FUNDAMENTALS OF ACCELERATOR OPTICS*)

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A. PARTICLE MOTION IN MAGNET SYSTEMS

1. TRAJECTORY EQUATIONS IN A FIXED COORDINATE SYSTEM

For guiding particle beams, we need bending and focusing. For charged particles, this is effectively done with electromagnetic fields which exert on the particles the Lorentz force

$$\dot{\vec{p}} = \frac{d}{dt}(m\vec{v}) = e(\vec{E} + \vec{v} \times \vec{B}) \quad \text{with} \quad E \text{ in } \frac{V}{m} \text{ and } Bv \approx 3\cdot10^8 \frac{m}{s}\cdot\frac{Vs}{m^2} \quad \text{for } v \approx c,\ B = 1T \qquad (1)$$

Thus a magnetic field of one Tesla gives the same bending force as an electric field of 300 million Volts per meter for relativistic particles with $v \approx c$. We therefore consider transverse magnetic fields only.

Since the relativistic mass is not changed by the magnetic field, we have

$$\dot{\vec{v}} = \frac{e}{m}(\vec{v} \times \vec{B}) \ . \qquad (1a)$$

Inserting the radius vector (Fig. 1)

$$\vec{r} = z\vec{z}_0 + x\vec{x}_0 + s\vec{s}_0$$
$$\vec{v} = \dot{\vec{r}} = \dot{z}\vec{z}_0 + \dot{x}\vec{x}_0 + \dot{s}\vec{s}_0$$
$$\dot{\vec{v}} = \ddot{\vec{r}} = \ddot{z}\vec{z}_0 + \ddot{x}\vec{x}_0 + \ddot{s}\vec{s}_0$$

into eq. (1a), we have

$$\begin{aligned} \ddot{z} &= \frac{e}{m}(\dot{x}B_s - \dot{s}B_x) \\ \ddot{x} &= \frac{e}{m}(\dot{s}B_z - \dot{z}B_s) \\ \ddot{s} &= \frac{e}{m}(\dot{z}B_x - \dot{x}B_z) \ . \end{aligned} \qquad (2)$$

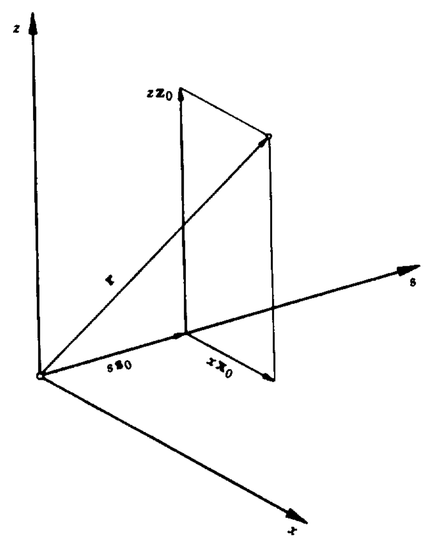


Fig. 1: Fixed Cartesian coordinate system {z, x, s}

We now set

$$\dot{z} = z'\dot{s} = \frac{dz}{ds}\frac{ds}{dt} \qquad \dot{x} = x'\dot{s}$$
$$\ddot{z} = z''\dot{s}^2 + z'\ddot{s} \qquad \ddot{x} = x''\dot{s}^2 + x'\ddot{s}$$

$$\ddot{s} = \frac{e}{m}\dot{s}\,(z'B_x - x'B_z)$$
$$v^2 = \dot{s}^2 + \dot{z}^2 + \dot{x}^2 = \dot{s}^2(1 + z'^2 + x'^2)$$

and obtain the exact trajectory equations in the fixed coordinate system {z, x, s}

* An abridged version of "Basic course on accelerator optics" published in CERN 85-19.

$$\boxed{\begin{aligned} z'' &= \frac{v}{\dot{s}}\frac{e}{p}\{x'B_s - (1 + z'^2)B_x + x'z'B_z\} \\ x'' &= -\frac{v}{\dot{s}}\frac{e}{p}\{z'B_s - (1 + x'^2)B_z + x'z'B_x\} \end{aligned}} \quad \text{with} \quad \frac{v}{\dot{s}} = \sqrt{1 + z'^2 + x'^2}\ . \qquad (3)$$

2. MOTION IN A HOMOGENEOUS FIELD $B_z(x)$ = CONST.

We insert $B_z \equiv B = \text{const}$, $B_x \equiv B_s \equiv 0$ into eqs. (3) and have, with $z_0' = 0$:

$$z'' = 0 \quad ; \quad \underbrace{\frac{x''}{(1 + x'^2)^{3/2}}}_{\text{curvature}} = -\underbrace{\frac{1}{\rho}}_{\text{radius of curvature}} = \frac{e}{p}B_z = \text{const.} \qquad (4)$$

The particle moves on a circle with radius ρ.

This can also be seen by setting the Lorentz force equal to the centrifugal force:

$$evB_z = -\frac{mv^2}{\rho} \quad ; \quad \frac{e}{p}B_z = -\frac{1}{\rho}\ .$$

Numerical evaluation of ρ (in m) for a given field B^* (in T) and momentum p^* (in GeV/c):

$$\boxed{-\frac{1}{\rho}\,[\mathrm{m}^{-1}] = \frac{eB^*\,\frac{\mathrm{Vs}}{\mathrm{m}^2}}{p^*\,\frac{10^9\,\mathrm{eV}}{0.2998\cdot 10^9\,\frac{\mathrm{m}}{\mathrm{s}}}} = 0.2998\,\frac{B^*[\mathrm{T}]}{p^*[\mathrm{GeV/c}]}} \qquad (5)$$

3. CURVED COORDINATE SYSTEM FOLLOWING A REFERENCE TRAJECTORY (Fig. 2)

We choose, in the horizontal plane $z \equiv 0$, a reference trajectory (center of beam). In order to describe particle trajectories in the vicinity of the reference trajectory, we introduce a right-handed rectangular system of coordinate vectors $\{z, x, s\}$ that follows this trajectory, with $\vec{s}$ pointing in its direction and $\vec{z}$ being orthogonal to the reference plane $z \equiv 0$. Within a small range of s, this system can be viewed as a cylindrical coordinate system $\{z, r, \vartheta\}$ with $r = \rho + x$ and $\vartheta = \frac{s}{\rho}$. For $\rho \to \infty$, the system transforms into the Cartesian coordinate system Fig. 1.

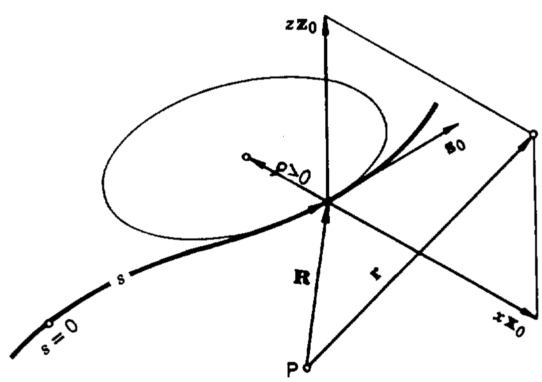


Fig. 2: Curved coordinate system {z,x,s}

4. FIELD EXPANSION IN THE CURVED COORDINATE SYSTEM, WITH $B_x = B_s = 0$ IN THE SYMMETRY PLANE $z \equiv 0$

We assume, at any given s, the field symmetry

$$B_z(z) = B_z(-z) \ ; \ B_x(z) = -B_x(-z) \ ; \ B_s(z) = -B_s(-z)\ .$$

The field then may be expanded as

$$B_z = \sum_{i,k=0}^{\infty} z^{2i} x^k a_{ik} \qquad \text{(even in z)}$$
$$B_x = z \cdot \sum_{i,k=0}^{\infty} z^{2i} x^k b_{ik} \qquad \text{(odd in z)} \qquad (6)$$
$$B_s = z \cdot \sum_{i,k=0}^{\infty} z^{2i} x^k d_{ik} \qquad \text{(odd in z)}$$

where the coefficients a_{ik}, b_{ik}, d_{ik} are functions of s.

