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CONTRIBUTIONS FROM DAUGHTER TRAJECTORIES IN UNEQUAL MASS SCATTERING

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ABSTRACT

We show that at $t = 0$ the asymptotic contribution from the leading daughter trajectories to certain unequal mass helicity amplitudes can be explicitly calculated using factorization properties, if the Toller quantum number M of the Regge poles requires the residue function in these helicity amplitudes to have more zeros at $t = 0$ than required by analyticity alone.

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For equal mass scattering, the contribution of the daughter trajectories at $t = 0$ relative to their parents can be calculated by a group theoretical method[1,2] or by a method based on factorization and analyticity[3,4]. For the unequal mass cases the problem is much more complicated. For example with spinless particles and $M = 0$ trajectories, only the singular parts of the daughter residues can be calculated. However, these singular parts do not contribute to the asymptotic behaviour of the scattering amplitude since they are cancelled in the full amplitude[5]. The part of the daughter residues which is not cancelled and thus contributes to the asymptotic behaviour is completely arbitrary[5].

In the reactions of unequal masses and general spins, the residues $\beta_{\lambda,\mu}(t)$ of a trajectory with $|\lambda| = M$ and/or $|\mu| = M$ can have the most singular t factors allowed by analyticity, factorization giving no additional dependence. For these amplitudes, the most singular part of all the daughter trajectories can be calculated from the analyticity requirement on the full amplitudes. However, as in the spinless case, the asymptotic contribution of the daughters cannot be explicitly calculated. Moreover for those residues requiring additional zeros due to factorization even the most singular parts of the first few daughter trajectories are not calculable from analyticity alone. The number of uncalculable residues is the same as the number of additional zeros. However, by relating them to the residues with helicity equal to M, all these undetermined daughter residues can be calculated. Due to the additional zeros at $t = 0$, the reduced residues of these daughter trajectories are regular at $t = 0$, and contribute directly to the asymptotic behaviour of the helicity amplitudes. In effect, these finite contributions from the daughters shift the original zero in the leading term away from $t = 0$. The new zero position depends upon $\alpha(0)$ in addition to the helicities and masses. These explicit asymptotic contributions from the daughter trajectories can in principle be studied experimentally.

We shall demonstrate our point in two special examples and then the results for general spins will be given at the end. We shall follow closely the notation of Ref. 4. A Regge pole contribution to the reduced helicity amplitude is given by

$$\bar{f}^{t}_{\mu',\mu}(s,t) = \beta_{\mu',\mu}(t)\, E^{\alpha}_{\mu,\mu'}(z_t) . \tag{1}$$

The expression for $E^{\alpha}_{\mu,\mu'}$ in terms of the hypergeometric function is

$$\begin{aligned} E^{\alpha}_{\mu,\mu'}(z) &= \left[\Gamma(-\alpha+\mu)\,\Gamma(-\alpha-\mu)\,\Gamma(-\alpha+\mu')\,\Gamma(-\alpha-\mu')\right]^{1/2} \\ &\quad \times \left[\Gamma(-2\alpha)\right]^{-1} \left[\tfrac{1}{2}(z-1)\right]^{\alpha-\mu} F\left(-\alpha+\mu,-\alpha+\mu';-2\alpha;\frac{2}{1-z}\right) \\ &\equiv a^{\alpha}_{\mu',\mu} \left[\tfrac{1}{2}(z-1)\right]^{\alpha-\mu} F\left(-\alpha+\mu,-\alpha+\mu',-2\alpha;\frac{2}{1-z}\right), \end{aligned} \tag{2}$$

for $\mu \geq \mu' \geq 0$. For other domains of μ and μ', the expression can be used after using the symmetry relation

$$\begin{aligned} E^{\alpha}_{\mu,\mu'}(z) &= (-1)^{\mu-\mu'} E^{\alpha}_{\mu',\mu}(z) = (-1)^{\mu-\mu'} E^{\alpha}_{-\mu,-\mu'}(z) \\ &= -e^{\mp i\pi(\alpha-\mu)} E^{\alpha}_{\mu,-\mu'}(z) . \end{aligned}$$

Therefore, we shall only give our calculation in one domain of μ and μ'.

