli V and b^i that appear in $S_{\rm hyper}$ and $S_{\rm bound}$. Those potentials involve the five–brane charges α_i, defined by eq. (3.13), that characterize the nonzero mode. The bulk potential in

[3]These relations are given for the normalization of the 11–dimensional action as in eq. (2.1). If instead the normalization of [14] is used the expression for $\alpha_{\rm GUT}$ gets rescaled to $a_{\rm GUT} = 2^{1/3}\left(\kappa^2/2v\right)(4\pi/\kappa)^{2/3}$ Otherwise the action and Bianchi identities are unchanged, except that in the expression (3.13) for α_i the RHS is multiplied by $2^{1/3}$.

the hypermultiplet part of the action arises directly from the kinetic term G^2 of the antisymmetric tensor field with the expression (3.14) for the nonzero mode inserted. It can therefore be interpreted as the energy contribution of the nonzero mode. The origin of the boundary potentials, on the other hand, can be directly seen from eq. (3.17) and the boundary actions (2.3). Essentially, they arise because the standard embedding leads to nonvanishing internal boundary actions due to the crucial factor 1/2 in front of the $\mathrm{tr}R^2$ terms. This is in complete analogy with the appearance of nonvanishing sources in the internal part of the Bianchi identity which led us to introduce the nonzero mode.

Before we discuss the relation to five–dimensional supergravity theories we would like to explain how gaugino condensation can be incorporated into the above action. In ref. [9] it was shown that gaugino bilinears in the 11–dimensional action can be grouped into a perfect square together with the antisymmetric tensor field kinetic term. Furthermore, it was pointed out in that paper, that the problem arising from the term quartic in the gauginos which is proportional to $\left[\delta(x^{11})\right]^2$, can be resolved by a redefinition of the antisymmetric tensor field that absorbs the condensate. The coupling of the condensate and the antisymmetric tensor is then shifted to the 11–dimensional Bianchi identity. Based on this observation, a smooth background with a gaugino condensate has been given in [9] and this was then used in ref. [26] to derive a well defined four–dimensional nonperturbative superpotential. In the five–dimensional setting, a covariantly constant condensate proportional to the holomorphic three–form Ω_{abc} on the Calabi–Yau manifold groups itself into a perfect square together with the corresponding zero mode; that is, the complex field ξ in the hypermultiplet. Clearly, this perfect square also involves a term proportional to $\left[\delta(x^{11})\right]^2$ and therefore leads to a similar problem to that in 11 dimensions. Of course, we can apply the same trick and redefine the field ξ. Then the condensate disappears from the five–dimensional action but leads to additional source terms in the Bianchi identity (3.24) for X_α. Having done this, one can expect to find smooth solutions of the five–dimensional theory in the presence of the condensate. More precisely, one should find a domain wall solution similar to the one we will present in section 5.

4 Relation to five-dimensional supergravity theories

As we have argued in the previous section, the five-dimensional effective action (3.22) should have $N = 1$, $D = 5$ supersymmetry in the bulk and $N = 1$, $D = 4$ supersymmetry on the boundary. In this section, we will rewrite the action in a supersymmetric form. This will allow us to complete the action (3.23) to include fermionic terms and give the supersymmetry transformations. One thing we will not do is complete the supersymmetry transformations to include the bulk and boundary couplings, but we assume a consistent completion is possible, as in eleven dimensions.

Of particular interest is the presence of potential terms in the bulk theory. Such terms are forbidden unless the theory is gauged; that is, unless some of the fields are charged under Abelian gauge fields $\mathcal{A}_\alpha$. In order to identify the supersymmetry structure of the theory in hand, we derive, in Appendix B, the general form of gauged $D = 5$ $N = 1$ with charged hypermultiplets, borrowing heavily from the work of Günaydin *et al.* [44, 50] and Sierra [51], and from the general theory of gauged $D = 4$, $N = 2$ supergravity as given, for instance, in [53].

Let us start by giving the $N = 1$ structure of the four-dimensional boundary theory. As discussed above, we have a set of chiral multiplets with scalar components C^{ip}, together with vector multiplets with gauge fields $A^{(i)}_\mu$. (The vectors live on both boundaries, but the chiral matter lives only on the E_6 boundary.) In addition, the scalars from the bulk (A, σ) and $(b^i, \mathcal{A}^i_{11})$ also form chiral multiplets. From the form of the theory on the boundaries we can give explicitly the functions determining the $N = 1$ theory. We have already given the form of the superpotential and the D-term on the E_6 boundary in equations (3.31) and (3.32). It is also easy to read off the Kähler potential on the E_6 boundary and the gauge kinetic functions on either fixed plane. We find, without care to correct normalizations,

$$K = G_{ij} C^{ip} \bar{C}^i_p \qquad f^{(n)} = V + i\sigma \tag{4.1}$$

The appearance of σ in the gauge kinetic function is not immediately apparent from the action (3.22). However, it is easy to show that on making the dualization of $C_{\alpha\beta\gamma}$ to σ, which is described in more detail below, the magnetic source in the Bianchi indentity (3.24) for $C_{\alpha\beta\gamma}$, becomes an electric source for σ. The result is that the gauge kinetic terms in the boundary action are modified to

$$-\frac{1}{16\pi\alpha_{\rm GUT}} \sum_{n=1}^{2} \int_{M_4^{(n)}} \sqrt{-g} \left[V \mathrm{tr} F^{(n)}_{\mu\nu} F^{(n)\mu\nu} - \frac{\sigma}{2} \epsilon^{\mu\nu\rho\sigma} \mathrm{tr} F^{(n)}_{\mu\nu} \mathrm{tr} F^{(n)}_{\rho\sigma} \right] \tag{4.2}$$

One notes that the expressions (4.1) include dependence on the bulk fields b^i and V, evaluated on the appropriate boundary. Further, we are considering the bulk multiplets as parameters, as their dynamics comes from bulk kinetic terms.

Now let us turn to the bulk theory. Our goal will be to identify the action (3.23) with the bosonic part of the general gauged theory discussed in appendix B. The gauged theory is characterized by a special Riemannian manifold $\mathcal{M}_V$ describing the vector multiplet sigma-model, a quaternionic manifold $\mathcal{M}_H$ describing the hypermultiplet sigma-model, and a set of Killing vectors and prepotentials on $\mathcal{M}_H$. These are the structures we must identify in the action (3.23).

We start by concentrating on the hypermultiplet structure. We have argued that, after dualizing the three-form potential $C_{\alpha\beta\gamma}$ to a scalar σ, the fields $(V, \sigma, \xi, \bar{\xi})$ represent the scalar components of a hypermultiplet. Concentrating on the kinetic terms let us make the dualization explicit. We find

$$G_{\alpha\beta\gamma\delta} = \frac{1}{\sqrt{2}} V^{-2} \epsilon_{\alpha\beta\gamma\delta}{}^{\epsilon} \left\{ \partial_\epsilon \sigma - i \left(\xi \partial_\epsilon \bar{\xi} - \bar{\xi} \partial_\epsilon \xi \right) - 2\epsilon(x^{11}) \alpha_i \mathcal{A}^i_\epsilon \right\} \, . \tag{4.3}$$

The kinetic terms can then be written in the form

$$h_{uv} D_\alpha q^u D^\alpha q^v \tag{4.4}$$

where $q^u = (V, \sigma, \xi, \bar{\xi})^u$ and

$$D_\alpha q^u = \left(\partial_\alpha V,\, \partial_\alpha \sigma - 2\epsilon(x^{11}) \alpha_i \mathcal{A}^i_\alpha,\, \partial_\alpha \xi,\, \partial_\alpha \bar{\xi} \right)^u \tag{4.5}$$

and the metric is given by

$$h_{uv} dq^u dq^v = \frac{1}{4V^2} dV^2 + \frac{1}{4V^2} \left[d\sigma + i(\xi d\bar{\xi} - \bar{\xi} d\xi) \right]^2 + \frac{1}{V} d\xi d\bar{\xi} \, . \tag{4.6}$$

This reproduces the well-known result that the universal multiplet classically parameterizes the quaternionic space $\mathcal{M}_H = SU(2,1)/U(2)$ [41].

In what follows, we would like to have an explicit realization of the quaternionic structure of $\mathcal{M}_H$. A review of quaternionic geometry is given in Appendix B. We will now give expressions for the quantities defined there, following a discussion given in [52]. Since we have a single hypermultiplet, the holonomy of $\mathcal{M}_H$ should be $SU(2) \times Sp(2) = SU(2) \times SU(2)$. To distinguish these, we will refer to the first factor as $SU(2)$ and the second as $Sp(2)$. Defining the symplectic matrix Ω_{ab} such that $\Omega_{12} = -1$, we have the vielbein

$$V^{Aa} = \frac{1}{\sqrt{2}} \begin{pmatrix} u & \bar{v} \\ v & -\bar{u} \end{pmatrix}^{Aa} \tag{4.7}$$

where we have introduced the one-forms

$$u = \frac{d\xi}{\sqrt{V}} \qquad v = \frac{1}{2V} \left(dV + i d\sigma + \xi d\bar{\xi} - \bar{\xi} d\xi \right) \tag{4.8}$$

and their complex conjugates $\bar{u}$ and $\bar{v}$. We find that the $SU(2)$ connection is given by

$$\omega^A{}_B = \begin{pmatrix} \frac{1}{4}(v - \bar{v}) & -u \\ \bar{u} & -\frac{1}{4}(v - \bar{v}) \end{pmatrix}^A{}_B \tag{4.9}$$

while the $Sp(2)$ connection is

$$\Delta^a{}_b = \begin{pmatrix} -\frac{3}{4}(v - \bar{v}) & 0 \\ 0 & \frac{3}{4}(v - \bar{v}) \end{pmatrix}^a{}_b \, . \tag{4.10}$$

The triplet of Kähler forms is given by

$$K^A{}_B = \begin{pmatrix} \frac{1}{2}(u \wedge \bar{u} - v \wedge \bar{v}) & u \wedge \bar{v} \\ v \wedge \bar{u} & -\frac{1}{2}(u \wedge \bar{u} - v \wedge \bar{v}) \end{pmatrix}^A{}_B \, . \tag{4.11}$$

With these definitions, one finds that the coset space $SU(2,1)/U(2)$, satisfies the conditions for a quaternionic manifold.

