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GENERAL CLASSIFICATION OF CONSPIRACY RELATIONS

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ABSTRACT

We present a study of constraints on helicity amplitudes when the external masses have arbitrary spins. The reaction $\pi N \rightarrow \rho N$ is studied in detail and we find two types of "conspiracy relations". The first one occurs at $t = 0$ when the masses of π and ρ are different; it is different from the Gribov-Volkov type as it vanishes when the masses of π and ρ are equal. The Gribov-Volkov type of relations is very different from the preceding one and occurs when the masses of π and ρ are the same. We show how all these relations can be found without explicit reference to the invariant amplitudes and argue that generally all conspiracy relations can be obtained directly from the Trueman-Wick crossing relation between helicity amplitudes.

I. INTRODUCTION

Due to the great utility of the Regge pole model, there has for a time been a great interest in the reggeizing and analytical continuation of helicity amplitudes corresponding to external particles with arbitrary spins. In a now classical paper, Wang¹⁾ has given a set of prescriptions for constructing helicity amplitudes free of kinematical singularities. There has, however, been some uneasiness about how to tackle problems leading to the so-called conspiracy relations. These are linear combinations of helicity amplitudes that must vanish at certain values of the kinematical variables of the process studied. This vanishing is necessary to avoid constructing reggeized amplitudes that violate Mandelstam analyticity.

We decided to treat a concrete sufficiently complicated and simple problem in detail to get an understanding of the different mechanisms that can lead one to construct models with wrong analytical properties. The example that we choose is $\mathcal{N} N \rightarrow \rho N$: in addition to reasonable high spins, it also contains three unequal masses. Still it is simpler to handle and we believe that what we have learnt should make work easier for those who want to reggeize high spin reactions. The plan of the paper is then to start from the specific example $\mathcal{N} N \rightarrow \rho N$ that we treat in the most pedestrian manner by calculating all helicity amplitudes in terms of the Mandelstam invariant functions (Section II). We then find, (Section III) two classes of constraints dependent on whether or not the \mathcal{N} and the ρ masses are equal or not.

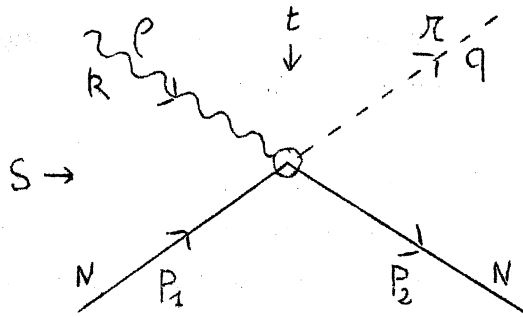
Next, we show how these constraints are obtained from the crossing relations of Trueman and Wick (Section IV). In Section V we then show how one can, from the crossing matrix, tackle the general spin problem.

II. GENERALITIES FOR $\pi N \rightarrow \rho N$ SCATTERING AMPLITUDE

a) Kinematics in the t channel

The general structure of the scattering amplitude for the process $\pi N \rightarrow \rho N$ is well known, because of its analogy with photoproduction or electroproduction²⁾. Nevertheless, our results depend so critically upon the kinematics, that to make our derivation as clear as possible, we shall define everything we need. In order to make the transposition to photoproduction trivial, we take the kinematics of the process $\rho N \rightarrow \pi N$, which accounts for the same thing in what we intend to do.

We shall call k , q , p_1 and p_2 , respectively the four-momenta of the ρ , π and nucleons, and m , μ and M their masses :



Then, in the centre-of-mass of the t channel, we write the four-vectors as

$$k = (\vec{k}, k_0) ; q = (\vec{k}, -\omega) ; p_1 = (\vec{P}, -E) ; p_2 = (\vec{P}, E)$$

where

$$|\vec{k}| = \frac{\sqrt{(t - (m + \mu)^2)(t - (m - \mu)^2)}}{2\sqrt{t}} \quad k_0 = \frac{t + m^2 - \mu^2}{2\sqrt{t}}$$

$$P = \frac{1}{2}\sqrt{t - 4M^2}$$

$$E = \frac{\sqrt{t}}{2}$$

$$\vec{k} \cdot \vec{p} = kP \cos \theta_t$$

$$\cos \theta_t = \frac{u - s}{4kP} = \frac{(u - s)\sqrt{t}}{\sqrt{t - 4M^2} \sqrt{(t - (m + \mu)^2)(t - (m - \mu)^2)}} \quad (1)$$

$$s = -(k + p_1)^2$$

$$u = -(k - p_2)^2$$

All these expressions are assumed to hold for $t > 4M^2$, with positive determination for the square roots. In the following we shall perform an analytic continuation to negative t following a path in the upper half plane of the complex t plane.