The field must obey Maxwell's equations which, in the curved (approximately cylindrical) coordinate system, demand

$$-\operatorname{curl}\vec{B} = \left\{\frac{\rho}{\rho + x}\frac{\partial B_x}{\partial s} - \frac{1}{\rho + x}B_s - \frac{\partial B_s}{\partial x}\,;\; \frac{\partial B_s}{\partial z} - \frac{\rho}{\rho + x}\frac{\partial B_z}{\partial s}\,;\; \frac{\partial B_z}{\partial x} - \frac{\partial B_x}{\partial z}\right\} \overset{!}{=} \{0;\,0;\,0\}$$

$$\operatorname{div}\vec{B} = \frac{\partial B_z}{\partial z} + \frac{\partial B_x}{\partial x} + \frac{\rho}{\rho + x}\frac{\partial B_s}{\partial s} + \frac{1}{\rho + x}B_x \overset{!}{=} 0$$

and yield, for the expansion coefficients, the recursion formulae

$$(k + 1)\,a_{i,k+1} = (2i + 1)\,b_{ik}$$
$$b'_{ik} = (k + 1)(d_{i,k+1} + \frac{1}{\rho}d_{ik})$$
$$a'_{ik} = (2i + 1)(d_{i,k} + \frac{1}{\rho}d_{i,k-1})$$
$$2(i + 1)(a_{i+1,k} + \frac{1}{\rho}a_{i+1,k-1}) + (k + 1)(b_{i,k+1} + \frac{1}{\rho}b_{ik}) + d'_{ik} = 0\,.$$

Using these formulae and writing the field in the symmetry plane $z \equiv 0$ as

$$\frac{e}{p}B_z(s) = h(s) + k(s)\cdot x + \frac{1}{2}m(s)\cdot x^2 + \frac{1}{6}n(s)\cdot x^3 + O(4)$$

with $h = \frac{e}{p}B_z = -\frac{1}{\rho}$ dipole

$k = \frac{e}{p}\frac{\partial B_z}{\partial x}$ quadrupole

$m = \frac{e}{p}\frac{\partial^2 B_z}{\partial x^2}$ sextupole

$n = \frac{e}{p}\frac{\partial^3 B_z}{\partial x^3}$ octupole,

the general field expansion with symmetry plane is, in the curved coordinate system

$$\boxed{\begin{aligned}
\frac{e}{p}B_z &= h + kx + \frac{1}{2}mx^2 - \frac{1}{2}\beta z^2 + \frac{1}{6}nx^3 - \frac{1}{2}\{h(\beta - 2m) + \alpha'' + n\}xz^2 + O(4)\\
\frac{e}{p}B_x &= kz + mxz + \frac{1}{2}nx^2z - \frac{1}{6}\{h(\beta - 2m) + \alpha'' + n\}z^3 + O(4)\\
\frac{e}{p}B_s &= h'z + \alpha'xz + (h\alpha' + \frac{1}{2}m')x^2z - \frac{1}{6}\beta' z^3 + O(4)
\end{aligned}} \qquad (7)$$

with $\alpha = \frac{1}{2}h^2 + k$ and $\beta = h'' - hk + m\,.$

5. LINEAR TRAJECTORY EQUATIONS IN THE CURVED COORDINATE SYSTEM

The time derivatives of the moving axes of the curved coordinate system are

$$\dot{\vec{z}}_0 = 0 \quad ; \quad \dot{\vec{x}}_0 = \frac{\dot{s}}{\rho}\,\vec{s}_0 \quad ; \quad \dot{\vec{s}}_0 = -\frac{\dot{s}}{\rho}\,\vec{x}_0$$

where $\dot{s}$ is the velocity of the particle projection on the reference orbit. Then

$$\vec{r} = z\vec{z}_0 + x\vec{x}_0 + R$$

$$\vec{v} = \dot{\vec{r}} = \dot{z}\vec{z}_0 + \dot{x}\vec{x}_0 + \dot{s}(1+\frac{x}{\rho})\vec{s}_0 \qquad (\text{since } \dot{R} = \dot{s}\vec{s}_0)$$

$$\dot{\vec{v}} = \ddot{\vec{r}} = \ddot{z}\vec{z}_0 + \{\ddot{x} - \frac{\dot{s}^2}{\rho}(1+\frac{x}{\rho})\}\,\vec{x}_0 + \{2\dot{x}\frac{\dot{s}}{\rho} + \ddot{s}(1+\frac{x}{\rho})\}\,\vec{s}_0 \,.$$

Setting again

$$\dot{z} = z'\dot{s} \qquad ; \qquad \dot{x} = x'\dot{s}$$

$$\ddot{z} = z''\dot{s}^2 + z'\ddot{s} \qquad ; \qquad \ddot{x} = x''\dot{s}^2 + x'\ddot{s}$$

and inserting into the Lorentz equation (1a) yields the trajectory equations

$$\boxed{\begin{aligned} z'' + \frac{\ddot{s}}{\dot{s}^2}z' &= \frac{v}{\dot{s}}\frac{e}{p}\{x'B_s - (1+\frac{x}{\rho})B_x\} \\ x'' + \frac{\ddot{s}}{\dot{s}^2}x' - \frac{1}{\rho}(1+\frac{x}{\rho}) &= -\frac{v}{\dot{s}}\frac{e}{p}\{z'B_s - (1+\frac{x}{\rho})B_z\} \end{aligned}} \tag{8}$$

with $\frac{v}{\dot{s}} = \sqrt{(1+\frac{x}{\rho})^2 + z'^2 + x'^2}$ and, by differentiation,

$$\frac{\ddot{s}}{\dot{s}^2} = -\frac{1}{2}\frac{(v^2/\dot{s}^2)'}{v^2/\dot{s}^2}\,.$$

We take here only the linear part of these equations, setting

$$\frac{v}{\dot{s}} \approx 1 + \frac{x}{\rho} \qquad ; \qquad \ddot{s} \approx 0$$

$$\frac{1}{p} \approx \frac{1}{p_0}(1 - \frac{\Delta p}{p_0})$$

$$\frac{e}{p}B_z \approx -\frac{1}{\rho} + kx \quad ; \quad \frac{e}{p}B_x \approx kz \quad ; \quad \frac{e}{p}B_s \approx 0$$

and have the linear trajectory equations in the curved coordinate system:

$$\boxed{\begin{aligned} z'' + kz &= 0 \\ x'' - (k - \frac{1}{\rho^2})x &= \frac{1}{\rho}\frac{\Delta p}{p_0} \end{aligned}} \tag{9}$$

6. GENERAL SOLUTION OF TRAJECTORY EQUATIONS IN TERMS OF PRINCIPAL TRAJECTORIES

In the general case where the bending strength $\frac{1}{\rho(s)}$ and the focusing strength $k(s)$ vary along the reference orbit, eqs. (9) are of Hill's type and describe an oscillatory motion with variable restoring force:

$$y'' + K(s) \cdot y = \frac{1}{\rho} \frac{\Delta p}{p} \, . \tag{9a}$$

The general solution of this equation can be written as

$$\begin{aligned} y(s) &= C(s) \cdot y_0 + S(s) \cdot y_0' + D(s) \cdot \frac{\Delta p}{p_0} \\ y'(s) &= C'(s) \cdot y_0 + S'(s) \cdot y_0' + D'(s) \cdot \frac{\Delta p}{p_0} \end{aligned} \tag{10}$$

where C and S are two independent solutions of the homogeneous equation, with initial conditions

$$\begin{pmatrix} C_0 & S_0 \\ C_0' & S_0' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \tag{10a}$$

and D(s) is a particular solution of the inhomogeneous equation for $\frac{\Delta p}{p_0} = 1$, with initial conditions

$$\begin{pmatrix} D_0 \\ D_0' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

C, S, and D are called principal trajectories (Cosinelike, Sinelike and Dispersion).
In matrix notation, the linear transformation (10) may be written as

$$\begin{pmatrix} y \\ y' \end{pmatrix}_s = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \begin{pmatrix} y \\ y' \end{pmatrix}_0 + \frac{\Delta p}{p_0} \cdot \begin{pmatrix} D \\ D' \end{pmatrix} \tag{10b}$$

or

$$\begin{pmatrix} y \\ y' \\ \frac{\Delta p}{p_0} \end{pmatrix}_s = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} y \\ y' \\ \frac{\Delta p}{p_0} \end{pmatrix}_0 . \tag{10c}$$

The determinant of the transformation matrices is independent of s, as seen by differentiation:

$$(CS' - SC')' = CS'' - SC'' = - K(CS - SC) = 0 .$$

Since its value is unity at $s = s_0$, owing to the chosen initial conditions, it stays unity throughout the system (good for numerical checks!).
The dispersion D(s) may be expressed in terms of C(s) and S(s):

$$\boxed{D = S \int_0^s \frac{1}{\rho} C d\tau - C \int_0^s \frac{1}{\rho} S d\tau} \tag{11}$$

$$D' = S' \int_0^s \frac{1}{\rho} C d\tau - C' \int_0^s \frac{1}{\rho} S d\tau$$

$$D'' = S'' \int_0^s \frac{1}{\rho} C d\tau - C'' \int_0^s \frac{1}{\rho} S d\tau + \frac{1}{\rho} \underbrace{(CS' - SC')}_{= 1} = - KD + \frac{1}{\rho} \, .$$

7. SOLUTION OF TRAJECTORY EQUATIONS IN A MAGNET WITH B(s) = CONST.

We assume that the magnet starts and ends abruptly with constant field within (hard edged model). The principal trajectories C and S then solve the harmonic oscillator equation

$$y'' + Ky = 0 \quad \text{with} \quad \begin{cases} K = k & = \text{const for } z \text{ (vert.)} \\ K = -(k - \frac{1}{\rho^2}) & = \text{const for } x \text{ (hor.)} \end{cases} . \tag{9a}$$

With $\varphi = s\sqrt{|K|}$ they are within the magnet

$$\boxed{\begin{array}{lll} \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = \begin{pmatrix} \cos\varphi & \frac{s}{\varphi}\sin\varphi \\ -\frac{\varphi}{s}\sin\varphi & \cos\varphi \end{pmatrix} & \text{for } K > 0 & \text{focusing} \\ \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} & \text{for } K = 0 & \text{drift space} \\ \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = \begin{pmatrix} \cosh\varphi & \frac{s}{\varphi}\sinh\varphi \\ \frac{\varphi}{s}\sinh\varphi & \cosh\varphi \end{pmatrix} & \text{for } K < 0 & \text{defocusing} \end{array}} \tag{12}$$