Let us first consider unequal mass spinless scattering with an exchange of an $M = 1$ Regge trajectory. We consider the mass configuration to be $s_0 = (m_a^2 - m_c^2)(m_b^2 - m_d^2) > 0$. For an $M = 1$ trajectory, factorization and analyticity require that[6] $\beta_{00}(t) \underset{t\to 0}{\sim} t^{-\alpha(0)}t$. Notice that we have one more zero here than required by analyticity alone. Without further use of factorization we would not be able to determine anything about the first daughter residue $\beta^{(1)}_{0,0}(t)$ near $t = 0$. Now let us couple a spinless channel to a channel with total helicity $\mu = 1$. The statement of factorization for parent residues β and 1st daughter residues $\beta^{(1)}$ is[7]

$$\begin{aligned} \beta_{0,0}\,\beta_{-1,1} &= (\beta_{0,1})^2 , \\ \beta^{(1)}_{0,0}\,\beta^{(1)}_{-1,1} &= (\beta^{(1)}_{0,1})^2 . \end{aligned} \tag{3}$$

Now $\beta_{0,1}$ and $\beta_{-1,1}$ have the most singular behaviour at $t = 0$ allowed by analyticity. Therefore a standard argument of analyticity[5,4] gives us the ratios $\beta^{(1)}_{0,1}/\beta_{0,1}$ and $\beta^{(1)}_{-1,1}/\beta_{-1,1}$. The result is

$$\frac{\beta^{(1)}_{-1,1}}{\beta_{-1,1}} = -\frac{(2\alpha-1)(2\alpha-2)}{2\alpha} , \tag{4}$$

$$\frac{\beta^{(1)}_{0,1}}{\beta_{0,1}} = -\frac{(2\alpha-1)(2\alpha-2)}{2\alpha}\left[\frac{\alpha-1}{\alpha+1}\right]^{1/2} . \tag{5}$$

From Eqs. (4), (5) and (3) we obtain

$$\frac{\beta^{(1)}_{0,0}}{\beta_{0,0}} = -\frac{(\alpha-1)(2\alpha-1)(2\alpha-2)}{2\alpha(\alpha+1)} . \tag{6}$$

Substituting these relations and Eq. (2) into the equation

$$f_{0,0}(s,t) = \beta_{0,0}(t)\, E^{\alpha}_{0,0}(z_t) + \beta^{(1)}_{0,0}\, E^{\alpha_1}_{0,0}(z_t) ,$$

we obtain

$$f_{0,0}(s,t) \propto \left(\frac{s}{s_0}\right)^{\alpha}\left[t + \frac{\alpha}{\alpha+1}\frac{s_0}{s} + \cdots\right] , \tag{7}$$

where we have used the fact that $\frac{1}{2}(z_t-1) \sim st/s_0$ for small t and large s. Thus we have determined the coefficient of the $s^{\alpha-1}$ term in the asymptotic expansion of $f_{00}(s,t)$ uniquely. In effect, the zero of the leading term at t = 0 is shifted to

$$t = -\frac{\alpha}{\alpha+1}\frac{s_0}{s} .$$

Note that this zero is not on the physical boundary.

This result can be applied immediately to the amplitude $f^{t}_{00,\frac{1}{2}\frac{1}{2}}(s,t)$ in $\pi N \to \rho N^*$. It happens that with pion exchange, $\alpha_\pi(0) \lesssim 0$, the original zero of the leading term at t = 0 is shifted with a negligible amount. M = 1 pion requires a zero near the forward direction, therefore, the inconsistency of an M = 1 pion with experimental results[8] remains after our more accurate calculation.

Let us now look at $\pi^- p$ backward scattering. Here the only known contributing Regge pole is Δ. Conventionally, the Toller quantum number $M = \frac{1}{2}$ is assigned[9] to Δ for phenomenological fitting. However, because of some suggestions[10] that $M = 3/2$ for Δ, it seems worthwhile to exploit its consequences. The amplitudes for the reactions are

$$\bar{f}^{\,t}_{\frac{1}{2},\frac{1}{2}}(s,t) = \beta_{\frac{1}{2},\frac{1}{2}}(t)\, E^{\alpha}_{\frac{1}{2},\frac{1}{2}}(z_t)\,,$$

$$\bar{f}^{\,t}_{-\frac{1}{2},\frac{1}{2}}(s,t) = \beta_{-\frac{1}{2},\frac{1}{2}}(t)\, E^{\alpha}_{\frac{1}{2},-\frac{1}{2}}(z_t)\,.$$

