U-branes and T^3 fibrations

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We describe eight-dimensional vacuum configurations with varying moduli consistent with the U-duality group $SL(2, \mathbb{Z}) \times SL(3, \mathbb{Z})$. Focusing on the latter less-well understood $SL(3, \mathbb{Z})$ properties, we construct a class of fivebrane solutions living on lines on a three-dimensional base space. The resulting U-manifolds, with five scalars transforming under $SL(3)$, admit a Ricci-flat Kähler metric. Based on the connection with special lagrangian T^3 fibered Calabi-Yau 3-folds, this construction provides a simple framework for the investigation of Calabi-Yau mirrors.

1. Introduction

U-duality has played an important role for understanding nonperturbative aspects of string theory [1,2]. This role is not just restricted to establishing the relationships between different theories, but also to constructing new vacua in string theory. The most dramatic example of this is F-theory [3], which is based on the $SL(2, \mathbb{Z})$ U-duality group of type IIB theory. The dilaton is allowed to jump according to $SL(2, \mathbb{Z})$, and the presence of seven-branes in the theory (which are direct generalizations of so called stringy cosmic strings [4]) ensures the existence of well-defined vacua. The jumps are encoded in a manifold that has a fibered structure where the fiber is the geometrization of the duality group. It has been shown recently [5] that this type of argument can be applied not just to the type IIB theory, but to other theories as well, and in particular to $N = 2$ theories in eight and seven dimensions. The structure of the moduli spaces, given by $SL(2, \mathbb{Z}) \times SL(3, \mathbb{Z}) \backslash SL(2) \times SL(3)/SO(2) \times SO(3)$ and $SL(5, \mathbb{Z}) \backslash SL(5)/SO(5)$ respectively, leads to arguments in favor of new higher-dimensional theories formulated on U-manifolds admitting $T^2 \times T^3$ and T^5 fibers. Putting aside the practical questions on construction of new vacua, in the following we try to understand the relation between supersymmetry and the fibration structure of the resulting U-manifolds.

We concentrate on the eight-dimensional case with the duality group $SL(2, \mathbb{Z}) \times SL(3, \mathbb{Z})$. While the first factor is well-understood, there is not much known about the solutions that respect the second. The five-dimensional piece of the moduli space parametrizes a three-torus at constant volume; that is, a supersymmetric three-cycle [6,7] in the resulting U-threefold (see also [8,9]). As a first step in understanding this, we construct a family of fivebrane solutions that transform consistently with $SL(3, \mathbb{Z})$. This family lives on a three-dimensional base that is topologically S^3, but each individual member is of real codimension two on the base (in agreement with the naive expectation that a fivebrane solution in eight dimensions should depend on only two transverse coordinates). In particular, the solutions all take the form of overlapping fivebranes, each individually preserving half of the supersymmetries, but together preserving only a quarter. This is in agreement with the expectation that intersecting branes break additional supersymmetries. As a result, this leaves us with a $D = 3$ $N = 2$ theory upon compactification on a $S^2 \times S^3$ base consistent with a full $SL(2, \mathbb{Z}) \times SL(3, \mathbb{Z})$ solution.

While the overlapping fivebranes are constructed based on first order supersymmetry equations, it turns out that these exact same equations are in perfect correspondence with

the conditions for the resulting U-manifold to admit a special lagrangian T^3 fibration. This should in fact come as no surprise, as $SL(3)/SO(3)$ is (at least locally) the moduli space of T^3. Based on this connection with T^3 fibered Calabi-Yau 3-folds, we give an explicit realization of mirror symmetry based on T-duality on the fibers. This picture is particularly nice, as it is manifest how the complex structure and Kähler deformations are interchanged in the mirror pairs.

In the next section we give an overview of U-scalars and the role of solvable Lie algebras in their classification. Then in section 3 we specialize to eight dimensions and the construction of the $SL(3, \mathbb{Z})$ based overlapping fivebrane solutions. This discussion is simplified by the use of first order Killing spinor equations. At this point we also indicate the straightforward $SL(n, \mathbb{Z})$ generalization. In section 4 we connect the U-manifold discussion with T^n fibered Calabi-Yau manifolds by demonstrating the correspondence of the first order equations with the special lagrangian conditions. This then enables us to consider the action of T-duality in generating mirror pairs. Finally, we conclude with some comments on global issues that are still not completely understood.

2. U-duality and scalar manifolds

We consider supersymmetric theories whose moduli spaces are given by homogeneous coset manifolds U_D/H_D, where U_D is a non-compact Lie group and H_D is a maximal compact subgroup. The moduli spaces in these theories are exact since supersymmetry protects again quantum corrections. The properties of the moduli space have been used to construct a large class of new vacua, in particular by introducing $(D - 3)$ branes carrying scalar charges. It has also been shown that they realize compact fibered manifolds, where the fiber captures the symmetries of the moduli space.

Our main focus is on maximal supersymmetric theories in eight and seven dimensions. Restricting to cosets of the type $SL(n)/SO(n)$, it is easy to notice that the scalar manifold $\mathcal{M}$ can be seen as originating from an "internal" T^n torus. Indeed, when the other fields are set to zero (note that these should respect U since it is the symmetry of the full theory), all the solutions can be understood from pure gravity in n dimensions higher, provided that the volume of the torus is kept fixed. As it turns out, these internal tori can always be realized as special lagrangian submanifolds of a larger U-manifold, and extended supersymmetric theories provide a natural environment for these. Although the construction developed in this paper is best suited for the case when the dimension of the

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cycle is half that of the manifold, this does not need to be the case — numerous examples of elliptically fibered n-folds have been discussed in F-theory literature. One may also consider T^3 fibrations in the manifolds of G_2 holonomy, but these will not be addressed here.

The concept of solvable algebras $Solv(U/H)$ that associate the scalars of the coset to group generators [10,11] has proven to be useful in analyzing the properties of U-scalars. This identification of coset manifolds with the group manifolds of solvable Lie algebras leads to replacing some of the notion of geometry of cosets by algebraic ones. In particular, this allows one to count the precise number of translational symmetries. As a matter of fact, the translational symmetries of U_D/H_D in D dimensions are classified by U_{D+1}, and, somewhat surprisingly, one finds that only half of the RR scalars have them (the $NSNS$ vs. RR division is classified by $O(9-D, 9-D)$). The scalar manifolds appearing in the toroidal compactifications with maximal supersymmetry are non-compact homogeneous spaces of maximal rank $11 - D$, and the associated solvable algebras have a special structure. It is of special importance that the type IIB theory in ten dimensions and $N = 2$ theories in eight and seven have these two special features. Even though most of our results are obtained explicitly for $SL(3, \mathbb{Z})$, the generalization to the $SL(5, \mathbb{Z})$ case is straightforward.