b) Invariant amplitudes

We expand the scattering amplitude on the eight following invariants

$$T = \bar{u}(p_2) \left(\sum_{i=1}^8 S_i B_i \right) u(p_1)$$

$$\begin{aligned} S_1 &= i \gamma_5 p_1 \cdot \epsilon & S_5 &= \gamma_5 \gamma \cdot k p_1 \cdot \epsilon \\ S_2 &= i \gamma_5 p_2 \cdot \epsilon & S_6 &= \gamma_5 \gamma \cdot k p_2 \cdot \epsilon \\ S_3 &= i \gamma_5 k \cdot \epsilon & S_7 &= \gamma_5 \gamma \cdot k k \cdot \epsilon \\ S_4 &= \gamma_5 \gamma \cdot \epsilon & S_8 &= i \gamma_5 \gamma \cdot k \gamma \cdot \epsilon \end{aligned} \quad (2)$$

and, as shown by Ball ²⁾, the functions B_i are free of kinematical singularities, and will be postulated to satisfy the Mandelstam representation.

c) Helicity amplitudes

As usual we define helicity amplitudes \bar{f} in the t channel free of kinematical singularities in s by

$$\bar{f}_{cdab} = \left[\frac{1 + \cos \theta_t}{2} \right]^{-\frac{|\lambda + \mu|}{2}} \left[\frac{1 - \cos \theta_t}{2} \right]^{-\frac{|\lambda - \mu|}{2}} f_{cdab} \quad (3)$$

with $\lambda = a - b$ and $\mu = c - d$, and helicity amplitudes with natural or unnatural parity \bar{f}^\pm by

$$\bar{f}_{cdab}^\pm = \bar{f}_{cdab} \pm \bar{f}_{-c-dab} \quad (4)$$

4.

After some algebraic manipulations, one obtains the helicity amplitudes in terms of the invariant functions B_i . The result is

$$\begin{aligned}
 \bar{f}_{01}^- &= \bar{f}_{\frac{1}{2}\frac{1}{2}10}^- = \frac{2\sqrt{2}iP}{M} [E(B_1+B_2) - Mk_0(B_5+B_6) + k_0 B_8] \\
 \bar{f}_{01}^+ &= \bar{f}_{\frac{1}{2}\frac{1}{2}10}^+ = -\frac{2\sqrt{2}ik}{M} E B_8 \\
 \bar{f}_{11}^- &= \bar{f}_{\frac{1}{2}-\frac{1}{2}10}^- = -\frac{2\sqrt{2}iP}{M} [B_4 + (\vec{k}\cdot\vec{p})(B_5+B_6)] \\
 \bar{f}_{11}^+ &= \bar{f}_{\frac{1}{2}-\frac{1}{2}10}^+ = \frac{2\sqrt{2}ik}{M} [\vec{p}^2(B_5+B_6) + M B_8] \\
 \bar{f}_{00}^- &= \bar{f}_{\frac{1}{2}\frac{1}{2}00}^- = -\frac{k_0}{M\sqrt{2}} \cos\theta_t \bar{f}_{01}^- + \frac{2ik}{Mm} [(\vec{k}\cdot\vec{p}) B_8 - E^2(B_1-B_2) + \\
 &\quad + M(B_4 + k_0 E(B_5 - B_6))] \\
 \bar{f}_{10}^+ &= \bar{f}_{\frac{1}{2}-\frac{1}{2}00}^+ = -\frac{4iP}{Mm} [k_0 B_4 + k_0(\vec{k}\cdot\vec{p})(B_5+B_6) + k^2 E(B_5 - B_6)]
 \end{aligned} \tag{5}$$

These expressions contain all information of interest concerning the kinematical singularities and constraints among helicity amplitudes. We shall now focus our attention on this.

III. CONSTRAINTS AMONG HELICITY AMPLITUDES

From Eq. (5) one now sees that for t going to zero, one gets two "conspiracy relations" among the amplitudes f , requiring only that the B_i 's are finite at this point³⁾.