So far our discussion has ignored the most important aspect of the hypermultiplet sigma-model. We note that the kinetic terms in (4.4) were in terms of a modified derivative (4.5), which included the gauge fields $\mathcal{A}^i_\alpha$. It appears that the hypermultiplet is charged under a $U(1)$ symmetry. Comparing with our discussion of gauged supergravity given in Appendix B, we see that this is indeed the case. The coset space $\mathcal{M}_H$ admits an Abelian isometry generated by the Killing vector

$$k = \partial_\sigma = iV^{-1}\left(\partial_v - \partial_{\bar{v}}\right) \ . \tag{4.12}$$

In general we can write the modified derivative (4.5) in the covariant form (B.22)

$$D_\alpha q^u = \partial_\alpha q^u + g\mathcal{A}^i_\alpha k^u_i \tag{4.13}$$

with

$$gk_i = -2\epsilon(x^{11})\alpha_i k = -2i\epsilon(x^{11})\alpha_i V^{-1}\left(\partial_v - \partial_{\bar{v}}\right) \ . \tag{4.14}$$

(Note that the gauge coupling is absorbed in α_i.) For consistency, the k^i_u should be writable in terms of a triplet of prepotentials given by (B.20). This is indeed the case and we find the prepotentials

$$g\mathcal{P}_i{}^A{}_B = \begin{pmatrix} -\frac{1}{4}i\epsilon(x^{11})\alpha_i V^{-1} & 0 \\ 0 & \frac{1}{4}i\epsilon(x^{11})\alpha_i V^{-1} \end{pmatrix}^A{}_B \ . \tag{4.15}$$

Thus it appears that the σ-component of the hypermultiplet is charged under each Abelian gauge field $\mathcal{A}^i_\alpha$, with a charge proportional to α_i. In particular, we can write the covariant derivative as

$$D_\alpha\sigma = \partial_\alpha\sigma + \frac{1}{4\sqrt{2}\pi}\left(\frac{4\pi}{\kappa_5}\right)^{2/3}\alpha_{\mathrm{GUT}}\,\epsilon(x^{11})\beta_i\mathcal{A}^i_\alpha \tag{4.16}$$

where β_i are integers characterizing the first Pontrjagin class of the Calabi-Yau.

If this interpretation is correct, the rest of the action should coincide with the general form for gauged supergravity given in Appendix B. It is clear that the vector multiplets are already in the correct form. Comparing the bosonic action (3.23) with the general form (B.25), we see that the gravitational and vector kinetic terms exactly match. (In the Appendix we have set the five-dimensional gravitational coupling v/κ^2 to unity.) The structure of the metric G_{ij} is identical, as is the appearance of Chern-Simons couplings. The compactification gives an interpretation of the numbers d_{ijk} in the Kähler potential (B.7) and (3.29). They are the Calabi-Yau intersection numbers.

The final check of this identification is to calculate the form of the potential. We have in general, from (B.29),

$$\begin{aligned} g^2 V &= -2g^2 G_{ij}\mathrm{tr}\mathcal{P}^i\mathcal{P}^j + 4g^2 b_i b_j \mathrm{tr}\mathcal{P}^i\mathcal{P}^j + \frac{g^2}{2}b^i b^j h_{uv}k^u_i k^v_j \\ &= \frac{1}{4}V^{-2}G^{ij}\alpha_i\alpha_j \ , \end{aligned} \tag{4.17}$$

exactly matching the derived potential.

Thus we can conclude that the bulk effective action is described by a set of Abelian vector multiplets coupled to a single charged hypermultiplet. The vector sigma-model manifold $\mathcal{M}_V$ has the general form described in Appendix B, but now the d_{ijk} in the Kähler potential (B.7) have the interpretation as Calabi-Yau intersection numbers. The hypermultiplet manifold $\mathcal{M}_H$ is the coset space $SU(2,1)/U(2)$. A $U(1)$ isometry, corresponding to the shift symmetry of the dualized three-form, is gauged. The charge of the hypermultiplet scalar field under each Abelian vector field A^i_α is given by α_i.

The appearance of gauged supergravity when non-zero modes are included has been seen before in the context of type II compactifications on Calabi-Yau manifolds to four-dimensions [54, 55]. It is natural to ask why this gauging arises. The appearance of a potential term is easy to interpret. We have included a non-zero four-form field strength G_{IJKL} on four-cycles of the Calabi-Yau. These contribute an energy proportional to the square of the field strength. For fixed total charge α_i (the integral of G over a cycle), the energy is reduced the larger the four-cycle. Thus it is no longer true that all points in Calabi-Yau moduli space have the same energy. As an example we see that the potential naturally drives the Calabi-Yau to large volume, minimizing the G^2 energy.

From the five-dimensional point of view, once we have a potential term, the theory must be gauged if it is to remain supersymmetric. We see that it is the dual of the five-dimensional three-form which is gauged. This arises because of the Chern-Simons term in eleven dimensions. Turning on non-zero modes, this term acts as an electric source for the five-dimensional three-form, though dependent on the gauge fields $\mathcal{A}^i$. Dualizing, the invariance $\sigma \to \sigma + \text{const}$ is a reflection of an absence of local electric charge. Thus it not surprising that the effect of the electric Chern-Simons terms is to modify this to a local gauge symmetry. We note that from this argument it can only ever be the five-dimensional three-form which becomes gauged by non-zero modes, whatever particular compactification to a $N = 1$ five-dimensional theory is considered.

We end this section by giving the the specific form of the fermionic supersymmetry variations. These are calculated using the general forms given in (B.30), (B.31) and (B.32), together with the explicit expressions for the vielbein, connections, Killing vectors and prepotentials given above. We find

$$\begin{aligned}
\delta\psi^A_\alpha &= \nabla_\alpha \epsilon^A + \frac{\sqrt{2}i}{8}\left(\gamma_\alpha{}^{\beta\gamma} - 4\delta^\beta_\alpha \gamma^\gamma\right) b_i \mathcal{F}^i_{\beta\gamma}\epsilon^A - P_\alpha{}^A{}_B \epsilon^B \\
&\quad - \frac{\sqrt{2}}{12} V^{-1} b^i \alpha_i \gamma_\alpha\, \epsilon(x^{11}) \tau_3{}^A{}_B \epsilon^B \qquad (4.18a)
\end{aligned}$$

$$\delta\lambda^{xA} = b^x_i \left(-\frac{1}{2} i\gamma^\alpha \partial_\alpha b^i \epsilon^A - \frac{1}{2\sqrt{2}} \gamma^{\alpha\beta} \mathcal{F}^i_{\alpha\beta}\epsilon^A - \frac{i}{2\sqrt{2}} V^{-1} \alpha^i \epsilon(x^{11}) \tau_3{}^A{}_B \epsilon^B \right) \qquad (4.18b)$$

$$\delta\zeta^a = -iQ_\alpha{}^A{}_B \gamma^\alpha \epsilon^B - \frac{i}{\sqrt{2}} b^i \alpha_i V^{-1} \epsilon(x^{11}) \tau_3{}^a{}_B \epsilon^B \qquad (4.18c)$$

where τ_i with $i = 1, 2, 3$ are the Pauli spin matrices and we have the matrices

$$
\begin{aligned}
P_\alpha{}^A{}_B &= \begin{pmatrix} \frac{\sqrt{2}i}{96} V \epsilon_{\alpha\beta\gamma\delta\epsilon} G^{\beta\gamma\delta\epsilon} & V^{-1/2}\partial_\alpha \xi \\ -V^{-1/2}\partial_\alpha \bar{\xi} & -\frac{\sqrt{2}i}{96} V \epsilon_{\alpha\beta\gamma\delta\epsilon} G^{\beta\gamma\delta\epsilon} \end{pmatrix}^A{}_B \\
Q_\alpha{}^A{}_B &= \begin{pmatrix} \frac{\sqrt{2}i}{48} V \epsilon_{\alpha\beta\gamma\delta\epsilon} G^{\beta\gamma\delta\epsilon} - \frac{1}{2} V^{-1}\partial_\alpha V & V^{-1/2}\partial_\alpha \xi \\ V^{-1/2}\partial_\alpha \bar{\xi} & \frac{\sqrt{2}i}{48} V \epsilon_{\alpha\beta\gamma\delta\epsilon} G^{\beta\gamma\delta\epsilon} + \frac{1}{2} V^{-1}\partial_\alpha V \end{pmatrix}^A{}_B
\end{aligned}
\tag{4.19}
$$

5 The domain wall solution

In this section, we would like to find the simplest BPS solutions of the five–dimensional theory, including the coupling to the potential terms induced by the nonzero mode. As we will see, these solutions provide the appropriate background for a reduction to four dimensions and can therefore be viewed as the "vacua" of the theory. After a general derivation of the solutions we will discuss several limiting cases of interest. In the next section one of these limiting cases will be used to derive the four–dimensional effective action to first nontrivial order.

5.1 The general solution

Let us first simplify the discussion somewhat by concentrating on the fields which are essential. Since we would like to find solutions that couple to the bulk potential terms we should certainly keep the hypermultiplet scalar V (the Calabi–Yau breathing mode) and the vector multiplet scalars b^i (the shape moduli). It turns out that those fields plus the five–dimensional metric are already sufficient. The action (3.22) can be consistently truncated to this reduced field content leading to

$$
\begin{aligned}
2\kappa_5^2 S_5 \;\; = \;\; & -\int_{M_5} \sqrt{-g} \left[R + G_{ij}\partial_\alpha b^i \partial^\alpha b^j + \frac{1}{2} V^{-2}\partial_\alpha V \partial^\alpha V + \frac{1}{2} V^{-2} G^{ij}\alpha_i \alpha_j + \lambda(\mathcal{K} - 6) \right] \\
& + 2\sqrt{2} \int_{M_4^{(1)}} \sqrt{-g}\, V^{-1}\alpha_i b^i - 2\sqrt{2} \int_{M_4^{(2)}} \sqrt{-g}\, V^{-1}\alpha_i b^i \; .
\end{aligned}
\tag{5.1}
$$

Note that we have explicitly added the Lagrange multiplier term which ensures the constraint (3.30) on b^i. For a finite Calabi–Yau volume V, that is, for an uncompactified internal space, the potential terms in this action do not vanish and, hence, flat space is not a solution of the theory. Therefore, the question arises of what the "vacuum" state of the theory is. A clue is provided by the fact that cosmological–type potentials in D dimensions generally couple to $D-2$ branes. This is well known from the eight–brane [59] which appears as a solution of the massive extension of type IIA supergravity [60] in ten dimensions. There, the eight–brane couples to a cosmological–type potential which consists of a single "cosmological" constant multiplied by a certain power of the dilaton. A way to understand to appearance of an eight–brane in this context is to dualize the cosmological constant to a nine–form antisymmetric tensor field which, according to the usual counting, should couple to an $8+1$–dimensional extended object. A systematic study of $D-2$

brane solutions in various dimensions using a generalized Scherk–Schwarz reduction can be found in ref. [61]. The present case is somewhat more complicated in that it involves $h^{1,1}$ scalar fields (as opposed to just the dilaton) and, correspondingly, $h^{1,1}$ constants α_i (as opposed to just one cosmological constant). Still, we can take a lead from the massive IIA example and dualize each of the constants α_i to a four–form antisymmetric tensor field. This would leave us with a theory that contains $h^{1,1}$ such antisymmetric tensor fields and, hence, a corresponding number of different types of three–branes that couple to those. The constants α_i can then be identified as the charges of these different types of three–branes. Since those constants are fixed in terms of the underlying theory (and are generically nonzero) one cannot really look for a "pure" solution which carries only one type of charge. Instead, what we are looking for is a multi–charged three–brane which is a mixture of the various different types as specified by the charges α_i. Clearly, the transverse space for a three–brane in five–dimensions is just one–dimensional. Given that the boundary source terms necessarily introduce dependence on the x^{11} coordinate this one–dimensional space can only be in the direction of the orbifold.