We see that indeed the determinant

$$CS' - SC' = 1 .$$

For the dispersion, we calculate in the focusing case (K > 0)

$$D = S\int_0^s \frac{1}{\rho} C d\tau - C\int_0^s \frac{1}{\rho} S d\tau = \frac{1}{\rho}[\frac{1}{\sqrt{K}}\sin\varphi \frac{1}{\sqrt{K}}\sin\varphi - \cos\varphi\{-\frac{1}{K}(\cos\varphi - 1)\}]$$

$$D = \frac{1}{\rho K}(1 - \cos\varphi)$$

$$D' = \frac{1}{\rho\sqrt{K}}\sin\varphi .$$

Similarly, in the defocusing case (K < 0)

$$D = \frac{1}{\rho}\ [\frac{1}{\sqrt{|K|}}\sinh\varphi \cdot \frac{1}{\sqrt{|K|}}\sinh\varphi - \cosh\varphi \cdot \frac{1}{|K|}(\cosh\varphi - 1)]$$

$$D = -\frac{1}{\rho\sqrt{|K|}}(1 - \cosh\varphi)$$

$$D' = \frac{1}{\rho\sqrt{|K|}}\sinh\varphi .$$

Thus, composed, for the dispersion, with

$$\varphi = s\sqrt{|K|} \quad \text{and} \quad \begin{cases} K = k & \text{in } z \\ K = -(k - \frac{1}{\rho^2}) & \text{in } x \end{cases}$$

$$\boxed{\begin{aligned} \begin{pmatrix} D \\ D' \end{pmatrix} &= \begin{pmatrix} \frac{1}{\rho K}(1 - \cos\varphi) \\ \frac{1}{\rho\sqrt{K}}\sin\varphi \end{pmatrix} && \text{for } K > 0 \text{ focusing} \\ \begin{pmatrix} D \\ D' \end{pmatrix} &= \begin{pmatrix} s^2/2\rho \\ s/\rho \end{pmatrix} && \text{for } K = 0 \\ \begin{pmatrix} D \\ D' \end{pmatrix} &= \begin{pmatrix} -\frac{1}{\rho|K|}(1 - \cosh\varphi) \\ \frac{1}{\rho\sqrt{|K|}}\sinh\varphi \end{pmatrix} && \text{for } K < 0 \text{ defocusing} \end{aligned}} \qquad (12a)$$

The overall transformation matrices M_x and M_z of the magnet are obtained from eqs. (12) with $s = \ell$ and $\varphi = \ell\sqrt{|K|}$, of course.

8. MAGNET TYPES

a) Synchrotron magnet ($\frac{1}{\rho} \neq 0$; $k \neq 0$)

This, in principle, is the most general linear magnet with bending strength $\frac{1}{\rho}$, the corresponding "weak focusing" strength $\frac{1}{\rho^2}$ and the quadrupole strength k superimposed (Fig. 3).

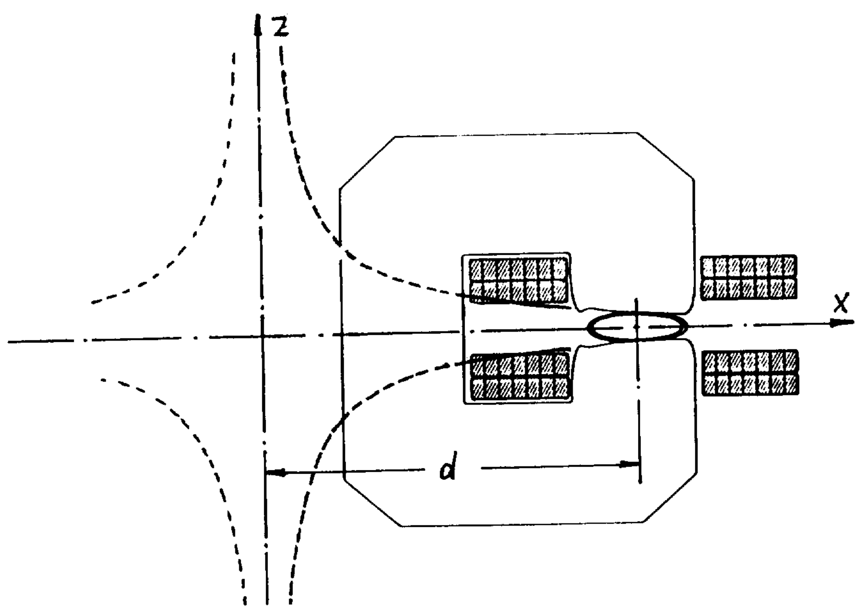


Fig. 3: Synchrotron magnet cross section

It is not much applied anymore in practice (little flexibility!). Its poles are hyperbolic, and it may be considered as a section of a quadrupole that is traversed by the beam at a distance d off-center. We have

$$k \cdot x = \frac{e}{p} \frac{\partial B_z}{\partial x} \cdot x = \frac{e}{p} B_z(x) = -\frac{1}{\rho(x)}$$

$$k \cdot d = -\frac{1}{\rho} \quad ; \quad d = -\frac{1}{\rho k}$$

for the characteristic distance d.

Note that, in our formulation, the entry and exit faces of the hard-edged magnet are perpendicular to the beam since we have assumed that the field is constant between $s = 0$ and $s = \ell$ in the curved coordinate system, independent of x and z.

b) <u>Quadrupole magnet $(\frac{1}{\rho} = 0;\ k \neq 0)$</u>

The beam passes through the center of the quadrupole, and there is no bending of the reference orbit.

The poles are hyperbolae (Fig. 4) given by

$$x \cdot z = \frac{1}{2} r_0^2 .$$

In terms of the field gradient

$$g = \frac{\partial B_z}{\partial x} = \frac{\partial B_x}{\partial z}$$

the quadruple strength is

$$k = 0.2998 \frac{g^*}{p^*} \frac{[\mathrm{T/m}]}{[\mathrm{GeV/c}]} .$$

At a given radius r, the modulus of the field strength is constant:

$$|B| = \sqrt{B_z^2 + B_x^2} = \sqrt{(gx)^2 + (gz)^2} = g \cdot r .$$

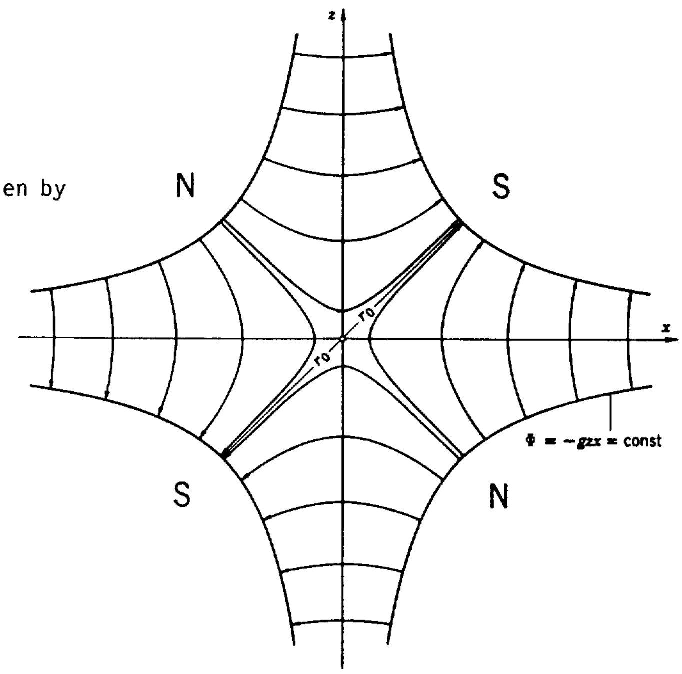


Fig. 4: Quadrupole field configuration

The transformation matrices are, for $k > 0$, from eqs. (12) with $\varphi = \ell \sqrt{|k|}$

$$M_x = \begin{pmatrix} \cosh\varphi & \frac{\ell}{\varphi}\sinh\varphi & 0 \\ \frac{\varphi}{\ell}\sinh\varphi & \cosh\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad ; \quad M_z = \begin{pmatrix} \cos\varphi & \frac{\ell}{\varphi}\sin\varphi & 0 \\ -\frac{\varphi}{\ell}\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} . \qquad (13)$$

For $k > 0$, there is hor. defocusing and vert. focusing;
for $k < 0$, " " hor. focusing " vert. defocusing.

c) Drift space ($\frac{1}{\rho} = 0;\ k = 0$)

The magnet is non-existent, and we have

$$M_x = M_z = \begin{pmatrix} 1 & \ell & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{14}$$

d) Sector magnet ($\frac{1}{\rho} \neq 0;\ k = 0$)

A homogeneous field bending magnet with cross section as e.g. in Fig. 5. In top view, the magnet is sector-shaped due to the orthogonal beam entry and exit (Fig. 6).

