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STABILITY OF A BUNCHED BEAM INTERACTING WITH A MATCHED LINE

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The collective stability of a bunched beam interacting with a matched line is studied in this paper. The functional dependences are found of the decay (growth) of the betatron and synchrotron excitations of arbitrary multiplicity on the characteristic parameters of the problem (plate length, bunch length, chromaticity).

INTRODUCTION

Experimental studies of the coherent stability of beams in storage rings have shown that there exist coherent effects such that their decay and growth do not depend on the choice of the working point for the particle oscillation frequency. This shows that these effects are due to interaction of the beam with low figure-of-merit systems where the fields excited by the beam damp in a time less then the period of the particle's revolution (so called "one-turn effects").

The first effect of this kind, so called "fast damping" of the beam's vertical oscillations (Ref. 1) was found on the installation VEPP-2. Its characteristic feature was that decrements of the oscillations were determined by the total charge of the bunch (Ne) and did not depend on the beam parameter:

$$\delta \sim \frac{N}{E} \tag{I.1}$$

where E is the particle's energy. This phenomenon was explained by the interaction of coherently oscillating beams with the main field wave in the matched lines (Ref. 2).

Some time later instabilities of transverse oscillations which could be attributed to the one-turn effect were found on the installations ACO and ADONE. Rates of increase of these instabilities are inversely proportional to the beam's length, depend on the chromaticity of the machine

$$\frac{\mathrm{d} \ln \nu}{\mathrm{d} \ln R_0}$$

and on the number of particles in the given bunch. The empirical relation of the threshold current on these parameters has the form (Ref. 3):

$$I_{\text{th}} \sim \frac{E l_B \Delta\nu}{1 - \dfrac{\mathrm{d} \ln \nu}{\mathrm{d} \ln R_0}},$$

where ν is the dimensionless betatron frequency, $\Delta\nu$ is the frequency spread, $2\pi R_0$ is the orbital circumference.

The explanation of the instability mechanism due to the $\nu(R_0)$ dependence was given by Pellegrini: it is the "head-tail" effect (Ref. 4). However, decrements of the oscillations obtained by him in the concrete examples disappear when either the length of the beam (l_B) or the machine's chromaticity goes to zero:

$$\delta \sim \frac{N l_B}{E} \frac{\mathrm{d} \ln \nu}{\mathrm{d} \ln R_0}. \tag{I.2}$$

In particular, there is no term in the decrement which corresponds to "fast damping" (the term which does not disappear when $l_B \to 0$) in case of matched lines. This result is due to the incomplete consideration of the beam's interaction with the low figure-of-merit element.

Theoretical study of the beam's interaction with matched lines and low figure-of-merit resonators was presented previously (Refs. 5,6,7). It was shown that it is possible to obtain damping of the main types (one-dimensional betatron and synchrotron type) of coherent oscillations of the beam by introducing matching lines while interaction with a low figure-of-merit resonator may lead to instability. The expressions for the decrements of the betatron oscillations were obtained in the small bunch length approximation and therefore, they did not contain terms of the type (I.2).

In this paper we study the stability of a beam of arbitrary length (but not shorter than the transverse dimension of the chamber) which is interacting with the matched line.

In the first section a general integral equation is obtained which determines the spectrum of the normal collective excitation of the beam near some stationary state when it is interacting with an outside system. The method of the kinetic equation in canonical variables which is used here (it was introduced in Ref. 5) allows one to approach a wide range of problems concerned with the collective motion of the beam. This method is especially effective when the collective interaction introduces only a small distortion of the particles' motion. Then the nonstationary part of the particles' phase space distribution which describes the normal collective excitation has the form:

$$\tilde{F} = \tilde{F}_m(I)\exp\left(i \sum_k m_k \psi_k - i\omega t\right), \omega \simeq \sum_k m_k \omega_k,$$

where I, ψ are action-phase variables of the stationary state in which, by definition, the distribution is homogeneous in phase, ω_k is the partial frequency of the nondisturbed particle motion, and m_k is the integer which defines the multiple character of the excitation. In particular the case $\sum_k |m_k| = 1$ corresponds to the case of dipole excitation. The expressions for the decrements of the model δ type distribution of the synchrotron oscillation amplitudes in the stationary state are studied in the second section. It is shown that the decrement of the arbitrary multiple excitation can in general be represented as a sum of two terms, one of which corresponds to the fast damping (Ref. 5), and the other depends on the machine's chromaticity ($\mathrm{d} \ln \nu/\mathrm{d} \ln R_0$).

The first term arises from excitation of the main wave by the transverse betatron motion and, therefore, is always positive. The second corresponds to excitation of the main wave by the longitudinal motion at the ends of the plates. The phase shift which is required for the instability is provided by the betatron frequency dependence on energy. The value of the decrement is determined by the first or second term depending on their ratio and the plate length and also on the type of excitation.

In the case of the vertical betatron excitation ($\omega = m_z\omega_z + \Delta$) the functional dependence of the decrement ($\delta = -Im\omega$) on the machine and beam parameters has the form

$$\delta \sim \frac{N}{\gamma}\frac{l}{2\pi R_0}\left(1 - \frac{8 l_B}{\pi^2 l \nu_z}\frac{\mathrm{d}\omega_z}{\mathrm{d}\omega_s}\right) \tag{I.3}$$

for the case where

$$\frac{l_B}{R_0}\left|\frac{\mathrm{d}\omega_z}{\mathrm{d}\omega_s}\right| \ll 1,$$

and the plate is "long" so that in the extreme relativistic case $l_B < l$, $\gamma^2 l_B \gg l$, with γ the usual relativistic mass parameter, ω_s the revolution frequency and l is the length of the plate. It is seen from (I.3) that when $\mathrm{d}\omega_z/\mathrm{d}\omega_s$ goes to zero, the decrement is determined by the first term, and in general instability occurs when

$$\frac{\mathrm{d}\omega_z}{\mathrm{d}\omega_s} > \frac{\nu_z \pi^2 l}{8 l_B}.$$

For the "short" plate case when

$$l_B \gg l, \gamma^2 l_B \gg l,$$

the expression for the decrement is given by

$$\delta \sim \frac{Nl}{\gamma l_B}\frac{l}{2\pi R_0}\left(1 + 2\frac{\mathrm{d}\ln \nu_z}{\mathrm{d}\ln R_0}\ln\frac{l_B}{l}\right). \tag{I.4}$$

It is evident, that when chromaticity is nonzero, the decrement's sign can be determined by the second term.

Two-dimensional synchrobetatron excitations ($\omega = m_z\omega_z + m_c\omega_c + \Delta$) are also studied in the paper. The characteristic feature of this type of excitation is that the beam participates in both transverse and longitudinal coherent oscillations with multiples m_z and m_c respectively. If we observe betatron oscillation by means of pickup electrodes, we will see that the frequency of the output signal is modulated by the frequency $m_c\omega_c$.

If the plate is longer than the beam ($l_B \ll |m_c| l$), then the decrement of the synchrobetatron excitation is proportional to the quantity

$$\delta \sim \frac{N}{\gamma}\frac{l_B}{\nu_z R_0}\frac{\mathrm{d}\omega_z}{\mathrm{d}\omega_s}\frac{1}{4m_c^2 - 1} \tag{I.5}$$

and the contribution of "fast damping" is negligible.

The expression for the decrement practically does not differ from (I.4) for the short plate.

The case with large chromaticity

$$\left(\frac{l_B}{R_0}\left|\frac{d\omega_z}{d\omega_s}\right| \gg 1\right)$$

may be interesting for strong-focusing machines. In this case the decrement is logarithmically proportional to the quantity

$$\delta \sim -\frac{N}{\gamma}\,\frac{R_0 \ln\left(\frac{l_B}{R_0}\left|m_z \frac{d\omega_z}{d\omega_s}\right|\right)}{l_B \frac{d\omega_z}{d\omega_s}} \qquad (I.6)$$

and one can neglect the "fast damping" contribution. When considering radial and longitudinal excitation, one has to take into account the coupling between these degrees of freedom in the accelerator. Without this coupling the expressions for the decrements of the radial betatron and synchrotron excitations are analogous to (I.3) and (I.6) with substitution of the indices, $z \to r$, and decrements of the synchrotron oscillations are small.

The coupling between the radial and the longitudinal motion leads to the redistribution of the decrements through the dependence of the coherent energy losses on the radial positions of the particles when the line is excited. As a result, radial betatron or synchrotron excitations may become, generally speaking, unstable.

This mechanism can be used for the damping of the synchrotron beam's oscillations. If decrement redistribution is done by matched lines, then for all excitations with multiplicity $|m_c| \leq m_{\max} = l_B/l_\perp$, where $l_\perp$ is the chamber's transverse dimension, decrements of the damping do not depend on m_c. (Oscillations in the separate bands are damped independently.) The decrement maximum is limited by the condition that radial betatron and synchrobetatron excitations are stable. The stability of the beam with smooth equilibrium distribution of amplitudes of the synchrotron oscillations is studied qualitatively in Sections III and IV of the paper.

For the extreme cases of short or long plates the integral equation which we study transforms into an integral equation with symmetrical and positive kernel. This allows one to study the stability of coherent oscillations in general, for arbitrary smooth distributions of the amplitudes of the synchrotron oscillations in the stationary state.

In the last part of the paper the solution of the dispersion equation is studied, taking into account the frequency dispersion of the betatron oscillations.

It follows from the results of the paper, that the use of matched plates can be particularly effective for damping of the main types of oscillations (one-dimensional betatron oscillations and synchrotron oscillations). The simultaneous stability (damping) of radial and axial synchrobetatron oscillations can be achieved if the machine chromaticity is sufficiently small.

I. METHOD

The condition of a beam interacting with an outside system we shall describe by the following equations:

$$\frac{\partial F}{\partial t} + \{H; F\} = 0$$
$$\Delta A - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} = -\frac{4\pi e}{c}\int d^3p\, vF + \frac{1}{c}\nabla\frac{\partial\varphi}{\partial t} \qquad (1)$$

$$\Delta\varphi = -4\pi e\int d^3pF;\ \mathrm{div}\, A = 0. \qquad (2)$$

Here A and φ are the vector and scalar potentials of the fields induced by the beam, which satisfy boundary conditions ($A_t = 0$, $\varphi = 0$) on the electrodes; $\{H;\,\}$ are Poisson brackets:

$$\{H;\,\} = \frac{\partial H}{\partial P}\frac{\partial}{\partial r} - \frac{\partial H}{\partial r}\frac{\partial}{\partial P}.$$

$P = p + e/c(A_0 + A)$ is the canonical momentum, and $p = m_0 v(1 - v^2/c^2)^{-1/2}$ is the kinetic momentum, A_0 is the vector potential of the focusing fields, H is the Hamiltonian describing single-particle motion in the focusing and beam-induced fields:

$$H = c\left[\left(P + \frac{e}{c}[A_0 + A]\right)^2 + m_0^2c^2\right]^{1/2}. \qquad (3)$$

m_0, e are the particle's mass and charge, c is the velocity of light, $F = F(P, r, t)$ is a particle distribution function normalized to the total number of particles in the bunch N:

$$\int d^3p\, d^3rF = \int d\Gamma F = N.$$

We shall study in this paper the stability of the beam's stationary states with respect to small coherent excitations. In the absence of coherent oscillations, the fields acting on the particle are periodic functions of time (with frequency equal to the frequency of revolution ω_s). Particles are oscillating near some equilibrium trajectory. This oscillation can be described in the action-phase variables (I, ψ), which are connected to P, r by some canonical transformation. In the stationary state I^k and $\psi_k - \omega_k t$ are integrals of the motion and in these variables the Hamiltonian depends only on I:

$$H_{st} \to H_{st}(I)$$

$$I^k(P, r, \theta) = \text{const.}$$

$$\dot{\psi}_k = \omega_k(I) = \frac{\partial H_{st}}{\partial I^k}. \qquad (k = 1, 2, 3)$$

Therefore, in the stationary state the distribution function which satisfies the system of equations (1), (2) will depend only on the action variables I.