This may be shown by inverting the preceding relations, or, more simply, by looking at the limit when $t \rightarrow 0^+$ of the helicity amplitudes.

One obtains

$$\lim_{t \rightarrow 0^+} \bar{f}_{01}^- = \frac{\sqrt{2}(m^2 - \mu^2)}{\sqrt{E}} [M(B_5 + B_6) - B_8] + O(\sqrt{E})$$

$$\lim_{t \rightarrow 0^+} \bar{f}_{11}^+ = -i \frac{\sqrt{2}(m^2 - \mu^2)}{\sqrt{E}} [M(B_5 + B_6) - B_8] + O'(\sqrt{E})$$

Then

$$\lim_{t \rightarrow 0^+} [i \bar{f}_{01}^- + \bar{f}_{11}^+] = O(\sqrt{E}) \quad (6)$$

and by a similar procedure

$$\lim_{t \rightarrow 0^+} [i \bar{f}_{10}^+ - 2 \bar{f}_{00}^-] = O(\sqrt{E}) \quad (7)$$

The two remaining amplitudes \bar{f}_{01}^+ and \bar{f}_{11}^- go to a constant.

One has now a strange situation: if we put the two boson masses equal and go to the limit $t=0$, then the two relations vanish and one obtains a third "conspiracy relation" which has nothing to do with the two preceding ones.

That relations (6) and (7) vanish is most easily seen in the following way. We define helicity amplitudes F_{ij} free of kinematical singularities by

$$F_{01} = \frac{\sqrt{E}}{P} \bar{f}_{01}^- \quad \bar{F}_{11} = \frac{\bar{f}_{11}^+}{k}$$

$$F_{00} = k t \bar{f}_{00}^- \quad F_{10} = \frac{\sqrt{E}}{P} \bar{f}_{10}^+$$

6.

then relation (6) becomes

$$\left[F_{01} - \frac{m^2 - \mu^2}{2M} F_{11} \right] \rightarrow 0 \quad (8.a)$$

and relation (7)

$$\left[\frac{M(m^2 - \mu^2)}{4} F_{10} + F_{00} \right] \rightarrow O(m^2 - \mu^2) \quad (8.b)$$

For $m = \mu$, this gives no relation.

Directly from expressions (5) one obtains

$$\begin{aligned} \bar{f}_{01}^- &\rightarrow 0 & \bar{f}_{11}^- &\rightarrow 2\sqrt{2} \left[B_4 + \frac{\Sigma - 2S}{4} (B_5 + B_6) \right] \\ \bar{f}_{01}^+ &\rightarrow 0 & \bar{f}_{11}^+ &\rightarrow 2\sqrt{2} \mu \left[M(B_5 + B_6) - B_8 \right] \\ \bar{f}_{10}^+ &\rightarrow 0 & \bar{f}_{00}^- &\rightarrow -\frac{2}{M} \left[\frac{\Sigma - 2S}{4} B_8 + M B_4 \right] \end{aligned} \quad (9)$$

where $\mu = m$ and $\Sigma = 2M^2 + 2\mu^2$.

It is clear that now the three helicity flip amplitudes vanish. But the three remaining non-flip amplitudes are not independent. In fact, they satisfy :

$$\left[\bar{f}_{11}^- - \frac{\Sigma - 2S}{4M\mu} \bar{f}_{11}^+ + \sqrt{2} \bar{f}_{00}^- \right] \rightarrow 0 \quad (10)$$

and this relation has nothing to do with (6) and (7).

The reason why these two sets of constraints are so different, and why there cannot be a continuation from one set to the other one is simply due to the fact that :

$$\lim_{m_1 \rightarrow m_2} \lim_{t \rightarrow 0} \left(k_0 = \frac{t + m_1^2 - m_2^2}{2\sqrt{t}} \right) \neq \lim_{t \rightarrow 0} \lim_{m_1 \rightarrow m_2} k_0$$

the left-hand side is infinite, the right-hand side is zero.

To complete this discussion, let us notice that :

- 1) when all four masses are equal, then the constraint equation (10) survives for $t \rightarrow 0$;
- 2) when all four masses are different, as for instance in $\pi N \rightarrow K^* \Sigma$ or in $\pi^- p \rightarrow e^0 n$ (if one takes into account the electromagnetic mass difference of p and n) then the helicity amplitudes become all independent for $t=0$. There is no longer any constraint of types (6), (7) or (10).