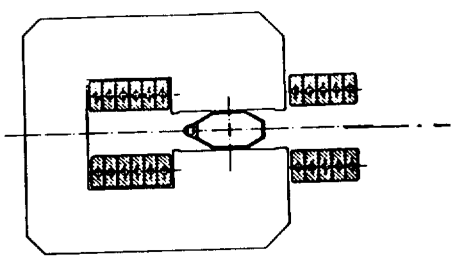


Fig. 5: Homogeneous field bending magnet

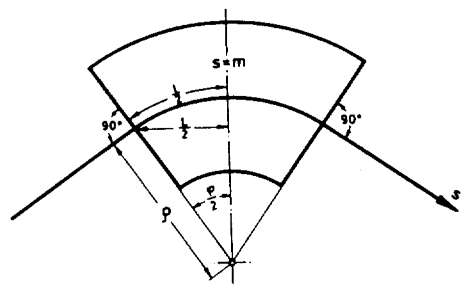


Fig. 6: Sector magnet

The transformation matrices are, from eqs. (12), with $\varphi = \frac{\ell}{\rho}$:

$$M_x = \begin{pmatrix} \cos\varphi & \rho\sin\varphi & \rho(1 - \cos\varphi) \\ -\frac{1}{\rho}\sin\varphi & \cos\varphi & \sin\varphi \\ 0 & 0 & 1 \end{pmatrix} \quad ; \quad M_z = \begin{pmatrix} 1 & \ell & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{15}$$

In the vertical, the sector magnet acts like a drift space, and in the horizontal like a focusing quadrupole of strength $\frac{1}{\rho^2}$.

9. EDGE FOCUSING

In practice, there are cases where the magnet face is not designed orthogonal to the beam. The magnet transformation, then, needs correction.

Let us assume that, at the magnet end, we superimpose a hard-edged "magnetic wedge" of angle δ (Fig. 7).

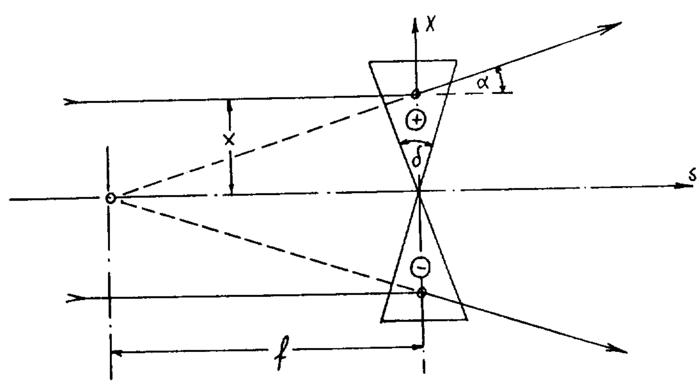


Fig. 7: "Magnetic wedge"

Then, $\alpha = \frac{x}{f} = \frac{x\tan\delta}{\rho}$; $\frac{1}{f} = \frac{1}{\rho}\tan\delta$.

Thus the thin magnetic wedge of Fig. 7, in the horizontal plane, acts as a thin defocusing lens of integrated strength $\frac{1}{\rho}\tan\delta$, and in the vertical plane as a focusing lens of same strength.

10. MAGNETS WITH EDGE FOCUSING

a) Symmetric zero gradient focusing magnet

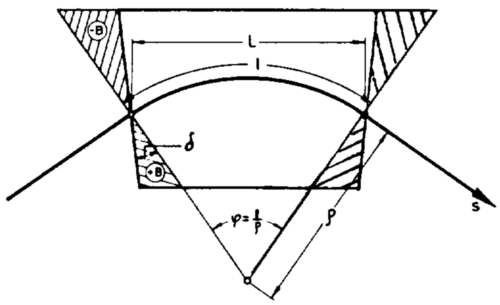


Fig. 8 : Homogeneous field magnet with nonorthogonal entry and exit (sector magnet with superimposed "magnetic wedges").

At each end of the sector magnet (Fig. 6) we superimpose a magnetic wedge of angle $\delta \leq \frac{\varphi}{2}$ (Fig. 7), making the magnet faces more parallel (Fig. 8). The horizontal transformation is then obtained by the matrix multiplications

$$\begin{pmatrix} 1 & 0 \\ \frac{1}{\rho}\tan\delta & 1 \end{pmatrix}\begin{pmatrix} \cos\varphi & \rho\sin\varphi \\ -\frac{1}{\rho}\sin\varphi & \cos\varphi \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{1}{\rho}\tan\delta & 1 \end{pmatrix} = \begin{pmatrix} \cos\varphi & \rho\sin\varphi \\ \frac{1}{\rho}\tan\delta\cos\varphi - \frac{1}{\rho}\sin\varphi & \tan\delta\sin\varphi + \cos\varphi \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{1}{\rho}\tan\delta & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\varphi + \tan\delta\sin\varphi & \rho\sin\varphi \\ \frac{1}{\rho}\{\frac{1}{\cos\delta}\sin(\delta-\varphi) + \frac{1}{\cos\delta}\cos(\delta-\varphi)\tan\delta\} & \frac{1}{\cos\delta}\cos(\delta-\varphi) \end{pmatrix} = \begin{pmatrix} \frac{\cos(\varphi-\delta)}{\cos\delta} & \rho\sin\varphi \\ -\frac{1}{\rho}\frac{\sin(\varphi-2\delta)}{\cos^2\delta} & \frac{\cos(\varphi-\delta)}{\cos\delta} \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 0 \\ \frac{1}{\rho}\tan\delta & 1 \end{pmatrix}\begin{pmatrix} \rho(1-\cos\varphi) \\ \sin\varphi \end{pmatrix} = \begin{pmatrix} \rho(1-\cos\varphi) \\ \tan\delta\,(1-\cos\varphi)+\sin\varphi \end{pmatrix} = \begin{pmatrix} \rho(1-\cos\varphi) \\ \frac{\sin(\varphi-\delta)+\sin\delta}{\cos\delta} \end{pmatrix}.$$

For the vertical

$$\begin{pmatrix} 1 & 0 \\ -\frac{1}{\rho}\tan\delta & 1 \end{pmatrix}\begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\frac{1}{\rho}\tan\delta & 1 \end{pmatrix} = \begin{pmatrix} 1 & \ell \\ -\frac{1}{\rho}\tan\delta & 1-\frac{\ell}{\rho}\tan\delta \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\frac{1}{\rho}\tan & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1-\varphi\tan\delta & \ell \\ -\frac{1}{\rho}\tan\delta\,(2-\varphi\tan\delta) & 1-\varphi\tan\delta \end{pmatrix}.$$

Summarizing the result, with $\varphi = \frac{\ell}{\rho}$, for the symmetric zero gradient focusing magnet, we have

$$M_x = \begin{pmatrix} \frac{\cos(\varphi-\delta)}{\cos\delta} & \rho\sin\varphi & \rho(1-\cos\varphi) \\ -\frac{1}{\rho}\frac{\sin(\varphi-2\delta)}{\cos^2\delta} & \frac{\cos(\varphi-\delta)}{\cos\delta} & \frac{\sin(\varphi-\delta)+\sin\delta}{\cos\delta} \\ 0 & 0 & 1 \end{pmatrix}; \quad M_z = \begin{pmatrix} 1-\varphi\tan\delta & \ell & 0 \\ -\frac{1}{\rho}\tan\delta(2-\varphi\tan\delta) & 1-\varphi\tan\delta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{16}$$

b) Rectangular magnet (special case of e., with $\delta = \frac{\varphi}{2}$)

When $\varphi \ll 1$, magnet faces are often made parallel for technical reasons (e.g. laminated magnets!). Then, with $\delta = \frac{\varphi}{2}$

$$M_x = \begin{pmatrix} 1 & \rho\sin\varphi & \rho(1-\cos\varphi) \\ 0 & 1 & 2\tan\frac{\varphi}{2} \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and, for } \varphi \ll 1 \quad M_z \approx \begin{pmatrix} \cos\varphi & \rho\sin\varphi & 0 \\ -\frac{1}{\rho}\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{17}$$

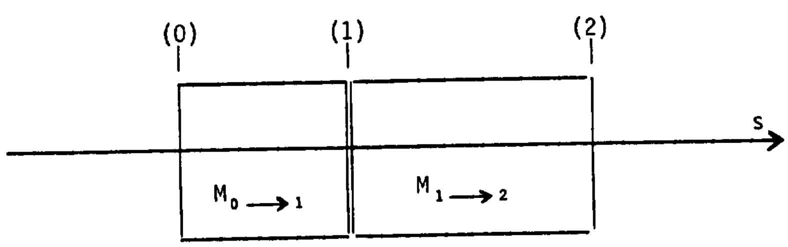
$$\text{since} \quad \tan\frac{\varphi}{2} = \frac{1-\cos\varphi}{\sin\varphi} = \frac{\sin\varphi}{1+\cos\varphi}.$$

Thus, in a rectangular magnet, the horizontal weak focusing of the sector magnet is exactly compensated by the edge focusing and is transferred into the vertical by the same amount.

11. PIECEWISE SOLUTION USING MATRIX FORMALISM

For a beam transport or accelerator system composed of magnets and drift spaces, we obtain the over-all transformation matrix by multiplying the matrices corresponding to each element in the correct order. We proceed by multiplying from the left, lines by columns.