In the excited state $F = F_{st}(I) + \tilde{F}(I, \psi, t)$ and $(A, \varphi) = (A, \varphi)_{st} + (\tilde{A}, \tilde{\varphi})$.

In order to study the stability of small excitations, the system of equations (1), (2) can be linearized by the deviations from the stationary state $(\tilde{F}, \tilde{A}, \tilde{\varphi})$. In the linear approximation $(H = H_{st} + \tilde{V} = H_{st} - e/c(v\tilde{A}) + e\tilde{\varphi})$. In the variables (I, ψ) the linearized equation for $\tilde{F}$ takes the form:

$$\frac{\partial \tilde{F}}{\partial t} + \omega_k \frac{\partial \tilde{F}}{\partial \psi_k} - \frac{\partial \tilde{V}}{\partial \psi_k}\frac{\partial F_{st}}{\partial I_k} = 0. \qquad (4)$$

Here potential $\tilde{A}$ and $\tilde{\varphi}$ satisfy equation (2), where F on the right side is to be replaced by $\tilde{F}$.

The normal solution of the system (4), (2) has the form;

$$X_\omega(I, \psi, \theta)\exp(-i\omega t)$$

where

$$X_\omega(I, \psi, \theta + 2\pi) = X_\omega(I, \psi + 2\pi, \theta) = X_\omega(I, \psi, \theta).$$

(Here the symbol X denotes any of the quantities $\tilde{F}, \tilde{A}$ or $\tilde{\varphi}$). In the absence of interaction $(N \to 0)$ the spectrum of the normal beam's oscillations

$$(\tilde{F} \sim \exp[-i\omega t + i m_k \psi_k])$$

is $\omega = m_k \omega_k$, m_k-integer.

If the beam's interaction with the induced fields is weak, i.e., weakly disturbs the particle's motion during a time of the order of the period of the oscillations $(\sim 2\pi/\omega_s)$, then the spectrum and the form of the excitation must be close to the undisturbed one. This means that the harmonic (Ref. 5) $F_{\omega_m} \sim \exp(i m_k \psi_k)$ makes the main contribution to the normal excitation F_ω near $m_k\omega_k(\omega = m_k\omega_k + \Delta, |\Delta| \ll \min\{\omega_k\})$ and the influence of the rest of the harmonics can be neglected.

Therefore, in order to determine the spectrum of the excitation in the first order of the interaction one can use the approximate equations:

$$(\omega - m_k\omega_k)F_{\omega_m} = -m_k \frac{\partial F_{st}}{\partial t^k} \tilde{V}_{\omega, m}(I) \qquad (5)$$

$$\Delta A - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} = -\frac{4\pi e}{c}\int d^3p\, v F_{\omega_m} e^{i m_k \psi_k - i\omega t} + \frac{1}{c}\nabla \frac{\partial \tilde{\varphi}}{\partial t} \qquad (6)$$

$$\Delta \tilde{\varphi} = -4\pi e \int d^3p\, F_{\omega_m} e^{i m_k \psi_k - i\omega t}; \ \operatorname{div} \tilde{A} = 0, \qquad (7)$$

where:

$$X_{\omega_m} = \int_0^{2\pi} \frac{d\theta_s}{2\pi} \int_0^{2\pi} \frac{d^3\psi}{(2\pi)^3} X_\omega(I, \psi, \theta) e^{-i m_k \psi_k}$$

The relative error arising in the determination of the shift $\Delta\omega_m = \omega - m_k\omega_k$ is of the order of $(|\Delta\omega_m|/|l\omega_s + p_k\omega_k|)$.

In what follows we will be interested in the effects arising from the interaction of the low-frequency beam's excitations with the main wave's field of the ideally-matched, doubly-coupled waveguide. The rest of the fields in the system in which the beam is at rest have quasistatic character and, therefore, will not be considered in what follows.

The potential of the "main TEM wave" field in an infinite double-coupled waveguide has a form:

$$A(r, t) = c\frac{A_0(r_\perp)}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk Q_k(t) \exp(iky) \qquad (8)$$

where $A_0(r_\perp)$ is proportional to the electric field from the potential difference U_0 between the electrodes:

$$A_0(r_\perp) = \sqrt{cZ_0}\frac{\mathscr{E}(r_\perp)}{U_0}; \ \Delta_\perp \mathscr{E}(r_\perp) = 0$$

Z_0 is the wave resistance, c-velocity of light.

In reality the beam is interacting with the section of the waveguide, the ends of which are loaded by the characteristic impedance (Z_0). For low-frequency oscillations $(\omega l_\perp/c \ll 1)$ the waveguide potential can be represented by the expression (8), where $A_0(r)$ is the real electrostatic field which falls off exponentially in a distance of the order of the transverse size of the chamber $(l_\perp)$ when going along the chamber from the waveguide terminals. It is essential that near the edges the electric field of the "main wave" has a longitudinal component. Hence it is possible to excite it by the longitudinal particle motion.

The quantity $Q_k(t)$ from (8) satisfies the equation

$$\ddot{Q}_k + c^2k^2Q_k = e\int \mathrm{d}\Gamma(vA_0e^{-iky})F_{\omega m}\exp(im_k\psi_k - i\omega t). \quad (9)$$

The solution of this equation can be written in the form

$$Q_k(t) = \sum_{n=-\infty}^{\infty} Q_{kn}\exp(-in\omega_s t - i\omega t) \quad (10)$$

and

$$Q_{kn} = e\frac{\int \mathrm{d}\Gamma(vA_0e^{iky})^*_{mn}F_{\omega m}(I)}{c^2k^2 - (\omega + n\omega_s)^2}. \quad (11)$$

Substituting (10) into (8) we get from (6) an integral equation for $F_{\omega m}$

$$\Delta\omega_m F_{\omega m} = e^2 m_k \frac{\partial F_{st}}{\partial I_k}\sum_n \int_{-\infty}^{\infty}\frac{\mathrm{d}k(vA_0e^{iky})_{mn}}{c^2k^2 - (\omega + n\omega_s)^2} \times \int \mathrm{d}\Gamma(vA_0e^{iky})^*_{mn}F_{\omega m}. \quad (12)$$

We will need formulas of the transformation from the particle's coordinate and momentum (r, P) to the action-phase variables (I, ψ) for further calculations. Far from the machine's resonances one can neglect the influence of the stationary induced fields on the particles' motion. The formulas have the usual form:

$$r = r_b + R_0\Psi(\theta)\frac{\Delta p_\parallel}{p_s};$$

$$\Delta p = p - p_s = \mu_c\dot{\varphi}_c; \; p_\perp = \frac{p_s}{R_0}r'_\perp$$

$$(r_b, z) = \frac{a_{r,z}}{2|f_{r,z}|_{\max}} \times \left[f_{r,z}(\theta)\exp\left(i\varphi_{r,z} + i\varphi_c\frac{\mathrm{d}\omega_{r,z}}{\mathrm{d}\omega_s}\right) + \text{c.c.}\right]$$

$$\frac{y}{R_0} = \theta = \theta_s + \varphi_c + \theta_b; \quad (13)$$

$$\theta_b = \frac{1}{R_0}(\Psi(\theta)r'_b - r_b\Psi'(\theta));$$

$$\varphi_c = \varphi\sin\psi_c; \; \dot{\psi}_k = \omega_k(p_s), \; k = r, z, c;$$

$$a^2_{r,z} = 8|f_{r,z}|^2\frac{I_{r,z}R_0}{p_s}.$$

Here index s indicates the quantities corresponding to the particle in the equilibrium orbit, $2\pi R_0$ is the perimeter of the machine, $\omega_0(p)$—the frequency of revolution of the particle $\mu_c^{-1} = (\mathrm{d}\omega_0/\mathrm{d}p)_s$—the mass of the synchronous motion, $f_{r,z}(\theta)$ are Floquet functions, normalized by the conditions

$$f_{r,z}(f'_{r,z} + i\nu_{r,z}f_{r,z})^* - \text{c.c.} = -2i,$$

where

$\nu_k = (\omega_k/\omega_0)_s$ is the number of betatron oscillations per revolution.

$|f_{r,z}|_{\max}$ is the maximum value of the Floquet function on the element of the periodicity.

$\Psi(\theta)$ is a forced, periodic solution of the equation

$$\left(\Psi'' + (1 - n(\theta))\frac{R_0^2}{R^2(\theta)}\Psi(\theta) = \frac{R_0}{R(\theta)}\right)$$

where

$n(\theta)$ is the index of the leading field decrease.

$R(\theta)$ is the radius of curvature of the orbit.

The modulation of the phases of the transverse oscillations is connected to the energy dependence of the frequencies ω_z and ω_r:

$$\omega_k(E) = \omega_0(E)\nu_k(E), \; k = r, z.$$

The calculation of the harmonics entering (12) can be simplified by the use of the fact that the field $\mathscr{E}(r)$ is the potential one: $\mathscr{E}(r) = -(U_0/\sqrt{cZ_0})\nabla U(r)$.

By the definition of the Fourier harmonic we have:

$$(vA_0e^{iky})_{mn} = \left(\overline{\frac{\mathrm{d}U(r(t))}{\mathrm{d}t}\exp[ikR_0\theta(t) - im_k\psi_k(t) - in\theta_s]}\right) \quad (14)$$

where the line means time averaging along the trajectory of the particle. Time integrating (14) by parts, we can rewrite this expression in the following form:

$$(vA_0e^{iky})_{mn} = -i\omega_s[kR_0 - n - m_k\nu_k]V_{mn}(I, k) \quad (15)$$

$$V_{mn}(I, k) = \int_0^{2\pi} \frac{d^3\psi}{(2\pi)^3} \int_{-\pi}^{\pi} \frac{d\theta_s}{2\pi} U(r_\perp(\theta), \theta) \times \exp[ikR_0\theta - im_k\psi_k - in\theta_s]. \quad (15a)$$

For the low-frequency field excitations ($kl_\perp \ll 1$) the azimuthal dependence U can be approximated by the expression

$$U(r_\perp, \theta) = \begin{cases} U(r_\perp); & |\theta| \le \dfrac{l}{2R_0} \\ 0; & |\theta| \ge \dfrac{l}{2R_0}, \end{cases} \quad (16)$$

where l is the plate's length and it is assumed that θ is measured from the center of the waveguide.

Then the harmonic V_{mn} can be calculated by expanding (15a) in a Taylor series in powers of the transverse oscillation amplitudes ($I_\perp$). The expression obtained is rather cumbersome, so we shall give it here when it is necessary for the particular values of m.

In formula (15) the term which is proportional to $(kR_0 - n)$, describes the beam interacting with the edge field, and the term, which is proportional to $m_k v_k$ describes the interaction over the length of the plate.

Note that when the beam is interacting with the system "without memory," the sum over n in (12) depends weakly on the exact value of ω and, therefore, to an accuracy of the order $|\Delta\omega_m|/\omega_k$ the frequency on the right side of (12) can be replaced by $m_k\omega_k$. We shall use the summation formula

$$\sum_{n=-\infty}^{\infty} B_n = \sum_{q=-\infty}^{\infty} \int_{-\infty}^{\infty} dn B(n) \exp(2\pi i q n) \quad (17)$$

when calculating the sum in n. Directly substituting (17) into (12) one can see that all the terms in the sum with $q \ne 0$ vanish since the expressions under the n-integrals do not have singularities in the plane of the complex variable n.

Physically this corresponds to complete damping of the induced fields during one period of the beam's revolution. Considering what was said, let us rewrite the integral equation for F_{ω_m} in the form

$$\Delta\omega_m F_{\omega_m} = e^2 m_k \frac{\partial F_{st}}{\partial I_k} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dn \times \frac{(kR_0 - n - m_k v_k)^2 V_{mn}}{c^2k^2 - (\omega_k m_k + n\omega_s + i\varepsilon)^2} \times \int d\Gamma V^*_{mn} F_{\omega_m}, \quad (18)$$

where $\Delta\omega_m = \omega - m_k\omega_k$, $\varepsilon \to 0$—determines the way of by-passing.