We take the baryon Regge pole channel to be the t-channel. Since the assignment of $M = 3/2$ for Δ gives $\beta_{\frac{1}{2},\frac{1}{2}}$ and $\beta_{-\frac{1}{2},\frac{1}{2}}$ one more zero than allowed by kinematical singularity, the first daughter residues can be determined by factorization. Consider the amplitudes $f^t_{\pm 3/2,3/2}$ in $\pi N^* \to \pi N^*$ and $f^t_{\pm\frac{1}{2},3/2}$ in $\pi N \to \pi N^*$. For these reactions $\frac{1}{2}(1 + z_t) \sim s t/s_0$ for small t and large s, where $s_0 = |(m_a^2 - m_c^2)(m_b^2 - m_d^2)|$. The contribution of a Regge pole to these amplitudes is

$$\bar{f}^{\,t}_{\pm\mu,\frac{3}{2}} = \beta_{\pm\mu,\frac{3}{2}}\; E^{\alpha}_{\frac{3}{2},\pm\mu}(z_t)$$

$$= \beta_{\pm\mu,3/2}\; a^{\alpha}_{\mu,3/2}\left[\tfrac{1}{2}(1+z_t)\right]^{\alpha-\frac{3}{2}}\left[1 - \frac{(\alpha-\frac{3}{2})(\alpha\pm\mu)}{2\alpha}\,\frac{2}{1+z_t} + \;\right],$$

where $\mu = 3/2$ or $\frac{1}{2}$, and $a^{\alpha}_{\mu,3/2}$ is given in Eq. (2). These residue functions take the most singular form allowed by analyticity: $\beta_{3/2,3/2} \sim t^{-\alpha}$, $\beta_{-3/2,3/2} \sim t^{-\alpha}\, t^{3/2}$, $\beta_{\frac{1}{2},3/2} \sim t^{-\alpha}\, t^{\frac{1}{2}}$ and $\beta_{-\frac{1}{2},3/2} \sim t^{-\alpha+1}$. Therefore, the most singular part of the first daughter residues $\beta^{(1)}(t)$ in these amplitudes can be determined at $t = 0$ by analyticity alone. We obtain

$$\beta^{(1)}_{\pm\mu,\frac{3}{2}} = -\,\beta_{\pm\mu,3/2}\left[\frac{(\alpha-\frac{3}{2})(\alpha\pm\mu)}{(\alpha+\frac{3}{2})(\alpha\mp\mu)}\right]^{\frac{1}{2}}\frac{1}{2\alpha}\,, \tag{8}$$

where $\mu = 3/2$ or $\frac{1}{2}$.

Now we use factorization to determine the first daughter residues in $f^t_{\frac{1}{2},\frac{1}{2}}$ and $f^t_{\frac{1}{2},-\frac{1}{2}}$. The factorizable residues are the residues with definite parity, namely $\beta^{\pm}_{\mu',\mu} = \beta_{\mu',\mu} \pm \beta_{-\mu',\mu}$. From the factorization relation $\beta^{\pm}_{\frac{1}{2},\frac{1}{2}}\,\beta^{\pm}_{3/2,3/2} = (\beta^{\pm}_{3/2,\frac{1}{2}})^2$, we obtain for the most singular part near $t \approx 0$:

$$\beta_{\frac{1}{2},\frac{1}{2}} = \left(\beta_{\frac{3}{2},\frac{1}{2}}\right)^2 \left(\beta_{\frac{3}{2},\frac{3}{2}}\right)^{-1},$$
$$\beta_{-\frac{1}{2},\frac{1}{2}} = 2\,\beta_{-\frac{1}{2},\frac{3}{2}}\,\beta_{\frac{1}{2},\frac{3}{2}}\left(\beta_{\frac{3}{2},\frac{3}{2}}\right)^{-1}. \tag{9}$$