From the above remarks it is now clear that the proper Ansatz for the type of solutions we are looking for is given by

$$\begin{aligned} ds_5^2 &= a(y)^2 dx^\mu dx^\nu \eta_{\mu\nu} + b(y)^2 dy^2 \\ V &= V(y) \\ b^i &= b^i(y) \ , \end{aligned} \tag{5.2}$$

where we use $y = x^{11}$ from now on. The equations of motion derived from the action (5.1) still contains the Lagrange multiplier λ. It can be eliminated using eqs. (A.19)–(A.21) from appendix A. A solution to the resulting equations of the form (5.2) is still somewhat hard to find, essentially due to the complication caused by the inclusion of all (1, 1) moduli and the associated Kähler structure. The trick is to express the solution in terms of certain functions $f^i = f^i(y)$ which are only implicitly defined rather than trying to find fully explicit formulae. It turns out that those functions are fixed by the equations

$$d_{ijk} f^j f^k = H_i \ , \quad H_i = 2\sqrt{2} k \alpha_i |y| + k_i \tag{5.3}$$

where k and k_i are arbitrary constants. Then the solution can be written as

$$\begin{aligned} V &= \left(\frac{1}{6} d_{ijk} f^i f^j f^k \right)^2 \\ a &= \tilde{k} V^{1/6} \\ b &= k V^{2/3} \\ b^i &= V^{-1/6} f^i \end{aligned} \tag{5.4}$$

where $\tilde{k}$ is another arbitrary constant. We should check that this solution is indeed a BPS state of the theory; that is, that it preserves four of the eight supercharges. For the reduced field content, the supersymmetry transformations (4.18) lead to the following Killing spinor equations

$$\delta\psi_\mu^A = 0 \quad : \quad \gamma_\mu \left(\frac{a'}{a}\gamma_{11}\epsilon^A - \frac{\sqrt{2}b}{6V} b^i \alpha_i \, \epsilon(y) \, {\tau_3}^A{}_B \epsilon^B \right) = 0 \tag{5.5a}$$

$$\delta\psi_{11}^A = 0 \quad : \quad \epsilon^{A\prime} - \frac{\sqrt{2}b}{12V} b^i \alpha_i \gamma_{11} \, \epsilon(y) \, {\tau_3}^A{}_B \epsilon^B = 0 \tag{5.5b}$$

$$\delta\lambda^{xA} = 0 \quad : \quad b^{i\prime}\gamma_{11}\epsilon^A + \frac{b}{\sqrt{2}V}\left(\alpha^i - \frac{2}{3}b^j\alpha_j b^i\right)\epsilon(y) \, {\tau_3}^A{}_B \epsilon^B = 0 \tag{5.5c}$$

$$\delta\zeta^a = 0 \quad : \quad \frac{V'}{V}\gamma_{11}\epsilon^A - \frac{\sqrt{2}b}{V} b^i \alpha_i \, \epsilon(y) \, {\tau_3}^A{}_B \epsilon^B = 0 \; , \tag{5.5d}$$

where the prime denotes the derivative with respect to y. These equations are satisfied for the solution (5.5) if the spinor ϵ^A takes the form

$$\epsilon^A = a^{1/2}\epsilon_0^A \; , \quad \gamma_{11}\epsilon_0^A = (\tau_3)^A{}_B \epsilon_0^B \; , \tag{5.6}$$

where ϵ_0^A is a constant spinor. As a result, the solution preserves indeed four supercharges.

As can be seen from eq. (5.3) the solution is described in terms of $h^{1,1}$ linear functions H_i. This follows the general pattern of p–brane solutions coupled to n different charges which can be expressed in terms of n harmonic functions on the transverse space. In our case the number of charges α_i is precisely $h^{1,1}$ and the transverse space is just one–dimensional leading to linear functions. Generally, elementary brane solutions have singularities at the location of the branes which have to be supported by brane worldvolume theories. The pure bulk theory does not impose any restrictions on the number and locations of these singularities. Correspondingly, if we would just consider the bulk part of the action (5.1) we could place an arbitrary number of parallel three–branes anywhere on the orbifold. However, the theory (5.1) involves two four–dimensional boundary actions which provide source terms that should be matched. This is possible, in the present case, because the height of the boundary potentials in (5.1) is set by the three–brane charges α_i. If we decide that the solution should have no further singularities other than those matched by the two boundaries we arrive at the specific form of the harmonic functions H_i in eq. (5.3). In fact, we have

$$H_i'' = 4\sqrt{2}k\alpha_i(\delta(y) - \delta(y - \pi\rho)) \; , \tag{5.7}$$

indicating sources at the orbifold planes $y = 0, \pi\rho$. Recall that we have restricted the range of y to $y \in [-\pi\rho, \pi\rho]$ with the endpoints identified. This explains the second delta–function at $y = \pi\rho$ in the above equation.

In conclusion, the solution (5.5) represents a multi–charged double domain wall (three–brane) solution with the two walls located at the orbifold planes. It preserves four–dimensional Poincaré invariance as well as four of the eight supercharges and has therefore the correct properties to make contact with four–dimensional $N = 1$ supergravity. More precisely, those theories should arise as a dimensional reduction of the five–dimensional theory on the domain wall background. In this sense, the solution (5.5) can be viewed as the vacuum state of the five–dimensional theory. In the next section we will carry out this reduction explicitly to the first nontrivial order in the charges α_i. As a result, we will obtain the Kähler potential, the superpotential and the gauge kinetic functions that specify the four–dimensional $N = 1$ supergravity. From the perspective of the four–dimensional theory the domain wall solution plays an interesting rôle. It is oriented precisely in the four uncompactified dimensions and carries the physical gauge and gauge matter fields. Therefore, at low energy four–dimensional space–time gets identified with the three–brane worldvolume. In this sense, our Universe lives on the worldvolume of a three–brane. Before we make this point more explicit we would like to discuss some examples and limiting cases of the general solution which will be useful in the following.

5.2 Universal solution

In ref. [5] we have presented a related three–brane solution which was less general in that it involved the universal Calabi–Yau modulus V only. Clearly, we should be able to recover this solution from eq. (5.5) if we consider the specific case $h^{1,1} = 1$. Then we have $d_{111} = 6$ and it follows from eq. (5.3) that

$$f^1 = \left(\frac{\sqrt{2}}{3}k\alpha_1|y| + k_1\right)^{1/2} . \tag{5.8}$$

Inserting this into eq. (5.5) provides us with the explicit solution in this case which is given by

$$\begin{aligned} a &= a_0 H^{1/2} \\ b &= b_0 H^2 \\ V &= b_0 H^3 . \end{aligned} \qquad H = \frac{\sqrt{2}}{3}\alpha|y| + c_0 , \quad \alpha = \alpha^1 \tag{5.9}$$

The constant a_0, b_0 and c_0 are related to the integration constants in eq. (5.5) by

$$a_0 = \tilde{k}k^{1/2} , \qquad b_0 = k^3 , \qquad c_0 = \frac{k_1}{k} . \tag{5.10}$$

Eq. (5.9) is indeed exactly the solution that was found in ref. [5]. It still represents a double domain wall. However, in contrast to the general solution it couples to one charge $\alpha = \alpha^1$ only. Geometrically, it describes a variation of the five–dimensional metric and the Calabi–Yau volume across the orbifold. The form of the solution (5.9) is typical for brane solutions that couple to one charge and, in fact, fits into the general scheme of domain walls in various dimensions [61].

One may ask if a structure as simple as the above universal solution is, in some way, also part of the general solution (5.5) even if $h^{1,1} > 1$. To see that this is indeed the case, we define constants

$\bar{\alpha}^i$ and α by

$$d_{ijk}\bar{\alpha}^j\bar{\alpha}^k = \frac{2}{3}\alpha_i \; , \qquad \alpha = 9\left(\frac{1}{6}d_{ijk}\bar{\alpha}^i\bar{\alpha}^j\bar{\alpha}^k\right)^{2/3} \; . \tag{5.11}$$

In addition, we choose the following special values for the integration constants k_i in eq. (5.3)

$$k_i = 6kc_0\frac{\alpha_i}{\alpha} \tag{5.12}$$

where c_0 is an arbitrary constant. Thanks to this specific choice, we can easily solve (5.3) for f^i. Inserting the result into eq. (5.5) gives the explicit solution

$$\begin{aligned} a &= a_0 H^{1/2} \\ b &= b_0 H^2 \; , \qquad H = \frac{\sqrt{2}}{3}\alpha|y| + c_0 \\ V &= b_0 H^3 \\ b^i &= 3\alpha^{-1/2}\bar{\alpha}^i \; . \end{aligned} \tag{5.13}$$

As before, a_0 and b_0 are constants expressed in terms of the integration constants in (5.5) as

$$a_0 = \tilde{k}k^{1/2} \; , \qquad b_0 = k^3 \; . \tag{5.14}$$

Hence, for arbitrary values of $h^{1,1}$, we have identified a special case of the general solution (5.5) where the fields a, b and V behave in exactly the same way as in the universal solution (5.9). The charge α which appears in this special solution is now a complicated function of the various charges α_i in the way defined by eq. (5.11). In addition, the shape moduli b^i are constant. Consequently, for this special solution the metric and the Calabi–Yau volume vary as in the universal solution while the shape of the Calabi–Yau space is fixed.

5.3 Another simple example

A nontrivial example where the domain wall solution can be obtained explicitly is provided by

$$h^{1,1} = 3 \; , \qquad d_{123} = 1 \; , \tag{5.15}$$

and $d_{ijk} = 0$ otherwise. The Kähler potential is then given by

$$\mathcal{K} = 6\, b^1 b^2 b^3 \; . \tag{5.16}$$

In a four–dimensional effective theory the real fields b^i are promoted to complex scalars. Then the Kähler potential (5.16) is associated with the coset space $[SU(1,1)/U(1)]^3$ [71] and describes the STU–model. Due to the simple structure of intersection numbers eq. (5.3) can be easily solved for the functions f_i resulting in

$$f^i = (H_1 H_2 H_3)^{1/2} H_i^{-1} \; . \tag{5.17}$$

Inserting into eq. (5.5) then gives the explicit solution

$$\begin{aligned} V &= (H_1 H_2 H_3)^{-1} \\ a &= \tilde{k}(H_1 H_2 H_3)^{-1/6} \qquad\qquad H_i = 2\sqrt{2}k\alpha_i|y| + k_i \\ b &= k(H_1 H_2 H_3)^{-2/3} \\ b^i &= (H_1 H_2 H_3)^{2/3} H_i^{-1} \end{aligned} \tag{5.18}$$

for $i = 1, 2, 3$. As before k, $\tilde{k}$ and k_i denote constants.