Specializing to the $SL(3, \mathbb{Z})$ case, one would like to understand the structure of the moduli space, and in particular its mapping on the base for the fivebranes. The study of the $SL(2, \mathbb{Z})$ case [4] reveals the special importance of the orbifold points and the uncontractable cycles on the moduli space. Much remains to be done in the study of global properties of the $SL(3, \mathbb{Z})$ case. But by focusing on local issues below, we nevertheless gain at least a partial understanding of the fibration structure of the U-manifold, and furthermore develop new insight on Calabi-Yau mirror symmetry.

3. Fivebranes on S^3

As a starting point for constructing the fivebrane solutions, we begin with a description of the effective action for the scalar fields. While the eight-dimensional $N = 2$ theory contains a total of seven scalars, we only consider the explicit action for the five scalars corresponding to the $SL(3, \mathbb{R})/SO(3)$ coset. These five scalars may be represented in

terms of a vielbein, V_{ai}, with determinant 1. Here a is a $SO(3)$ index, while i is a $SL(3)$ index. Since V is essentially a coset representative, we define

$$(\partial_\mu V V^{-1})^{ab} = P_\mu^{(ab)} + Q_\mu^{[ab]}, \tag{3.1}$$

where P_μ is symmetric in the $SO(3)$ indices, and is used in constructing the kinetic term for the scalars. Q_μ is antisymmetric and is a composite $SO(3)$ connection. Due to the $SO(3)$ invariance, it is clear that V contains only five scalar degrees of freedom.

In terms of the vielbein V, the eight-dimensional effective action for the scalar fields coupled to gravity may then be written as

$$\mathcal{L} = \frac{1}{2\kappa^2}\sqrt{-g}[R - \mathrm{Tr}P_\mu P^\mu + \cdots], \tag{3.2}$$

and gives rise to the following equations of motion:

$$\mathcal{D}^\mu P_\mu = 0$$
$$R_{\mu\nu} = \mathrm{Tr}P_\mu P_\nu, \tag{3.3}$$

where $\mathcal{D}$ is the Lorentz and $SO(3)$ covariant derivative, so that

$$\mathcal{D}_\mu P_\nu = \nabla_\mu P_\nu + [Q_\mu, P_\nu]. \tag{3.4}$$

In principle, we are interested in fivebrane solutions that solve the above equations of motion. However, in contrast with the $SU(1,1)/U(1)$ F-theory with a complex moduli space, since the $SL(3,\mathbb{R})/SO(3)$ space is odd-dimensional, complex geometry no longer plays a dominant role in constructing the solutions. This is a rather crucial difference between the $SL(3,\mathbb{Z})$ U-dual solutions and the $SL(2,\mathbb{Z})$ solutions, and forces us to develop new methods in the present case.

With five scalars coupled to gravity, the second order equations of motion are rather cumbersome to examine. Fortunately we may use supersymmetry as a guide, and examine the first order Killing spinor equations. Recall that a single fivebrane solution may be constructed based on preserving exactly half of the eight-dimensional supersymmetries. Furthermore, by now many cases of overlapping branes are well understood from a supersymmetric point of view. The rest of this section details the construction of overlapping fivebranes, and their relation to half-supersymmetry projections.

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The supersymmetry of the theory given by (3.2) may be obtained in several manners. A direct compactification of type IIB theory to eight dimensions will give an explicit parametrization of the vielbein V along with its supersymmetry properties. For completeness this reduction is presented in the Appendix. On the other hand, making use of T duality, we may equally well relate the action to M-theory compactified on a 3-torus, in which case the $SL(3, \mathbb{R})/SO(3)$ coset is directly related to the symmetries of the compactification T^3. We will have more to say about this later. In either case, the eight-dimensional fermions are pseudo-Majorana, and transform as a doublet under $SO(3)$. Using T^a to denote representation matrices for the spinor representation of $SO(3)$, the resulting supersymmetry transformations on the fermions are given by

$$\delta \chi^a = -\tfrac{1}{2} \gamma^\mu P_\mu^{ab} T^b \epsilon$$
$$\delta \psi_\mu = \mathcal{D}_\mu \epsilon \equiv [\nabla_\mu + \frac{1}{4} Q_\mu^{ab} T^{ab}] \epsilon. \tag{3.5}$$

Note that the spin-1/2 fermions, χ^a, carry an additional vector index a of $SO(3)$.

A basic fivebrane solution, preserving exactly half of the supersymmetries, may be obtained by demanding the vanishing of $\delta \chi$ and $\delta \psi_\mu$ on a set of Killing spinors, ϵ, such that $P\epsilon = 0$. For a fivebrane with transverse directions x^1 and x^2, the $\tfrac{1}{2}$-SUSY projection takes the form

$$P = \tfrac{1}{2}(1 + \gamma^{\overline{12}} T^{12}). \tag{3.6}$$

Here $\gamma^{\overline{1}}$ and $\gamma^{\overline{2}}$ denote γ-matrices with tangent space indices. From the form of this projection, it is clear that we have related the rotational $SO(2)$ symmetry in the x^1-x^2 plane with a $SO(2)$ subgroup of the $SO(3)$ automorphism group. This is a general property of $D-3$ branes constructed with scalars varying over a two-dimensional base. In fact, the projection (3.6) and its resulting supersymmetry properties is the basis for constructing the $SL(2, \mathbb{Z})$ solution of F-theory [4]. In that case the situation is quite clear, as parallel sevenbranes (all satisfying the identical projection P) may be combined to give an elliptically fibered $K3$ surface.

In the $SL(3)/SO(3)$ case, however, we see that each fivebrane picks out a specific embedding of $SO(2)$ within $SO(3)$, with considerable freedom on how this embedding may be done. Thus, making full use of $SO(3)$ invariance, the $\tfrac{1}{2}$-SUSY projection generalizes to

$$P = \tfrac{1}{2}(1 + \gamma^{\overline{12}} \Lambda_{1a} \Lambda_{2b} T^{ab}), \tag{3.7}$$

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where $\Lambda_{ab}(x)$ is a $SO(3)$ rotation matrix, allowing for different fivebrane orientations at different points on the base.