In order to prove this last point, which looks at first sight rather astonishing, we should write down explicitly the relations between helicity amplitudes and invariant functions in the four unequal mass case. This we shall not do, because in Section V we shall prove it in a more direct and more general way. But in a very intuitive way, let us notice that the factor E which represents the nucleon energy is zero when $t \rightarrow 0$, because $E = \sqrt{t}/2$, but tends to infinity when the two fermions have different masses, as $E_1 = ((t+m_1^2-m_2^2)/(2\sqrt{t}))$, (even if the mass difference is 10^{-1} eV !).

We now try to get some insight into the various types of constraints, and we first show the connection between our constraints and the Gribov-Volkov ⁴⁾ type of constraints.

IV. CONSTRAINTS IN THE FORWARD DIRECTION

The first "conspiracy relation" historically found is the well-known relation of Gribov and Volkov. This relation was found in NN scattering, that is, in the four equal mass case, for t going to zero. But the important point to notice is that, **in that case, $t=0$ corresponds to forward scattering in the s channel.** In fact

$$\cos\theta_s = 1 + \frac{2t}{s-4M^2}$$

Gribov and Volkov found their "conspiracy relation" by inverting the relation between helicity amplitudes in the t channel and invariant functions, requiring the latter to be regular around $t=0$, and they gave a very natural explanation of their relation, by noticing that for $\cos\theta_s=+1$, among the five independent amplitudes in the s channel, only three survive, namely those with helicity non flip. Then, because the number of independent amplitudes should be the same at that point in all three channels, one should get two relations between helicity amplitudes in the t channel. One of these relations is the conspiracy relation, the other one is the vanishing of one amplitude.

All this has been explicitly demonstrated by Abers and Teplitz ⁵⁾, using the crossing relation of Trueman and Wick ⁶⁾ .:

$$f_{\lambda\mu}^s = \sum_{\lambda'\mu'} M_{\lambda\mu}^{\lambda'\mu'} f_{\lambda'\mu'}^t \quad (11)$$

Because of formula (3), which is a consequence of total angular momentum conservation, $f_{\lambda\mu}^s(\cos\theta_s=+1) = 0$ if $\lambda \neq \mu$.

In the case of NN scattering, there are two helicity flip amplitudes $f_{\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}}^s$ and $f_{\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}}^s$. The vanishing of $f_{\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}}^s$ leads to the G.V. relation

$$f_{++++}^t - f_{++--}^t - f_{+-+-}^t + f_{+--+}^t = 0,$$

The vanishing of $f_{\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}}^s$ leads to nothing new.

All this is well known, but we have recalled it in order to give some insight into the following.

The conclusion of this discussion is that the constraints between helicity amplitudes f^t in the t channel, through the T.W. crossing relation, due to the vanishing of the helicity flip amplitudes f^s for $\cos\theta_s=+1$, seem to be a very general property which should be met in all scattering processes $a+b \rightarrow c+d$. In the equal mass case $\cos\theta_s=+1$ corresponds to $t=0$. But in the unequal mass case $\cos\theta_s=+1$ leads to a relation between s and t , $s=s(t)$. Thus in the unequal mass case the helicity amplitudes of the t channel should satisfy a number of relations along a curve $s=s(t)$. We shall show that these relations are satisfied in a trivial manner.

We return to our $\pi N \rightarrow \rho N$ problem, and note that $t=0$ does not correspond to forward direction in the s channel. But nevertheless we found a constraint at $t=0$, and it is apparent in Eq. (5) that there is no other singular point except for $p=0$ or $k=0$. The assumed constraint for $\cos\theta_s=+1$ seems to be rather hidden in our case.

The answer is that there is no constraint in the unequal mass case for $\cos\theta_s=+1$.