Example:



$$M_{0\to2} = M_{1\to2}\cdot M_{0\to1} = \begin{pmatrix} C_2 & S_2 & D_2 \\ C_2' & S_2' & D_2' \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} C_1 & S_1 & D_1 \\ C_1' & S_1' & D_1' \\ 0 & 0 & 1 \end{pmatrix}$$

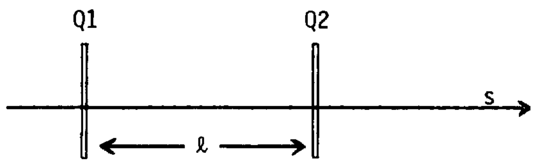
$$= \begin{pmatrix} C_2C_1 + S_2C_1' & C_2S_1 + S_2S_1' & C_2D_1 + S_2D_1' + D_2 \\ C_2'C_1 + S_2'C_1' & C_2'S_1 + S_2'S_1' & C_2'D_1 + S_2'D_1' + D_2' \\ 0 & 0 & 1 \end{pmatrix}.$$

12. THIN LENS QUADRUPOLE DOUBLET

Whenever the length of a quadrupole is small as compared to its focal length, i.e. for $\ell \ll \frac{1}{|k|\ell}$ or $\ell^2|k| \ll 1$, it may be represented by a thin lens positioned at its center. From the quadrupole transformation eq. (13) it is seen that, with constant strength $k\ell = \frac{1}{f}$ and the length ℓ approaching zero, the matrices assume the simple form

$$M_{x,z} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

As the simplest example of "strong focusing" or "alternating gradient focusing", we write down the transformation of a quadrupole doublet in thin lens approximation:



focal strength: $\frac{1}{f_1}$ $\frac{1}{f_2}$

$$M = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f_2} & 1 \end{pmatrix} \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f_1} & 1 \end{pmatrix} = \begin{pmatrix} 1-\frac{\ell}{f_1} & \ell \\ -\frac{1}{f^*} & 1-\frac{\ell}{f_2} \end{pmatrix} \tag{18}$$

where $$\frac{1}{f^*} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{\ell}{f_1 f_2}.$$

With $f_1 = -f_2 = f$, for example, we have $\frac{1}{f^*} = \frac{\ell}{f^2}$ in x and z, and the doublet is equally focusing in both planes. This is because all trajectories entering parallel to the axis will have a larger amplitude in the focusing lens than in the defocusing lens and will therefore be bent more strongly toward the axis than away from it.

B. BEAM MOTION IN MAGNET SYSTEMS

13. TRAJECTORIES IN TERMS OF AMPLITUDE AND PHASE FUNCTIONS

There is another quite powerful way of formulating the solution of Hill's equation

$$y'' + K y = 0 \tag{9a}$$

by writing the trajectory in quasi-harmonic form

$$y(s) = \sqrt{\varepsilon}\,\sqrt{\beta(s)}\,\cos(\Phi(s) - \Phi_0) \tag{19}$$

where the amplitude $\sqrt{\beta(s)}$ and phase $\Phi(s)$ of this "betatron oscillation" vary as function of s (nonlinearly). β is called the "amplitude function", $\Phi(s)$ the (closely related) "phase function", and ε a constant called the "emittance" of the trajectory.

By differentiation, we have with

$$\Delta\Phi = \Phi - \Phi_0 \quad ; \quad \alpha = -\frac{1}{2}\beta' \quad ; \quad \gamma = \frac{1+\alpha^2}{\beta} \tag{20}$$

$$y = \sqrt{\varepsilon}\sqrt{\beta}\cos\Delta\Phi$$

$$y' = \sqrt{\varepsilon}\{\frac{\beta'}{2\sqrt{\beta}}\cos\Delta\Phi - \sqrt{\beta}\,\Phi'\sin\Delta\Phi\} = -\sqrt{\varepsilon}\{\frac{\alpha}{\sqrt{\beta}}\cos\Delta\Phi + \sqrt{\beta}\,\Phi'\sin\Delta\Phi\}$$

$$y'' = -\sqrt{\varepsilon}\{\frac{\alpha'\sqrt{\beta} - \alpha\frac{\beta'}{2\sqrt{\beta}}}{\beta}\cos\Delta\Phi + \frac{\beta'}{2\sqrt{\beta}}\Phi'\sin\Delta\Phi + (\frac{\beta'}{2\sqrt{\beta}}\Phi' + \sqrt{\beta}\Phi'')\sin\Delta\Phi + \sqrt{\beta}\Phi'^2\cos\Delta\Phi\}$$

$$y'' = -\frac{\sqrt{\varepsilon}}{\sqrt{\beta}}\{(\alpha' + \frac{\alpha^2}{\beta} + \beta\Phi'^2\}\cos\Delta\Phi + (\beta'\Phi' + \beta\Phi'')\sin\Delta\Phi \overset{!}{=} -\frac{\sqrt{\varepsilon}}{\sqrt{\beta}}K\beta\cos\Delta\Phi \,.$$

Thus $\quad 0 = \beta'\Phi' + \beta\Phi'' = (\beta\Phi')'$

We set $\quad \beta\Phi' = \text{const} = 1 \quad$, i.e.

$$\Phi' = \frac{1}{\beta} \quad ; \quad \Delta\Phi = \Phi - \Phi_0 = \int_0^s \frac{1}{\beta}\,ds \tag{21}$$

$\frac{ds}{\beta} = d\Phi$ is the local phase advance of the betatron oscillation.

From y", we then also have

$$\alpha' + \frac{1+\alpha^2}{\beta} \overset{!}{=} K\beta \quad \text{or}$$

$$\alpha' + \gamma - K\beta = 0 \quad \text{or} \quad \frac{1}{2}\beta'' + K\beta - \frac{1+\frac{1}{4}\beta'^2}{\beta} = 0 \tag{22}$$

as a differential equation for the amplitude function $\beta(s)$.

Introducing the "envelope"

$$E(s) = \sqrt{\varepsilon}\sqrt{\beta(s)} \tag{23}$$

we have by differentiation

$$E = \sqrt{\varepsilon}\sqrt{\beta} \qquad = \frac{\sqrt{\varepsilon}}{\sqrt{\beta}}\cdot\beta$$

$$E' = \sqrt{\varepsilon}\frac{\beta'}{2\sqrt{\beta}} \qquad = -\frac{\sqrt{\varepsilon}}{\sqrt{\beta}}\cdot\alpha$$

$$E'' = -\sqrt{\varepsilon}\,\frac{\alpha'\sqrt{\beta} - \alpha\frac{\beta'}{2\sqrt{\beta}}}{\beta} \qquad = -\frac{\sqrt{\varepsilon}}{\sqrt{\beta}}\cdot(\alpha' + \frac{\alpha^2}{\beta})\,.$$

Thus we obtain the "envelope equation"

$$\boxed{E'' + KE - \frac{\varepsilon^2}{E^3} = -\frac{\sqrt{\varepsilon}}{\sqrt{\beta}}\cdot(\alpha' + \frac{\alpha^2}{\beta} - K\beta + \frac{1}{\beta}) = 0} \tag{24}$$

14. CALCULATION OF AMPLITUDE FUNCTION FROM TWO ORTHOGONAL TRAJECTORIES

In a general magnet system, the amplitude function $\beta(s)$ is not uniquely given, but depends on two parameters, e.g. β_0 and $\alpha_0 = -\frac{1}{2}\beta_0'$ at the entrance of the system. It becomes unique only in a periodic system, for instance in an accelerator ring when, among all possible solutions, the periodic beta is selected. $\beta(s)$ can be calculated by solving eqs. (22) or (24) with initial conditions β_0, α_0, but this is never done in practice since there is a much simpler way. By choosing two orthogonal trajectories

$$\begin{pmatrix} y_1 \\ y_1' \end{pmatrix} = \begin{pmatrix} \sqrt{\varepsilon}\sqrt{\beta}\cos(\Phi - \Phi_0) \\ -\frac{\sqrt{\varepsilon}}{\sqrt{\beta}}(\sin\Delta\Phi + \alpha\cos\Delta\Phi) \end{pmatrix} \; ; \; \begin{pmatrix} y_2 \\ y_2' \end{pmatrix} = \begin{pmatrix} \sqrt{\varepsilon}\sqrt{\beta}\sin(\Phi - \Phi_0) \\ \frac{\sqrt{\varepsilon}}{\sqrt{\beta}}(\cos\Delta\Phi - \alpha\sin\Delta\Phi) \end{pmatrix}$$

with any value Φ_0 and given values β_0, α_0 and transforming them through the system by matrix multiplication, $\beta(s)$ is obtained as

$$\boxed{\varepsilon\beta = y_1^2 + y_2^2 = E^2} \tag{25}$$

15. AMPLITUDE FUNCTION AND PHASE PLANE ELLIPSE

It is the particular value of the amplitude function that it is closely related to an ellipse in the $\{y,y'\}$ phase plane and is thus able to describe the motion of a beam, i.e. a family of trajectories instead of individual trajectories only.

Writing

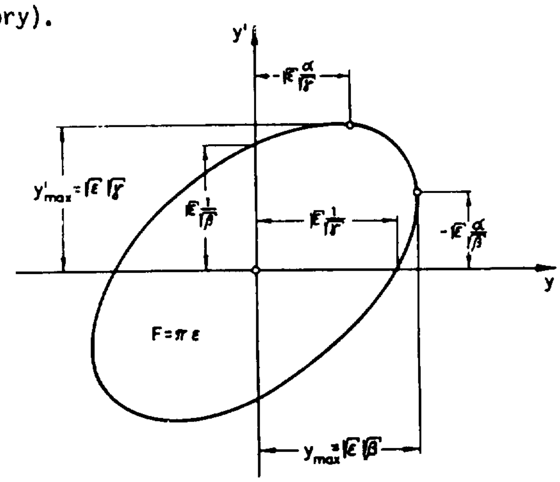
$$\boxed{\begin{pmatrix} y \\ y' \end{pmatrix} = \begin{pmatrix} \sqrt{\varepsilon}\sqrt{\beta}\cos(\Phi - \Phi_0) \\ -\frac{\sqrt{\varepsilon}}{\sqrt{\beta}}(\sin(\Phi - \Phi_0) + \alpha\cos(\Phi - \Phi_0)) \end{pmatrix}} \tag{26}$$

this is the parametic representation of an ellipse in the $\{y,y'\}$ phase plane; if the phase parameter Φ_0 varies by 2π, the point $\{y,y'\}$ moves once around the ellipse which is centered about the origin $\{0,0\}$ (reference trajectory).

