Integrable multiparametric quantum spin chains

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Abstract

Using Reshetikhin's construction for multiparametric quantum algebras we obtain the associated multiparametric quantum spin chains. We show that under certain restrictions these models can be mapped to quantum spin chains with twisted boundary conditions. We illustrate how this general formalism applies to construct multiparametric versions of the supersymmetric t-J and U models.

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I. INTRODUCTION

The advent of quantum algebras [1,2] precipitated many new results in the area of integrable models. Likewise their supersymmetric counterparts quantum superalgebras [3–6] facilitate systematic treatments of integrable models which accommodate both bosonic and fermionic degrees of freedom. An important subclass are those which may be interpreted as describing systems of correlated electrons for their obvious physical applications in condensed matter physics. Among these are the supersymmetric (SUSY) $t - J$ model [7,8] and supersymmetric generalizations of the Hubbard model [9,10]. Particularly, it would be beneficial if these models could provide some knowledge of the transitions between the metallic, insulating and superconducting phases. It has long been known that an insight into these properties may be gained by studying the effects of the boundary conditions on such models (e.g. see [11,12]). Subsequently several authors have studied electronic models with twisted boundary conditions [16–19].

Integrable models with twisted boundary conditions may be formulated within the framework of quantum (super)algebras. This notion of twisting is more general than that usually treated in electronic models whereby a twisted boundary condition is thought of as the introduction of a phase factor into the periodicity of the model. Here our twisted boundary conditions correspond to more general transformations of the local states and have their origins in the underlying symmetry of the model [20]. For a given R-matrix $R(u)$ any matrix M satisfying $[M_1 M_2, R(u)] = 0$ allows an integrable model to be constructed with the trace over the auxiliary space weighted by M. This is what we will refer to as a twisted boundary condition.

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The work of Reshetikhin [21] on multiparametric quantum (super)algebras permits a natural way to construct integrable models also dependent upon additional free parameters. In fact the model of Perk and Schultz [22] may be formulated within this framework [23]. We will make it apparent that there exists under suitable conditions a integrability preserving mapping between these multiparametric models and those with twisted boundary conditions. Some particular cases have been studied previously [24–27]. We show that these methods may be applied to any model with an underlying quantum (super)algebra symmetry. By establishing this correspondence between models with twisted boundary conditions and multiparametric models, it is reasonable to expect that the multiparametric solutions may provide suitable test models for describing the various phases associated with correlated electron systems.

As examples we will consider the cases of the SUSY $t-J$ model [8] and the SUSY U model [10]. As well as giving the Hamiltonians for the integrable multiparametric generalizations of these two models, we have also determined the corresponding Bethe ansatz equations which provide the starting point for an investigation into their thermodynamics.

The paper is organized as follows. In section II we present the general construction of multiparametric spin chains and their relation to models with twisted boundary conditions. In section III we illustrate how our general formalism applies to construct multiparametric versions of the SUSY t-J and U models. The Bethe ansatz equations of the models are also obtained. A summary of our main results is presented in section IV.

II. GENERAL CONSTRUCTION

Using Reshetikhin's construction [21] for multiparametric quantum algebras, it is straightforward to obtain the associated multiparametric quantum spin chain. Here we demonstrate that under appropriate constraints these models may be transformed to quantum spin chains with twisted boundary conditions; i.e. the additional parameters arising from Reshetikhin's construction may be mapped to the boundary.

Let (A, Δ, R) denote a quasitriangular Hopf (super)algebra where Δ and R denote the co-product and R-matrix respectively. Suppose that there exists an element $F \in A \otimes A$ such that

$$(\Delta \otimes I)(F) = F_{13}F_{23}, \qquad (I \otimes \Delta)(F) = F_{13}F_{12},$$

$$F_{12}F_{13}F_{23} = F_{23}F_{13}F_{12}, \qquad F_{12}F_{21} = I \qquad (1)$$

Theorem 1 of [21] states that (A, Δ^F, R^F) is also a quasitriangular Hopf algebra with co-product and R-matrix respectively given by

$$\Delta^F = F_{12}\Delta F_{21}, \qquad R^F = F_{21}RF_{21}. \qquad (2)$$

In the case that (A, Δ, R) is an affine quantum (super)algebra we have from [21] that F can be chosen to be

$$F = \exp \sum_{i<j} (H_i \otimes H_j - H_j \otimes H_i)\, \phi_{ij} \qquad (3)$$

where $\{H_i\}$ is a basis for the Cartan subalgebra of the affine quantum (super)algebra and the ϕ_{ij}, $i < j$ are arbitrary complex parameters. For our purposes we will extend the Cartan subalgebra by an additional central extension (not the usual central charge) H_0 which will act as a scalar multiple of the identity operator in any representation.