The function V_{mn} can be represented in the form

$$V_{mn}(a_r, a_z, \varphi) = \frac{a_r^{|m_r|} a_z^{|m_z|}}{|m_r|!\,|m_z|!} g_{mn}(\varphi, k) + O(I_{\perp 0}^{|m_r| + |m_z| + 1/2}),$$

where

$$g_{mn}(\varphi, k) = \left. \frac{\partial^{|m_z| + |m_r|} V_{mn}(I, k)}{\partial a_r^{|m_r|} \partial a_z^{|m_r|}} \right|_{a_r, a_z = 0}$$

depends only on the amplitude of the synchrotron oscillations φ: $I_{\perp 0}^{1/2}$—maximal cross size of the beam.

In this paper we shall only consider the cases when, in the stationary state, the betatron oscillations are nonlinear, and the nonlinearity of the synchrotron oscillations can be neglected. Then, with the help of the substitution

$$F_{\omega_m}(a_z, a_r, \varphi) = a_z^{|m_z|} a_r^{|m_r|} \chi_m(\varphi) [\omega - m_k\omega_k(I)]^{-1}$$

leaving in (18) lower powers of the transverse oscillation amplitudes, we will get the equation for $\chi_m(\varphi)$:

$$\Omega_m \chi_m(\varphi) = \xi_m(\varphi) \int_{-\infty}^{\infty} dx\, dn \times \frac{(x - n - m_k v_k)^2 g_{mn}(\varphi, x)}{x^2 - \beta^2(n + m_k v_k + i\varepsilon)^2} \times \int_0^{\infty} d\varphi' \varphi' \chi_m(\varphi') g_{mn}(\varphi', x) \quad (18a)$$

The notations:

$$\Omega_m^{-1} = \frac{1}{A_{m_\perp}} \left\langle m_k \frac{\partial}{\partial I^k} \frac{a_r^{2|m_r|} a_z^{2|m_z|}}{\omega - m_k\omega_k(I)} \right\rangle \quad (18b)$$

$$A_{m_\perp} = \left\langle m_k \frac{\partial}{\partial I^k} [a_r^{2|m_r|} a_z^{2|m_z|}] \right\rangle$$

$$\xi_m(\varphi) = \begin{cases} -\dfrac{Ne^2 m_c \omega_s^2 R_0}{c^2} \dfrac{\partial\rho}{\partial I_c}, & m_r = m_z = 0 \\ -\dfrac{Ne^2\beta^2\rho(\varphi)}{R_0} \dfrac{A_{m_\perp}}{(|m_r|!\,|m_z|!)^2}, & m_z, m_r \ne 0, \end{cases}$$

$\beta = v_s/c$ were introduced here.

Brackets $\langle\ \ \rangle$ mean averaging $F_0(I_\perp)$ (assuming that $F_{st}(I_r, I_z, I_c)$ can be factored)

$$F_{st}(I_\perp, I_c) = F_0(I_\perp)\rho(I_c).$$

In the absence of frequency dispersion (in the stationary state, oscillations are linear), the spectrum of the normal collective excitations coincides with the eigenvalue spectrum of the equation (18a):

$$\omega - m_k\omega_k = \Omega_m.$$

If the oscillations are nonlinear in the stationary state, the frequencies of the normal oscillations ω can be found from the dispersion equation (18b) after determination of the eigenvalues of the equation (18a).

The solution of the integral equation (18a) with some smooth distribution $\rho(I_c)$, requires, in general, application of numerical methods. We will consider a number of cases in which one can follow the qualitative dependence of the normal oscillation spectrum on the characteristic parameters of the problem (length of plates, length of bunch, $d\omega_k/d\omega_s$, etc.).

II. MODEL SOLUTIONS

Let us consider for simplicity only those excitations for which the collective betatron oscillations in the beam are one-dimensional, i.e., $m_r \cdot m_z = 0$, $m_r^2 + m_z^2 > 0$.

II-1. *Axial and Axial-Longitudinal Excitations*

First let $m_r = 0$ (axial-longitudinal excitations). The formula for $\Delta\omega_m$ can be easily obtained for the model distribution

$$\rho(\varphi) = \delta(\varphi^2 - \varphi_0^2) \tag{19}$$

where φ_0 is connected to the "length" of the beam by $l_B = 2R_0\varphi_0$.

In the case under consideration the function $g_{mn}(\varphi, k)$ is equal to

$$g_{mn} = \frac{1}{(2\pi)^{3/2}} J_{m_c}\left(\varphi\left[n + m_z \frac{d\omega_z}{d\omega_s}\right]\right) \times \frac{B_n^{(z)}(u)}{2^{|m_z|}} \frac{\partial^{|m_z|}U}{\partial_z^{|m_z|}}\bigg|_{r_\perp = 0} \tag{20}$$

$$B_n^{(z)}(u) = \int_{-l/2R_0}^{l/2R_0} d\theta \bar{f}_z(\theta)^{m_z} \exp[i\theta(u - n)], \tag{20a}$$

where the notation $\bar{f}_z(\theta) = f_z(\theta)/|f_z|_{max}$ is introduced.

Substituting (19) and (20) into (18a) and considering the ultrarelativistic case ($\gamma^2 l_B \gg l$, $\gamma = E_s/m_0c^2$), we will find decrements ($\delta = -\mathrm{Im}\,\omega$) of the axial-longitudinal excitations:

$$\delta_m = N\delta_0 \int_{-\infty}^{\infty} dn J_{m_c}^2\left(\varphi_0\left[n + m_z \frac{d\omega_z}{d\omega_s}\right]\right) \times (m_z\nu_z + n)|B_n^{(z)}(-n - m_z\nu_z)|^2. \tag{21}$$

Here $J_m(x)$ is the Bessel function of order m,

$$\delta_0 = \frac{r_0 c |f_z|_{max}^2 \operatorname{sign}(m_z)}{2\pi^2\gamma\Gamma^2(|m_z|)} \times \left\langle\left(\frac{a_z}{2}\right)^{2|m_z|-2}\right\rangle\left(\frac{\partial^{|m_z|}U}{\partial z^{|m_z|}}\bigg|_{r_\perp=0}\right)^2$$

$r_0 = e^2/m_0c^2$ is the classical radius of the particle. Formula (21) can be simplified in two extreme cases.

a) $l_B \ll |m_c|l$—short bunch. After integration over n we obtain

$$\delta = N\delta_0 m_z\left(\frac{\pi l}{2R_0}\Phi_1 + \frac{4l_B}{\pi R_0} \times \left[\Phi_2 - \frac{d\nu_z}{d\ln R_0}\Phi_3\right]\frac{1}{4m_c^2 - 1}\right). \tag{22}$$

Factors Φ_1, Φ_2, and Φ_3 are respectively

$$\Phi_1 = J_{m_c}^2\left(\frac{m_z l_B}{2}\frac{d\nu_z}{dR_0}\right)\frac{R_0}{e}\int_{\theta_1}^{\theta_2} d\theta \bar{f}_z^{2|m_z|-2}(\theta)|f_z|_{max}^{-2};$$

$$\Phi_2 = |f_z|_{max}^2 = \begin{cases} \dfrac{3|m_z| - 2}{2|m_z| - 2}|\bar{f}_z(\theta_2)|^{2|m_z|-2} + \dfrac{|m_z| - 2}{2|m_z| - 2}|\bar{f}_z(\theta_1)|^{2|m_z|-2}, & |m_z| > 1 \\ 2 + \ln\left(\dfrac{|f_z(\theta_2)|}{|f_z(\theta_1)|}\right), & |m_z| > 1 \end{cases}$$

$$\Phi_3 = |\bar{f}_z(\theta_2)|^{2|m_z|} + |\bar{f}_z(\theta_1)|^{2|m_z|},$$

where $\theta_1 = -l/2R_0$, $\theta_2 = l/2R_0$—coordinates of the plate's ends.

The formula (22) differs from the corresponding formula from Ref. 4 by the presence of terms proportional to Φ_1 and Φ_2.

The first term in (22), which is proportional to the plate's length, is the decrement of the damping of the coherent beam's motion due to the energy radiation into the matched line arising from the transverse beam motion. Note that this term is always positive and corresponds to the decrement of the "fast damping" obtained previously in Ref. 5. The second term, which does not depend on the plate's length, corresponds to the so-called "head-tail" effect. The sign of this term depends on the type of the excitation and on the sign and value of the quantity $d\nu_z/d \ln R_0$. Physically it appears due to the excitation of the (TEM) wave by the longitudinal motion near the edges of the plate, where the electric field of the "main wave" has a longitudinal component.

If $m_c \neq 0$, factor Φ_1 is proportional to the quantity

$$\left(\frac{m_z l_B}{2}\frac{d\nu_z}{dR_0}\right)^{2|m_c|},$$

therefore, synchrobetatron excitations may appear to be unstable. However, an excitation with $cm_c = 0$ can always be made stable by choosing a sufficiently long plate.

Let us note specially, that the value of the "edge" terms is proportional to $2|m_z|$—the extent to which Floquet's function is modulated on the line's edges. This circumstance can be of major importance for choosing a plate location in a machine with large beats of the Floquet function (machines with small values of the β-function).

In the axially symmetrical case,

$$\frac{d\nu_z}{d\ln R_0} = \frac{n(1+n)}{2\nu_z}; f_z = \nu_z^{-1/2}; 0 \leq n < 1$$

if the leading field is linear (here n is the field decrease index). The stability condition has the form:

$$\begin{aligned} &1)\ m_c = 0; \frac{8 l_B}{\pi^2 l \nu_z}\frac{d\omega_z}{d\omega_s} = \frac{4 l_B}{\pi^2 l}(1-n) < 1 \\ &2)\ m_c \neq 0; \frac{1}{\nu_z}\frac{d\omega_z}{d\omega_s} = \frac{1-n}{2} > 0, \end{aligned} \tag{23}$$

i.e., in this case axial-longitudinal excitations are stable. More general stability conditions (for arbitrary focusing) can be obtained directly from (22).

Let us in addition present the formula for the decrement for the most important type of oscillations, where $|m_z| = 1$:

$$\begin{aligned} \delta_m = \frac{N r_0 c}{2\gamma}\left(\frac{\partial U}{\partial_z}\bigg|_{r_\perp = 0}\right)^2 \Bigg\{ &\frac{l}{2\pi R_0} J_{m_c}^2\left(\frac{l}{2}\frac{d\nu_z}{dR_0}\right) \\ &+ \frac{2 l_B}{\pi^3 R^0}\left[2 + \ln\frac{|f_z(\theta_2)|}{|f_z(\theta_1)|} - \frac{d\nu_z}{d\ln R_0} \right. \\ &\times \left.(|\bar{f}_z(\theta_1)|^2 + |f_z(\theta_2)|^2)\right]\frac{1}{4m_c^2 - 1}\Bigg\}. \end{aligned} \tag{24}$$

If the machine's chromaticity is high

$$\frac{l_B}{R_0}\left|m_z\frac{d\omega_z}{d\omega_s}\right| \gg 1$$

(as is the case in strong-focusing machines), then the decrement of the short bunch can be estimated with logarithmic accuracy by the formula

$$\delta = -\frac{N\delta_0 R_0}{\pi l_B m_z \dfrac{d\omega_z}{d\omega_s}} \ln\left(\frac{l_B}{R_0}\left|m_z\frac{d\omega_z}{d\omega_s}\right|\right). \tag{22a}$$

In order to obtain (22a) we assumed that the leading field is axially uniform.

Note, that the decrement δ is inversely proportional to the length of the bunch l_B and to the quantity $d\omega_z/d\omega_s$, which is connected to the chromaticity of the machine (the l_B dependence of the numerator of (22a) is a slow, logarithmic one).

b) $l_B \gg |m_c|$—long bunch. Let us assume for the simplicity, that the leading field is axially uniform. Then we will get from (21)

$$\begin{aligned} \delta = N\delta_0 \frac{l}{2\pi R_0}\frac{m_z \nu_z l}{l_B} \\ \times\left(1 + 2\frac{d\ln\nu_z}{d\ln R_0}\ln\left[\frac{l_B}{l(|m_z|+1)}\right]\right). \end{aligned} \tag{25}$$

It is evident that instability can arise when the machine's chromaticity $(d\ln\nu_z/d\ln R_0)$ is negative. We note that the decrement in (25) is inversely proportional to the length of the bunch l_B. It is so because $l_B \gg |m_c| l$, and the main contribution to the integral (21) comes from the harmonic interval $\Delta n \sim R_0/l_B$. The contribution of each harmonic from this interval is of the order of $(l/2R_0)^2$, so $\delta \sim l^2/l_B R_0$.