In the $\pi N \rightarrow \rho N$ problem we have six independent helicity amplitudes in the s channel

$$f_{1/2-1/2}^s = f_{1/2 0 1/2}^s \quad f_{3/2-1/2}^s = f_{1-1/2 0 1/2}^s \quad f_{-1/2-1/2}^s = f_{0 1/2 0 1/2}^s$$

$$f_{1/2 1/2}^s = f_{1/2 0 -1/2}^s \quad f_{3/2 1/2}^s = f_{1-1/2 0 -1/2}^s \quad f_{-1/2 1/2}^s = f_{0 1/2 0 -1/2}^s$$

For $\cos\theta_s=+1$ only $f_{\frac{11}{22}}^s$ and $f_{-\frac{1}{2}-\frac{1}{2}}^s$ are non-zero. Thus we expect four relations between helicity amplitudes in the t channel. In fact, there are four relations, but they are completely trivial, because in that case $\cos\theta_s=+1$ corresponds to $\cos\theta_t=\pm 1$, and due to the factor $(1+\cos\theta_t)^{|\lambda+\mu|/2} (1-\cos\theta_t)^{|\lambda-\mu|/2}$ in $f_{\lambda\mu}^t$

$$f_{\lambda\mu}^t (\cos\theta_t = \pm 1) = 0 \quad \text{if} \quad \lambda \pm \mu \neq 0$$

Explicitly, the following helicity amplitudes vanish :

$$f_{01}^t = f_{\frac{1}{2}\frac{1}{2}10} \quad \text{and} \quad \begin{cases} f_{11}^t = f_{\frac{1}{2}\frac{1}{2}10} \\ f_{10}^t = f_{\frac{1}{2}\frac{1}{2}00} \end{cases} \quad \text{for } \cos\theta_t = -1 \quad \begin{cases} f_{-11}^t = f_{\frac{1}{2}\frac{1}{2}10} \\ f_{00}^t = f_{\frac{1}{2}\frac{1}{2}00} \end{cases} \quad \text{for } \cos\theta_t = +1$$

There is no "conspiracy relation" of the Gribov-Volkov type.

On the other hand, if we make the two boson masses equal, $m = \mu$, then $\cos\theta_s = +1$ corresponds to $t=0$, and as $\cos\theta_t = -(2(s-M^2 - \mu^2)/4\mu M) \neq \pm 1$ one does not find trivial zeros in $f_{\lambda\mu}^t$. Application of the T.W. crossing relation leads to the "conspiracy relation" of Eq. (10).

For example

$$f_{dcba}^s = f_{-\frac{1}{2}1\frac{1}{2}0}^s = \frac{\cos\chi_c}{2} \left[\frac{\sin\chi_a + \chi_b}{2} f_{01}^- - \cos\frac{\chi_d + \chi_b}{2} f_{11}^+ \right] + \frac{1}{2} \left[\frac{\sin\chi_d - \chi_b}{2} f_{01}^+ + \cos\frac{\chi_d - \chi_b}{2} f_{11}^- \right] + \frac{\sin\chi_c}{2\sqrt{2}} \left[-f_{00}^- \frac{\sin\chi_d + \chi_b}{2} + f_{10}^+ \cos\frac{\chi_d + \chi_b}{2} \right] \quad (12)$$

For $\cos\theta_s = +1$, then $\chi_c = \chi_b = \chi_d = \pi/2$ and $f_{-\frac{1}{2}1\frac{1}{2}0}^s = 0$. This gives

$$f_{11}^- - \frac{\sqrt{2}}{2} f_{00}^- = 0 \quad (13)$$

If one uses

$$f_{11}^- = \frac{\cos\theta_t}{2} f_{11}^+ - \frac{1}{2} f_{11}^- \quad \text{and} \quad \cos\theta_t (t=0) = \frac{\Sigma - 2s}{4\mu\mu}$$

then Eq. (13) leads to Eq. (10)

$$\left[\bar{f}_{11}^- - \frac{\sum -2s}{4m\mu} \bar{f}_{11}^+ + \sqrt{2} \bar{f}_{00}^- \right] \rightarrow 0$$

Let us now show how we can obtain the $t=0$ conditions (6) and (7) on the $f_{\lambda\mu}^t$'s when $m \neq \mu$, from the crossing relations, without making reference to the invariant amplitudes $\{B_i\}$. Evidently, at $t=0$, $\cos\theta_s \neq \pm 1$, and we have no zeros on the left-hand side of the crossing relation (11). However, the left-hand side is regular when t is continued from the small negative value of t (when $\cos\theta_s = +1$) to $t=0$. Inspection of the crossing matrix $\mathcal{M}_{\lambda\mu}^{\lambda'\mu'}$ [T.W.'s Eqs. (42) and (43)] shows, however, that some of the elements of this are singular. By looking up which combinations of $f_{\lambda\mu}^t$ are multiplied by these singular factors, and requesting their product to be regular, we recover Eqs. (6) and (7). We shall return to this in the next Section.