While a single fivebrane preserves half of the supersymmetries, it is well known that solutions with less supersymmetries may be constructed by overlapping multiple branes. From a supersymmetry point of view, this corresponds to finding a set of commuting projections, $P_{(i)}$, each of the form (3.7), so that the only remaining supersymmetries are those preserved by the complete set of $P_{(i)}$. For the case at hand, it is easy to see that, with a two dimensional base, it is impossible to construct solutions based on more than one projection. Equivalently, this indicates that supersymmetric parallel fivebranes must have identical $SO(2)$ orientations within $SO(3)$, resulting in a solution preserving exactly half of the supersymmetries.

The full U-symmetry of the fivebrane construction is brought out only when the base is enlarged to three dimensions. In this case, we may overlap two fivebranes, constructed with $e.g.$ the individual (commuting) projections

$$
\begin{aligned}
P_{(1)} &= \tfrac{1}{2}(1 + \gamma^{\overline{12}} T^{12}) \\
P_{(2)} &= \tfrac{1}{2}(1 + \gamma^{\overline{23}} T^{23}).
\end{aligned}
\tag{3.8}
$$

Note that we have made a rigid identification between the two $SO(3)$'s in this case; as we see later, this cannot be the most general solution, but nevertheless serves as a useful example. While a third projection, $P_{(3)} = \tfrac{1}{2}(1 + \gamma^{\overline{13}} T^{13})$, may be constructed, its addition does not kill any more supersymmetries. This is easily seen since $P_{(3)}$ may be expressed as $P_{(3)} = P_{(1)} P_{(2)} + (1 - P_{(1)})(1 - P_{(2)})$, and hence gives no further content than the combination of $P_{(1)}$ and $P_{(2)}$ alone.

As an example of the above construction, we now turn to an explicit parametrization of the $SL(3)/SO(3)$ vielbein V_{ai} in terms of five scalars. Using $SO(3)$ invariance, we may write V in the upper triangular form

$$
V = e^{\Phi_1/\sqrt{3}}
\begin{bmatrix}
1 & a & b \\
0 & e^{-(\sqrt{3}\Phi_1 - \Phi_2)/2} & ce^{-(\sqrt{3}\Phi_1 - \Phi_2)/2} \\
0 & 0 & e^{-(\sqrt{3}\Phi_1 + \Phi_2)/2}
\end{bmatrix},
\tag{3.9}
$$

where Φ_1 and Φ_2 are the two dilatonic scalars. This parametrization is motivated by the structure of the $SL(3)$ generators, in particular with the dilatons corresponding to the Cartan generators λ^3 and λ^8.

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At this stage it is instructive to see the explicit form of the Lagrangian. Defining the $SL(3)$ roots

$$\alpha_1 = \frac{\sqrt{3}}{2}\Phi_1 - \tfrac{1}{2}\Phi_2$$
$$\alpha_2 = \Phi_2,$$

(3.10)

we find

$$\mathcal{L} = \frac{1}{2\kappa^2}\sqrt{-g}[R - \tfrac{1}{3}(\partial\alpha_1)^2 - \tfrac{1}{3}(\partial\alpha_2)^2 - \tfrac{1}{3}(\partial\alpha_{12})^2$$
$$- \tfrac{1}{2}e^{2\alpha_1}(\partial a)^2 - \tfrac{1}{2}e^{2\alpha_2}(\partial c)^2 - \tfrac{1}{2}e^{2\alpha_{12}}(\partial b - c\partial a)^2],$$

(3.11)

where $\alpha_{12} \equiv \alpha_1 + \alpha_2$ was introduced for convenience and to highlight the nature of the $SL(3)$ symmetry. From here we see that only scalars a and b possess translational invariance, as expected.

In contrast to the parallel fivebrane solution of [4], here it is not possible to separate the behavior of the gravity fields from that of the scalars. One may anticipate that this is the case since the external metric of a $D - 3$ brane contains a deficit angle. Hence when several branes are overlapped they would necessarily be affected by the presence of the others which share only part of the transverse directions. A natural choice for the metric on the base is given by

$$ds^2 = e^{2\phi_1(x)}dx_1^2 + e^{2\phi_2(x)}dx_2^2 + e^{2\phi_3(x)}dx_3^2.$$

(3.12)

We note that making this choice has essentially forced us to consider only "rigid" fivebranes — those with a globally fixed orientation between the two $SO(3)$'s.

Demanding that the supersymmetry variations (3.5) vanish for Killing spinors satisfying (3.8), we end up with a set of first order equations taking the form

$$\partial_1 a = -e^{\phi_1-\phi_2-\phi_3}\partial_2 e^{\phi_3-\alpha_1}$$
$$\partial_2 a = e^{-\phi_1+\phi_2+\phi_3}\partial_1 e^{-\phi_3-\alpha_1}$$
$$\partial_1 b - c\partial_1 a = -e^{\phi_1+\phi_2-\phi_3}\partial_3 e^{-\phi_2-\alpha_{12}}$$
$$\partial_3 b \qquad = e^{-\phi_1+\phi_2+\phi_3}\partial_1 e^{-\phi_2-\alpha_{12}}$$
$$\partial_2 c = -e^{\phi_1+\phi_2-\phi_3}\partial_3 e^{-\phi_1-\alpha_2}$$
$$\partial_3 c = e^{-\phi_1-\phi_2+\phi_3}\partial_2 e^{\phi_1-\alpha_2},$$

(3.13)

in addition to the conditions

$$\partial_3 a = \partial_2 b - c\partial_2 a = \partial_1 c = 0,$$

(3.14)

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and

$$\partial_i[\phi_1 + \tfrac{2}{3}\alpha_1 + \tfrac{1}{3}\alpha_2] = 0 \qquad i = 2, 3$$
$$\partial_1[\phi_2 + \tfrac{1}{3}\alpha_1 - \tfrac{1}{3}\alpha_2] = 0$$
$$\partial_3[\phi_2 - \tfrac{1}{3}\alpha_1 + \tfrac{1}{3}\alpha_2] = 0 \qquad\qquad (3.15)$$
$$\partial_i[\phi_3 + \tfrac{1}{3}\alpha_1 + \tfrac{2}{3}\alpha_2] = 0 \qquad i = 1, 2.$$

While superficially these equations may appear quite formidable, they actually have a very simple structure dictated by $SL(3)/SO(3)$ symmetry considerations. In (3.13) the three sets of equations, for non-dilatonic scalars a, b, and c respectively, are essentially generalized Cauchy-Riemann-like equations, corresponding to individual fivebranes built out of the pairs of fields $(a, e^{-\alpha_1})$, $(b, e^{-\alpha_{12}})$, and $(c, e^{-\alpha_2})$. Of course α_{12} is not independent, so the equations do not separate, but are quite subtlely intertwined. This is simply a result of $SL(3)$ not being large enough to allow two independent $SL(2)$ fivebranes. Alternatively we note that there are two possibilities for preserving a quarter of the supersymmetries, these being $K3 \times K3$ or CY_3, corresponding to $SL(2) \times SL(2)$ or $SL(3)$ brane configurations respectively.