II-2. Radial-Longitudinal Excitations ($m_z = 0$)

Collective excitations of this type can differ considerably from the axial-longitudinal ones because of the relation between the radial and longitudinal motion of particles in storage rings. Hence one has to take into account the modulation of the azimuth by betatron motion when calculating $(g_{mn}(\varphi, k)(m = \{m_r, m_c, 0\})$. The function $g_{mn}(\varphi, k)$ can be represented by a sum

$$g_{mn}(\varphi, k) = g_{mn}^{(1)}(\varphi, k) + g_{mn}^{(2)}(\varphi, k). \tag{26}$$

Here $g_{mn}^{(1)}$ can be obtained from (20) by changing index z into r, and $g_{mn}^{(2)}(\varphi, k)$, which describes the influence of the radial-longitudinal coupling, is equal to:

$$g_{mn}^{(2)} = \frac{i|m_r|u}{(2\pi)^{3/2}} B_n^{(2)}(u) J_{m_c}\left(\varphi\left[n + m_r \frac{d\omega_r}{d\omega_s}\right]\right) \times \left.\frac{\partial^{|m_r|-1} U}{\partial r^{|m_r|-1}}\right|_{r_\perp = 0}, \tag{27}$$

where

$$B_n^{(2)}(u) = \int_{\theta_1}^{\theta_2} d\theta \bar{f}_r^{m_r - 1} \times [\Psi(\theta)(\bar{f}'_r + i\nu_r \bar{f}_r) - \bar{f}_r(\theta)\Psi'(\theta)]\exp(i\theta[u - n]).$$

After substitution of (27) and (19) into (18a) we obtain the formula for the decrements ($\delta = \delta_m^{(1)} - \delta_m^{(2)}$ in which $\delta_m^{(1)}$ is obtained from (21) by substitution for the index z by r, and $\delta_m^{(2)}$ is equal to

$$\delta_m^{(2)} = N\delta'_0 \int_{-\infty}^{\infty} n\, dn(m_r \nu_r + n) J_m^2 \times \left.\left(\varphi_0\left[n + m_r \frac{d\omega_r}{d\omega_s}\right]\right) \mathrm{Im}[B_n^{(r)}(u) B_n^{(2)*}]\right|_{u = -n - m_r \nu_r}, \tag{28}$$

where

$$\delta'_0 = \frac{|m_r| r_0 c |f_r|^2_{\max}}{2\pi_\gamma^2 R_0 \Gamma^2(|m_r|)} \left\langle \left(\frac{a_r}{2}\right)^{2|m_r| - 2} \right\rangle \times \left.\frac{\partial}{\partial r}\left(\frac{\partial^{|m_r|-1} U}{\partial r^{|m_r|-1}}\right)^2\right|_{r_\perp = 0}$$

The n-integral in formula (28) is logarithmically divergent. However, in the real case, where the length of the edge field fall-off is finite, the integrand in (28) is multiplied by a factor which cuts off the integral at n of the order of $n_{\max} \sim R_0/l_\perp$. For an order of magnitude estimate of the decrement there is no need to know this factor explicitly. In particular one can simply replace the infinite limits of integration in (28) by finite ones with $|n| < n_{\max}$.

Let us present decrement formulas in two limiting cases:

a) $l_B < |m_c| l$, $$\delta_m^{(2)} = \frac{N\delta'_0}{\pi l_B} \Phi_4 \ln\left[\frac{l_B}{l_\perp(|m_c| + 1)}\right] \tag{29}$$

where

$$|f_r|^2_{\max} \Phi_4 = [\Psi(\theta_2)|\bar{f}_r(\theta_2)|^{2|m_r| - 2} + \Psi(\theta_1)|f_r(\theta_1)|^{2|m_r| - 2}]/2.$$

Note, that $\delta_m^{(2)}$ does not depend on $d\omega_r/d\omega_s$ and it decreases the decrement of the radial oscillations when

$$\Phi_4 \left.\frac{\partial}{\partial r}\left(\frac{\partial^{|m_r|-1} U}{\partial r^{|m_r|-1}}\right)^2\right|_{r_\perp = 0} > 0. \tag{30}$$

If $\delta_m^{(1)}$ is positive, then we see by comparing (29) and (22) that the radial oscillations ($m_c = 0$) can become unstable when

$$\bar{\Psi} \frac{l_\perp R_0}{l l_B} > 1$$

and radial-phase ones when

$$\bar{\Psi} \frac{l_\perp R_0}{l_B^2} > 1.$$

Here the increments of the normal oscillations are inversely proportional to the bunch length l_B. Let us also present the expression for the decrement of the most important type of oscillations $|m_r| = 1$

$$\delta_m^{(2)} = \frac{N r_0 c}{4\pi^3 \gamma} \frac{\Psi(\theta_1) + \Psi(\theta_2)}{l_B} \times \left(\left.\frac{\partial u^2}{\partial r}\right|_{r_\perp = 0}\right) \ln\left[\frac{l_B}{l_\perp(|m_c| + 1)}\right] \tag{31}$$

b) $l_B \gg |m_c| l$. For the axially-symmetrical machine ($\Psi \equiv \bar{\Psi} = \text{const}$)$\delta_m^{(2)}$ has the form

$$\delta_m^{(2)} = \frac{N r_0 c |m_r|}{2\pi^3 \gamma \Gamma^2(|m_r|)} \frac{\bar{\Psi}}{l_B} \ln \frac{l}{l_\perp} \left.\frac{\partial}{\partial r}\left(\frac{\partial^{|m_r|-1} U}{\partial r^{|m_r|-1}}\right)^2\right|_{r_\perp = 0}. \tag{32}$$

It is necessary to note that the appearance of the large logarithms in the formulas (25), (29), (32) is the specific feature of the distribution (19). This is connected to the fact that decrements in the formulas (25), (29), (32) determine sums of the decrements of the normal beam excitations for the arbitrary, smooth distributions $\rho(\varphi)$. When the numbers of the excitation are large, separate decrements decrease slowly with the growth of the excitation number $p(\delta(p) \sim 1/p)$, so that the sum is logarithmically large.

II-3. Synchrotron Excitation

For excitation with $\omega \simeq m_c \omega_c$; $m_r = m_z = 0$ equation (18a) can be written to an accuracy of the order of $(a_\perp/l_\perp)$ ($a_\perp$ is the transverse size of the beam) as

$$\lambda\chi_m(y) = \int_{-\infty}^{\infty} \mathrm{d}x L\left(\frac{x}{\varphi_0}\right) \times \left(U_0^2 + \frac{R_0\varphi_0\mu_c m_c}{p_s x}\,\overline{\Psi}\,\frac{\partial U_0^2}{\partial r}\right) J_{m_c}(xy) \times \int_0^{\infty} \mathrm{d}y' \left(\frac{\mathrm{d}q(y')}{\mathrm{d}y'}\right) J_{m_c}(xy')\chi_m(y') \quad (33)$$

where

$$\lambda = \frac{(\omega - m_c\omega_c)\mu_c\omega_c 2\pi^2 R_0^2\varphi_0^3}{Ne^2 m_c}, \; x = n\varphi_0$$

and

$$q\left(\frac{\varphi}{\varphi_0}\right) = \varphi_0^2\rho\left(\frac{\varphi}{\varphi_0}\right); \; \int_0^{\infty} \mathrm{d}y\, y q(y) = 1.$$

Here we use the assumption that the leading field is axially uniform and that the bunch moves parallel to the waveguide axis.

Because of the smallness of the synchrotron frequency ($\omega_c \ll \omega_0$) the function L can be represented in the form:

$$L(x) = \frac{\varphi_0}{4}\,\frac{\sin\left(\dfrac{4lx}{l_B}\right)}{x} + \frac{i\varphi_0}{2}\,\frac{\sin^2\left(\dfrac{2lx}{l_B}\right)}{x}; \; l_B = 2R_0\varphi_0$$

In this section we will obtain the solution (33) for the case when $q(y)$ has a form of a "step":

$$q(y) = \begin{cases} 2, & 0 \le y \le 1 \\ 0, & y > 1 \end{cases} \quad (34)$$

From (33) we immediately get the dispersion equation:

$$\lambda = -2\int_{-\infty}^{\infty} \mathrm{d}x L\left(\frac{x}{\varphi_0}\right) \times \left[U_0^2 + \frac{R_0\varphi_0\mu_c m_c\omega_c}{p_s x}\,\overline{\Psi}\,\frac{\partial U_0^2}{\partial r}\right] J_{m_c}^2(x). \quad (35)$$

Im $L(x)$ is an odd function of x. Hence the first term in (35), which is due to the full energy losses, contributes only to the real parts of the frequency shift. The second term, which is proportional to the gradient of the losses, determines the decrements of the excitations, which are equal to

$$\delta_m = \frac{2Nr_0 c}{\gamma\pi^2}\,\frac{m_c^2}{l_B}\,\overline{\Psi}\,\frac{\partial U_0^2}{\partial r}\int_0^{\infty}\frac{\mathrm{d}x}{x^2}\,J_{m_c}^2(x)\sin^2\left(\frac{2lx}{l_B}\right). \quad (36)$$

The last expression can be significantly simplified in two limiting cases:

a) $l_B \ll |m_c|l$—the "long" plate. If this is true, then the sine squared in (36) is oscillating rapidly and can be replaced by the average value:

$$\delta_m \cong \frac{Nr_0 c}{\gamma\pi^3}\,\frac{\overline{\Psi}}{l_B}\,\frac{\partial U_0^2}{\partial r}. \quad (37)$$

The condition for damping of the coherent synchrotron oscillations is

$$\overline{\Psi}\,\frac{\partial U_0^2}{\partial r} > 0. \quad (38)$$

This means that plates have to be located on the outer side of the equilibrium orbit when $\overline{\Psi} > 0$.

The clear physical meaning of this condition is that the modulation of the coherent losses should be such that when energy increases the losses decrease. Let us especially point out one notable peculiarity of this damping method: the decrement value does not depend on the multipole number for all the excitations $|m_c| < l_B/l_\perp$. (The limitation on m_c appears because one has to take into account excitation of the other wave types when $|m_c| \gtrsim l_B/l_\perp$. The explanation of this fact is that the length of the radiation formation is equal to the "length" of the edge

$$l_p \sim l_\perp \ll \frac{l_B}{|m_c|},$$

i.e., separate "bunches" radiate independently.)

b) $l_B \gg |m_c|l$—the "short" plate. In this case the region $|m_c| < x < l_B/l$ gives the main contribution to the integral in (36). Then with logarithmic precision we will get

$$\delta_m = \frac{8Nr_0 c}{\gamma\pi^3}\left(\frac{m_c l}{l_B}\right)^2 \frac{\overline{\Psi}}{l_B}\,\frac{\partial U_0^2}{\partial r}\ln\left(\frac{l_B}{|m_c|l}\right). \quad (39)$$

The condition for stability of the excitations is given by the inequality (38), as in the case of the short beam.

III. THE QUALITATIVE PICTURE OF THE EXCITATION SPECTRUM FOR SMOOTH $\rho(\varphi)$.

We will consider in this section a number of limiting cases when the equation (18a) can be transformed into an equation with a real symmetrical kernel. The eigenvalue spectra of these equations are well known (see Ref. 8, for example), which makes it possible to examine the stability of the excitations for the relatively wide class of smooth functions $\rho(\varphi)$.