We have worked out explicitly, on a specific example, namely the $\pi N \rightarrow \rho N$ process, the various mechanisms of constraints among helicity amplitudes. We now try to generalize this to the general case for any masses and any spins.

V. THE GENERAL PROBLEM OF CONSTRAINTS AMONG HELICITY AMPLITUDES

In order to exhibit and characterize the various kinds of constraints among helicity amplitudes, we shall need only the two following properties.

- 1) Each helicity amplitude may be written

$$f_{\lambda\mu}^p = \left[\frac{1 + \cos\theta_p}{2} \right]^{\frac{|\lambda+\mu|}{2}} \left[\frac{1 - \cos\theta_p}{2} \right]^{\frac{|\lambda-\mu|}{2}} \bar{f}_{\lambda\mu}^p, \quad p = s, u \text{ or } t \quad (14)$$

where $\bar{f}_{\lambda\mu}^p$ is free of kinematical singularities in t .

- 2) The helicity amplitudes f^s in the s channel and f^t in the t channel are related through the T.W. crossing relation

$$f_{cdab}^s = \sum_{A'B'C'D'} d_{A'a}^{s_a}(\lambda_a) d_{b'b}^{s_b}(\lambda_b) d_{c'c}^{s_c}(\lambda_c) d_{d'd}^{s_d}(\lambda_d) f_{c'A'B'b'}^t \quad (15)$$

a) The forward direction constraints

It is well known that the boundary of the physical domain for $\cos^2\theta_s=+1$, $\cos^2\theta_u=+1$ or $\cos^2\theta_t=+1$, is given in all three channels by the same curve in the (s,t,u) plane, namely

$$\Phi(s,t) = 0$$

where

$$\begin{aligned} \Phi(s,t) = & stu - s(m_b^2 - m_d^2)(m_a^2 - m_c^2) - t(m_a^2 - m_b^2)(m_c^2 - m_d^2) - \\ & - (m_a^2 m_d^2 - m_c^2 m_b^2)(m_a^2 + m_d^2 - m_c^2 - m_b^2) \end{aligned} \quad (16)$$

If θ_s is the c.m. angle for the process $a+b \rightarrow c+d$, then $\cos\theta_s=+1$ implies $\cos\theta_t=\pm 1$, when $m_a \neq m_c$, θ_t being the c.m. angle of $\bar{d}+b \rightarrow c+\bar{a}$. There is an ambiguity in sign in the definition of $\cos\theta_t$. We choose it for simplicity to be $\cos\theta_t=+1$ when $\cos\theta_s=+1$.

In this kinematical situation

$$f_{\lambda\mu}^s = 0 \quad \text{if } \lambda - \mu \neq 0$$

$$f_{\lambda'\mu'}^t = 0 \quad \text{if } \lambda' - \mu' \neq 0$$

But there are exactly the same number of helicity amplitudes in each channel satisfying these relations because

$$\lambda - \mu = \lambda_a - \lambda_b - (\lambda_c - \lambda_d) = \lambda_d - \lambda_b - (\lambda_c - \lambda_a) = \lambda' - \mu'$$

We are then led to the following statement :

If θ_s is the c.m. angle of the process $a+b \rightarrow c+d$, then $\cos\theta_s = \pm 1$ implies no constraints among helicity amplitudes in the t channel $\bar{d}+b \rightarrow c+\bar{a}$, when $m_c \neq m_a$.

If $m_c = m_a$ and $m_d = m_b$, then the Trueman-Wick crossing relation leads to relations among helicity amplitudes of the Gribov-Volkov type.

b) The $t=0$ constraints

A. Since all kinematical singularities in t of $f_{\lambda\mu}^s$ are contained in the $(1 \pm \cos\theta_s)$ factors, and as

$$\cos\theta_s = \left[2st + s^2 - s \sum m_i^2 + (m_a^2 - m_b^2)(m_c^2 - m_d^2) \right] / \mathcal{Y}_{ab} \mathcal{Y}_{cd}$$

with

$$\mathcal{Y}_{ab}^2 = [s - (m_a - m_b)^2] [s - (m_a + m_b)^2]$$

$$\mathcal{Y}_{cd}^2 = [s - (m_c - m_d)^2] [s - (m_c + m_d)^2]$$

it is clear that $f_{\lambda\mu}^s$ has no pole for $t=0$.