Special pair of orthogonal trajectories:

$$\underline{\Phi_0 = \Phi} \qquad\qquad \underline{\Phi_0 = \Phi + \frac{\pi}{2}}$$

$$\begin{pmatrix} y \\ y' \end{pmatrix}_1 = \begin{pmatrix} \sqrt{\varepsilon}\sqrt{\beta} \\ -\alpha\frac{\sqrt{\varepsilon}}{\sqrt{\beta}} \end{pmatrix} ; \; \begin{pmatrix} y \\ y' \end{pmatrix}_2 = \begin{pmatrix} 0 \\ \frac{\sqrt{\varepsilon}}{\sqrt{\beta}} \end{pmatrix} \tag{27}$$



Beam ellipse in terms of amplitude function β.

Let us assume that into a magnet system a particle beam is injected which, at the entrance, is given by a family of initial conditions or a cluster of points in the $\{y,y'\}$ phase plane, centered about the reference trajectory $\{0,0\}$. We may then, by choosing β_0, α_0 and ε, tailor an ellipse that closely surrounds this cluster and thus represents the "edge" of the beam. By then following this ellipse through the system, it tells us the properties of the beam at each point. Thereby (see Fig. 11) the area of the ellipse

$$\pi \cdot \sqrt{\varepsilon}\sqrt{\beta} \cdot \frac{\sqrt{\varepsilon}}{\sqrt{\beta}} = \pi\varepsilon$$

stays constant, which means that the particle density in phase plane stays constant (Liouville's theorem) or, in other words, the product of beam width (or height) times the angular spread on axis stays constant.

The beta function is the ratio of beam width over the on-axis angular spread.

In a periodic system, for instance in an accelerator ring, a phase plane ellipse corresponding to the <u>periodic</u> amplitude function is the line on which a particle with the corresponding betatron amplitude will migrate and reappear in successive revolutions. If this particle has the maximum betatron amplitude in the beam, it will mark the "edge" of the beam, and the quantity $E = \sqrt{\varepsilon}\sqrt{\beta}$ will be the beam width (or height) at that place, as given by betatron oscillations. Therefore, $E(s)$ is called the beam envelope.

From the parametric ellipse representation eq. (26) we can obtain the coordinate representation of the ellipse

$$\boxed{\gamma \cdot y^2 + 2\alpha \cdot yy' + \beta \cdot y'^2 = \varepsilon} \tag{28}$$

which is seen to be valid by inserting eqs. (26) into it.

16. <u>CALCULATION OF AMPLITUDE FUNCTION FROM PRINCIPAL TRAJECTORIES</u>

By inserting the inverse trajectory transformation

$$\begin{pmatrix} y_0 \\ y_0' \end{pmatrix} = \begin{pmatrix} S' & -S \\ -C' & C \end{pmatrix} \begin{pmatrix} y \\ y' \end{pmatrix}$$

into the ellipse equation (28), we have at point s_0

$$\gamma_0 y_0^2 + 2\alpha_0 y_0 y_0' + \beta_0 y_0'^2$$

$$= \gamma_0(S'y - Sy')^2 + 2\alpha_0(S'y - Sy')(-C'y + Cy') + \beta_0(-C'y + Cy')^2$$

$$= \underbrace{(C'^2\beta_0 - 2C'S'\alpha_0 + S'^2\gamma_0)}_{\gamma} y^2 + 2\underbrace{(-CC'\beta_0 + (S'C + SC')\alpha_0 - SS'\gamma_0)}_{\alpha} yy' + \underbrace{(C^2\beta_0 - 2CS\alpha_0 + S^2\gamma_0)}_{\beta} y'^2 .$$

$$\stackrel{!}{=} \gamma \cdot y^2 + 2\alpha \cdot yy' + \beta \cdot y'^2$$

Thus ß, $\alpha = -\frac{1}{2}\beta'$ and $\gamma = \frac{1+\alpha^2}{\beta}$ can be calculated from the principal trajectories C, S by the linear 3x3 transformation

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix} = \begin{pmatrix} C^2 & -2CS & S^2 \\ -CC' & CS'+SC' & -SS' \\ C'^2 & -2C'S' & S'^2 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \alpha_0 \\ \gamma_0 \end{pmatrix}. \tag{29}$$

In a drift space with

$$\begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$$

beta is a quadratic function of s, α a linear function and γ is constant:

$$\begin{aligned} \beta &= \beta_0 - 2\alpha_0 s + \gamma_0 s^2 \\ \alpha &= \alpha_0 - \gamma_0 s \\ \gamma &= \gamma_0 = \text{const.} \end{aligned} \tag{29a}$$

17. PRINCIPAL TRAJECTORIES IN TERMS OF AMPLITUDE AND PHASE FUNCTIONS

By subjecting the trajectory representation eqs. (26) to the initial conditions eqs. (10a), the principal trajectories may be written, with $\Delta\Phi = \Phi - \Phi_0$

$$\boxed{\begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{\beta}}{\sqrt{\beta_0}}(\cos\Delta\Phi + \alpha_0 \sin\Delta\Phi) & \sqrt{\beta_0}\sqrt{\beta}\sin\Delta\Phi \\ -\frac{1}{\sqrt{\beta_0}\sqrt{\beta}}\{(\alpha-\alpha_0)\cos\Delta\Phi + (1+\alpha\alpha_0)\sin\Delta\Phi\} & \frac{\sqrt{\beta_0}}{\sqrt{\beta}}(\cos\Delta\Phi - \alpha\sin\Delta\Phi) \end{pmatrix}} \tag{30}$$

This form of the transformation matrix is very useful in practical accelerator work.

In a periodic system, e.g. an accelerator ring, we have for the periodic amplitude function in the matrix for one period or revolution

$$\beta = \beta_0 \quad ; \quad \alpha = \alpha_0 .$$

Then, in a symmetry point of an accelerator where $\alpha = -\frac{1}{2}\beta' = 0$, the revolution matrix assumes the very simple form

$$\begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = \begin{pmatrix} \cos\mu & \beta\sin\mu \\ -\frac{1}{\beta}\sin\mu & \cos\mu \end{pmatrix}$$

where $\mu = 2\pi\nu$ is the phase advance per revolution and ν (often called Q-value) the number of betatron oscillations per turn.

When Eq. (30) is applied to a periodic structure $\beta = \beta_0$, $\alpha = \alpha_0$ and the result can be expressed as,

$$\begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos \Delta\phi + \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} \sin \Delta\phi$$

$$= I \cos \Delta\phi + J \sin \Delta\phi \ .$$

It is quickly verified that $J^2 = -I$, so that the expression $I \cos \Delta\phi + J \sin \Delta\phi$ has the properties of a complex number and De Moivre's formula can be applied. This makes it easy to express the transfer matrix for an arbitrarily large number of cycles.

$$\begin{pmatrix} C & S \\ C' & S' \end{pmatrix}^n = (I \cos \Delta\phi + J \sin \Delta\phi)^n = I \cos (n\Delta\phi) + J \sin (n\Delta\phi) \ . \tag{31}$$

The beam will be stable if all terms in Eq. (31) remain bounded as $n \to \infty$, i.e. if $\Delta\phi$ is real or $|\cos \Delta\phi| < 1$. By referring back to Eq. (30) with $\beta = \beta_0$ and $\alpha = \alpha_0$ this condition can be expressed in terms of the matrix elements as,

$$\boxed{\text{Stability if,} \quad \left| \frac{1}{2} \, \text{Trace} \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \right| < 1} \tag{32}$$

18. PERIODIC DISPERSION IN AN ACCELERATOR RING

Particles with a relative momentum deviation $\frac{\Delta p}{p_0} \neq 0$ are less or more strongly bent than the reference particle and therefore move about a closed orbit that deviates from the reference orbit. A general formula for this off-momentum closed orbit may be derived with the tools at hand. We demand that the dispersion trajectory closes upon itself after one revolution of length L. Using eq. (11) and the notations

$$\int_s^{s+L} \frac{1}{\rho(\tau)} C(\tau) d\tau = \mathcal{C} \quad ; \quad \int_s^{s+L} \frac{1}{\rho(\tau)} S(\tau) d\tau = \mathcal{S}$$

and writing

$$C(s+L) = C \quad ; \quad S(s+L) = S \quad ; \quad \text{etc.}$$

then

$$\begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \begin{pmatrix} D \\ D' \end{pmatrix} + \begin{pmatrix} S\mathcal{C} - C\mathcal{S} \\ S'\mathcal{C} - C'\mathcal{S} \end{pmatrix} = \begin{pmatrix} D \\ D' \end{pmatrix}$$

or

$$(C-1)D + \quad S \ D' = C\mathcal{S} - S\mathcal{C}$$
$$C' \ D + (S'-1)D' = C'\mathcal{S} - S'\mathcal{C}$$

yielding

$$\{(S'-1)(C-1) - SC'\}D = (S'-1)(C\mathcal{S} - S\mathcal{C}) - S(C'\mathcal{S} - S'C)$$

or, with $S'C-SC'=1$

$$\{2 - (C+S')\}D = \mathcal{S} + S\mathcal{C} - C\mathcal{S} \ .$$

Now from eq. (30)

$$2 - (C+S') = 2 - \mathrm{trace}\,M = 2 - 2\cos\mu = 2(1-\cos 2\pi\nu) = 4\sin^2\pi\nu$$

where μ is the phase advance per revolution and ν the betatron number.