Let us assume for simplicity that the leading field is axially symmetrical and that $\rho(\varphi)$ depends only on one parameter—the "width" φ_0, i.e.,

$$\rho(\varphi) = \frac{q\left(\dfrac{\varphi}{\varphi_0}\right)}{\varphi_0^2}; \quad \int_0^\infty \mathrm{d}y\, yq(y) = 1. \tag{40}$$

Let us put $\overline{\Psi} = 0$ for the time being. Then equation (18a) can be written in the form

$$\lambda\chi_m(y) = \int_0^\infty \mathrm{d}y'\, y'q(y')K_\perp\left(\frac{y}{y'}\right)\chi_m(y'), \tag{41}$$

where

$$\lambda = -\frac{\omega - m_k\omega_k}{N\delta_1}\varphi_0;$$

$$\delta_1 = \frac{r_0 c}{2\pi^2\gamma\Gamma^2(|m_r|)\Gamma^2(|m_z|)} \times \left\langle\left(\frac{a_z}{2}\right)^{2|m_z|-2}\left(\frac{a_r}{2}\right)^{2|m_r|-2} \times \left[\frac{\operatorname{sign}(m_r)}{v_r}\left(\frac{a_z}{m_z}\right)^2 + \frac{\operatorname{sign}(m_z)}{v_z}\left(\frac{a_r}{m_r}\right)^2\right]\right\rangle \times \left(\left.\frac{\partial^{|m_r|+|m_z|}U}{\partial r^{|m_r|}\,\partial z^{|m_z|}}\right|_{r_\perp=0}\right)^2;$$

$$K_\perp\left(\frac{y}{y'}\right) = \int_{-\infty}^\infty \mathrm{d}x L_\perp\left(\frac{x}{\varphi_0} - m_k\frac{\mathrm{d}\omega_k}{\mathrm{d}\omega_s}\right)J_{m_c}(xy)J_{m_c}(xy')$$

and the function $L_\perp(x)$ is equal to

$$L_\perp = -\frac{l}{2R_0}\frac{m_k v_k}{2x + m_k v_k} - \frac{x\sin\dfrac{l}{R_0}(2x + m_k v_k)}{(2x + m_k v_k)^2} + i\frac{2(x + m_k v_k)\sin^2\dfrac{l}{2R}(2x + m_k v_k)}{(2x + m_k v_k)^2}. \tag{42}$$

It is evident that the kernel $K_\perp(y/y')$ is symmetrical: $K_\perp(y/y') = K_\perp(y'/y)$.

III-1. Betatron Excitations ($m_c = 0$).

Let us first consider one-dimensional, say axial, excitation of a short bunch $l_B \ll l$. Neglecting quantities of the order

$$\frac{l_B}{l}\left|\frac{\mathrm{d}v_z}{\mathrm{d}\ln R_0}\right| \ll 1, \frac{l_B v_z}{l} \ll 1$$

let us rewrite $K_\perp(y/y')$ in the form

$$K_\perp\left(\frac{y}{y'}\right) = i\frac{\varphi_0 m_z v_z l}{R_0}k\left(\frac{y}{y'}\right);$$

$$k\left(\frac{y}{y'}\right) = \int_0^\infty \frac{\mathrm{d}x}{x^2}\sin^2 x\, J_0\left(\frac{xyl_B}{2l}\right)J_0\left(\frac{xy'l_B}{2l}\right).$$

Then (41) takes the form

$$\Lambda_m\chi_m = \int_0^\infty \mathrm{d}y'y'q(y')k\left(\frac{y}{y'}\right)\chi_m(y');$$

$$\Lambda_m = -\frac{i\lambda R_0}{m_z v_z l\varphi_0},$$

where the function $q(y)$ is positive by definition. All eigenvalues of this equation are positive, as is shown in the Appendix.

Therefore in this approximation the decrements of one-dimensional excitation, which are equal to

$$\delta = \Lambda\frac{Nr_0 c|m_z|}{\pi\gamma\Gamma^2(|m_z|)}\left(\frac{l}{2\pi R_0}\right) \times \left\langle\left(\frac{a_z}{2}\right)^{2|m_z|-2}\right\rangle\left(\left.\frac{\partial^{|m_z|}U}{\partial z^{|m_z|}}\right|_{r_\perp=0}\right)^2$$

are positive. The eigenvalues satisfy the inequality $\Lambda \le \pi/2$ (see Appendix A6). Hence the decrements $\delta < \delta_0$, where δ_0 is the decrement of the one-dimensional betatron excitations obtained in Ref. 5.

When $l_B/l|\mathrm{d}\omega_z/\mathrm{d}\omega_s|$ is of the order of unity the numerical solution of (41) is needed for examination of the excitational stability.

In case of the two-dimensional betatron excitations ($|m_r m_z| > 0$) the coherent oscillations can become unstable. The decrements of the excitations are:

$$\delta = \Lambda' N\delta_1(m_r v_r + m_z v_z)\frac{l}{R_0}, \Lambda' > 0.$$

From here we immediately get the condition for stability of the oscillations

$$(m_z v_z + m_r v_r)\left\langle\left(\frac{a_z}{2}\right)^{2|m_z|-2}\left(\frac{a_r}{2}\right)^{2|m_r|-2} \times \left[\frac{\operatorname{sign}(m_z)}{v_z}\left(\frac{a_r}{m_r}\right)^2 + \frac{\operatorname{sign}(m_r)}{v_r}\left(\frac{a_z}{m_z}\right)^2\right]\right\rangle > 0$$

which coincides with the one obtained in Ref. 5.

III-2. Axial-longitudinal excitations ($m_r = 0$).

a) The short bunch $l_B \ll |m_c| l$. In this case to accuracy of the order of $l_B/2|\mathrm{d}\nu_z/\mathrm{d}R_0| \ll 1$, the kernel $K_\perp(y/y')$ is:

$$K_\perp \cong \left[\frac{2l}{R_0}\left(\frac{\mathrm{d}\nu_z}{\mathrm{d}\ln R_0} - \frac{\nu_z}{2}\right) + \frac{i}{2}\frac{\mathrm{d}\omega_z}{\mathrm{d}\omega_s}\right] \times m_z \varphi_0^2 \int_0^\infty \frac{\mathrm{d}x}{x^2} J_{m_c}(xy) J_{m_c}(xy').$$

Therefore, the decrements of the excitations can be written in the form

$$\delta_1 = \Lambda \frac{N r_0 c |m_z| \varphi_0}{4\pi^2 \gamma \Gamma^2(|m_z|)} \frac{\mathrm{d}\omega_z}{\mathrm{d}\omega_s} \left\langle \left(\frac{a_z}{2}\right)^{2|m_z|-2} \right\rangle \times \left(\frac{\partial^{|m_z|} U}{\partial z^{|m_z|}}\bigg|_{r_\perp = 0}\right)^2, \tag{43}$$

where the Λ's are the eigenvalues of the equation

$$\Lambda \chi = \int_0^\infty \frac{\mathrm{d}x}{x^2} J_{m_c}(xy) \int_0^\infty \mathrm{d}y' y' q(y') J_{m_c}(xy') \chi_m(y'). \tag{44}$$

It is easy to see that they are real positive numbers (see Appendix). Therefore the condition for stability of the excitations has the form

$$\frac{\mathrm{d}\omega_z}{\mathrm{d}\omega_s} > 0, \tag{45}$$

which is the same as for δ-like distribution. If the excitation is unstable it is of interest to find the region to which the numerical value of the maximum increment belongs. According to the formula (43) $\delta_{\max}$ is determined by the maximum eigenvalue $\Lambda_{\max}$ of equation (44). The inequality for this number is given in the Appendix (A6).

This estimate depends on the form of the distribution and on the form of the trial function $p_m(y)$. For example, for the distribution $q = 2\exp(-y^2)$,

$$p_m = \left(\frac{2}{|m_c|!}\right)^{1/2} y^{|m_c|+1/2} \exp\left(-\frac{y^2}{2}\right)$$

we get the following inequality for $\Lambda_{\max}$:

$$\left(\frac{\pi}{2}\right)^{1/2} \frac{2^{-2|m_c|}(2|m_c|-1)!!}{|m_c|!(2|m_c|-1)} \le \Lambda_{\max} \le \frac{2}{\sqrt{\pi}} \frac{1}{4m_c^2 - 1}.$$

b) Finally, let us consider the axial-longitudinal excitation of the long bunch $l_B \gg |m_c| l$. In this case, the main contribution to the bunch-line interaction is from the region of harmonics with the frequency of revolution ($n = x/\varphi_0$):

$$|n| < \frac{R_0}{l}. \tag{46}$$

Then the kernel $K_\perp(y/y')$ can be written as

$$K_\perp\left(\frac{y}{y'}\right) \simeq \frac{i}{2}\left(\frac{l}{R_0}\right)^2 \frac{\mathrm{d}\nu_z}{\mathrm{d}\ln R_0} \int_0^\infty \mathrm{d}x J_{m_c}(xy) J_{m_c}(xy'). \tag{47}$$

Equation (47) can be reduced to the equation with positive symmetrical kernel by the standard procedure (see above). Hence the stability of the excitations is determined by the sign of chromaticity only ($\mathrm{d}\nu_z/\mathrm{d}\ln R_0$).

The decrements of the excitations

$$\delta = \Lambda \frac{N r_0 c}{\pi \gamma \Gamma^2(|m_z|)} \left(\frac{l}{2\pi R_0}\right) \frac{|m_z| l}{l_B} \frac{\mathrm{d}\nu_z}{\mathrm{d}\ln R_0} \times \left\langle \left(\frac{a_z}{2}\right)^{2|m_z|-2} \right\rangle \left(\frac{\partial^{|m_z|} U}{\partial z^{|m_z|}}\bigg|_{r_\perp = 0}\right)^2 \tag{48}$$

can be expressed through the eigenvalues of the equation

$$\Lambda \chi_m(y) = \int_0^\infty \mathrm{d}x J_{m_c}(xy) \int_0^\infty \mathrm{d}y' y' q(y') J_{m_c}(xy') \chi_m(y')$$

which is easy to obtain from (41) and (47).

The corresponding estimate of the maximum increment will be given below (see Section III-4).

III-3. Radial-longitudinal excitations ($m_z = 0$).

As we said above, this type of excitation differs from the axial-longitudinal ones by the influence of the coupling of the radial and longitudinal motion in storage rings on the collective oscillations of the bunch.

Taking into account the modulation of the azimuth θ by the radial oscillations, let us rewrite (18a) in the form

$$\lambda_m \chi_m = \int_0^\infty \mathrm{d}y' y' q(y') \left[K_\perp\left(\frac{y}{y'}\right) - K_\parallel\left(\frac{y}{y'}\right)\right] \chi_m(y'), \tag{49}$$

where λ_m and $K_{\perp}(y/y')$ can be obtained from (33) by substituting r for z, and where $K_{\parallel}$ is

$$K_{\parallel}\left(\frac{y}{y'}\right) = \delta_{\parallel} \int_0^{\infty} \mathrm{d}x \sin^2\left(\frac{2lx}{l_B}\right) J_{m_c}(xy) J_{m_c}(xy'),$$

$$\delta_{\parallel} = 2i \frac{m_r \overline{\Psi}}{R_0}$$

$$\times \left[\frac{\partial}{\partial r} \left(\frac{\partial^{|m_r|-1} U}{\partial r^{|m_r|-1}} \right)^2 \bigg|_{r_\perp = 0} \bigg/ \left(\frac{\partial^{|m_r|} U}{\partial r^{|m_r|}} \bigg|_{r_\perp = 0} \right)^2 \right]. \quad (50)$$

The equality (50) can be obtained by assuming that $K_{\parallel}(y/y')$ describes the bunch's interaction with the edge fields when the harmonics of the frequency of revolution with $|n| > R_0/\min\{l_B, l\}$ are of importance.