5.4 The linearized solution

In the next section we would like to calculate the four–dimensional effective theory by reducing on the general domain wall solution (5.5). Clearly, to be able to do this we need an explicit form for this solution. So far, we have obtained explicit formulae only for specific examples. Unfortunately, the quadratic equations (5.3) can generally not be solved exactly. However, we can still try to find a sensible approximate solution. Since the appearance of the domain walls is triggered by the charges α_i one obvious approach is to find an approximate solution as an expansion in $\alpha_i|y|$. This is what we will do in the present subsection. We will restrict ourselves to the first nontrivial order; that is, to the terms linear in $\alpha_i|y|$, although in principle one could continue the procedure and find higher order terms as well. The result will be the basis for the calculation of the four–dimensional effective action up to order α_i in the next section.

Under which conditions does an expansion in $\alpha_i|y|$ actually make sense? Inspection of the right hand side of eq. (5.3) shows that one should require

$$\left|\frac{k_i}{k}\right| \gg 2\sqrt{2}\pi\rho|\alpha_i| \tag{5.19}$$

for all $i = 1, \cdots h^{1,1}$. In the following, we assume that the integration constants k_i, k are chosen such that these conditions are satisfied. Eventually, those integration constants become functions of the four–dimensional moduli. Then eq. (5.19) represents a constraints on the four–dimensional moduli space. We will return to this point later.

In order to solve eq. (5.3) up to linear order in $\alpha_i|y|$ we start with the Ansatz

$$f^i = A^i|y| + B^i \tag{5.20}$$

with yet unknown constants A^i, B^i. Inserting this into (5.3) leads to the relations

$$k_i = d_{ijk}B^j B^k , \qquad k\alpha_i = \frac{1}{\sqrt{2}} d_{ijk}A^j B^k \tag{5.21}$$

which fix A^i and B^i in terms of the integration constants and the charges. Before we explicitly calculate the solution we would like to introduce a new set of integration constants V_0, $\hat{R}_0$, b_0^i which is better adapted to their rôle as four–dimensional moduli. This new set is defined by the conditions

$$\begin{aligned} V \to V_0 \quad &, \quad a \to 1 \\ b \to \hat{R}_0 \quad &, \quad b^i \to b_0^i \end{aligned}$$

for $\alpha_i \to 0$. Inserting the Ansatz (5.20) into eq. (5.5) and comparing the α_i independent parts with the above conditions it is easy to find the equations

$$\begin{aligned} V_0 = B^2 \quad &, \quad b_0^i = b^{-1/3} B^i \\ \hat{R}_0 = k B^{4/3} \quad &, \quad \tilde{k} B^{1/3} = 1 \end{aligned}$$

with

$$B = \frac{1}{6} d_{ijk} B^i B^j B^k \tag{5.22}$$

which relate the old and the new integration constants. The final missing piece of information necessary is an expression for the constants A^i in eq. (5.20). They are fixed by the second eq. (5.21). Fortunately, this equation can be solved explicitly using the relations (A.19) from appendix A. We find

$$A^i = -\frac{V_0^{1/6}}{\sqrt{2}} \frac{\hat{R}_0}{V_0} (\alpha^i - b_0^j \alpha_j b_0^i) \ . \tag{5.23}$$

Using these results we finally get for the solution to order $\alpha_i |y|$

$$\begin{aligned} V &= V_0 \left(1 + \frac{\sqrt{2}\hat{R}_0}{V_0} b_0^i \alpha_i (|y| - \pi\rho/2) \right) \\ a &= 1 + \frac{\sqrt{2}\hat{R}_0}{6V_0} b_0^i \alpha_i (|y| - \pi\rho/2) \\ b &= \hat{R}_0 \left(1 + \frac{2\sqrt{2}\hat{R}_0}{3V_0} b_0^i \alpha_i \right) (|y| - \pi\rho/2) \\ b^i &= b_0^i - \frac{\hat{R}_0}{\sqrt{2}V_0} \left(\alpha^i - \frac{2}{3} b_0^j \alpha_j b_0^i \right) (|y| - \pi\rho/2) \ . \end{aligned} \tag{5.24}$$

Note that we have performed the shift $|y| \to |y| - \pi\rho/2$ which we can always do by a suitable redefinition of the constants. The rationale for this shift is that we would like the orbifold integral over the corrections in (5.24) to vanish. Then the moduli V_0, $\hat{R}_0$ and b_0^i equal the average values of the corresponding fields over the orbifold, that is

$$\langle V \rangle_{11} = V \quad , \quad \langle a \rangle_{11} = 1 \tag{5.25a}$$

$$\langle b \rangle_{11} = \hat{R}_0 \quad , \quad \langle b^i \rangle_{11} = b_0^i \tag{5.25b}$$

where the average is defined as

$$\langle f \rangle_{11} = \frac{1}{2\pi\rho} \int_{-\pi\rho}^{\pi\rho} dy f(y) \tag{5.26}$$

for any function $f = f(y)$. As we will see in the next section (and has been demonstrated for the universal case in ref. [8]) with this convention the moduli part of the four–dimensional Kähler potential takes its usual form; that is, it is not affected by corrections of order α_i.

In terms of the new moduli, the condition (5.19) for the validity of the linear approximation can be written in the form

$$\bar{\epsilon} \ll \left| \frac{1}{4} d_{ijk} b_0^j b_0^k (\beta_i)^{-1} \right| , \qquad \bar{\epsilon} = \epsilon \frac{\hat{R}_0}{V_0} \tag{5.27}$$

where

$$\epsilon = \frac{1}{16} \left(\frac{4\pi}{\kappa_5} \right)^{2/3} \rho \, \alpha_{\rm GUT} , \qquad \beta_i = -\frac{1}{8\pi^2} \int_{C_i} {\rm tr} R \wedge R \, . \tag{5.28}$$

Here we have used the definition (3.13) of the charges α_i. Note that $\bar{\epsilon}$ is a dimensionless quantity computed from the 11–dimensional Newton constant, the Calabi–Yau volume and the orbifold radius. This is the quantity discussed by Banks and Dine [4] and eq. (5.27) is the generalization of the validity condition discussed in their paper [4]. As they explain, for the "physical" values of κ, $\rho \hat{R}_0$ and $v^{2/3} V_0$ (that is, the values that match the Newton constant and the grand unification coupling and scale) $\bar{\epsilon}$ is of order one and, hence, not a very good expansion parameter. Of course, in our case things depend on more details like the topological values d_{ijk}, β_i of the Calabi–Yau space and the values of the Calabi–Yau shape moduli b_0^i. Nevertheless, it seems likely that the approximation is not particularly good at the physical point. Therefore it would be very desirable to go beyond the linear approximation in order to get a reliable low energy theory. For the purpose of the present paper, we assume that we are in a part of the moduli space where eq. (5.27) holds.

How does the linearized solution (5.24) relate to Witten's 11–dimensional solution [3] which was also constructed up to linear order in the charge $\alpha_i = O(\kappa^{2/3})$? The explicit form of this 11–dimensional solution was given in ref. [8] as an expansion in terms of Calabi–Yau harmonics. Comparison with the expressions given in this paper shows that eq. (5.24) coincides with the zero mode part of the 11–dimensional solution. For the universal case, a similar relation was demonstrated in ref. [5]. Our general solution (5.5) therefore represents an exact generalization of the original linear solution which holds to all orders in $\kappa^{2/3}$. Of course "exactness" merely indicates that it is an exact solution of the low energy effective theory which itself has higher-dimension

[4] This can be seen by expressing $\hat{R}_0$ in terms of the orbifold radius R_0 as measured with the 11–dimensional Einstein frame metric which gives $\hat{R}_0 = R_0 V_0^{1/3}$ and using equation (3.26).

correction terms which have not been included. Nevertheless it is a tantalizing question to which extend it can be used to calculate corrections to the four–dimensional theory beyond the linear order.

6 Four–dimensional physics from five dimensions

As an application of the results presented so far we would now like to calculate the effective four–dimensional theory which results from the reduction on the domain–wall background to lowest nontrivial order. This shows how the properties of the domain wall affect physically relevant quantities in four dimensions such as the Kähler potential and the gauge kinetic functions.

In order to establish the correspondence between four– and five–dimensional fields let us first look at lowest order; that is, at the terms independent on the domain wall charges α_i. In this case, the bulk zero modes simply coincide with the Z_2 even fields. Therefore, we have

$$\begin{aligned} ds^2 &= \hat{R}_0^{-1} g_{\mu\nu} dx^\mu dx^\nu + \hat{R}_0^2 (dx^{11})^2 \\ V &= V_0 \\ b^i &= b_0^i \end{aligned} \tag{6.1}$$

where $g_{\mu\nu}$ is the four–dimensional Einstein frame metric. Hence, the surviving fields are $h^{1,1}+1$ four–dimensional scalars $\hat{R}_0$, V_0 and b_0^i (recall that the b^i satisfy the constraint (3.30))which describe the orbifold radius, the Calabi–Yau volume and the Calabi–Yau shape, respectively. Furthermore, the following zero modes arise from the even components of the antisymmetric tensor fields

$$\begin{aligned} \mathcal{A}_{11}^i = \chi^i \quad &, \quad \mathcal{F}_{\mu 11}^i = \partial_\mu \chi^i \\ C_{\mu\nu 11} = \frac{1}{6} B_{\mu\nu} \quad &, \quad G_{\mu\nu\rho 11} = H_{\mu\nu\rho} = 3\partial_{[\mu} B_{\nu\rho]} \; . \end{aligned} \tag{6.2}$$

As a result, we have additional $h^{1,1}$ scalars χ^i and the two–form $B_{\mu\nu}$ with fields strength $H_{\mu\nu\rho}$. The latter can be dualized to a scalar σ_0 in the usual manner as

$$H_{\mu\nu\rho} = V_0^{-2} \epsilon_{\mu\nu\rho}{}^\sigma \partial_\sigma \sigma_0 \; . \tag{6.3}$$

How do these scalar fields fit into four–dimensional chiral multiplets? A straightforward reduction to lowest order shows that the dilaton S and the $h^{1,1}$ T–moduli T^i should be defined as

$$S = V_0 + i\sqrt{2}\sigma_0 \; , \qquad T^i = \hat{R}_0 b_0^i + i\sqrt{2}\chi^i \; . \tag{6.4}$$

In addition to those fields arising from the bulk we of course keep the boundary gauge and gauge matter fields.