These relations hold for both the parents and the daughters. From Eqs. (8) and (9) we obtain

$$\bar{f}^{\,t}_{\frac{1}{2},\frac{1}{2}}(s,t) \propto \left(\frac{s}{s_0}\right)^{\alpha-\frac{1}{2}} t^{-\frac{1}{2}} \left[t - \frac{\alpha+\frac{1}{2}}{\alpha+\frac{3}{2}}\frac{s_0}{s} + \cdots\right], \tag{10}$$

where $t^{-\frac{1}{2}}$ is just the t singularity of $\bar{f}^{\,t}_{\frac{1}{2},\frac{1}{2}}$, and

$$\bar{f}^{\,t}_{-\frac{1}{2},\frac{1}{2}}(s,t) \propto \left(\frac{s}{s_0}\right)^{\alpha-\frac{1}{2}} \left[t - \frac{\alpha-\frac{1}{2}}{\alpha+\frac{3}{2}}\frac{s_0}{s} + \cdots\right]. \tag{11}$$

The dominating amplitude near the backward direction is $\bar{f}^{\,t}_{\frac{1}{2},\frac{1}{2}}$ and for $\alpha_\Delta(0) \approx 0.2$ its zero is shifted outward almost to the physical boundary. The zero of the less dominating amplitude is shifted further inside the physical region from $t = 0$. For phenomenological determination of the quantum number M of the Δ trajectory, our calculation offers a more accurate parametrization.

Finally we give a summary of results for general spins and masses.

I. Unequal mass to unequal mass reactions: Here we classify three cases:

I.1 $M \geq \mu \geq \mu' \geq 0$. From analyticity alone we obtain $\beta_{\pm\mu',\mu}(t) \sim t^{-\alpha}(\sqrt{t})^{\mu\mp\eta\mu'}$, where $\eta = \mp 1$ if $(m_a^2 - m_c^2)(m_b^2 - m_d^2) \gtrless 0$. If factorization is used together with analyticity we obtain $\beta_{\pm\mu',\mu}(t) \sim t^{-\alpha}(\sqrt{t})^{\mu\mp\eta\mu'}\, t^{M-\mu}$. A calculation similar to that used in the previous examples gives:

$$\bar{f}^{\,t}_{\pm\eta\mu',\mu} \propto \left(\sqrt{t}\right)^{-(\mu\pm\mu')} t^{M-\mu-1} \left(\frac{s}{s_0}\right)^{\alpha-\mu} \left[t - \eta\frac{(\alpha\pm\mu')(M-\mu)}{\alpha+M}\frac{s_0}{s} + \cdots\right], \tag{12}$$

where $s_0 = \left|(m_a^2 - m_c^2)(m_b^2 - m_d^2)\right| > 0$. The result corresponds to

$$\frac{\beta^{(1)}_{\pm\eta\mu',\mu}}{\beta^{(1)}_{\pm\eta\mu',\mu}} = -\frac{(2\alpha-1)(2\alpha-2)}{2\alpha}\left[\frac{(\alpha+\mu)(\alpha\pm\mu')}{(\alpha-\mu)(\alpha\mp\mu')}\right]^{\frac{1}{2}}\frac{\alpha-M}{\alpha+M} \quad (13)$$

For $(M-\mu) > 1$ one can calculate all the leading $(M-\mu)$ terms. Here we write only the first two terms.

I.2 $0 \leq M \leq \mu' \leq \mu$. Analyticity requires $\beta_{\eta\mu',\mu} \sim t^{-\alpha}(\sqrt{t})^{\mu-\mu'}$. Factorization imposes no additional zeros on $\beta_{-\eta\mu',\mu}$. However, after factorization $\beta_{\eta\mu',\mu} \sim t^{-\alpha}(\sqrt{t})^{\mu-M}(\sqrt{t})^{\mu'-M}$. Therefore, we can calculate only the lower order terms in $\bar{f}_{\eta\mu',\mu}$:

$$\bar{f}_{\eta\mu',\mu} \propto (\sqrt{t})^{-(\mu+\mu')}(t)^{\mu'-M-1}\left(\frac{s}{s_0}\right)^{\alpha-\mu}\left[t-\eta\frac{(\alpha-\mu)(\mu'-M)}{\alpha-M}\frac{s_0}{s}+\dots\right] \quad (14)$$

The corresponding 1st daughter residue is given by

$$\frac{\beta^{(1)}_{\eta\mu',\mu}}{\beta_{\eta\mu',\mu}} = -\frac{(2\alpha-1)(2\alpha-2)}{2\alpha}\left[\frac{(\alpha-\mu)(\alpha-\mu')}{(\alpha+\mu)(\alpha+\mu')}\right]^{\frac{1}{2}}\frac{\alpha+M}{\alpha-M} \quad (15)$$

I.3 $0 \leq \mu' \leq M \leq \mu$. No additional zeros in the residues result from factorization. Therefore, the lower asymptotic terms cannot be calculated.