The character of the excitation spectrum is determined by the quantity

$$\xi_m = \overline{\Psi} \frac{l_\perp R_0}{l_B l^{(m)}},$$

where

$$l^{(m)} = \begin{cases} l, & m_c = 0 \\ l_B, & m_c \neq 0. \end{cases}$$

If ξ_m is small ($|\xi_m| \ll 1$), then it is easy to see that the influence of $K_{\parallel}$ on the solution (49) is small and it can be accounted for by perturbation theory. In this case the stability of the excitations is determined by the properties of the kernel $K_{\perp}(y/y')$, so that the stability conditions have the same form as in the previous section. Let us calculate the distortion of the $K_{\perp}$ spectrum because of $K_{\parallel}$. Let us assume for simplicity that the eigenvalues of $K_{\perp}(y/y')$ are non-degenerate. Then we have in first order perturbation theory

$$(\lambda_m - \lambda_{m_0})_k = \delta_{\parallel} \int_0^{\infty} \mathrm{d}x \sin^2\left(\frac{2lx}{l_B}\right)$$
$$\times \left[\int_0^{\infty} \mathrm{d}y y q(y) J_{m_c}(xy) \chi^k_{m0}(y) \right]^2, \quad (51)$$

where λ^k_{m0}, χ^k_{m0} can be determined from (49) when $K_{\parallel}(y/y') \equiv 0$, and $\chi^k_{m0}(y)$ satisfies the normalization conditions

$$\int_0^{\infty} \mathrm{d}y y q(y) \chi^k_{m0} \chi^{k'}_{m0} = \delta_{kk'}.$$

Comparing (51) with (42), (43) or (48) we find that the ratio $|\lambda_m - \lambda_{m0}|_k / |\lambda_{m0}|_k$ is of the order of magnitude of $|\xi|$. In the other limiting case ($|\xi| \gg 1$) solutions (49) are mainly determined by $K_{\parallel}(y/y')$, and $K_{\perp}$ can be considered as a small perturbation.

Without the perturbation ($K_{\perp} \equiv 0$) the spectrum of the excitations is determined by the equation

$$\Lambda \chi_m = \int_0^{\infty} \mathrm{d}x \sin^2\left(\frac{2lx}{l_B}\right) J_{m_c}(xy)$$
$$\times \int_0^{\infty} \mathrm{d}y' y' q(y') J_{m_c}(xy') \chi_m(y'), \quad (52)$$

where

$$\lambda_{ml} = \delta_{\parallel} \Lambda.$$

The last equation can be transformed into the equation with the positive symmetrical kernel by the substitution $\tilde{\chi}(y) = \chi(y)(q/y)^{1/2}$. Therefore all the eigenvalues of (52) are positive. The decrements of the exitations are

$$\delta = -\Lambda \frac{N r_0 c |m_r|}{2\pi^2 \gamma \Gamma^2(|m_r|)} \frac{\overline{\Psi}}{l_B}$$
$$\times \left\langle \left(\frac{a_r}{2}\right)^{2|m_r|-2} \right\rangle \frac{\partial}{\partial r} \left(\frac{\partial^{|m_r|-1} U}{\partial r^{|m_r|-1}} \bigg|_{r_\perp = 0} \right)^2, \quad (53)$$

where Λ is the eigenvalue of (47). The stability condition ($\delta > 0$) has the form

$$\overline{\Psi} \frac{\partial}{\partial r} \left(\frac{\partial^{|m_r|-1} U}{\partial r^{|m_r|-1}} \bigg|_{r_\perp = 0} \right)^2 < 0. \quad (54)$$

The eigenvalues of equation (52) depend, generally speaking, on the parameter $l_B/1$. However, it can be seen from (52) that the dependence is weak, and it disappears completely in the limit $l_B/l \to 0$.

III-4. Synchrotron excitations.

In this case the integral equation has the form of (33). Note, that if $\rho(\varphi)$ depends only on one parameter, then the condition

$$\frac{\mathrm{d}q}{\mathrm{d}y} < 0, 0 \leq y < \infty$$

is needed for the convergence of the normalization integral for q. For such a distribution $q(y)$ the stability condition for the synchrotron excitations coincides with (38) and does not depend on the form of the amplitude distribution of the synchrotron oscillations in both limiting cases of the extremely short ($l_B \ll |m_c| l$) and extremely long ($l_B \gg |m_c| l$) bunches.

First let $l_B \ll |m_c| l$. Then equation (33) can be rewritten in the form

$$\Lambda\chi = \int_0^\infty \frac{dx}{x^2} J_{m_c}(xy) \int_0^\infty dy'y' \left| \frac{dq}{dy'} \right| \chi_m(y') J_{m_c}(xy'), \tag{55}$$

where

$$\Lambda = i \frac{(\omega - m_c\omega_c)\pi^2 l_B \gamma}{N r_0 c m_c^2} \left(\overline{\Psi} \frac{\partial U_0^2}{\partial r} \right)^{-1}. \tag{56}$$

Equation (55) can be easily reduced† to the integral equation with a positive symmetrical kernel. Hence all its eigenvalues are positive numbers. The decrements of the excitations can be expressed through the eigenvalues of equation (55) by the formula

$$\delta_m(k) = \Lambda(k) \frac{N r_0 c m_c^2}{\pi^2 \gamma l_B} \overline{\Psi} \frac{\partial U_0^2}{\partial r}; \; k = 1, 2, 3, \ldots. \tag{57}$$

Here k is the number of the solution of (53). It can be seen from here that $\delta_m(k)$ is positive if (38) is valid.

Values of Λ have an upper limit of (see A6)

$$\Lambda(k) \leq \frac{1}{\pi(m_c^2 - \frac{1}{4})} \int_0^\infty dy q(y). \tag{58}$$

The integral in the numerator of (58) depends rather weakly on the particular form of the function $q(y)$. Hence one can say that $\delta_m(k)$ in (55) has an upper limit of δ_m from (37).

If the bunch is longer than the plate $l_B \gg |m_c| l$, the equation (33) also can be transformed (to within the terms of the order of $|m_c| l/l_B$) into an equation with a real positive kernel. Hence the stability condition in this case has the form (38).

IV. SHORT-WAVE EXCITATIONS.

We will derive here the spectrum of excitations due to the interaction of the bunch with the high-frequency part of the induced fields ($|n| \gg |m_c|/\varphi_0$). Let us first consider the axial-longitudinal excitations. Let us introduce the unknown function

$$C_m(x) = \int_0^\infty dy y q(y) J_{m_c}(xy) \chi_m(y)$$

† See Appendix.

and rewrite the equation (41) in the form

$$\Lambda_m C_m(x) = \int_0^\infty dx' \Phi(x') g\left(\frac{x}{y'}\right) C_m(x'),$$

where

$$g\left(\frac{x}{y'}\right) = \int_0^\infty dy y q(y) J_{m_c}(xy) J_{m_c}(x'y) \tag{41a}$$

$$\Phi(x) = L_\perp\left(\frac{x}{\varphi_0} - m_z \frac{d\omega_z}{d\omega_s}\right) + L_\perp\left(-\frac{x}{\varphi_0} - m_z \frac{d\omega_z}{d\omega_s}\right) \tag{59}$$

and $L_\perp(x)$ is determined by the formula (42).

In the high-frequency range ($x \gg |m_c|$) $g(x/x')$ has a sharp maximum as a function of x (the width of this maximum is of the order of unity) at $x' \simeq x$, and it drops fast as x' departs from x.†

The short-wave part of the spectrum can be obtained in two limiting cases:

a) $l_B \ll l$. Then we can neglect fast oscillations of $\Phi(x)$ over the length $\Delta x \sim 1$, which correspond to the substitution

$$\Phi(x) \simeq \overline{\Phi(x)} = \int_{\Delta x \cong 1} dx' \Phi(x + \Delta x').$$

b) In the opposite extreme case ($l_N \gg l$) the change of $\Phi(x')$ in the distance $\Delta x \cong 1$ can be neglected: $\Phi(x') \to \Phi(x)$—as can be seen from (42).

Therefore, in the region under consideration we can write approximately

$$g\left(\frac{x}{x'}\right) \simeq \frac{1}{x} \delta(x - x'), \quad x \gg |m_c| + 1. \tag{60}$$

Substituting (60) into (41a) we will obtain the spectrum of the excitations in the short-wave region

$$\Lambda_m(x) = \begin{cases} \dfrac{\overline{\Phi(x)}}{x}, & l_B \ll l \\ \dfrac{\Phi(x)}{x}, & l_B \gg l. \end{cases} \tag{61}$$

† For example, for

$$q(y) = 2\exp(-y^2), \quad g\left(\frac{x}{x'}\right) \sim \frac{1}{x} \exp - \frac{(x - x')^2}{4}, \quad x \gg |m_c| + 1;$$

for

$$q(y) = (1 + y^2)^{-1}$$

$$g\left(\frac{x}{x'}\right) \sim \frac{1}{x} \exp(-|x - x'|), \quad x \gg |m_c| + 1$$

The weakening of the correlation between harmonics of the frequency of revolution $C_m(x)$ when $x \gg |m_c| + 1$ means physically that the normal excitations due to the interaction of the bunch with the high-frequency part of the induced fields are close to "plane waves."

$$F_m(I_\perp, I_c) \sim (\exp[i(n\varphi_c - \omega t)])_{m_c}, |n| \gg \frac{|m_c|}{\varphi_0}.$$

The "distance" between the separate excitations is of the order of the width of $g(x/x')(\Delta x' \sim 1)$, which corresponds to $\Delta n \sim 1/\varphi_0, (x = n\varphi_0)$.

The decrement calculations are simplified in two extreme cases:

a) $l_B \ll l$. The functions $\overline{\Phi(x)}$ can be written in the form

$$\Phi(x) = \begin{cases} \dfrac{i\varphi_0^2}{2x^2} m_z \dfrac{d\omega_z}{d\omega_s}, & x \gg |m_z| v_z \varphi_0, \varphi_0 \left| m_z \dfrac{d\omega_z}{d\omega_s} \right| \\ -\dfrac{i}{2}\left(m_z \dfrac{d\omega_z}{d\omega_s}\right)^{-1}, & x \ll \varphi_0 \left| m_z \dfrac{d\omega_z}{d\omega_s} \right|. \end{cases}$$

Substituting $\overline{\Phi(x)}$ in (61) we get the expression for the decrements ($\delta = -\mathrm{Im}(\omega)$)

$$\delta = \begin{cases} \dfrac{N\delta_0\varphi_0}{2x^3} m_z \dfrac{d\omega_z}{d\omega_s}, & x \gg |m_c| \gg \varphi_0 \left| m_z \dfrac{d\omega_z}{d\omega_s} \right| \\ -\dfrac{N\delta_0}{2x}\left(m_z \varphi_0 \dfrac{d\omega_z}{d\omega_s}\right)^{-1}, & \\ & \varphi_0 \left| m_z \dfrac{d\omega_z}{d\omega_s} \right| \gg x \gg |m_c| + 1. \end{cases} \quad (62)$$

It is easy to obtain sums of decrements by integration of (62) over x ($|m_c| \neq 0$):

$$\sum_{x \simeq |m_c|}^{\infty} \delta \simeq \int_{|m_c|}^{\infty} dx \delta(x)$$

$$= \begin{cases} N\delta_0 \dfrac{\varphi_0 m_z}{4m_c^2} \dfrac{d\omega_z}{d\omega_s}; & |m_c| > \varphi_0 \left| m_z \dfrac{d\omega_z}{d\omega_s} \right| \\ -\dfrac{N\delta_0 \ln\left(\dfrac{\varphi_0}{|m_c|}\left| m_z \dfrac{d\omega_z}{d\omega_s}\right|\right)}{2\varphi_0 m_z \dfrac{d\omega_z}{d\omega_s}}; & \\ & \varphi_0 \left| m_z \dfrac{d\omega_z}{d\omega_s} \right| > |m_c|. \end{cases}$$

To the good approximation the latter formulas coincide (when $m_c \neq 0$) with (22) and (22a), respectively. This congruence is due to the fact that the main contribution to the synchrobetatron excitation of the short bunch comes from the interaction with the edge fields, which are very nonlinear in θ. Therefore one can expect that the high-frequency spectrum directly adjoins the low-frequency one.