Thus

$$4\sin^2\pi\nu\cdot D(s) = \int_s^{s+L} \frac{1}{\rho(\tau)} S(\tau)d\tau + S(s+L)\int_s^{s+L} \frac{1}{\rho(\tau)} C(\tau)d\tau - C(s+L)\int_s^{s+L} \frac{1}{\rho(\tau)} S(\tau)d\tau$$

with

$$C(\tau) = \frac{\sqrt{\beta(\tau)}}{\sqrt{\beta(s)}}(\cos\Delta\Phi + \alpha(s)\sin\Delta\Phi) \quad ; \quad S(\tau) = \sqrt{\beta(s)}\sqrt{\beta(\tau)}\sin\Delta\Phi$$

where $\Delta\Phi = \Phi(\tau) - \Phi(s)$,

$$C(s+L) = \cos 2\pi\nu + \alpha(s)\sin 2\pi\nu \quad ; \quad S(s+L) = \beta(s)\sin 2\pi\nu$$

according to eq. (30). Using the relation

$$\sin\Delta\Phi + \sin(2\pi\nu - \Delta\Phi) = 2\sin\pi\nu\cos(\Delta\Phi - \pi\nu)$$

we finally have

$$\boxed{D(s) = \frac{\sqrt{\beta(s)}}{2\sin\pi\nu}\int_s^{s+L} \frac{1}{\rho(\tau)}\sqrt{\beta(\tau)}\cos(\Phi(\tau) - \Phi(s) - \pi\nu)d\tau} \tag{33}$$

Between bending magnets, the dispersion looks just like an on-energy particle trajectory, receiving an additional kick only in each bending magnet.

19. MOMENTUM COMPACTION

In an accelerator, the relative variation of closed orbit length with relative momentum deviation is called the "momentum compaction factor α"

$$\alpha = \frac{p}{L}\frac{dL}{dp}.$$

With the differential trajectory length

$$d\sigma = \frac{\rho + x}{\rho}\,ds$$

the circumferential length of the trajectory x(s) is

$$L + \Delta L = \oint (1 + \frac{x}{\rho})ds.$$

Since the particle with momentum deviation $\frac{\Delta p}{p_0}$ moves around the closed orbit $\frac{\Delta p}{p_0}\cdot D(s)$, the momentum compaction factor is

$$\boxed{\alpha = \frac{1}{L}\int \frac{D}{\rho}\,ds} \tag{34}$$

20. STRONG FOCUSING HIGH ENERGY ACCELERATOR; SIMPLIFIED MODEL

Practically all very high energy accelerator rings are now being built in a similar fashion. They may somewhat differ in the arrangement of straight sections, but in the arcs they all have a periodic sequence of quadrupole magnets of alternating polarity (FODO-channel), and between them the bending magnets that cover of the order of, say, 80 % of the length. Since the straight sections are short as compared to the arcs, the optical properties of the ring are essentially given by the parameters chosen for the regular arc cell. The regular cell may be represented by a very simple model, making use of the following observations:

- The radius of curvature of the bending magnets is large as compared to the focal length of the quadrupoles; the weak focusing of the magnets may therefore be ignored, and it is irrelevant whether they are of the sector or the rectangular magnet type.
- For a given bending angle, the linear optic does not depend on the length of the bending magnets, which may therefore be assumed to extend from one quadrupole to the next.
- The F- and D-quadrupoles are usually of similar strength. For simplicity, the strengths are here assumed to be equal, and the quadrupoles are treated by the thin lens approximation.

The simplified regular half cell is shown in Fig. 9. It is given by only 3 parameters:

ℓ	half cell length = bending magnet length
$\frac{1}{\rho}$	strength of bending magnet
$\pm \frac{1}{f}$	strength of half quadrupole, integrated

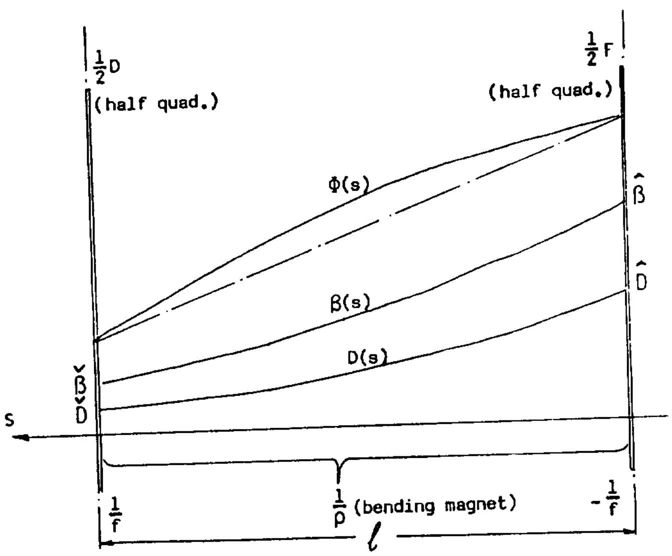


Fig. 9 : Optical functions in the simplified half cell model, schematic.

We calculate the optical properties in terms of these parameters, with $\tau = \frac{\ell}{\rho}$:

$$\begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix} \left\{ \begin{matrix} \overset{\text{sector magnet}}{\begin{pmatrix} \cos\tau & \rho\sin\tau \\ -\frac{1}{\rho}\sin\tau & \cos\tau \end{pmatrix}} \\ \underset{\text{rectangular mag.}}{\begin{pmatrix} 1 & \rho\sin\tau \\ 0 & 1 \end{pmatrix}} \end{matrix} \right\} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} = \left\{ \begin{matrix} \begin{pmatrix} 1-\sin\Psi & f\sin\Psi \\ -\frac{1}{f}\left(1-\cancel{\frac{f^2}{\rho^2}}\right)\sin\Psi & 1+\sin\Psi \end{pmatrix} \\ \begin{pmatrix} 1-\frac{\ell}{f} & \ell \\ -\frac{\ell}{f^2} & 1+\frac{\ell}{f} \end{pmatrix} \end{matrix} \right\}$$

where $\sin\Psi = \frac{\rho}{f}\sin\tau \approx \frac{\ell}{f}$.

Equating this to the matrix representation eq. (30)

$$\begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = \begin{pmatrix} \left\{\frac{\check{\beta}}{\hat{\beta}}\right\}^{1/2}\cos\Phi & \left\{\hat{\beta}\check{\beta}\right\}^{1/2}\sin\Phi \\ -\left\{\hat{\beta}\check{\beta}\right\}^{-1/2}\cdot\sin\Phi & \left\{\frac{\hat{\beta}}{\check{\beta}}\right\}^{1/2}\cos\Phi \end{pmatrix} \qquad (35)$$

yields $\quad CS' = \cos^2\Psi \overset{!}{=} \cos^2\Phi \quad ; \quad SC' = -\sin^2\Psi \overset{!}{=} -\sin^2\Phi$

$$\boxed{\sin\Phi = \sin\Psi = \frac{\rho}{f}\sin\tau \approx \frac{\ell}{f}} \qquad (36)$$

i.e. the betatron phase advance per half cell is given by $\sin\Phi = \frac{\ell}{f}$.

Further

$$\left.\begin{aligned} \frac{S'}{C} &= \frac{1+\sin\Phi}{1-\sin\Phi} \overset{!}{=} \frac{\hat{\beta}}{\check{\beta}} \\ -\frac{S}{C'} &= f^2 \overset{!}{=} \hat{\beta}\cdot\check{\beta} \end{aligned}\right. \Longrightarrow \boxed{\begin{aligned} \hat{\beta} &= f\left(\frac{1+\sin\Phi}{1-\sin\Phi}\right)^{1/2} = f\,\frac{1+\sin\Phi}{\cos\Phi} \\ \check{\beta} &= f\left(\frac{1-\sin\Phi}{1+\sin\Phi}\right)^{1/2} = f\,\frac{1-\sin\Phi}{\cos\Phi} \end{aligned}} \qquad (37)$$