Before turning to overlapping brane solutions, we note that the ansatz gives three $SL(2)/SO(2)$ special cases, obtained by setting either $b = c = 0$, $a = b = 0$ or $a = c = 0$. For example, in the first case, we would find the Cauchy-Riemann equation

$$\partial_i a(x_1, x_2) = -\epsilon_{ij}\partial_j e^{-\alpha_1(x_1, x_2)}, \qquad\qquad (3.16)$$

which is solved by complex analytic functions $\tau(z)$ where $\tau = a + ie^{-\alpha_1}$ and $z = x_1 + ix_2$. Thus in this case we have essentially reproduced the $SL(2)$ solution of [4]. One key difference, though, is that by picking a "rigid" fivebrane orientation, the present ansatz gives rise to the non-modular invariant relation between the metric and scalar fields, $\phi_1 = \phi_2 = -\alpha_1/2 = \alpha_2$ (or $\phi = \log \tau_2$ in the notation of [4]), indicating that the global properties are not fully addressed in this "rigid" ansatz. In principle this issue is solved by picking a more flexible ansatz allowing more freedom in the fivebrane orientation. It is this point that allows considerably more freedom in the solutions, giving rise to a much richer structure of T^3 fibered CY_3's than the corresponding case of K3.

As anticipated, we observe that the metric fields ϕ_μ do not separate from the dilatons in the fivebrane ansatz. Using the relation between metric fields and dilatons, (3.15), as a hint, we may further restrict the solution by introducing the three quantities

$$\phi_a(x_1, x_2) \qquad \phi_b(x_1, x_3) \qquad \phi_c(x_2, x_3), \qquad\qquad (3.17)$$

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so that the metric ansatz now takes the new form

$$\phi_1 = -\tfrac{1}{2}(\phi_a + \phi_b) \qquad \phi_2 = -\tfrac{1}{2}(\phi_a + \phi_c) \qquad \phi_3 = -\tfrac{1}{2}(\phi_b + \phi_c). \tag{3.18}$$

To remain consistent with (3.15), the dilatons then must have the form

$$\alpha_1 = \tfrac{1}{2}(2\phi_a + \phi_b - \phi_c)$$
$$\alpha_2 = \tfrac{1}{2}(-\phi_a + \phi_b + 2\phi_c). \tag{3.19}$$

The beauty behind this choice of fields is that the Cauchy-Riemann-like equations, (3.13), now take the simplified form

$$\partial_1 a = -e^{\phi_c - \phi_b}\partial_2 e^{-\phi_a}$$
$$\partial_2 a = \partial_1 e^{-\phi_a}$$
$$\partial_1 b - c\partial_1 a = -e^{-\phi_a}\partial_3 e^{-\phi_b}$$
$$\partial_3 b = e^{-\phi_c}\partial_1 e^{-\phi_b} \tag{3.20}$$
$$\partial_2 c = -\partial_3 e^{-\phi_c}$$
$$\partial_3 c = e^{\phi_a - \phi_b}\partial_2 e^{-\phi_c},$$

indicating the connection between a, b, c and ϕ_a, ϕ_b, ϕ_c respectively. This clearly shows the relation between the individual fivebranes, their transverse directions, and its effect on the metric fields through (3.18). For example, the fivebrane corresponding to (a, ϕ_a) is transverse in x_1 and x_2, and does not affect the metric component ϕ_3 in the longitudinal x_3 direction. Viewed in terms of a three-dimensional base, this picture is one of overlapping fivebranes living on one-dimensional lines on the base.

In order to consider the overlapping solution of two fivebranes, we set both $c = 0$ and $\phi_c = 0$. As a result, (3.20) gives rise to the second-order equations

$$(\partial_1^2 + e^{-\phi_b}\partial_2^2)e^{-\phi_a} = 0$$
$$(\partial_1^2 + e^{-\phi_a}\partial_3^2)e^{-\phi_b} = 0, \tag{3.21}$$

which is an overlapping brane solution with x_1 being the common transverse direction in the solution. However, in addition to these transverse laplacians, we also have the consistency requirement that $\partial_3 e^{-\phi_b}\partial_2 e^{-\phi_a} = 0$, so that either ϕ_a or ϕ_b must be a function of x_1 only. Taking the latter case, the resulting fivebrane solution is described by the functions

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$\phi_a(x_1, x_2)$ and $\phi_b(x_1)$, and corresponds to an overlapping fivebrane and smeared fivebrane solution.

While it appears that the "rigid" ansatz only leads to a solution with smeared out branes, it is anticipated that a more general $SO(3)$ ansatz would allow true overlapped fivebrane solutions. On the other hand, from a more general point of view, the solution may be viewed as a mapping of the three-dimensional base into the five-dimensional moduli space. While this space is locally $SL(3)/SO(3)$, it is in fact an orbifold since points must be identified under action of the $SL(3, \mathbb{Z})$ U-duality. It is thus the pullback of the orbifold singularities that may be related to the fivebrane configuration on the base. In particular, singular lines, with codimension two on the base, are then identified with the $SL(3)$ fivebranes.