It follows from this, that the formulas (62), if extrapolated to the low-frequency region $x \simeq |m_c|$, should give (to the order of magnitude) the maximal decrement (increment) of the axial-longitudinal excitations of the short bunch†:

$$\delta_{\max} \simeq \begin{cases} \dfrac{N\delta_0 m_z \varphi_0}{2|m_c|^3} \dfrac{d\omega_z}{d\omega_s}, & |m_c| > \varphi_0 \left| m_z \dfrac{d\omega_z}{d\omega_s} \right| \\ -\dfrac{N\delta_0}{2|m_c|}\left(m_z \varphi_0 \dfrac{d\omega_z}{d\omega_s}\right)^{-1}, & \\ & \varphi_0 \left| m_z \dfrac{d\omega_z}{d\omega_s} \right| > |m_c|. \end{cases}$$

b) $l_B \gg l|m_c|$. The region $l_B/l \gg x \gg |m_c|$ is the most interesting for estimation of the maximal increment of the oscillations in this case. Using (59) we can write $\Phi(x)$ in this region in the form:

$$\Phi(x) = i\left(\frac{l}{R_0}\right)^2 m_z \frac{dv_z}{d \ln R_0}; \frac{l_B}{l} \gg x \gg |m_c|.$$

Then the maximal increment can be estimated from the formula:

$$\delta_{\max} = 2N\delta_0\left(\frac{l}{l_B}\right)\left(\frac{l}{2R_0}\right) m_z \frac{dv_z}{d \ln R_0} \frac{1}{|m_c| + 1}.$$

For radial-longitudinal excitation the function $\Phi(x)$ in equation (41a) should be replaced by

$$\Phi(x) - \delta_{\|} \sin^2\left(\frac{2lx}{l_B}\right).$$

Then the high-frequency excitational decrements can be written in the form:

$$\delta(x) = \delta^{(1)}(x) - \delta^{(2)}(x),$$

where $\delta^{(1)}$ is obtained from the axial-longitudinal excitational decrement by substituting r in place of index z, and $\delta^{(2)}$ describes the influence of the radial-longitudinal coupling.

† For the excitations with $m_c = 0$ such an extrapolation is not justified because in this case the major contribution to the decrements of the long-wave excitations comes from the harmonics $n \lesssim R/l \ll 1/\varphi_0$.

For a short bunch ($l_B \ll l$) the quantity $\delta^{(2)}(x)$ can be calculated from the formula

$$\delta^{(2)}(x) \simeq \frac{\delta_0^{(2)}}{2x}, \frac{l_B}{l_\perp} \gg x \gg |m_c|, \qquad (63)$$

where

$$\delta_0^{(2)} = \frac{N r_0 c |m_r|}{\pi^2 \gamma l_B \Gamma^2(|m_r|)} \left\langle \left(\frac{a_r}{2}\right)^{2|m_r|-2} \right\rangle \Psi \times \frac{\partial}{\partial r} \left(\frac{\partial^{|m_r|-1} U}{\partial r^{|m_r|-1}} \bigg|_{r_\perp = 0} \right)^2.$$

The maximal decrement, to an order of magnitude, is equal to

$$\delta_{\max}^{(2)} \simeq \frac{\delta_0^{(2)}}{2(|m_c| + 1)}.$$

If the bunch is longer than the plate ($l_B \gg l|m_c|$) then

$$\delta^{(2)}(x) \simeq \begin{cases} \delta_0^{(2)} x \left(\dfrac{2l}{l_B}\right), & \dfrac{l_B}{l} \gg x \gg |m_c| \\ \dfrac{\delta_0^{(2)}}{2x}, & \dfrac{l_B}{l_\perp} \gg x \gg \dfrac{l_B}{l}. \end{cases} \qquad (63a)$$

Then the maximal decrement, to an order of magnitude, is:

$$\delta_{\max}^{(2)} \simeq \frac{N r_0 c |m_r|}{\pi^2 \gamma l_B \Gamma^2(|m_r|)} \left(\frac{l}{l_B}\right) \left\langle \left(\frac{a_r}{2}\right)^{2|m_r|-2} \right\rangle \times \frac{\partial}{\partial r} \left(\frac{\partial^{|m_r|-1} U}{\partial r^{|m_r|-1}} \bigg|_{r_\perp = 0} \right)^2.$$

V. ON THE INFLUENCE OF THE FREQUENCY SPREAD

We will limit ourself to the consideration of only those cases where the frequency spread in the stationary state is determined by the nonlinearity of the transverse motion only, and nonlinearity of the longitudinal motion can be neglected. Then, for excitation of arbitrary multipolarity, the dispersion equation [see (18b)] has the form

$$1 = -\frac{\Omega_m}{A_m} \int \mathrm{d}I_r\, \mathrm{d}I_z \times \frac{I_r^{|m_r|} I_z^{|m_z|} \left(m_r \dfrac{\partial F_0}{\partial I_r} - m_z \dfrac{\partial F_0}{\partial I_z} \right)}{\omega - m_z \omega_z(I_r, I_z) - m_r \omega_r(I_r\, I_z)}, \qquad (64)$$

where A_m is the normalization constant equal to

$$\langle I_r^{|m_r|-1} I_z^{|m_z|-1} (m_z |m_z| I_r + m_r |m_r| I_z) \rangle.$$

Ω_m is the eigenvalue of equation (18a) with the number m [which is the solution of equation (64) without spread].

Despite the seeming complexity of equation (64) it can easily be reduced to the standard form. In order to do so let us introduce new variables

$$\mathscr{E} = \sum_{i=r,z} m_i[\omega_i(I_r, I_z) - \omega_i(0, 0)],\ x = x(I_r, I_z), \qquad (65)$$

where the variable $x(I_r, I_z)$ is chosen so that the Jacobian of the transformation (65) is equal to one

$$\frac{\partial(\mathscr{E}, x)}{\partial(I_r, I_z)} = 1.$$

Then equation (64) transforms into

$$1 = \Omega_m \int_{-\infty}^{\infty} \mathrm{d}\mathscr{E} \frac{g(\mathscr{E})}{\omega - \omega_0 - \mathscr{E}}, \qquad (66)$$

where $\omega_0 = m_r \omega_r(0, 0) + m_z \omega_z(0, 0)$, and the quantity $g(\mathscr{E})$, which corresponds to the "effective" density of the frequency distribution, is determined by the equality

$$g(\mathscr{E}) = -\frac{1}{A_m} \int \mathrm{d}I_r\, \mathrm{d}I_z \int_{-\infty}^{\infty} \mathrm{d}x' \delta(\mathscr{E} - \mathscr{E}(I_r, I_z)) \times \delta(x' - x'(I_r, I_z)) I_r^{|m_r|} I_z^{|m_z|} \times \left(m_r \frac{\partial F_0}{\partial I_r} + m_z \frac{\partial F_0}{\partial I_z} \right). \qquad (67)$$

By definition the function $g(\mathscr{E})$ is normalized to unity:

$$\int_{-\infty}^{\infty} \mathrm{d}\mathscr{E} g(\mathscr{E}) = 1.$$

Equation (66) can be studied by standard methods. Using the Nyquist criterion it is easy to determine that all the roots of (66) will lie in the lower semi-plane of (ω) if the inequality

$$1 - \pi g(\mathscr{E}_i) \frac{|\Omega_m|^2}{\operatorname{Im} \Omega_m} > 0 \qquad (68)$$

holds. Here $\mathscr{E}_i$ are real numbers which can be determined from the equation

$$\operatorname{Im} \Omega_m \fint_{-\infty}^{\infty} \frac{\mathrm{d}\mathscr{E} g(\mathscr{E})}{\mathscr{E}_i - \mathscr{E}} - \operatorname{Re} \Omega_m \pi g(\mathscr{E}_i) = 0.$$

Here $⨍_{-\infty}^{\infty}$ means that integral should be taken in the sense of its principal value.

In particular, the inequality (68) is valid automatically, if the inequality

$$1 - \pi \frac{|\Omega_m|^2}{\operatorname{Im} \Omega_m} g_{\max} > 0 \qquad (68a)$$

is valid. Here $g_{\max}$ is the highest value of g on the whole interval of $\mathscr{E}$ variation. Therefore, inequality (68a) can be considered as a condition which ensures the damping of the coherent oscillations.

This condition means that one can ensure the stability of the coherent motion by choosing the parameters of the outside system so that the complex coherent frequency shift Ω_m lies inside the circle:

$$\left| \Omega_m - \frac{i}{2\pi g_{\max}} \right| < \frac{1}{2\pi g_{\max}}.$$

We note, that the conditions (68), (68a) are not, generally speaking, necessary ones. Hence their violation does not necessarily mean that oscillations are unstable.

The conditions for stability of coherent oscillations in the presence of spread can also be obtained by examination of equation (66) near the instability threshold $(\omega - \omega_0 \to \omega_{\text{th}} + i\varepsilon,\ \varepsilon > 0)$ (Ref. 5). Then equation (66) breaks into two equations

$$\begin{aligned} \pi g(\omega_{\text{th}}) &= \frac{\operatorname{Im} \Omega_m}{|\Omega_m|^2} \\ P(\omega_{\text{th}}) &\equiv ⨍_{-\infty}^{\infty} \frac{d\mathscr{E} g(\mathscr{E})}{\omega_{\text{th}} - \mathscr{E}} = \frac{\operatorname{Re} \Omega_m}{|\Omega_m|^2}. \end{aligned} \qquad (69)$$

The system of equations (69) determines in parametric form the stability region boundary in the plane of the complex variable Ω_m for the given distribution function $g(\mathscr{E})$. The location of the stability region with respect to the boundary is determined by the relation (Ref. 5)

$$\left[\frac{\operatorname{Im} \Omega'_m}{|\Omega'_m|^2} - \pi g(\omega_{\text{th}}) \right] \frac{\partial P}{\partial \omega_{\text{th}}} > 0. \qquad (70)$$

In this relation Ω'_m is a point on the plane of the complex variable Ω_m, near the stability region boundary, and ω_{th} corresponds to a point on the boundary curve.

Equations (69) can be used for the calculation of the threshold current and coherent frequency shift of the instability threshold when parameters of the outside system are given. For this it is more convenient to rewrite (69) in the form

$$\begin{aligned} \frac{\pi g(\omega_{\text{th}})}{P(\omega_{\text{th}})} &= \frac{\operatorname{Im} \Omega_m}{\operatorname{Re} \Omega_m} \\ N_{\text{th}} &= \frac{\operatorname{Im} \overline{\Omega}_m}{R |\overline{\Omega}_m|^2} \frac{1}{\pi g(\omega_{\text{th}})}, \end{aligned} \qquad (69a)$$

where the notation $\Omega_m = N \overline{\Omega}_m$ is introduced.

For further analysis it is necessary to define concretely the form of the density $g(\mathscr{E})$. Consider, for example, two-dimensional synchrobetatron excitations $(\omega \simeq m_z \omega_z + m_c \omega_c)$. Let us assume also that in the stationary state the nonlinearity of the motion is determined by the cubic nonlinearity of the leading field. Then

$$\omega_z(I_r, I_z) = \omega_z(0) + \frac{\partial \omega_z}{\partial I_z} I_z + \frac{\partial \omega_z}{\partial I_r} I_r. \qquad (71)$$

The form of the distribution function $g(\mathscr{E})$ depends considerably on which dimension of the beam (vertical or radial) determines the frequency spread.