For the dispersion we have with $\tau = \frac{\ell}{\rho}$

$$\begin{pmatrix} D \\ D' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix} \left\{ \begin{matrix} \begin{pmatrix} \rho(1-\cos\tau) \\ \sin\tau \end{pmatrix} \\ \begin{pmatrix} \rho(1-\cos\tau) \\ 2\tan\frac{\tau}{2} \end{pmatrix} \end{matrix} \right\} = \left\{ \begin{matrix} \begin{pmatrix} \rho(1-\cos\tau) \\ \frac{\rho}{f}(1-\cos\tau)+\sin\tau \end{pmatrix} \\ \begin{pmatrix} \rho(1-\cos\tau) \\ \frac{\rho}{f}(1-\cos\tau)+2\tan\frac{\tau}{2} \end{pmatrix} \end{matrix} \right\} \quad \begin{pmatrix} \frac{1}{2}\frac{\ell^2}{\rho} \\ \frac{\ell}{\rho}\left(1+\frac{\ell}{2f}\right) \end{pmatrix}$$

and, looking for the periodic solution with $D' = D_0' = 0$:

$$\begin{pmatrix} \check{D} \\ D' = 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1-\frac{\ell}{f} & \ell & \frac{1}{2}\frac{\ell^2}{\rho} \\ -\frac{\ell}{f^2} & 1+\frac{\ell}{f} & \frac{\ell}{\rho}\left(1+\frac{\ell}{2f}\right) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{D} \\ D_0' = 0 \\ 1 \end{pmatrix}$$

$$
\begin{aligned}
(1-\tfrac{\ell}{f})\hat{D} + \tfrac{1}{2}\tfrac{\ell^2}{\rho} &= \check{D} \\
-\tfrac{\ell}{f^2}\hat{D} + \tfrac{\ell}{\rho}(1+\tfrac{\ell}{2f}) &= 0 \\
\hline
\hat{D} - \tfrac{\ell f}{\rho} &= \check{D}
\end{aligned}
$$

$$
\Longrightarrow \boxed{\begin{aligned}
\hat{D} &= \frac{f^2}{\rho}(1 + \frac{1}{2}\frac{\ell}{f}) = \frac{f^2}{\rho}(1 + \frac{1}{2}\sin\Phi) \\
\check{D} &= \frac{f^2}{\rho}(1 - \frac{1}{2}\frac{\ell}{f}) = \frac{f^2}{\rho}(1 - \frac{1}{2}\sin\Phi)
\end{aligned}}
\qquad
\frac{\rho}{\ell}\frac{\hat{D}}{\ell} = \frac{1 + \frac{1}{2}\sin\Phi}{\sin^2\Phi} . \tag{38}
$$

Interpreting these results, we see from eq. (35) that periodic solutions β, D only exist for $\sin\Phi < 1$, i.e. $f > \ell$. $\hat{\beta}$ and $\check{\beta}$ are shown in Fig. 10 as functions of the half cell phase advance Φ for a given half cell length ℓ, and also $\hat{D}$ and $\check{D}$.
The phase advance with the smallest value of $\hat{\beta}$, for given ℓ, requires the least beam aperture; we obtain it by differentiating

$$\frac{\hat{\beta}}{\ell} = \frac{1 + \sin\Phi}{\sin\Phi\cos\Phi} \qquad \text{with respect to } \Phi:$$

$$\cos\Phi\cdot\sin\Phi\cos\Phi - (1 + \sin\Phi)(\cos^2\Phi - \sin^2\Phi) \overset{!}{=} 0$$

$$\sin^2\Phi(2 + \sin\Phi) = 1 \quad ; \qquad \sin\Phi = \frac{\ell}{f} = 0.618 \qquad \begin{aligned} \Phi &= 38.17^\circ \\ \min(\hat{\beta}) &= 3.33\ \ell . \end{aligned}$$

In practice, full cell phase advances are chosen between, say, 45° and 90°.

Knowing the amplitudes of β and D in the thin quadrupoles of the half cell, we also know, from the strength of the quadrupole, their slope at the quadrupole entrance and can thus calculate, as a function of $\sigma = \frac{s}{\ell}$, their shape within the half cell, using eq. (29a) for β and eq. (12a) for D. We shall not do this explicitely here, but just give the result:

$$
\boxed{\begin{aligned}
\beta(\sigma) &= \underbrace{f\,\frac{1-\sin\Phi}{\cos\Phi}}_{\check{\beta}}\{1 + 2\sin\Phi\cdot\sigma + 2\tan^2\Phi(1+\sin\Phi)\cdot\sigma^2\} \\
\alpha(\sigma) &= -\frac{1}{2}\beta' = -\frac{\check{\beta}}{f}\{1 + 2\,\frac{\sin\Phi(1+\sin\Phi)}{\cos^2\Phi}\cdot\sigma\} \\
D(\sigma) &= \frac{f^2}{\rho}\{1 - \frac{1}{2}\sin\Phi + \sin\Phi(1-\frac{1}{2}\sin\Phi)\cdot\sigma + \frac{1}{2}\sin^2\Phi\cdot\sigma^2\} \\
D'(\sigma) &= \frac{f}{\rho}\{1 - \frac{1}{2}\sin\Phi + \sin\Phi\cdot\sigma\}
\end{aligned}} \tag{39}
$$

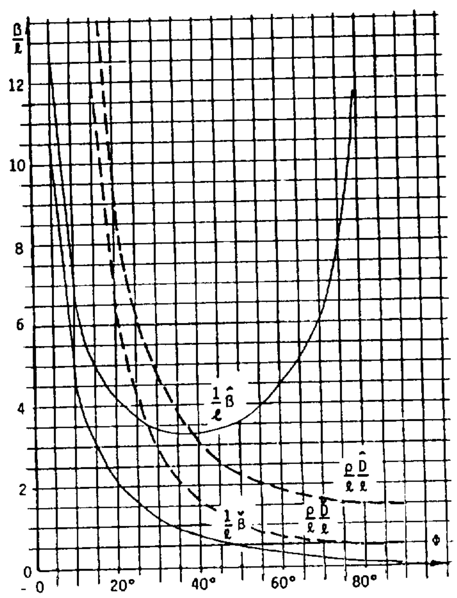


Fig. 10: Normalized β's and D's as functions of Φ

For the momentum compaction factor in a machine built of these half cells, we have from eq. (34)

$$\alpha = \frac{1}{L}\oint \frac{D}{\rho}\,ds \approx \frac{2\pi\rho}{2\pi R}\cdot\frac{1}{\rho}\cdot\frac{1}{2}(\hat{D}+\check{D}) = \frac{f^2}{R\rho} \tag{40}$$

where R is the mean radius of the machine including straight sections.

Another optical quantity of great practical interest is the "chromaticity" of our model machine, i.e. the variation of betatron tune ν with relative momentum deviation $\delta = \frac{\Delta p}{p_o}$

$$\xi = \frac{d\nu}{d\delta} = \frac{n}{2\pi}\frac{d\Phi}{d\delta} \tag{41}$$

where n is the number of half cells in the machine. ξ is the chromaticity of the arcs only; the straight sections will give an additional chromaticity contribution.

$$\sin\Phi = \frac{\ell}{f} \quad ; \quad \frac{1}{f} = \frac{1}{f_o}(1-\delta) = k\,\ell_q\,.$$

By differentiation

$$\cos\Phi\, d\Phi = -\frac{\ell}{f_o}d\delta = -\sin\Phi\, d\delta$$

$$\frac{d\Phi}{d\delta} = -\tan\Phi = -\frac{1}{2f}(\hat{\beta}-\check{\beta}) = -\frac{1}{2}(k\hat{\beta}\ell_q - k\check{\beta}\ell_q)$$

which suggest the general formulation

$$\xi = -\frac{1}{4\pi}\oint k\beta\, ds\,. \tag{42}$$

21. "NECKTIE" STABILITY DIAGRAM

We now allow the focusing and defocusing thin quadrupoles in the half cell to have different strengths and then, with

$$\frac{\ell}{f_1} = F \quad \text{and} \quad -\frac{\ell}{f_2} = D$$

have from eq. (18) the transformation matrix

$$\begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = \begin{pmatrix} 1-F & \ell \\ -\frac{1}{\ell}(F-D+FD) & 1+D \end{pmatrix}$$

and from eq. (35)

$$-C'S = \sin^2\Phi \overset{!}{=} F - D + FD\,.$$

For stable beam motion, we require

$$\begin{array}{lcl} 0 \le \sin^2\Phi_x \le 1 & & 0 \le F - D + FD \le 1 \\ & \Longrightarrow & \\ 0 \le \sin^2\Phi_z \le 1 & & 0 \le D - F + FD \le 1 \end{array}$$

which yields for the limits of the stable region

$$\sin\Phi_x = 1 \curvearrowright F = 1 \quad ; \quad \sin\Phi_z = 1 \curvearrowright D = 1$$

$$\sin\Phi_x = 0 \curvearrowright F = \frac{D}{1+D} \quad ; \quad \sin\Phi_z = 0 \curvearrowright D = \frac{F}{1+F} \, .$$

These limits are shown in Fig. 11. The stable region indeed has the shape of a necktie.

We see that

$F \ll 1$; $D \ll 1$	requires	$F \approx D$
$F \approx 1$	"	$\sim \frac{1}{2} < D < 1$
$D \approx 1$	"	$\sim \frac{1}{2} < F < 1$
$F = D$	"	$0 < F, D < 1.$

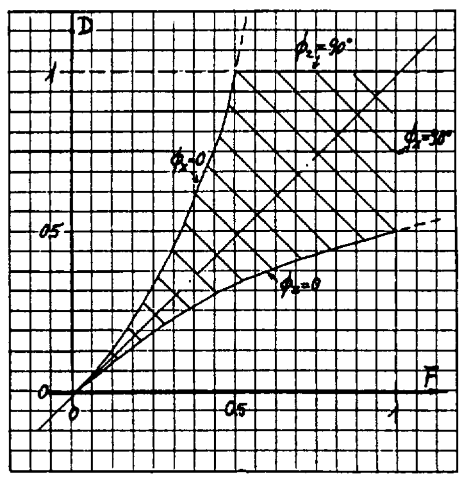


Fig. 11: Necktie stability diagram.