Suppose that π is a loop representation of the affine quantum superalgebra. We let $R(u)$, $R^F(u)$ be the (super)matrix representatives of R and R^F respectively, which both satisfy the Yang-Baxter equation

$$R_{12}(u - v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u - v).$$

For the supersymmetric case it is necessary to impose the multiplication rule

$$(a \otimes b)(c \otimes d) = (-1)^{[b][c]}(ac \otimes bd)$$

for homogeneous supermatrices a, b, c, d where $[a] = 0$ if a is even and $[a] = 1$ if a is odd [28]. However with an appropriate redefinition of its matrix elements $R(u)$ satisfies the

usual (non-graded) Yang-Baxter equation (e.g. see [29]). Throughout we adopt this latter convention.

If $R(u)|_{u=0} = P$ with P the permutation operator then $R^F(u)\big|_{u=0} = P$ as a result of (1). We may construct the transfer matrix

$$
\begin{aligned}
t^F(u) &= \mathrm{str}_0\left(\pi^{\otimes(N+1)}\left(I \otimes \Delta_N^F\right) R_{01}^F\right) \\
&= \mathrm{str}_0\left(R_{0N}^F(u) R_{0(N-1)}^F(u)....R_{01}^F(u)\right)
\end{aligned}
\tag{4}
$$

where Δ_N^F is defined recursively through

$$
\begin{aligned}
\Delta_N^F &= \left(I \otimes I.... \otimes \Delta^F\right) \Delta_{N-1}^F \\
&= \left(\Delta^F \otimes I.... \otimes I\right) \Delta_{N-1}^F.
\end{aligned}
\tag{5}
$$

The subscripts 0 and 1,2,...,N denote the auxiliary and quantum spaces respectively and str_0 is the supertrace over the zeroth space. From the Yang-Baxter equation it follows that the multiparametric transfer matrices $t^F(u)$ form a commuting family. The associated multiparametric spin chain Hamiltonian is given by

$$
\begin{aligned}
H^F &= \left(t^F(u)\right)^{-1} \frac{d}{du} t^F(u)\bigg|_{u=0} \\
&= \sum_{i=1}^{N-1} h_{i,i+1}^F + h_{N1}^F
\end{aligned}
\tag{6}
$$

with

$$
h^F = \frac{d}{du}\, P R^F(u)\big|_{u=0}.
$$

Through use of (1) we may alternatively write

$$
t^F(u) = \mathrm{str}_0\left(\pi^{\otimes(N+1)}\left(I \otimes J_N\right)\left[(I \otimes \Delta_N)(F_{10} R_{01} F_{10})\right]\left(I \otimes J_N\right)^{-1}\right)
$$

with

$$
\begin{aligned}
J_N &= G_{N-1} G_{N-2}....G_1, \\
G_i &= F_{iN} F_{i(N-1)}....F_{i(i+1)}.
\end{aligned}
\tag{7}
$$

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We now define a new transfer matrix

$$t(u) = J_N^{-1} t^F(u) J_N$$

$$= \mathrm{str}_0 \left(\pi^{\otimes(N+1)} \left(I \otimes \Delta_N \right) \left(F_{10} R_{01} F_{10} \right) \right) \qquad (8)$$

where we have employed the convention to let F denote both the algebraic object and its (super)matrix representative. Through further use of (1) we may show that

$$t(u) = \mathrm{str}_0 \left(F_{10} F_{20} .... F_{N0} R_{0N}(u) R_{0(N-1)}(u) .... R_{01}(u) F_{10} .... F_{N0} \right)$$

and the associated Hamiltonian is given by

$$H = \left(t(u) \right)^{-1} \frac{d}{du} t(u) \Big|_{u=0}$$

$$= \sum_{i=1}^{N-1} h_{i,i+1} + \left(F_{N(N-1)} .... F_{N1} \right)^2 h_{N1} \left(F_{1N} .... F_{(N-1)N} \right)^2 \qquad (9)$$

where

$$h = \frac{d}{du} PR(u) \Big|_{u=0} .$$

The above Hamiltonian describes a closed system where instead of the usual periodic boundary conditions we now have a more general type of boundary condition. The boundary term in the above Hamiltonian is a global operator; i.e. it acts non-trivially on all sites. However we can in fact interpret this term as a local operator which couples only the sites labelled 1 and N. It can be shown that the boundary term commutes with the local observables $h_{i,i+1}$ for $i \neq 1$, $N-1$. This situation is analogous to the closed quantum (super)algebra invariant chains discussed in [30].