Our next goal is to incorporate corrections of order α_i into the picture. The domain–wall solution to this order has been explicitly worked out in the previous subsection. From eq. (5.24) we have

$$
\begin{aligned}
ds^2 &= (1+\tilde{a})\hat{R}_0^{-1} g_{\mu\nu} dx^\mu dx^\nu + \hat{R}_0^2 (1+\tilde{\gamma})(dx^{11})^2 \\
V &= V_0(1+\tilde{h}) \\
b^i &= b_0^i + \tilde{b}^i
\end{aligned}
\tag{6.5}
$$

where all quantities with a tilde are of order α_i. They are given by

$$
\begin{aligned}
\tilde{a} &= \frac{2\sqrt{2}\hat{R}_0}{3V_0} b_0^i \alpha_i (|y| - \pi\rho/2) \\
\tilde{\gamma} &= 2\tilde{a} \\
\tilde{h} &= \frac{3}{2}\tilde{a} \\
\tilde{b}^i &= -\frac{\hat{R}_0}{\sqrt{2}V_0}\left(\alpha^i - \frac{2}{3} b_0^j \alpha_j b_0^i\right)(|y| - \pi\rho/2) \; .
\end{aligned}
\tag{6.6}
$$

These equations already contain much of the essential information necessary to compute the corrections of order α_i. For a complete reduction, however, one should deal with a number of additional complications. First of all, the corrections to the metric given above induce corrections to the zero modes of the antisymmetric tensor fields. Further corrections appear if one considers gauge and gauge matter fields on the boundary as we do in the present context. Those fields lead to boundary source terms in the Bianchi identities as well as in the Einstein equation which have to be carefully integrated out. In practise, this means that one has to solve the Bianchi identities (3.24) and the linearized Einstein equations with gauge and gauge matter sources. This leads to correction terms in the fields $G_{\alpha\beta\gamma\delta}$, $\mathcal{F}^i_{\alpha\beta}$ and X_α as well as in the metric that involve gauge and gauge matter fields. For the universal case, all these corrections have been determined in ref. [8]. Finally, if one wishes to include gaugino condensates one should solve the Bianchi identity (3.24) for X_α in the presence of the condensate as explained in the end of section 3. In an, equivalent, 11–dimensional framework this has been done in ref. [26]. Here, we will not repeat the procedures described in these papers but rather just state the final result.

The four–dimensional $N = 1$ supergravity theory with chiral fields $Y^\imath$ is specified by a Kähler potential $K = K(Y, \bar{Y})$, a holomorphic superpotential $W = W(Y)$ and a holomorphic gauge kinetic function $f = f(Y)$. To fix the normalizations let us state the relevant terms in the bosonic part of the component action

$$
\begin{aligned}
S &= -\frac{1}{2\kappa_P^2}\int_{M_4}\sqrt{-g}R \\
&\quad -\int_{M_4}\sqrt{-g}\left[K_{\imath\bar{\jmath}}\partial_\mu Y^\imath\partial^\mu\bar{Y}^{\bar{\jmath}} + e^{\kappa_P^2 K}\left(K^{\imath\bar{\jmath}}D_\imath W\overline{D_\jmath W} - 3\kappa_P^2|W|^2\right) + \text{D-terms}\right] \\
&\quad -\frac{1}{4g_{\text{GUT}}^2}\int_{M_4}\sqrt{-g}\left[\text{Re}f(Y)\text{tr}F^2 + \text{Im}f(Y)\text{tr}F\tilde{F}\right] \ . \qquad (6.7)
\end{aligned}
$$

Here $K_{\imath\bar{\jmath}} = \frac{\partial^2 K}{\partial Y^\imath \partial \bar{Y}^{\bar{\jmath}}}$ is the Kähler metric and $D_\imath W = \partial_\imath W + \frac{\partial K}{\partial Y^\imath}W$ is the Kähler covariant derivative acting on the superpotential. The dual field strength $\tilde{F}_{\mu\nu}$ is defined as $\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\sigma\rho}F^{\sigma\rho}$. The four–dimensional Planck constant κ_P and the four–dimensional gauge coupling g_{GUT} are defined in terms of the constants of the underlying five-dimensional theory as

$$
\begin{aligned}
\kappa_P^2 &= 8\pi G_N = \frac{\kappa_5^2}{2\pi\rho} \qquad (6.8a) \\
g_{\text{GUT}}^2 &= 4\pi\alpha_{\text{GUT}} \ . \qquad (6.8b)
\end{aligned}
$$

To have the kinetic terms of the matter fields normalized as in the above action we should perform the rescaling

$$
C \to \frac{g_{\text{GUT}}}{\sqrt{2}}C \ . \qquad (6.9)
$$

Then we find for the Kähler potential

$$
K = -\kappa_P^{-2}\ln(S+\bar{S}) + \kappa_P^{-2}K_T + Z_{ij}C^{ip}\bar{C}^j_p \qquad (6.10)
$$

where

$$
\begin{aligned}
K_T &= -\ln\left(\frac{1}{6}d_{ijk}(T^i+\bar{T}^i)(T^j+\bar{T}^j)(T^k+\bar{T}^k)\right) \qquad (6.11) \\
Z_{ij} &= \exp(-K_T/3)\left(K_{Tij} - \epsilon\beta_k\frac{1}{S+\bar{S}}\tilde{\Gamma}^k_{Tij}\right) \qquad (6.12)
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{\Gamma}^k_{Tij} &= \Gamma^k_{Tij} + \frac{1}{3}K_{Tij}(T^k+\bar{T}^k) \qquad (6.13) \\
K_{Tij} &= \frac{\partial^2 K_T}{\partial T^i \partial \bar{T}^j} \ , \qquad \Gamma^k_{Tij} = K_T^{kl}\frac{\partial K_{Tjl}}{\partial T^i} \ . \qquad (6.14)
\end{aligned}
$$

The matter field superpotential is given by the usual expression

$$
W = \frac{\sqrt{2}}{3}g_{\text{GUT}}d_{ijk}f_{pqr}C^{ip}C^{jq}C^{kr} \qquad (6.15)
$$

and the gauge kinetic functions take the form

$$
\begin{aligned}
f^{(1)} &= S + \epsilon\beta_i T^i \qquad (6.16a) \\
f^{(2)} &= S - \epsilon\beta_i T^i \ . \qquad (6.16b)
\end{aligned}
$$

In addition, if the gauginos of the hidden E_8 group condense one finds the nonperturbative superpotential [26]

$$W_{\rm gaugino} \sim \exp\left[-\frac{6\pi}{b_0\alpha_{\rm GUT}}(S-\epsilon\beta_i T^i)\right] \ . \tag{6.17}$$

We recall that the range of the $(1,1)$ indices is $i,j,k,\cdots = 1,\cdots,h^{1,1}$. Therefore we have $h^{1,1}$ moduli T^i and the same number of matter fields C^{ip}, each in the fundamental representation **27** of E_6, labeled by the indices $p,q,r,\cdots = 1,\cdots,27$. From eqs. (3.13) and (5.28) the quantities ϵ and β_i are defined by

$$\epsilon = \frac{1}{16}\left(\frac{4\pi}{\kappa_5}\right)^{2/3}\rho\,\alpha_{\rm GUT}\ , \qquad \beta_i = -\frac{1}{8\pi^2}\int_{C_i}{\rm tr}R\wedge R\ , \tag{6.18}$$

where C_i are the four–cycles of the Calabi–Yau space. Note that ϵ is a dimensionless quantity, while β_i are topological integers that can be computed for a given Calabi–Yau manifold. The interpretation of the moduli S, T^i in terms of the underlying geometry is encoded in the eqs. (6.4) and (5.25). All moduli fields are dimensionless and they measure the form of the internal manifold relative to the dimensionful quantities v and ρ that we have introduced.

In the previous subsection we have discussed a validity condition for the linear approximation that has been used to derive the above four–dimensional theory. For convenience, we would now like to translate this condition into four–dimensional language. From eq. (5.27) and the definition of the moduli (6.4) we find

$$\left|(S+\bar S)\frac{\partial K_T}{\partial T^i}\right| \gg |2\epsilon\beta_i| \ . \tag{6.19}$$

The above four–dimensional theory is only valid in the region of moduli space where these inequalities are satisfied for all i. A violation of the conditions implies that corrections of quadratic and higher order in $\epsilon\beta_i$ to the action can no longer be ignored.

The four–dimensional theory we have derived above is a generalization of the results for the universal case, obtained in ref. [8] and ref. [19, 27], to the full $(1,1)$ sector of the theory. As such it represents a new result within the M–theory context. The terms independent of $\epsilon\beta_i$ coincide with the effective theory computed from the weakly coupled heterotic string using geometrical methods [62, 63] or conformal field theory methods [64]. In addition, we have two corrections of order $\epsilon\beta_i$, one to the matter field metric Z_{ij}, eq. (6.12), and a threshold correction to the gauge kinetic functions (6.16). To our knowledge, a derivation of these additional terms for the full $(1,1)$ sector from weakly coupled heterotic string theory does not exist in the literature. This is understandable as the correction terms of order $\epsilon\beta_i$ are tiny in the weakly coupled region of the moduli space whereas they can be sizable or even of order one in the strongly coupled region. For a careful comparison between the strong and weak coupling limits of the heterotic string in relation to their four–dimensional effective actions we refer the reader to ref. [8]. At this point, all we would like to demonstrate is that we recover the universal result of [8] for the case $h^{1,1} = 1$. In this case, $d_{111} = 6$ and we set $T = T^1$ and $\beta = \beta^1$. Then we find from eqs. (6.11) and (6.12)

$$K_T = -3\ln(T + \bar{T}) \ , \qquad Z_{T\bar{T}} = \frac{3}{T + \bar{T}} + \frac{\epsilon\beta}{S + \bar{S}} \ , \tag{6.20}$$

and for the gauge kinetic functions

$$f^{(1)} = S + \epsilon\beta T \ , \qquad f^{(2)} = S - \epsilon\beta T \ . \tag{6.21}$$

This is indeed the result obtained in ref. [8]. The terms of order $\epsilon\beta$ in this universal theory have important consequences for soft supersymmetry breaking terms as has been pointed out in ref. [19, 25, 26]. Phenomenological implications of the generalized result will be discussed elsewhere.

Acknowledgments A. L. would like to thank Jens Erler for helpful discussions. A. L. is supported in part by a fellowship from Deutsche Forschungsgemeinschaft DFG. A. L. and B. A. O. are supported in part by DOE under contract No. DE-AC02-76-ER-03071. D. W. is supported in part by DOE under contract No. DE-FG02-91ER40671.

Appendix

A The Calabi–Yau Kähler moduli space

In this appendix we are going to review the structure of the $(1,1)$ moduli space of a Calabi–Yau three–fold as needed for the purpose of the present paper. In addition, we collect some related formulae which are used in our calculations. In the presentation, we are following ref. [65, 66, 67].