II. Unequal-mass to equal-mass reactions: Here the convenient amplitudes to discuss are the redefined helicity amplitudes[2)]

$$\begin{aligned}\hat{f}^{t}_{\mu;s\lambda}(s,t) &= \sum_{\substack{b\,d\\ d-b=\lambda}} (-1)^{s_b-b}\, C(s_d,s_b,s;d,-b)\, \bar{f}^{t}_{c\mu;db} \\ &= \hat{\beta}_{\mu;s\lambda}(t)\, \bar{E}^{\alpha}_{\lambda,\mu}(z_t)\end{aligned} \quad (16)$$

where μ is the helicity for the unequal-mass vertex, λ the helicity for the equal-mass vertex, and s the total spin in the "north pole frame" of the equal mass vertex[2)]. For $\mu \neq M$, and $s \geq M$, there are additional $(\sqrt{t})^{|M-\mu|}$ type zeros introduced at the unequal-mass vertex. Through factorization

$\hat{\beta}_{\mu,S\lambda}(t) = \sqrt{\beta_{\eta\mu,\mu}(t)}\, d^{\alpha,M}_{\alpha,S,\lambda}(\pi/2)$ where $\beta_{\eta\mu,\mu}$ is the more singular residue of $\beta_{\pm\mu,\mu}$. The $d^{\alpha,M}_{\alpha,S,\lambda}(\pi/2)$ are defined in Ref. 2 and can be calculated explicitly. The ratio of the first daughter residue to that of the parent is

$$\frac{\hat{\beta}^{(1)}_{\mu,s\lambda}}{\hat{\beta}_{\mu,s\lambda}} = \left[\frac{\beta^{(1)}_{\eta\mu,\mu}}{\beta_{\eta\mu,\mu}}\right]^{\frac{1}{2}} \frac{d^{\alpha,M}_{\alpha-1,s\lambda}\left(\frac{\pi}{2}\right)}{d^{\alpha,M}_{\alpha,s\lambda}\left(\frac{\pi}{2}\right)} ,$$

where $\beta^{(1)}_{\eta\mu,\mu}/\beta_{\eta\mu,\mu}$ is given in Eqs. (13) and (15).

Notice that the t = 0 zero constraint equations are automatically satisfied by these couplings. Their application to $\gamma N \to \pi N$ may be interesting.

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REFERENCES

1) M. Toller, Nuovo Cimento 53A, 671 (1968) and references therein.

2) D.Z. Freedman and J.M. Wang, Phys. Rev. 160, 1560 (1969).

3) J.C. Taylor, Nucl. Phys. B3, 504 (1969);
J.B. Bronzan and C.E. Jones, Phys. Rev. Letters 21, 564 (1968);
P. Di Vecchia and F. Drago, Phys. Rev. 178, 2329 (1969).

4) J.M. Wang and L.L. Wang, to be published in Phys. Rev.

5) D.Z. Freedman and J.M. Wang, Phys. Rev. 153, 1596 (1969).

6) See for example P.H. Frampton, Nucl. Phys. B7, 507 (1968), see also Ref. 4.

7) Factorization holds only for residues with definite parity. Here we have dropped the parity superscript because these residues happen to satisfy factorization in the order of t we consider.

8) Aachen-Berlin-CERN Collaboration, Phys. Letters 27B, 174 (1968).

9) V. Barger and D. Cline, Phys. Rev. Letters 21, 392 (1968).

10) Theoretically a $M = {}^3\!/_2$ Δ trajectory has been suggested by G. Domokos and S. Kövesi-Domokos, Nuovo Cimento 60A, 437 (1969). Experimentally there seems to be some structure near the backward direction of π^-p scattering [J. Orear et al., Phys. Rev. Letters 21, 389 (1968)].