From the above construction we may also yield models with twisted boundary conditions by an appropriate choice of F. Recall that we extend the Cartan subalgebra by the central element H_0. Let this element act as cI in the representation π where c is some complex number. If we now choose $\phi_{ij} = 0$ for $i \neq 0$ in the expression (3) the matrix F factorizes as $F = M_1^{-1} M_2$ with

$$M = \exp \left(\sum_{i=1}^{l} c \phi_{0i} H_i \right)$$

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and l is the rank of the underlying quantum (super)algebra $U_q(g)$. Using the fact that the R-matrix satisfies

$$[R(u), \ I \otimes H_i + H_i \otimes I] = 0, \qquad i = 1, 2, ..., l$$

tells us that

$$[R(u), \ M_1 M_2] = 0.$$

In this case the Hamiltonian (9) reduces to

$$H = \sum_{i=1}^{N-1} h_{i,i+1} + M_1^{2N} h_{N1} M_1^{-2N} \tag{10}$$

which is precisely the form for a system with twisted boundary conditions (see [20]).

III. EXAMPLES

In this section we illustrate how our formalism applies to construct a multiparametric version of the SUSY t-J model [7] and the SUSY U model [10]. Both models are $gl(2/1)$ invariant and their formulation through the quantum inverse scattering method can be found in [8] and [31], respectively. The first model describes electrons with nearest-neighbor hopping and spin exchange interaction on a chain, while the second can be considered an extension of the Hubbard model with additional pair-hopping and bond-charge interaction terms. These models are of interest because of their possible connection with high-Tc superconductivity. In order to turn our discussion more general, we will in fact handle with their anisotropic or q-deformed versions [32], [33,34]. Of course, in the rational limit $q \to 1$ all results reduce to their corresponding isotropic ones.

A. The supersymmetric t-J model

We begin by introducing the multiparametric $U_q(g\ell(2/1))$ R-matrix, which in terms of a generic spectral parameter x and a deformation parameter q reads

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$$R^F(x) = \begin{pmatrix} a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & t_1^2 b & 0 & c_- & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & t_2^2 b & 0 & 0 & 0 & c_- & 0 & 0 \\ 0 & c_+ & 0 & \frac{b}{t_1^2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & t_3^2 b & 0 & c_- & 0 \\ 0 & 0 & c_+ & 0 & 0 & 0 & \frac{b}{t_2^2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_+ & 0 & \frac{b}{t_3^2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & w \end{pmatrix} \tag{11}$$

where

$$a = xq - \frac{1}{xq}, \quad b = x - \frac{1}{x}, \quad c_+ = x(q - \frac{1}{q}), \quad c_- = \frac{1}{x}(q - \frac{1}{q}), \quad w = -\frac{x}{q} + \frac{q}{x} \tag{12}$$

and t_1, t_2, t_3 are independent parameters written in terms of $\phi_{01}, \phi_{02}, \phi_{12}$ as

$$\begin{aligned} t_1 &= \exp(-2\phi_{01} + \phi_{02} + \phi_{12}) \\ t_2 &= \exp(-\phi_{01} + \phi_{02} + \phi_{12}) \\ t_3 &= \exp(\phi_{01} - \phi_{12}) \end{aligned} \tag{13}$$

The above matrix was already presented by Perk and Schultz [22] when studying a multicomponent generalization of the six-vertex model.