While the above discussion has focused on $SL(3)$ solutions, we note that cosmic-string-type solutions can be easily generalized to other extended supersymmetric theories despite the fact that writing down explicit first-order equations for the scalars generalizing (3.13), (3.14) and (3.15) may seem terribly involved. Let us recall the decomposition of the solvable Lie algebra for U/H ($Solv(U/H) \sim T\mathcal{M}_{scalars}$):

$$Solv(U/H) = \mathcal{H} \oplus \Phi^+(U), \tag{3.22}$$

where $\mathcal{H}$ is the Cartan piece, while $\Phi^+(U)$ is the positive part of the root space of U. We have seen that the "dilatonic" (exponentiated) scalars in the Cartan subalgebra in turn appear in combinations corresponding to the positive roots of the duality group. Bearing in mind that only the primitive roots of the duality group correspond to independent "dilatons", we now see that in each case the number of pairs (a_i, e^{α_i}) used to build individual solutions is equal to the number of positive roots of the duality group. The nested structure of the further decomposition of (3.22) allows for more simplifications in analyzing the solutions. For example, for the $D = 8$ $SL(3) \times SL(2)$ case, the Cartan piece is three-dimensional, and we find that $\Phi^+(U) = \Phi^+(E_2) \oplus \mathcal{D}_2^+$ where $\Phi^+(E_2)$ is the one-dimensional root space of the nine-dimensional U-duality group (corresponding to the RR scalar in type IIB), and $\mathcal{D}_2^+$ is the weight space of the $SL(3) \times SL(2)$ irreducible representation to which the nine-dimensional vectors are assigned (corresponding to the three-dimensional abelian ideal, or in other words, the scalars with translational symmetries).

4. T^3 fibrations and Calabi-Yau manifolds

In the previous section we have constructed a class of overlapping fivebrane solutions with varying $SL(3)/SO(3)$ scalars. We now show that these solutions have a natural interpretation in terms of a T^3 fibered Calabi-Yau manifold. In particular, this fivebrane solution provides a simple system where such T^3 fibrations may be studied in detail.

We begin by recalling that the $SL(2)$ F-theory solution may be described in terms of a $K3$ fibration where a T^2 of constant volume but varying shape is fibered over a S^2 base. In particular, this solution is given in terms of a function $\tau(z)$ that maps z, the complex coordinate on the sphere, to τ, the modular parameter of T^2. Locally, any analytic map $\tau(z)$ solves the equations of motion and preserves exactly half of the supersymmetry. However it is the global properties that give rise to the intricacies of the solution. In particular, exactly 24 strings are required to give $c_1(m(S^2)) = 2$, leading to a fibered $K3$ surface (where m defines a map from the base S^2 into the moduli space).

In the present case, since $SL(3)/SO(3)$ is locally the moduli space for T^3 at constant volume, there is a similar picture of the fivebrane solution in terms of a T^3 fibration. As we have shown in the previous section, an overlapping fivebrane solution preserving one quarter of the supersymmetry naturally lives on a three-dimensional base (which is expected to be topologically S^3 [8]). Since the mapping is from an odd-dimensional base to a five-dimensional moduli space, unlike the $SL(2)$ case, there are no natural set of complex coordinates to work with. On the other hand, for the complete space to correspond to a Calabi-Yau 3-fold, there must exist a choice of complex structure relating pairs of real coordinates. While the fivebrane ansatz, (3.13), appears to give three sets of Cauchy-Riemann conditions, suggesting an intertwined set of complex coordinates on the base ($x_1 + ix_2$, $x_2 + ix_3$ and $x_3 + ix_1$), we find that this is in fact not a natural choice. Instead, the choice that is consistent with a T^3 fibration is to pair each of the real coordinates on the base with a corresponding coordinate on the internal T^3. We now demonstrate this in some detail.

Working with real geometry, we denote the coordinates on the base as x^μ, $\mu = 1, 2, 3$. Since the internal space is T^3, we introduce a set of three periodic internal coordinates, $\xi^i = \xi^i + 1$ with $i = 1, 2, 3$. Combining the metric (3.12) on the base with the $SL(3)$ invariant form of the metric on T^3, a natural choice for the six-dimensional metric is simply

$$ds^2 = e^{2\phi_\mu(x)}(dx^\mu)^2 + d\xi^i M_{ij}(x)d\xi^j, \tag{4.1}$$

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and defines a T^3 fibration over S^3. Note that this metric is block diagonal between the base and internal space, so that it describes T^3 fibers that are always perpendicular to the base.

In order to better understand the T^3 fibration, it is essential to show that the manifold defined in this manner is in fact complex. As mentioned above, we seek a complex structure relating internal and base coordinates in pairs, so that at least locally the line element becomes

$$ds^2 = dz^\mu d\overline{z}^\mu. \tag{4.2}$$

Comparing this with (4.1), and using the definition $M = V^T V$, we see that natural complex coordinates are then of the form

$$z^\mu = e^{\phi_\mu} x^\mu + i\delta_{\mu a} V_{ai} \xi^i, \tag{4.3}$$

corresponding to a complex structure given by

$$J^M{}_N = \begin{bmatrix} 0 & -e^{-\phi_\mu} \delta_{\mu a} V_{ai} \\ V_{ib}^{-1} \delta_{b\nu} e^{\phi_\nu} & 0 \end{bmatrix}. \tag{4.4}$$

Note that this choice of complex structure involves some arbitrariness in pairing up the coset and base $SO(3)$ symmetries; the particular form was chosen above to agree with the fivebrane ansatz (3.13).

Since J is a function of x_μ and hence varies over the base, it is important to check that it is actually integrable. This check is easily performed by examining the Nijenhuis tensor corresponding to J. Since the Nijenhuis tensor is constructed in terms of first derivatives of J, we find a set of first order equations as an integrability condition on the complex structure. Remarkably these conditions are a subset of (3.13), so that in fact the complex structure is integrable for the fivebranes constructed in the previous section.

Using the complex structure (4.4), we are now able to connect the real parametrization of the fibered space with complex geometry. In particular, the Kähler form,

$$k_{MN} \equiv g_{MP} J^P{}_N = \begin{bmatrix} 0 & -e^{\phi_\mu} \delta_{\mu a} V_{ai} \\ V_{ja}^T \delta_{\nu a} e^{\phi_\nu} & 0 \end{bmatrix}, \tag{4.5}$$

may be determined to be covariantly constant for fivebranes solving (3.13). In fact, verification that $\nabla_M k_{NP} = 0$ requires full use of all 15 conditions of the supersymmetric ansatz, (3.13), (3.14) and (3.15).

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Finally, to show that the fibered space is a Calabi-Yau 3-fold (at least locally), we need to verify that (4.1) is not only Kähler, but also Ricci flat. Since the Ricci tensor involves second derivatives, the calculation is somewhat more tedious. Nevertheless, explicit verification shows that the overlapping fivebrane solution indeed gives a Ricci flat Kähler metric (4.1).