We consider a Calabi–Yau three–fold X with coordinates x^A, metric g_{AB} and volume V given by [5]

$$V = \int_X \sqrt{g} \; . \tag{A.1}$$

Its Kähler form ω_{AB} is defined in terms of the metric as

$$\omega_{a\bar{b}} = i g_{a\bar{b}} \; . \tag{A.2}$$

The cohomology group $H^{1,1}$ with dimension $h^{1,1}$ has a basis of harmonic $(1,1)$ forms which we call $\{\omega_{iAB}\}$. Since the Kähler form is similarly a harmonic $(1,1)$ form it can be expanded in terms of this basis as

$$\omega_{AB} = a^i \omega_{iAB} \tag{A.3}$$

where a^i are the Kähler moduli. For the metric $G_{ij}(a)$ on the Kähler moduli space one finds

$$G_{ij}(a) = \frac{1}{2V} \int_X \omega_i \wedge (*\omega_j) \tag{A.4}$$

To express this metric explicitly as a function of the moduli we introduce the Kähler potential

$$\mathcal{K}(a) = \int_X \omega \wedge \omega \wedge \omega \; . \tag{A.5}$$

Inserting the expansion (A.3) it is easy to see that

$$\mathcal{K}(a) = d_{ijk} a^i a^j a^k \tag{A.6}$$

where d_{ijk} are the intersection numbers defined by

$$d_{ijk} = \int_X \omega_i \wedge \omega_j \wedge \omega_k \; . \tag{A.7}$$

In addition, it is useful to define the quantities

$$\mathcal{K}_i(a) = \int_X \omega_i \wedge \omega \wedge \omega = d_{ijk} a^j a^k \tag{A.8}$$

$$\mathcal{K}_{ij}(a) = \int_X \omega_i \wedge \omega_j \wedge \omega = d_{ijk} a^k \; . \tag{A.9}$$

[5] Unlike in the main text, we omit the dimensionful coordinate volume v in this appendix for simplicity.

Then the metric (A.4) can be expressed in terms of the moduli as

$$G_{ij}(a) = -\frac{1}{2}\frac{\partial}{\partial a^i}\frac{\partial}{\partial a^j}\ln\mathcal{K}(a) = -3\left(\frac{\mathcal{K}_{ij}(a)}{\mathcal{K}(a)} - \frac{3}{2}\frac{\mathcal{K}_i(a)\mathcal{K}_j(a)}{\mathcal{K}(a)^2}\right) . \tag{A.10}$$

We note two useful properties of the metric which follow directly from the above explicit form, namely

$$G_{ij}(a)a^j = \frac{3}{2}\frac{\mathcal{K}_i(a)}{\mathcal{K}(a)} , \qquad G_{ij}(a)a^i a^j = \frac{3}{2} . \tag{A.11}$$

Let us also introduce the connection on the moduli space

$$\Gamma^k_{ij}(a) = \frac{1}{2}G^{kl}(a)\frac{\partial G_{ij}(a)}{\partial a^l} \tag{A.12}$$

which reads explicitly

$$\Gamma^k_{ij}(a) = -\frac{3}{2}G^{kl}\left(\frac{d_{ijl}}{\mathcal{K}(a)} - 9\frac{\mathcal{K}_{(i}(a)\mathcal{K}_{jl)}(a)}{\mathcal{K}(a)^2} + 9\frac{\mathcal{K}_i(a)\mathcal{K}_j(a)\mathcal{K}_l(a)}{\mathcal{K}(a)^3}\right) . \tag{A.13}$$

By straightforward computation one finds the simple property

$$\Gamma^k_{ij}(a)a^i = -\delta^k_j . \tag{A.14}$$

In the reduction from eleven to five dimensions we encounter a number of integrals over the Calabi–Yau space which we would like to express as functions of the moduli. We find for those integrals

$$\begin{aligned} \int_X \sqrt{g} &= V = \frac{1}{6}\mathcal{K}(a) \\ \int_X \sqrt{g}\omega_{ia\bar{b}}\omega_j{}^{a\bar{b}} &= 2VG_{ij}(a) \\ \int_X \sqrt{g}g^{a\bar{b}}\omega_{ia\bar{b}}g^{c\bar{d}}\omega_{ic\bar{d}} &= -2VG_{ij}(a) - \mathcal{K}_{ij}(a) . \end{aligned} \tag{A.15}$$

In the five–dimensional effective theory that results from the reduction on the Calabi–Yau space the volume modulus V becomes part of the universal hypermultiplet whereas the shape moduli fall into vector–multiplets. It is therefore appropriate to scale out the volume V and to define the shape moduli b^i as

$$b^i = V^{-1/3}a^i . \tag{A.16}$$

Of course they are not independent but have to vary such that the volume is unchanged. This is expressed by the constraint

$$\mathcal{K}(b) \equiv d_{ijk}b^i b^j b^k = 6 \tag{A.17}$$

which follows directly from their definition and the fact that $V = \mathcal{K}(a)/6$. The metric for these shape moduli is then simply given by

$$G_{ij}(b) = V^{2/3}G_{ij}(a) \tag{A.18}$$

In the main text we will often drop the argument of the metric and other related quantities. Whenever we do so it is to be understood that we are referring to $G_{ij}(b)$. The other quantities introduced above scale in a simple way if a^i is replaced by b^i. Using those scalings and and the constraint (A.17) one can easily rewrite all equations in terms of b^i. In particular, if we introduce the shape moduli b_i with lowered index defined by $b_i = G_{ij}(b)b^j$ the properties (A.11) can be rewritten as

$$b_i = \frac{1}{4}\mathcal{K}_i(b)\;, \qquad b_i b^i = \frac{3}{2}\;. \tag{A.19}$$

If the moduli are viewed as fields in the five–dimensional uncompactified space with coordinates x^α those equations lead the differential equations

$$b_i \partial_\alpha b^i \;=\; 0 \tag{A.20}$$

$$b_i \left(\nabla_\alpha \nabla^\alpha b^i + \Gamma^i_{jk}(b)\partial_\alpha b^i \partial^\alpha b^k \right) \;=\; 0\;, \tag{A.21}$$

which are useful to simplify the five–dimensional equations of motion.

B Einstein-Maxwell $D = 5$, $N = 1$ supergravity with gauged hypermultiplets

The purpose of this appendix is to give a summary of the general form of the coupling of matter and Abelian gauge fields to gravity in five dimensions which preserves $N = 1$ supersymmetry. There are three types of multiplet: a gravitational multiplet with a graviton, two gravitinos and a vector field; a vector multiplet with a vector field, two gauginos and a real scalar; and a matter hypermultiplet with two spinors and four scalar fields. The general coupling of Abelian multiplets to gravity, but with none of the fields charged under the Abelian symmetry, was first given by Günaydin *et al.* [44]. The same authors generalized this to the gauged case where some of the fields became charged in a subsequent paper [50]. The coupling to uncharged matter hypermultiplets was introduced by Sierra [51]. In the following we will generalize to the case where the matter multiplets become charged, giving the general gauging of the theory with Abelian vector multiplets. We will not consider general gauging of a non-Abelian vector multiplet sector.

B.1 Gamma matrices and symplectic spinors

Let us start by defining our conventions for spinors in five dimensions. A general spinor has four complex components. However, the spinor and its complex conjugate can be arranged into a pair which satisfies a twisted reality condition. Following [44], we define the gamma matrix algebra

$$\{\gamma^m, \gamma^n\} = -2\,\eta^{mn} = -2\,\mathrm{diag}\,(-++++)^{mn} \tag{B.1}$$

If C is the charge conjugation matrix, so that $\gamma^{m\mathrm{T}} = C\gamma^m C^{-1}$, one finds $C^2 = -1$ so that the spinors cannot be taken to be real. However, suppose we have a symplectic metric Ω_{ab} in a $2n$-dimensional vector space such that we raise and lower indices by

$$v^a = \Omega^{ab} v_b \qquad v_a = v^b \Omega_{ba} \qquad \Omega_{ab} = -\Omega_{ba} \tag{B.2}$$

One can then arrange n independent spinors and their conjugates into a $2n$-dimensional vector of symplectic spinors λ_a satisfying a reality condition

$$\bar{\lambda}^a \equiv {\lambda_a}^\dagger \gamma_0 = \left(\Omega^{ab}\lambda_b\right)^{\mathrm{T}} C \tag{B.3}$$

Here one is using the fact that Ω squares to minus one to cancel the minus one in $C^2 = -1$ and so provide a reality condition.

The simplest supersymmetry is $N = 1$ with a single supercharge. Writing the charge as a pair of symplectic spinors we see that the superalgebra must have a $Sp(2) = SU(2)$ R-symmetry automorphism group, corresponding to rotations in the two-dimensional symplectic space. We will denote these symplectic indices, in the fundamental of $SU(2)$, by $A, B, C, \dots$. The indices are raised and lowered following the conventions of (B.2) where we write ϵ_{AB} for the metric Ω_{ab} with $\epsilon_{12} = \epsilon^{12} = 1$.

B.2 Coupling of gravitational and Abelian vector multiplets

In this section we will describe the general ungauged coupling of n_V Abelian vector multiplets and the gravitational multiplet following ref. [44]. From the vector multiplets we have n_V real scalars which we will label ϕ^x while there are no scalars in the gravitational multiplet. The general kinetic energy of these scalars is described by a sigma-model where the scalars can be interpreted as coordinates on some Riemannian manifold $\mathcal{M}_V$. As usual, the form of the manifold is restricted by supersymmetry, and encodes all the information to describe the general coupling of the vector and gravitational multiplets. In particular, generally no independent function, which may provide a potential is available. Elucidating the structure of this manifold will be the main object of this section.

First, though, we summarize the component fields of the multiplets. Aside from the scalar ϕ^x, the vector multiplets contain $2n_V$ gauginos and n_V vector fields. The spinors λ^{Ax} are symplectic-real with respect to the $SU(2)$ R-symmetry labeled by A and transform as vectors in the tangent space of $\mathcal{M}_V$. The vector fields $\mathcal{A}^x_\alpha$ are also vectors in the tangent space. The gravitational multiplet is given by a fünfbein ${e_\alpha}^m$ a pair of gravitinos ψ^A_α, symplectic-real under the $SU(2)$ R-symmetry, and a graviphoton $\mathcal{A}_\alpha$. We will find that the vector fields and graviphoton naturally group together, so will often be labeled by $\mathcal{A}^i_\alpha$ where $i = 0, 1, \dots, n_V$.