Next we construct the transfer matrix $t^F(x)$ according to eq.(4) from which we find the associated multiparametric Hamiltonian (see eq.(6)) on a one-dimensional periodic lattice

$$H^F = \sum_{i=1}^{N-1} h_{i,i+1}^F + h_{N1}^F,$$

where

$$\begin{aligned} h_{i,i+1}^F =& -(\frac{1}{t_2^2} c_{i\uparrow}^\dagger c_{i+1\uparrow} + t_2^2 c_{i+1\uparrow}^\dagger c_{i\uparrow} + \frac{1}{t_3^2} c_{i\downarrow}^\dagger c_{i+1\downarrow} + t_3^2 c_{i+1\downarrow}^\dagger c_{i\downarrow}) \\ & - 2\left[\frac{1}{t_1^2} S_i^+ S_{i+1}^- + t_1^2 S_i^- S_{i+1}^+ + 2\cos\gamma \ (S_i^z S_{i+1}^z - \frac{n_i n_{i+1}}{4})\right] \\ & + i\sin(\gamma)(n_i - n_{i+1}) - i\sin(\gamma)(n_i S_{i+1}^z - S_i^z n_{i+1}) - \cos\gamma n_i + 2\cos\gamma \end{aligned} \tag{14}$$

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Above $c_{i\pm}^{(\dagger)}$'s are spin up or down annihilation (creation) operators, the $\vec{S}_i$'s spin matrices, the n_i's occupation numbers of electrons at lattice site i and γ is the anisotropy parameter ($q = e^{i\gamma}$). A similar version of a multiparametric SUSY t-J model has already been discussed in ref. [35]. Notice that here it emerges systematically from our general construction. By setting $t_1, t_2, t_3 \to 1$ in eq. (14), the usual terms of the anisotropic SUSY t-J model [32] can be recovered.

The Hamiltonian (14) can be exactly solved through the algebraic nested Bethe ansatz method. This procedure is carried out in two steps and the Bethe ansatz equations are given by

$$
t_1^{2(N-M_2)} t_2^{2M_2} t_3^{-2M_2} \left(\frac{a(x_k^{(1)})}{b(x_k^{(1)})}\right)^N \prod_{i=1}^{M_1} \frac{a(x_i^{(1)}/x_k^{(1)})}{b(x_i^{(1)}/x_k^{(1)})} \frac{b(x_k^{(1)}/x_i^{(1)})}{a(x_k^{(1)}/x_i^{(1)})} \prod_{j=1}^{M_2} \frac{b(x_j^{(2)}/x_k^{(1)})}{a(x_j^{(2)}/x_k^{(1)})} = -1 \ , \ k = 1, \dots M_1,
$$

$$
t_1^{-2(N-M_1)} t_2^{2(N-M_1)} t_3^{2M_1} (-1)^{M_2} \prod_{i=1}^{M_1} \frac{a(x_k^{(2)}/x_i^{(1)})}{b(x_k^{(2)}/x_i^{(1)})} \prod_{j=1}^{M_2} \frac{a(x_j^{(2)}/x_k^{(2)})}{b(x_j^{(2)}/x_k^{(2)})} \frac{b(x_k^{(2)}/x_j^{(2)})}{w(x_k^{(2)}/x_j^{(2)})} = 1 \ , \ k = 1, \dots M_2 \quad (15)
$$

where $x_k^{(m)} (m = 1, 2 \ ; \ k = 1, ..., M_m)$ denote the Bethe ansatz parameters, N is the number of lattice sites, M_1 is the number of holes plus down spins and M_2 is the number of holes. We see from (15) that the additional parameters t_1, t_2, t_3 have the meaning of external fields (see e.g. [36,37])

Following the approach presented in the previous section, we perform the transformation (8) and then set $t_3 = \frac{t_2}{t_1}$ (or $\phi_{12} = 0$, see eq. (13)) in order to find the anisotropic SUSY t-J model with twisted boundary conditions (10)

$$
H = \sum_{i=1}^{N-1} h_{i,i+1} + h_{N1},
$$

where $h_{i,i+1} = \lim_{\{t_1, t_2, t_3 \to 1\}} h_{i,i+1}^F$ and

$$
h_{N,1} = -\left[t_2^{2N} c_{N\uparrow}^\dagger c_{1\uparrow} + \frac{1}{t_2^{2N}} c_{1\uparrow}^\dagger c_{N\uparrow} + \left(\frac{t_2}{t_1}\right)^{2N} c_{N\downarrow}^\dagger c_{1\downarrow} + \left(\frac{t_1}{t_2}\right)^{2N} c_{1\downarrow}^\dagger c_{N\downarrow} \right]
$$
$$
- 2\left[t_1^{2N} S_N^+ S_1^- + \frac{1}{t_1^{2N}} S_N^- S_1^+ + 2\cos\gamma \ (S_N^z S_1^z - \frac{n_N n_1}{4}) \right]
$$
$$
+ i \sin(\gamma)(n_N - n_1) - i\sin(\gamma)(n_N S_1^z - S_N^z n_1) - \cos\gamma \, n_N + 2\cos\gamma \quad (16)
$$

B. The supersymmetric U model

Let us now construct a multiparametric version of the anisotropic SUSY U model, which has been proposed recently as a new integrable model for correlated electrons (see ref. [33,34] for more details).