So far we have only considered the local properties of this fibered space. The global properties will clearly be related to the mapping of the S^3 base into the 5-dimensional moduli space of T^3, the latter being the orbifold $SL(3,\mathbb{Z})\backslash SL(3,\mathbb{R})/SO(3)$. While some of the periods are obvious (*e.g.* the 1-cycles $a \to a+1$, $b \to b+1$ and $c \to c+1; b \to b+a$) the overall picture is not so straightforward. Nevertheless, we expect the T^3 fibers to degenerate on singular *lines* on S^3, corresponding to the loci of the fivebranes. These global issues are currently under investigation[1].

Writing down the metric (4.1) and complex structure (4.4) in fact provides an explicit construction of a T^3 fibration of CY_3 [6,7,8]. We recall that, for such a fibration, the T^3 must in fact be a special Lagrangian submanifold, which essentially means that the pullback of the Kähler form must vanish, and the pullback of the holomorphic Calabi-Yau form must give the real volume form on T^3. For the fivebrane solution, the first condition is trivially satisfied since k_{ij}, the restriction of the Kähler form (4.5) to T^3, is automatically vanishing. For the Calabi-Yau form, we appeal to (4.3) to write

$$\Omega = i \prod_\mu (e^{\phi_\mu} dx^\mu + i\delta_{\mu a} V_{ai} d\xi^i), \tag{4.6}$$

which is by construction holomorphic with respect to the complex structure (4.4), and squares to the six-dimensional volume form. One may also verify that Ω is closed, provided the complex structure is integrable. When restricted to the T^3 fiber, the Calabi-Yau form simply becomes $\Omega_{ijk} = \epsilon_{abc} V_{ai} V_{bj} V_{ck} = \det V \epsilon_{ijk}$, which is indeed the expected result (since $\det V = 1$).

In retrospect, this correspondence is not surprising at all, since the fivebrane solution was explicitly constructed based on supersymmetry requirements of the eight-dimensional

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[1] There seems to be an appropriate generalization of the usual $SL(2;\mathbb{Z})$ fundamental domain; in this $SL(3,\mathbb{Z})$ case it would be given by $-1/2 \leq a,b,c, \leq 1/2$ and $|\vec{\lambda}_2| \geq |\vec{\lambda}_1| \geq 1$, where the period vectors are given by $\vec{\lambda}_1 = (a, \exp(-\alpha_1), 0)$ and $\vec{\lambda}_2 = (b, c\exp(-\alpha_1), \exp(-\alpha_{12}))$. It is the boundaries of this domain and their intersections that give rise to various fixed lines in the moduli space.

theory. Viewed from a M-theory point of view, the $SL(3, \mathbb{Z})$ part of the U-duality group is exactly the symmetry of the eleven dimensional theory compactified to eight dimensions on T^3. Thus while in the type IIB picture the T^3 is not physical, in M-theory it is certainly present. Constructing a $SL(3)$ solution preserving a quarter of the supersymmetries is equivalent in the M-theory picture to a compactification to five-dimensions on a Calabi-Yau three-fold. Of course it is important to realize that the complete U-manifold involves not just the $SL(3, \mathbb{Z})$, but also the $SL(2, \mathbb{Z})$ part of the U-duality group, where the $SL(2, \mathbb{Z})$ is rather obscure in the M-theory language.

Mirror symmetry as T-duality

An implicit formulation of the mirror space to a Calabi-Yau manifold was found in [7], where it was argued that every Calabi-Yau that has a mirror necessarily has a supersymmetric T^3 fibration. Moreover, the mirror symmetry turns out to be just T-duality on the T^3 fibers. Thus all the U-manifolds appearing here should have mirrors, as they are constructed explicitly as T^3 fibrations.

We already saw that supersymmetry "knows" about the fibered structure on the U-manifold. As it turns out, it also "knows" about the mirror U-manifold: T-duality on the fibers is nothing but a discrete symmetry of the coset representative

$$T : V \rightarrow (V^{-1})^T, \tag{4.7}$$

corresponding to T-duality on all three abelian cycles of T^3. The way the fibration metric is written in (4.1) makes T-duality on the fibers particularly simple. Since the metric is of diagonal form, the duality on the internal part does not generate any torsion, and hence its action is given by $M \rightarrow M^{-1}$. This is guaranteed to be the correct T-duality from the M-theory point of view since in that case M *is* exactly the internal Kaluza-Klein metric. From a string theory point of view, this inversion precisely corresponds to the interchange of winding and momenta modes for all three compact dimensions[2]. Note, however, that while (4.7) is contained in the $O(3, 3; \mathbb{Z})$ T-duality group appropriate to a string compactification, it is not part of $SL(3, \mathbb{Z})$. Thus the inversion of the vielbein is

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[2] More precisely, the string interpretation is obtained only upon further compactification of M-theory on S^1.

a unique discrete symmetry of the scalar manifold $\mathcal{M}$, and lies outside of the U-duality group[3].

It is interesting to study the behavior of the fibration under T-duality. Suppressing all indices, we may rewrite the complex structure (4.4) and Kähler form (4.5) in the block form

$$J = \begin{bmatrix} 0 & -j^{-1} \\ j & 0 \end{bmatrix} \qquad k = \begin{bmatrix} 0 & -\kappa^T \\ \kappa & 0 \end{bmatrix}, \tag{4.8}$$

where $j = V^{-1} \cdot e^\phi$ and $\kappa = V^T \cdot e^\phi$. Here e^ϕ denotes the diagonal matrix of metric factors on the base, $e^\phi = diag(e^{\phi_1}, e^{\phi_2}, e^{\phi_3})$. In this form, it immediately follows that the action of T-duality, (4.7), interchanges κ with j,

$$T : \quad \kappa \leftrightarrow j, \tag{4.9}$$

with corresponding interchange between J and k. As a result, this gives an explicit realization of Calabi-Yau mirror symmetry, where deformations of complex structure, δj, and Kähler class, $\delta \kappa$, are interchanged between the original manifold and its mirror. While this has been a local statement, the picture nevertheless continues to hold in general, since any globally well defined δj on the S^3 base may equally well apply to $\delta \kappa$ and vice versa. Of course, the T^3 fibration structure is crucial, as it is the discrete freedom of choice for the vielbein, (4.7), which corresponds to the choice between two mirror U-manifolds.