B. On the other hand, one has, for example, for the angles entering the crossing matrix

$$\sin\chi_a = 2m_a [\Phi(s,t)]^{1/2} / \mathcal{Y}_{ab} \mathcal{Z}_{ac}$$

with

$$\mathcal{Z}_{ac} = [t - (m_a + m_c)^2] [t - (m_a - m_c)^2]$$

If $m_a = m_c$, then $\sin\chi_a$ and $\cos\chi_a$ have a pole for $t=0$, and $d_{\lambda\mu}^J(\chi_a)$ has a pole of order n , with

$$n = \frac{|\lambda + \mu| + |\lambda - \mu|}{4} + \frac{J - \sup(|\lambda|, |\mu|)}{2} = \frac{J}{2}$$

because

$$d_{\lambda\mu}^J(\chi_a) = N_J \left[\frac{1 + \cos\chi_a}{2} \right]^{\frac{|\lambda + \mu|}{2}} \left[\frac{1 - \cos\chi_a}{2} \right]^{\frac{|\lambda - \mu|}{2}} P_{J-M}^{|\lambda - \mu|, |\lambda + \mu|}(\cos\chi_a)$$

where $P_{J-M}^{\alpha\beta}$ is a Jacobi polynomial of order $J-M$, and $M = \sup(|\lambda|, |\mu|)$.

Because $f_{\lambda\mu}^S$ should not have this pole in t^{-n} , it is necessary that some combinations of helicity amplitudes f^t go to zero as t^n .

Let us work out explicitly this for $\pi N \rightarrow \rho N$. We express the $f_{-\frac{1}{2}1\frac{1}{2}0}^S$ helicity amplitude in terms of helicity amplitudes in the t channel. This has been given in (12). $\cos\chi_c$ and $\sin\chi_c$ are finite when $t \rightarrow 0$, but χ_d and χ_b are not.

$$\cos\chi_b = \frac{[(s + M^2 - m^2)t - 2M^2(\mu^2 - m^2)]}{\mathcal{L}_{ab} \mathcal{G}_{bd}}$$

$$\cos\chi_d = \frac{[-(s + M^2 - \mu^2)t - 2M^2(\mu^2 - m^2)]}{\mathcal{L}_{cd} \mathcal{G}_{bd}}$$

$$\sin\chi_b = \frac{2M\sqrt{\Phi}}{\mathcal{L}_{ab} \mathcal{G}_{bd}}$$

$$\sin\chi_d = \frac{2M\sqrt{\Phi}}{\mathcal{L}_{cd} \mathcal{G}_{bd}}$$

(17)

$$\mathcal{L}_{ab} = 2\sqrt{s} q_s$$

$$\mathcal{L}_{cd} = 2\sqrt{s} k_s$$

$$\mathcal{G}_{bd} = 2\sqrt{t} p_t = \sqrt{t} \sqrt{t - 4M^2}$$

After some algebraic manipulations, it may be shown that

$$\cos(\chi_d + \chi_b) = \frac{t^2 A(s) + t B(s) + 8M^4(\mu^2 - m^2)^2}{4s k_s q_s (t - 4M^2) t}$$

and

$$\cos(\chi_b - \chi_d) = \frac{t^2 A'(s) + t B'(s)}{4s k_s q_s (t - 4M^2) t}$$

(18)

Now, when $t \rightarrow 0$, $\cos(\chi_d + \chi_b)$ is going to infinity like $1/t$ and $\cos(\chi_b - \chi_d)$ behaves as a function of s . From this we deduce that $\cos((\chi_d + \chi_b)/2)$ and $\sin((\chi_d + \chi_b)/2)$ are going to infinity like $1/\sqrt{t}$, with

$$\cos \frac{\chi_d + \chi_b}{2} \sim i \sin \frac{\chi_d + \chi_b}{2}$$

From Eq. (12), then, it is necessary that

$$\begin{aligned} \bar{f}_{11}^+ + i \bar{f}_{01}^- &\sim \sqrt{t} \\ i \bar{f}_{10}^+ - 2 \bar{f}_{00}^- &\sim \sqrt{t} \end{aligned} \quad (19)$$

These relations are exactly the same as those deduced in Eqs. (6) and (7), from the invariant amplitudes [the factor two in (19) is introduced when one goes from the f to the \bar{f}]. One may convince oneself that the two relations (19) are really independent, by looking at other relations with the f^S 's, where the same combinations of helicity amplitudes always occur, but with different multiplicative factors.