We begin by recalling the trigonometric R-matrix associated with the one parameter family of four-dimensional representations of $U_q(g\ell(2/1))$

$$
\begin{aligned}
R(x) &= P\check{R}(x), \\
\check{R}(x) &= \frac{q^x - q^{2\alpha}}{1 - q^{x+2\alpha}} P_1 + P_2 + \frac{1 - q^{x+2\alpha+2}}{q^x - q^{2\alpha+2}} P_3.
\end{aligned}
\tag{17}
$$

Here x and q are, respectively, the spectral and deformation parameters and α is a free parameter which arises from the underlying representation. P is the permutation operator and $P_i, i = 1, 2, 3$ are projectors whose explicit form can be found in [34].

We find the corresponding multiparametric R-matrix

$$R^F(x) = \begin{pmatrix}
* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & t_1^2* & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & t_2^2* & 0 & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & t_1^2 t_2^2* & 0 & 0 & t_1 t_2 t_3* & 0 & 0 & \frac{t_1 t_2}{t_3}* & 0 & 0 & * & 0 & 0 & 0 \\
0 & * & 0 & 0 & \frac{*}{t_1^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & t_1 t_2 t_3* & 0 & 0 & t_3^2* & 0 & 0 & * & 0 & 0 & \frac{t_3}{t_1 t_2}* & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & t_1^2 t_3^2* & 0 & 0 & 0 & 0 & 0 & * & 0 & 0 \\
0 & 0 & * & 0 & 0 & 0 & 0 & 0 & \frac{*}{t_2^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{t_1 t_2}{t_3}* & 0 & 0 & * & 0 & 0 & \frac{*}{t_3^2} & 0 & 0 & \frac{*}{t_1 t_2 t_3} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & *\frac{t_2^2}{t_3^2} & 0 & 0 & * & 0 \\
0 & 0 & 0 & * & 0 & 0 & \frac{t_3}{t_1 t_2}* & 0 & 0 & \frac{*}{t_1 t_2 t_3} & 0 & 0 & \frac{*}{t_1^2 t_2^2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & \frac{*}{t_1^2 t_3^2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & 0 & *\frac{t_3^2}{t_2^2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & *
\end{pmatrix}$$

$$(18)$$

where t_1, t_2 and t_3 are independent parameters also given by eq. (13) and "$*$" denote the elements of the R-matrix (17), which can be obtained from the projectors given in [34]. We do not write them explicitly here since we will not need them later. Notice that here, in contrast to the previous case (see eq.(11)), the new parameters t_1, t_2, t_3 occupy also non-diagonal positions. This is a peculiarity of higher representations and can also be verified for other higher spin models (e. g. spin 1 XXZ chain). In fact, our prescription for the element F of the multiparametric R matrix is particularly interesting in these cases, where it is not obvious how to construct R^F.

Next we construct the transfer matrix $t^F(x)$ according to eq.(4) from which we find the

multiparametric version of the anisotropic SUSY U model (see eq.(6))

$$H^F = \sum_{i=1}^{N-1} h^F_{i,i+1} + h^F_{N1},$$

where

$$h^F_{i,i+1} = -\xi c^+_{i\uparrow} c_{i+1\uparrow} [-\eta^{-1}\nu^+]^{n_{i\downarrow}} [-\eta^{-1}\nu^-]^{n_{i+1\downarrow}} - h.c.$$

$$-\rho c^+_{i\downarrow} c_{i+1\downarrow} [-\eta\nu^-]^{n_{i\uparrow}} [-\eta\nu^+]^{n_{i+1\uparrow}} - h.c.$$

$$+[\alpha]^{-1}_q \xi\rho c^+_{i\downarrow} c^+_{i\uparrow} c_{i+1\uparrow} c_{i+1\downarrow} + h.c.$$

$$+[\alpha]^{-1}_q (n_{i\uparrow} n_{i\downarrow} + n_{i+1\uparrow} n_{i+1\downarrow})$$