5. Comments and conclusions

Starting from an attempt to construct U-branes transforming non-trivially under $SL(3, \mathbb{Z})$, we have ended up with an explicit *local* description of T^3 fibered Calabi-Yau 3-folds. The first order equations resulting from the Killing spinor equations correspond in the Calabi-Yau picture to the special lagrangian conditions. By explicit construction, the complex coordinates of the Calabi-Yau 3-fold are built from the coordinates on the base each paired with a coordinate on the T^3 fiber. This may be contrasted with the $SL(2, \mathbb{Z})$ "stringy cosmic string" construction of [4], where $K3$ is described in terms of complex tori

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[3] A simple way to see this is to note that if the scalar matrix M and its inverse were related under $SL(3, \mathbb{Z})$, this relation would have the form $M^{-1} = \Omega^T M \Omega$ where Ω is a $SL(3, \mathbb{Z})$ matrix. Since Ω has integer coefficients, it cannot vary continuously as the five scalars are varied. Thus every entry in M^{-1} must be able to be written as some linear combination of the terms already present in M. However a simple calculation of the inverse shows that this is not in general possible.

fibered over CP^1. Since the two sets of complex coordinates used in [4] and here are not related by an analytic map, our construction simply corresponds to a different choice of the complex structure. However (as pointed out in [7]) this description of $K3$ may be related to the present case by a rotation of complex structure. In fact, it is the (hyper-Kähler) $K3$ case that is special; general mirror pairs of Calabi-Yau n-folds presumably involve T^n fibrations, giving a complex structure of the form (4.4) without any freedom of rotation of complex structure.

Another notable difference in the construction of an elliptically fibered $K3$ from the one presented here is that, as mentioned before, the $K3$ case only involves parallel branes. One way to see this is that for the $SL(2)$ case there is a unique supersymmetry projection operator, the analogue of (3.7), which transforms as a singlet under the $SO(2)$ automorphism group, resulting in a unique configuration preserving half of the supersymmetries. In the more general case, the freedom to pick different $SO(n)$ orientations for the branes leads to many possibilities for "closing" the manifold. We observe that each additional overlapping brane reduces the supersymmetry by a half, so that the seven dimensional $SL(5, \mathbb{Z})$ case preserves only a sixteenth of the supersymmetry, and involves even fancier intersections of branes with different orientations[4]. Once again this shows that $K3$ is a somewhat degenerate case, as it is the only one where the $D-3$ branes are completely parallel. It is of some interest to better understand the relation between the distribution of branes on the base and the global properties of the manifold such as the number of deformations of complex structure and Kähler class.

One may speculate that, even with a complicated overlapping fivebrane configuration, it would nevertheless be possible to study individual branes within the solution, each given by a single "dilaton-axion" pair $(a_i, e^{-\alpha_i})$ with a specific $SO(2)$ orientation on the base in the notation of section 3. Moreover, one would hope to evaluate the energy per volume for the individual branes in this fashion and obtain a relation analogous to the elliptically fibered case [12], where for the base B:

$$c_1(B) = -\frac{1}{12} \sum N_a \delta_2(D_a), \tag{5.1}$$

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[4] In general, $SL(n)$ solutions are expected to preserve anywhere from $1/2$ to $1/2^{n-1}$ supersymmetries, with the latter corresponding to true $SL(n)$ solutions that do not factorize into separate pieces. For example, with $SL(5)$ we would have a family of manifolds of the form $K3 \times T^6$, $CY_3 \times T^4$, $CY_4 \times T^2$ and finally CY_5.

which plays an important role in F-theory constructions. Physically speaking, this is what one expects from the fivebranes on S^3. Indeed, the effect of the branes on the curvature seems to be localized, but the complicated global features such as the deficit angle of the solutions do not allow an analysis of an isolated solution. Further exploration of the properties of the moduli space should hopefully lead to progress in this direction.

We reiterate that much remains to be done to achieve a better understanding of the global features of the U-brane configurations. Nevertheless, the profound relation between supersymmetry and geometry as well as the remarkable emerging picture of mirror symmetry makes this study worthwhile.

Appendix A.

In this appendix we make an explicit connection between the U-scalars in eight dimensions and the original type IIB theory. Starting in ten dimensions, while the type IIB theory contains a 4-form potential with self-dual field strength, and hence does not admit a conventional Lagrangian formulation, for our purposes the 4-form potential may be ignored, as it does not give rise to any scalars in eight dimensions. As a result, it is sufficient to focus on the truncated type IIB Lagrangian, written in natural string coordinates as [13]

$$
\begin{aligned}
\mathcal{L}_{D=10} = \sqrt{-G^{(10)}}e^{-2\Phi^{(10)}}[&R_{G^{(10)}} + 4(\partial_M \Phi^{(10)})^2 - \frac{1}{2\cdot 3!}(H_{MNP}^{(1)})^2 \\
&- e^{\Phi^{(10)}}(\frac{1}{2}(\partial_M \ell)^2 + \frac{1}{2\cdot 3!}(H_{MNP}^{(2)} - \ell H_{MNP}^{(1)})^2)],
\end{aligned}
\tag{A.1}
$$

with $H^{(i)} = dB^{(i)}$. In this form, there is a clear division between the $NSNS$ fields, $\{G^{(10)}, B^{(1)}, \Phi^{(10)}\}$, and the RR fields $\{B^{(2)}, \ell\}$ (ignoring the RR 4-form potential).

For the type IIB theory in ten dimensions, we may consider the fermions as pairs of Majorana-Weyl spinors, transforming as a doublet under SO(2). In this case, introducing

the Pauli matrices σ^i acting on the SO(2) index, the supersymmetries may be written as

$$\delta\Lambda = \tfrac{1}{2}[\Gamma^M\partial_M\Phi^{(10)} - \tfrac{1}{12}H^{(1)}_{MNP}\Gamma^{MNP}\sigma^3]\eta$$
$$+ \tfrac{1}{2}e^{\Phi^{(10)}}[\Gamma^M\partial_M\ell i\sigma^2 + \tfrac{1}{12}(H^{(2)} - \ell H^{(1)})_{MNP}\Gamma^{MNP}\sigma^1]\eta$$
$$\delta\Psi_M = [\nabla - \tfrac{1}{8}H^{(1)}_{MNP}\Gamma^{NP}\sigma^3]\eta$$
$$- \tfrac{1}{8}e^{\Phi^{(10)}}[\Gamma^N\partial_N\ell i\sigma^2 + \tfrac{1}{6}(H^{(2)} - \ell H^{(1)})_{NPQ}\Gamma^{NPQ}\sigma^1]\Gamma_M\eta, \tag{A.2}$$

again with a clear split between the $NSNS$ and RR fields.