It is remarkable to notice that if the four masses are different, then $\cos \chi_b$ and $\cos \chi_d$ are no longer singular for $t=0$, so that there is no constraint at this point and we make the general statement :

When the four masses of a scattering process $a+b \rightarrow c+d$ are all different, there is no constraint among helicity amplitudes for $t=0$. There are constraints at this point only when three kinds of different masses occur with $m_a \neq m_c$ and $m_b = m_d$, and these constraints have nothing to do with the forward type of constraints.

VI. CONCLUSIONS

It results from the preceding discussion that there are two types of constraints among helicity amplitudes in the t channel :

- i) the first type of constraints occurs only in reactions $a+b \rightarrow c+d$ with pairwise equal external masses ($m_a = m_c$, $m_b = m_d$), in the s channel forward direction (or backward direction), and is due to the vanishing of helicity flip amplitudes f^S ;
- ii) the second type of constraints occurs at those points where the rotation matrices are infinite, i.e., when

$$[t - (m_b + m_d)^2] [t - (m_b - m_d)^2] = 0$$

They are usually known as "threshold conditions", see for instance Jones ⁷⁾, but occur at $t=0$ only when $m_b = m_d$, and $m_a \neq m_c$.

We summarize the various possibilities in the following table :

masses	type of constraint for $t = 0$	other constraints	number of constraint points
$m_a = m_c$ $m_b = m_d$	$\cos\theta_s = \pm 1$ type I	$q_t = 0$	3
$m_b \neq m_d$	$t = 0$ type II	$q_t = 0$; $p_t = 0$	4
$m_a \neq m_c$ $m_b = m_d$	$t = 0$ type II	$q_t = 0$; $p_t = 0$	4
$m_b \neq m_d$	none	$q_t = 0$; $p_t = 0$	4

In all cases these two types of constraints can be computed directly from the Trueman-Wick crossing relation, without reference to invariant amplitudes.

Therefore, one may avoid the tremendous algebraic machinery of going from the invariant amplitudes to the helicity amplitudes and then inverting the obtained relations in order to get the constraints. It is not necessary either to go from the helicity frame to the usual partial wave frame to obtain the threshold conditions. This last procedure involves rather complicated Clebsch-Gordan coefficient calculations and further assumptions on the threshold behaviour of partial wave amplitudes. In our opinion, this is a non-negligible progress. Nevertheless, it seems that the general formalism developed by Wang ¹⁾ in order to extract the kinematical singularities, loses some of its general interest because one is forced to evaluate explicitly the crossing relations, which, at the same time, gives both the kinematical singularities of helicity amplitudes and the constraints among them.

ADDENDUM

After this paper was written, we have been informed that the whole problem of kinematical singularities, crossings and constraints among helicity amplitudes, has been reviewed and solved by G. Cohen-Tannoudji, A. Morel and H. Navelet from Saclay. Among other powerful results, these authors have reached the same conclusion as ours with regard to the constraint problem, using transversity helicity amplitudes. Nevertheless, they found that some phases between matrix elements of the crossing matrix of Trueman and Wick have to be changed, so that the explicit constraint relations derived from the T.W. formula might be wrong.

We are much indebted to Dr. Cohen-Tannoudji for having informed us of his results prior to publication, as well as for very illuminating discussions.

R E F E R E N C E S

- 1) L.L.C. Wang, Phys.Rev. 142, 1187 (1966).
- 2) J.S. Ball, Phys.Rev. 124, 2014 (1961);
N. Dombey, Nuovo Cimento 31, 1025 (1964).
- 3) M.B. Halpern, Princeton preprint (1967) and
B. Diu and M. Le Bellac, Orsay preprint (1967).
- 4) V.N. Gribov and D.V. Volkov, Soviet Phys. JETP 17, 720 (1963).
- 5) E. Abers and V. Teplitz, Los Angeles preprint (1967).
- 6) T.L. Trueman and G.C. Wick, Ann.Phys. 26, 322 (1964).
- 7) H.F. Jones, Imperial College London preprint (1966).