The structure of $\mathcal{M}_V$ is as follows. (Here we will use slightly different normalizations from that of [44] in order to match the form which naturally arises in the dimensional reduction of eleven-dimensional supergravity on a Calabi-Yau.) One starts with a $n_V + 1$-dimensional space $\mathcal{C}$ with coordinates a^i. A metric $G_{ij}(a)$ on this space by

$$G_{ij}(a) = -\frac{1}{2}\partial_i\partial_j\mathcal{K}(a) \tag{B.4}$$

where the partial derivative is with respect to the coordinates a^i and $\mathcal{K}$ is a homogeneous polynomial of degree three

$$\mathcal{K} = d_{ijk}a^i a^j a^k \tag{B.5}$$

(One notes that this metric becomes degenerate for certain values of a^i.) One then imagines restricting to the space $K = 6$. This n_V-dimensional manifold with a metric induced from the pull-back of the metric $G_{ij}(a)$ is the sigma-model manifold $\mathcal{M}_V$. It is precisely the same structure described by equations (A.5), (A.10) and (A.17) in the previous appendix, where the metric was derived from compactification on a Calabi-Yau. However here we have no geometrical interpretation of the symmetric tensor d_{ijk}.

Let us write b^i for the values of the coordinates in $\mathcal{C}$ restricted to the subspace $\mathcal{M}_V$. Since the scalar fields ϕ^x are coordinates on $\mathcal{M}_V$, we have $b^i = b^i(\phi^x)$. We can then write the induced metric on $\mathcal{M}_V$ as

$$g_{xy} = b^i_x b^i_x G_{ij} \tag{B.6}$$

where $b^i_x = \partial b^i/\partial\phi^x$ and G_{ij} is the metric on $\mathcal{C}$ evaluated at

$$\mathcal{K} = d_{ijk}b^i b^j b^k = 6 \tag{B.7}$$

(In general, when we drop the argument of G_{ij} it is to be understood as the restriction of the metric to $\mathcal{M}_V$.) We can also write the inverse matrix G^{ij} in terms of the inverse g^{xy}. One finds

$$G^{ij} = b^i_x b^i_x g^{xy} + \frac{2}{3}b^i b^j \tag{B.8}$$

In the following we will adopt the convention that all i, j, k indices are raised and lowered with the metric G_{ij} and x, y, z indices are raised and lowered with g_{xy}. There is a Levi-Civita connection on $\mathcal{M}_V$, giving the covariant derivative

$$D_x v^y = \partial_x v^y + \Gamma^y{}_{xz}v^z \tag{B.9}$$

The corresponding Riemann curvature is highly constrained due to the particular form of the metric $G_{ij}(a)$ given in (B.4).

The full action and supersymmetry variations will be given at the end of this appendix. Here let us simply summarize the transformation properties of the component fields with respect to $\mathcal{M}_V$. As we have already mentioned, the n_V scalar fields ϕ^x are interpreted as coordinates on $\mathcal{M}_V$. We will sometimes write them implicitly in terms of b^i. The $n_V + 1$ vector fields $\mathcal{A}^i_\alpha$, including the graviphoton, live naturally in the tangent space to $\mathcal{C}$ at the subspace $\mathcal{M}_V$. Thus their indices are raised and lowered with G_{ij}. Clearly the vectors with components in the $\mathcal{M}_V$ tangent space are the fields of the vector multiplet, while the graviphoton $\mathcal{A}_\alpha$ is given by the component orthogonal to $\mathcal{M}_V$. Thus, since $b_i b^i_x = 0$, we have the decomposition

$$\mathcal{A}^x_\alpha = b^x_i \mathcal{A}^i_\alpha \, , \qquad \mathcal{A}_\alpha = \frac{2}{3} b_i \mathcal{A}^i_\alpha \, . \tag{B.10}$$

The $2n_V$ symplectic gauginos λ^{Ax} live in the tangent space of $\mathcal{M}_V$. In particular, the spacetime covariant derivative of λ^{Ax} must include a contribution from the Levi-Civita connection on $\mathcal{M}_V$,

$$D_\alpha \lambda^{Ax} = \nabla_\alpha \lambda^{Ax} + \partial_\alpha \phi^y \Gamma^x{}_{yz} \lambda^{Az} \tag{B.11}$$

where ∇_α is the conventional spacetime covariant derivative with spin-connection. The graviton and gravitino are both scalars on the manifold $\mathcal{M}_V$.

B.3 Coupling of hypermultiplets

We now turn to the form of the general coupling of uncharged matter hypermultiplets to the Abelian-vector-gravity system described above. This was first given by Sierra in [51]. With n_H hypermultiplets we have $4n_H$ scalar fields q^u ($u = 1, \dots, 4n_V$) and $2n_H$ symplectic-real fermions ζ^a ($a = 1, \dots, 2n_V$). Again the coupling is fixed by giving the form of the scalar field sigma-model. Fortunately one finds that the sigma model for the vector multiplet and hypermultiplet scalars factorizes as $\mathcal{M}_V \times \mathcal{M}_H$, with no cross couplings between the two types of field. The vector multiplet sigma-model manifold remains exactly as described in the previous section. Meanwhile, one finds, as for $N = 2$ theories in four dimensions, that the hypermultiplet scalars parameterize a quaternionic manifold. As for the coupling of vector multiplets, there is in general no independent function describing a potential, unless the theory admits gauging.

The structure of quaternionic geometry in supersymmetry was first given in [68]. Here we will follow the discussion in [53]. The quaternionic manifold is a Riemannian manifold with holonomy $Sp(2) \times Sp(2n_H)$. In the context of supersymmetry, the $Sp(2)$ group is the $SU(2)$ R-symmetry. All tangent space indices can thus be decomposed into the holonomy groups. Thus if q^u are coordinates on the manifold, we have, for instance, vielbein one-forms

$$V^{Aa} = V^{Aa}_u dq^u \tag{B.12}$$

where A is an $SU(2)$ index and a an index in the fundamental of $Sp(2n_H)$. Let us write Ω_{ab} for the symplectic metric preserved by $Sp(2n_H)$. The metric on the quaternionic manifold can then be written as

$$h_{uv} = V_u^{Aa} V_v^{Bb} \Omega_{ab} \epsilon_{AB} \tag{B.13}$$

In fact, more generally we have

$$\begin{aligned} V_u^{Aa} V_{va}^{B} + V_v^{Aa} V_{ua}^{B} &= h_{uv} \epsilon^{AB} \\ V_u^{Aa} V_{vA}^{b} + V_v^{Aa} V_{uA}^{b} &= \frac{1}{n_H} h_{uv} \Omega^{ab} \end{aligned} \tag{B.14}$$

Since the holonomy has a product structure, the spin-connection compatible with the metric also decomposes into an $SU(2)$ connection and a $Sp(2n_H)$ connection. Let us write the corresponding one-forms as $\omega^A{}_B$ and $\Delta^a{}_b$.

The metric on the quaternionic manifold is hermitian with respect to any of three complex structures $J^A{}_B$ which fill out the adjoint representation of $SU(2)$. As such they satisfy the quaternionic algebra under matrix multiplication. There is correspondingly a triplet of Kähler forms in the adjoint representation

$$K^A{}_B = (K^A{}_B)_{uv} dq^u \wedge dq^v \tag{B.15}$$

The Kähler forms are closed with respect to the $SU(2)$ connection. Thus, dropping the $SU(2)$ indices

$$dK + \omega \wedge K = 0 \tag{B.16}$$

Finally one finds that the $SU(2)$ curvature is proportional to the Kähler forms. Using the normalization of [53], we have

$$R = d\omega + \omega \wedge \omega = -K \tag{B.17}$$

One notes that in the rigid supersymmetry limit this condition becomes $R = 0$ and the quaternionic manifold becomes a hyper-Kähler manifold.

As in the case of the vector multiplet, we would like to describe the transformation properties of the various component fields on $\mathcal{M}_H$. As mentioned above, the hypermultiplet scalar fields q^u are coordinates on $\mathcal{M}_H$. Unlike the gauginos in the vector multiplet, one finds that the fermions ζ^a in the matter multiplet are symplectic-real not with respect to the $SU(2)$ symmetry but rather with respect to the $Sp(2n_H)$ symmetry, satisfying the reality condition given in (B.2). From this we see that ζ^a live in the $Sp(2n_H)$ part of the tangent bundle. Consequently, as for the gauginos, there is a corresponding correction piece in the spacetime covariant derivative from the $Sp(2n_H)$ connection

$$D_\alpha \zeta^a = \nabla_\alpha \zeta^a + \partial_\alpha q^u \Delta_u{}^a{}_b \zeta^b \tag{B.18}$$

The bosonic fields in the vector and gravitational multiplets are all scalars on $\mathcal{M}_H$. However, one recalls that both the gauginos λ^{Ax} and the gravitinos ψ^A_α were doublets under $SU(2)$. Thus they are in the $SU(2)$ part of the $\mathcal{M}_H$ tangent space. Consequently, their covariant derivatives get a correction, when coupled to matter hypermultiplets, from the $SU(2)$ connection on $\mathcal{M}_H$. Namely we have

$$\begin{aligned} D_\alpha \lambda^{Ax} &= \nabla_\alpha \lambda^{Ax} + \partial_\alpha \phi^y \Gamma^x{}_{yz} \lambda^{Az} + \partial_\alpha q^u \omega_u{}^A{}_B \lambda^{Ax} \\ D_\alpha \psi^A_\beta &= \nabla_\alpha \psi^A_\beta + \partial_\alpha q^u \omega_u{}^A{}_B \psi^A_\beta \end{aligned} \tag{B.19}$$

B.4 Gauging the hypermultiplet sector

We now turn to the question of how this structure is modified when the matter fields become charged under the Abelian vector symmetries. Again we will follow closely the description of the four-dimensional case given in [53]. For the matter scalars to be charged we must have an Abelian isometry of $\mathcal{M}_H$ which can become gauged.

Suppose $\mathcal{M}_H$ admits $n_V + 1$ Abelian isometries (one for each vector field). There is a Killing vector k^u_i for each isometry. If the isometries are to preserve the quaternionic structure of $\mathcal{M}_H$, each Killing vector can be written in terms of a $SU(2)$ triplet of prepotentials $\mathcal{P}_i{}^A{}_B$. These are sections of the adjoint $SU(2)$-bundle on $\mathcal{M}_H$ with derivatives related to k^u_i. In general we have

$$k^u_i K_{uv} = \partial_v \mathcal{P}_i + [\omega_v, \mathcal{P}_i] \tag{B.20}$$

so that in particular

$$k^u_i = -\frac{1}{3} \mathrm{tr} K^{uv} \partial_v \mathcal{P}_i \tag{B.21}$$

where $\mathrm{tr} AB = A^A{}_B B^B{}_A$. These prepotentials are the analogs of the D-term prepotentials of $N = 1$ supersymmetry. For Abelian isometries, there is an ambiguity in the definition of $\mathcal{P}_i$. They may be shifted by constants matrices $\mathcal{C}_i$ satisfying $[\mathcal{C}_i, \mathcal{C}_j] = 0$. These constants are the extension of Fayet-Iliopoulos terms to $N = 1$ $D = 5$ supergravity.