$$+q^{\alpha+1}(n_{i\uparrow} + n_{i\downarrow} - 1) + q^{-\alpha-1}(n_{i+1\uparrow} + n_{i+1\downarrow} - 1) \tag{19}$$

and $\xi = \frac{t_3^2}{t_2^2}$, $\eta = \frac{t_1 t_3}{t_2}$, $\rho = \frac{1}{t_1^2 t_3^2}$, $\nu^\pm = sgn(\alpha)q^{\pm\frac{1}{2}}(\frac{[\alpha+1]_q}{[\alpha]_q})^{\frac{1}{2}}$. In the $h.c.$ terms one should notice that the parameters $t_i \rightarrow t_i^{-1}, i = 1,2,3$. Equation (19) is a generalization of the anisotropic SUSY U model [34], which can be recovered in the limit $t_1, t_2, t_3 \rightarrow 1$.

This model can be exactly solved by means of the algebraic nested Bethe ansatz method and the Bethe ansatz equations are given by

$$t_1^{2(N-N_2)} t_2^{2N_2} t_3^{-2N_2} \left(\frac{x_k^{(1)} q^{\alpha+1} - q^{-1}}{x_k^{(1)} q^{-\alpha} - 1}\right)^N = \prod_{j\neq k}^{N_2} \frac{x_k^{(1)} - x_j^{(2)} q^2}{q(x_k^{(1)} - x_j^{(2)})} \quad ; \qquad k = 1,\ldots N_1$$

$$t_1^{-2(N-N_1)} t_2^{2(N-N_1)} t_3^{2N_1} \prod_i^{N_1} \frac{x_i^{(1)} - x_k^{(2)} q^2}{q(x_i^{(1)} - x_k^{(2)})} = \prod_{j\neq k}^{N_2} \frac{-x_j^{(2)} + x_k^{(2)} q^2}{x_k^{(2)} - x_j^{(2)} q^2} \quad ; \qquad k = 1,\ldots N_2 \tag{20}$$

where $x_k^{(m)} (m = 1,2 ; k = 1,...,N_m)$ are the Bethe ansatz parameters, N_1 is the total number of spins and N_2 is the number of spins down.

According to the approach presented in the previous section, we perform the transformation (8) and then choose $t_3 = \frac{t_2}{t_1}$ in order to find the anisotropic SUSY U model with twisted boundary conditions, which yields

$$H = \sum_{i=1}^{N-1} h_{i,i+1} + h_{N1},$$

where $h_{i,i+1}$ denotes the local terms of the anisotropic SUSY U model [34] and

$$h_{N,1} = -t_1^{2N} c_{N\uparrow}^+ c_{1\uparrow} (-\nu^+)^{n_{N\downarrow}} (-\nu^-)^{n_{1\downarrow}} - h.c.$$

$$-t_2^{2N} c_{N\downarrow}^+ c_{1\downarrow} (-\nu^-)^{n_{N\uparrow}} (-\nu^+)^{n_{1\uparrow}} - h.c.$$

$$+[\alpha]_q^{-1} (t_1 t_2)^{-2N} \, c_{N\downarrow}^+ c_{N\uparrow}^+ c_{1\uparrow} c_{1\downarrow} + h.c.$$

$$+[\alpha]_q^{-1} (n_{N\uparrow} n_{N\downarrow} + n_{1\uparrow} n_{1\downarrow})$$

$$+q^{\alpha+1} (n_{N\uparrow} + n_{N\downarrow} - 1) + q^{-\alpha-1} (n_{1\uparrow} + n_{1\downarrow} - 1) \tag{21}$$

IV. CONCLUSIONS

In this paper we have demonstrated a correspondence between multiparametric spin chains and models with twisted boundary conditions in the expectation that this connection will provide further insight into the description of phase transitions of such integrable systems. Our approach can be applied to any model with an underlying quantum (super)algebra symmetry. We are particularly interested in models which describe systems of correlated electrons and have studied the SUSY $t - J$ and U models as examples.

Another important class of integrable models are those associated with the Temperley-Lieb algebra. In [38] Zhang proposes a systematic method to generate multiparametric extensions of these models. It is possible to adapt the techniques employed in this paper to establish a mapping from models based on the Temperley-Lieb algebra with twisted boundary conditions to associated multiparametric generalizations. With respect to correlated electron systems, an example based on the Temperley-Lieb algebra has been described in [39,40] to which this procedure can be applied.

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