Upon T^2 reduction to eight dimensions, both $H^{(1)}$ and $H^{(2)}$ give rise to scalars. Combined with $\Phi^{(10)}$, ℓ and the three scalars parametrizing T^2, this gives a total of seven eight-dimensional scalars. Since we are only interested in the scalar sector, dimensional reduction is especially straightforward. In order to end up in an eight-dimensional Einstein frame, we take

$$G_{MN} = \begin{bmatrix} e^{2\phi/3}g_{\mu\nu} & \\ & G_{ij} \end{bmatrix}, \tag{A.3}$$

where the T^2 metric is

$$G_{ij} = \frac{e^{-\sigma}}{\tau_2}\begin{bmatrix} 1 & \tau_1 \\ \tau_1 & |\tau|^2 \end{bmatrix}, \tag{A.4}$$

and the eight-dimensional dilaton is given by

$$\phi = \Phi^{(10)} + \tfrac{1}{2}\sigma. \tag{A.5}$$

Finally, defining $B^{(k)}_{ij} = b^{(k)}\epsilon_{ij}$, we end up with

$$\mathcal{L}_{D=8} = \sqrt{-g}[R - \frac{1}{2}\frac{|\partial_\mu\tau|^2}{\tau_2^2} - \frac{2}{3}(\partial_\mu\phi)^2 - \frac{1}{2}(\partial_\mu\sigma)^2$$
$$- \frac{1}{2}e^{2\sigma}(\partial_\mu b^{(1)})^2 - \frac{1}{2}e^{2\phi-\sigma}(\partial_\mu\ell)^2 - \frac{1}{2}e^{2\phi+\sigma}(\partial_\mu b^{(2)} - \ell\partial_\mu b^{(1)})^2], \tag{A.6}$$

which may be compared with (3.11). As a result, we identify the $SL(3)$ scalars as

$$\{a, b, c\} = \{b^{(1)}, b^{(2)}, \ell\} \qquad \{\alpha_1, \alpha_2\} = \{\sigma, \phi - \tfrac{1}{2}\sigma\}, \tag{A.7}$$

and the $SL(2)$ scalars as simply the complex structure of the T^2, namely $\{\tau_1, \tau_2\}$.

Working out the supersymmetry of (A.6) is somewhat more involved. Upon dimensional reduction, the eight-dimensional dilatino (λ) is shifted according to

$$\lambda = e^{\phi/6}(\Lambda - \tfrac{1}{2}\Gamma^i\Psi_i), \tag{A.8}$$

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where the exponential factor accounts for transforming from the string to the Einstein frame. This latter transformation also shifts the gravitino so that

$$\psi_\mu = e^{-\phi/6}\Psi_\mu - \tfrac{1}{3}\gamma_\mu\lambda. \tag{A.9}$$

The resulting eight-dimensional gravitino variation becomes

$$\begin{aligned}
\delta\psi_\mu =&[\nabla_\mu - \frac{i}{4}\frac{\partial_\mu\tau_1}{\tau_2}\gamma^9]\epsilon\\
&- \frac{1}{4}[e^\sigma\partial_\mu b^{(1)}(\gamma^9 i\sigma^3) + e^{\phi-\frac{1}{2}\sigma}\partial_\mu\ell(i\sigma^2) + e^{\phi+\frac{1}{2}\sigma}(\partial_\mu b^{(2)} - \ell\partial_\mu b^{(1)})(\gamma^9 i\sigma^1)]\epsilon,
\end{aligned} \tag{A.10}$$

where $\epsilon = e^{-\phi/6}\eta$ and $(\gamma^9)^2 = 1$ is the chirality operator in eight dimensions ($\gamma^9 \equiv i\gamma^0\gamma^1\cdots\gamma^7$). In performing the reduction of (A.2), we have made use of the fact that the ten-dimensional spinors have definite chirality, $\Gamma^{11}\eta = \eta$. Comparing (A.10) to (3.5) (making use of the $SL(3)$ interpretation (A.7)), we find the $SO(3)$ generators

$$T^a = \{\sigma^2, -\gamma^9\sigma^1, \gamma^9\sigma^3\}, \tag{A.11}$$

resulting in

$$\delta\psi_\mu = [\nabla_\mu - \frac{i}{4}\frac{\partial_\mu\tau_1}{\tau_2}\gamma^9 + \frac{1}{4}Q_\mu^{ab}T^{ab}]\epsilon. \tag{A.12}$$

For the remaining spin-$\frac{1}{2}$ fields, the dilatino may be combined with the "internal" components of the gravitino, Ψ_8 and Ψ_9, in the combination

$$\chi^a = \left\{\begin{array}{l}
\sigma^2(-\frac{1}{3}\lambda + \frac{1}{2}e^{\phi/6}\Gamma^i\Psi_i + \frac{1}{3}e^{\phi/6}(\Gamma^8\Psi_8 - \Gamma^9\Psi_9))\\
\gamma^9\sigma^1(-\frac{1}{3}\lambda - \frac{1}{2}e^{\phi/6}\Gamma^i\Psi_i + \frac{1}{3}e^{\phi/6}(\Gamma^8\Psi_8 - \Gamma^9\Psi_9))\\
-\gamma^9\sigma^3(\quad\frac{2}{3}\lambda \qquad\qquad + \frac{1}{3}e^{\phi/6}(\Gamma^8\Psi_8 - \Gamma^9\Psi_9))
\end{array}\right\}, \tag{A.13}$$

with resulting variation

$$\delta\chi^a = -\frac{1}{2}\gamma^\mu P_\mu^{ab}T^b\epsilon - \frac{1}{6}\gamma^\mu[\frac{\partial_\mu\tau_2}{\tau_2} - i\frac{\partial_\mu\tau_1}{\tau_2}\gamma^9]T^a\epsilon. \tag{A.14}$$

The supersymmetry variations (A.12) and (A.14) indicate the form of the additional $SL(2)$ contributions that were ignored in (3.5). The above dimensional reduction demonstrates the explicit correspondence between the type IIB fields and the U-scalars of the eight dimensional theory.

Finally, we note that the U-duality group in D dimensions has a convenient type IIB-inspired decomposition [11] that makes the original ten-dimensional $SL(2)_U$ symmetry apparent:

$$E_{r+1} \to SL(2, \mathbb{R})_U \otimes GL(r, \mathbb{R}), \tag{A.15}$$

where r stands for the number of compact dimensions. Note that only the first factor in (A.15) mixes RR and $NSNS$ states; $GL(r, \mathbb{R})$ is just the isometry group of the classical moduli space of the T^r torus.

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