The procedure of gauging is now to make these isometry transformations local functions in spacetime. The simplest modification is that the spacetime derivative of the scalar fields q^u, coordinates in $\mathcal{M}_H$, becomes a gauge covariant derivative. Writing g for the general gauge coupling (this could be different for each Abelian isometry) we have

$$\partial_\alpha q^u \longrightarrow D_\alpha q^u = \partial_\alpha q^u + g \mathcal{A}^i_\alpha k^u_i \tag{B.22}$$

However, because isometries in $\mathcal{M}_H$ also produce a mapping between tangent space vectors, we are in addition gauging both the $SU(2)$ and the $Sp(2n_H)$ holonomy groups. Thus the corresponding connections in the covariant derivatives are modified. Equations (B.18) and (B.19) are modified to

$$\begin{aligned} D_\alpha\zeta^a &= \nabla_\alpha\zeta^a + \left(D_\alpha q^u \Delta_u{}^a{}_b + g\mathcal{A}^i_\alpha \partial_u k^v_i V^{uAa} V_{vAb}\right)\zeta^b \\ D_\alpha\lambda^{Ax} &= \nabla_\alpha\lambda^{Ax} + \partial_\alpha\phi^y \Gamma^x{}_{yz}\lambda^{Az} + \left(D_\alpha q^u \omega_u{}^A{}_B + g\mathcal{A}^i_\alpha \mathcal{P}_i{}^A{}_B\right)\lambda^{Bx} \\ D_\alpha\psi^A_\beta &= \nabla_\alpha\psi^A_\beta + \left(D_\alpha q^u \omega_u{}^A{}_B + g\mathcal{A}^i_\alpha \mathcal{P}_i{}^A{}_B\right)\psi^{Bx} \end{aligned} \tag{B.23}$$

Perhaps the most important aspect of the gauging is that it introduces a potential into the action. The potential is the analog of the D-term potential in $N = 1$, $D = 4$ supergravity and is accompanied by fermion mass terms.

B.5 The gauged action and supersymmetry transformations

Having identified the structure of the sigma-models for the vector multiplets and matter hypermultiplets, and discussed how the gauging of charged matter is introduced, we finally turn to the form of the gauged action and the supersymmetry transformations.

The action can be split into kinetic terms, fermion mass terms, potential and finally four-fermi terms which we will not give. We write

$$S = \int_{M_5} \sqrt{-g}\left(\mathcal{L}_{\text{kinetic}} + \mathcal{L}_{\text{four fermi}} + \mathcal{L}_{\text{fermi mass}} - g^2 V\right) \tag{B.24}$$

The kinetic terms are given by, with $\kappa_5 = 1$,

$$\begin{aligned} \mathcal{L}_{\text{kinetic}} = &-\frac{1}{2}R - \frac{1}{2}G_{ij}\partial_\alpha b^i \partial^\alpha b^j - \frac{1}{2}G_{ij}\mathcal{F}^i_{\alpha\beta}\mathcal{F}^{j\alpha\beta} - \frac{1}{12\sqrt{2}}d_{ijk}\epsilon^{\alpha\beta\gamma\delta\epsilon}\mathcal{A}^i_\alpha \mathcal{F}^j_{\beta\gamma}\mathcal{F}^k_{\delta\epsilon} - h_{uv}D_\alpha q^u D^\alpha q^v \\ &- \frac{1}{2}\bar{\psi}^A_\alpha \gamma^{\alpha\beta\gamma}D_\beta\psi_{\gamma A} - \frac{1}{2}\bar{\lambda}^{Ax}\gamma^\alpha D_\alpha \lambda_{Ax} - \frac{1}{2}\bar{\zeta}^a\gamma^\alpha D_\alpha\zeta_a \\ &+ \frac{i}{4\sqrt{2}}\left(\bar{\psi}^A_\gamma \gamma^{\alpha\beta\gamma\delta}\psi_{\delta A} + 2\bar{\psi}^{A\alpha}\psi^\beta_A - \bar{\lambda}^{Ax}\gamma^{\alpha\beta}\lambda_{Ax} - \bar{\zeta}^a\gamma^{\alpha\beta}\zeta_a\right) b_i \mathcal{F}^i_{\alpha\beta} \\ &+ \frac{1}{2\sqrt{2}}\left(\bar{\lambda}^A_x \gamma^\alpha\gamma^{\beta\gamma}\psi_{\alpha A}\right) b^x_i \mathcal{F}^i_{\beta\gamma} - \frac{i}{8\sqrt{2}}\left(\bar{\lambda}^{Ax}\gamma^{\alpha\beta}\lambda^y_A\right) d_{ijk} b^i_x b^j_y \mathcal{F}^i_{\alpha\beta} \\ &- \frac{i}{2}\left(\bar{\lambda}^A_x\gamma^\alpha\gamma^\beta\psi_{\alpha A}\right) b^x_i \partial_\beta b^i + i\left(\bar{\zeta}_a\gamma^\alpha\gamma^\beta\psi_{\alpha A}\right) V^{Aa}_u D_\beta q^u \end{aligned} \tag{B.25}$$

Here $\mathcal{F}^i = d\mathcal{A}^i$ is the two-form Abelian field strength. All the derivatives $D_\alpha\lambda_{Ax}$ etc. are the full covariant derivatives including gauging terms, as given in equations (B.22) and (B.23). We see the appearance of the sigma-model metrics on $\mathcal{M}_V$ and $\mathcal{M}_H$ in the scalar fields kinetic terms. We have written the vector multiplet scalar fields in terms of the constrained b^i throughout. We could equally well have used the unconstrained fields ϕ^x. The sigma-model would then appear as

$$\frac{1}{2}G_{ij}\partial_\alpha b^i \partial^\alpha b^j = \frac{1}{2}\left(G_{ij}b^i_x b^j_y\right)\partial_\alpha\phi^x\partial^\alpha\phi^y = \frac{1}{2}g_{xy}\partial_\alpha\phi^x\partial^\alpha\phi^y \tag{B.26}$$

where we have used equation (B.6). One notes how the gauge fields $\mathcal{A}^i$ naturally couple to the $\mathcal{C}$ metric G_{ij}, and in addition that there is a Chern-Simons term $\mathcal{AFF}$ depending on the d_{ijk} which define the $\mathcal{C}$ metric.

The fermion mass terms can be written as

$$\begin{aligned}\mathcal{L}_{\text{fermi mass}} = &- igS^{AB}\bar{\psi}_{\alpha A}\gamma^{\alpha\beta}\psi_{\beta B} - gW_x^{AB}\bar{\lambda}_A^x\gamma^a\psi_{\alpha B} - gN_a^A\bar{\zeta}^a\gamma^\alpha\psi_{\alpha A} \\ &- igM_{AxBy}\bar{\lambda}_{xA}\lambda_{yB} - igM^{aAx}\bar{\zeta}_a\lambda_{Ax} - igM^{ab}\bar{\zeta}_a\zeta_b\end{aligned} \tag{B.27}$$

where all these terms depend on the Killing vectors and prepotentials

$$\begin{aligned}S_{AB} &= \frac{1}{\sqrt{2}}b_i\mathcal{P}^i{}_{AB} \\ W^x_{AB} &= -\sqrt{2}b_i^x\mathcal{P}^i{}_{AB} \\ N^{Aa} &= -\frac{1}{\sqrt{2}}V_u^{Aa}b^ik_i^u \\ M_{AxBy} &= -\frac{3}{\sqrt{2}}d_{ijk}b^{ix}b^{jx}\mathcal{P}^k{}_{AB} - 3\sqrt{2}b_x^ib_y^jG_{ij}\left(b_k\mathcal{P}^k{}_{AB}\right) \\ M^{aAx} &= -\frac{1}{\sqrt{2}}V_u^{Aa}b^{ix}k_i^u \\ M^{ab} &= \frac{1}{4\sqrt{2}}V_u^{Aa}V_v^{Bb}\epsilon_{AB}\left(b^i\nabla^{[u}k_i^{v]}\right)\end{aligned} \tag{B.28}$$

Here $\nabla^u k_i^v$ is the covariant derivative of the Killing vector on the $\mathcal{M}_H$ quaternionic manifold.

Finally we have the potential

$$V = -2G_{ij}\text{tr}\mathcal{P}^i\mathcal{P}^j + 4b_ib_j\text{tr}\mathcal{P}^i\mathcal{P}^j + \frac{1}{2}b^ib^jh_{uv}k_i^uk_j^v \tag{B.29}$$

where $\text{tr}AB = A^A{}_BB^B{}_A$. Clearly the potential vanishes if there is no gauging. We also note that it is possible to have a contribution to the potential from pure Fayet-Iliopoulos terms. It is specified by $k_i^u = 0$ and $\mathcal{P}^i$ =const which satisfies eq. (B.20).

To complete the theory we also list the supersymmetry transformations. The supersymmetry parameter ϵ^A is an $SU(2)$ symplectic real spinor. The fermion variations

$$\delta\psi_\alpha^A = D_\alpha\epsilon^A - \frac{i}{6\sqrt{2}}\left(\gamma_\alpha{}^{\beta\gamma} - 4\delta_\alpha{}^\beta\gamma^\gamma\right)b_i\mathcal{F}^i_{\beta\gamma}\epsilon^A + \frac{2ig}{3}S^{AB}\gamma_\alpha\epsilon_B \tag{B.30}$$

$$\delta\lambda^{Ax} = b_i^x\left(\frac{i}{2}\gamma^\alpha\partial_\alpha b^i + \frac{1}{2\sqrt{2}}\gamma^{\alpha\beta}\mathcal{F}^i_{\alpha\beta}\right)\epsilon^A + gW^{xAB}\epsilon_B \tag{B.31}$$

$$\delta\zeta^a = -iV_u^{Aa}\gamma^\alpha D_\alpha q^u\epsilon_A - gN^{aA}\epsilon_A \tag{B.32}$$

include corrections from the gauging. The bosonic variations on the other hand receive no corrections, and read

$$\delta e_\alpha{}^m = \frac{1}{2}\bar{\epsilon}^A \gamma^m \psi_{\alpha A} \tag{B.33}$$

$$\delta \mathcal{A}^i_\alpha = \frac{1}{2\sqrt{2}} b^i_x \bar{\epsilon}_A \gamma_\alpha \lambda^{Ax} - \frac{i}{2\sqrt{2}} b^i \bar{\psi}^A_\alpha \epsilon_A \tag{B.34}$$

$$\delta b^i = -\frac{i}{2} b^i_x \bar{\epsilon}_A \lambda^{Ax} \tag{B.35}$$

$$\delta q^u = \frac{i}{2} V^u_{Aa} \bar{\epsilon}^A \zeta^a \tag{B.36}$$

This completes the description of the general coupling of Abelian vector multiplets to charged matter in $D = 5$ supergravity.

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