Let us call the spread "intrinsic" if

$$\left| \frac{\partial \omega_z}{\partial I_z} \right| \langle I_z \rangle \gg \left| \frac{\partial \omega_z}{\partial I_r} \right| \langle I_r \rangle.$$

In this case the dispersion equation can be written

$$1 = -\frac{\Omega_m}{m_z \alpha_{zz} A_{m_z}} \cdot \int_0^{\infty} \frac{d\mathscr{E} \cdot \mathscr{E}^{|m_z|} \dfrac{\partial F_0}{\partial \mathscr{E}}}{t - \mathscr{E}}, \qquad (66a)$$

where $\alpha_{zz} = \partial \omega_z / \partial I_z$, $t = (\omega - \omega_0)/\alpha_{zz}$. Then $g(\mathscr{E})$ is given by

$$g(\mathscr{E}) = -\frac{\mathscr{E}^{|m_z|}}{m_z \alpha_{zz} A_{m_z}} \cdot \frac{\partial F_0}{\partial \mathscr{E}} \cdot \theta(\mathscr{E}) \qquad (72)$$

where

$$\theta(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0. \end{cases}$$

In the other extreme case (Fig. 1):

$$\left| \frac{\partial \omega_z}{\partial I_z} \right| \langle I_z \rangle \ll \left| \frac{\partial \omega_z}{\partial I_r} \right| \langle I_r \rangle$$

we will call the spread "external". Then the function $g(\mathscr{E})$ is:

$$g(\mathscr{E}) = \frac{F_0(\mathscr{E}) \theta(\mathscr{E})}{m_z \alpha_{zr}}. \qquad (73)$$

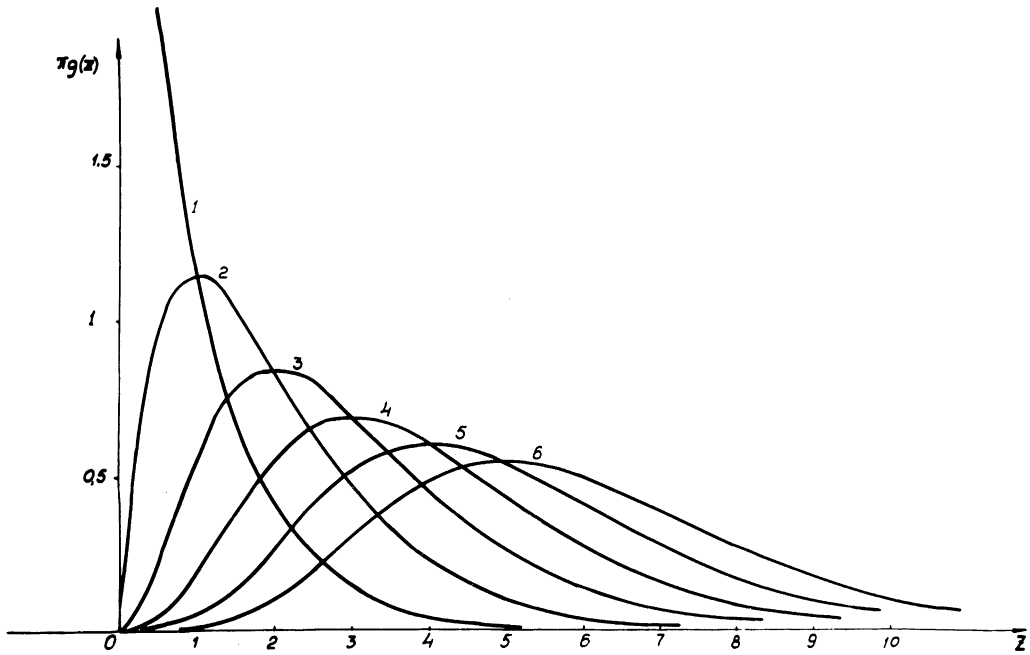

FIGURE 1 Effective frequency distribution for the beam with the exponential square amplitude distribution of betatron oscillations in the stationary state [see (72), (73)]. Curve (1) corresponds to the "external" spread. Curves (2), (3), (4), (5), (6) correspond to the "intrinsic" spread and multipole numbers $m_z = 1, 2, 3, 4, 5$, correspondingly.

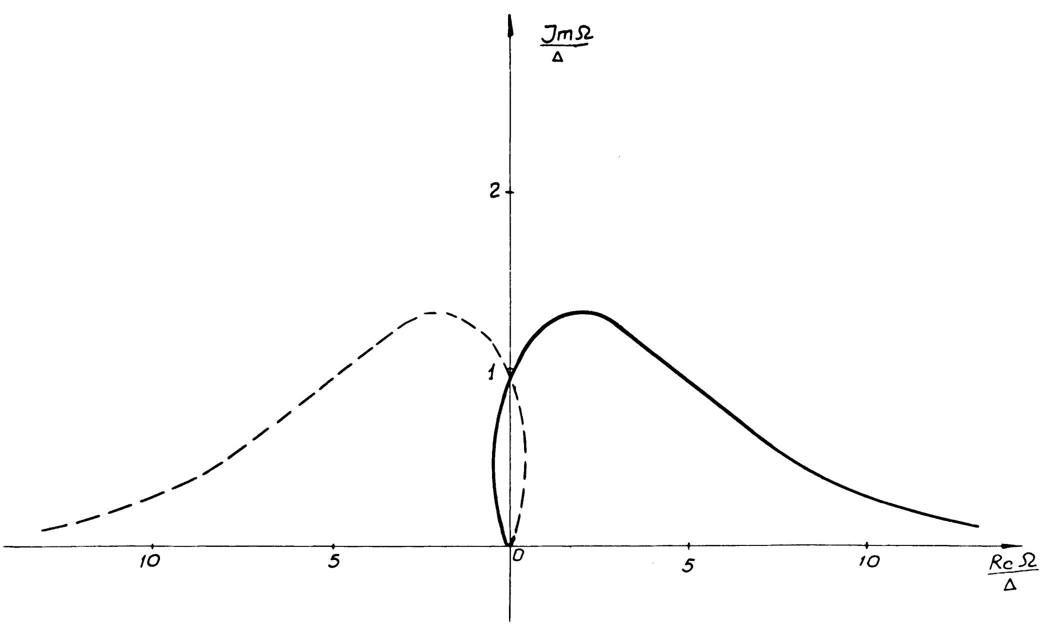

FIGURE 2. The stability region boundary for the distribution (1) (Figure 1). Solid curve corresponds to $\alpha_{zr} > 0$, dotted line corresponds to $\alpha_{zr} < 0$.

It is seen from (72) that the form of the frequency distribution is determined, generally speaking, by the multipole number m_z (m_r in the case of radial-longitudinal excitations) besides particle distribution by the oscillation amplitudes. Note in particular, that the "effective" width of the distribution $g(\mathscr{E})$ grows approximately as $|m_z|^{1/2}$. In addition, $g(\mathscr{E})$ has at least one maximum approximately at $\mathscr{E} \simeq |m_z|\langle I_z\rangle$ for smooth $F_0(I)$.

An important fact is that $g(\mathscr{E})$ in general is not symmetric with respect to its maxima. Therefore, the stability region boundary which is given by the equation

$$\frac{\operatorname{Im}\Omega_m}{|\Omega_m|^2} = -\frac{1}{|m_z\alpha_{zz}|}\cdot\frac{\mathscr{E}^{|m_z|}}{A_{m_z}}\cdot\frac{\partial F_0}{\partial\mathscr{E}}\cdot\theta(\mathscr{E}) \qquad (74)$$
$$\frac{\operatorname{Re}\Omega_m}{|\Omega_m|^2} = \frac{P(\mathscr{E})}{m_z\alpha_{zz}}$$

is not symmetric with respect to the axis Im Ω_m (see Fig. 2 and 3). In particular, it coincides with the axis Re Ω_m when

$$\operatorname{Re}\Omega_m \le \operatorname{Re}\Omega_m^0 = \frac{m_z\alpha_{zz}}{P(0)}. \qquad (75)$$

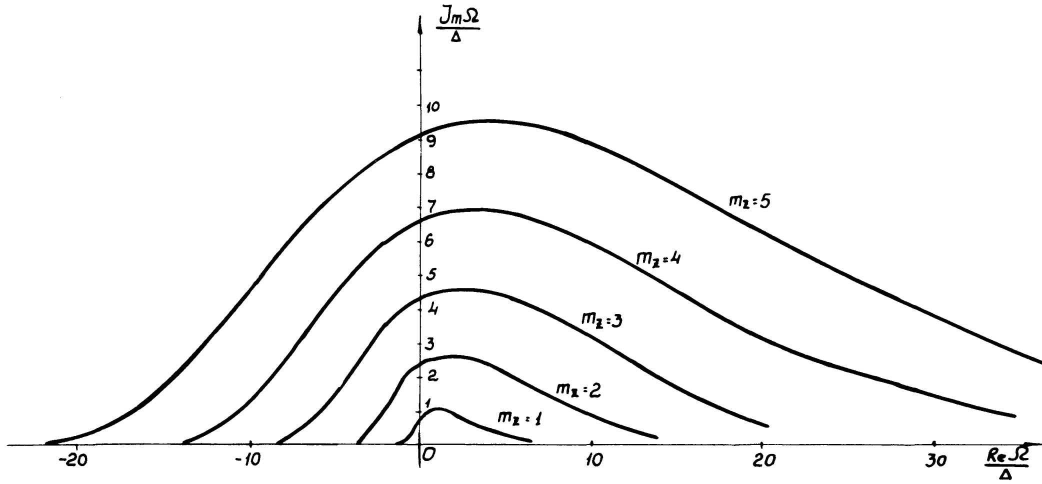

FIGURE 3a. The stability region boundary for the effective distributions (2), (3), (4), (5), (6) (Figure 1), $\alpha_{zz} > 0$.

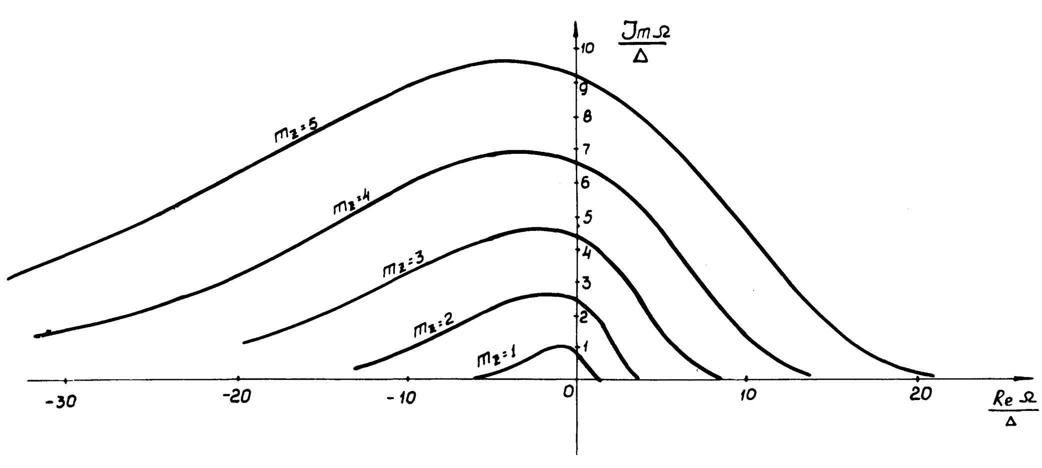

FIGURE 3b. The same for $\alpha_{zz} < 0$.

Therefore, if the coherent frequency shift introduced by the system is such that inequality (75) is valid, then stabilization of the coherent oscillations by frequency spread is impossible.

It is seen from equation (74) that the stability region reflects with respect to the axis $\mathrm{Im}\,\Omega_m$ when α_{zz} changes its sign. The stability region does not transform into itself then, because of its asymmetry with respect to $\mathrm{Im}\,\Omega_m$. This means (in case of the "intrinsic" spread) that for the outside systems with

$$|\mathrm{Re}\,\Omega_m| > |\mathrm{Re}\,\Omega_m^0|$$

the coherent oscillations can become unstable after the change of sign α_{zz}.

The integral $P(\mathscr{E})$ diverges logarithmically when $\mathscr{E} \to 0$ in case of the external spread. However, this only influences the value of the boundary coherent frequency shift $\mathrm{Re}\,\Omega_m^0$, which no longer can be calculated from (75). Qualitatively the results are the same.

The value of the instability threshold current can be calculated easily for monotone $F_0(I)$ in the case of $|\mathrm{Im}\,\Omega_m| \gg |\mathrm{Re}\,\Omega_m|$. Then $g(\omega_{\mathrm{th}}) \gg P(\omega_{\mathrm{th}})$, which means that the roots of the first equation (69a) must be located near $\bar{\omega}_m$ (which corresponds to the maximum of $g(\mathscr{E})$), i.e., $|\omega_{\mathrm{th}} - \bar{\omega}_m| \ll (\overline{\Delta\omega_m^2})^{1/2}$. Therefore, the value of the threshold current can be estimated by the formula

$$N_{\mathrm{th}} \simeq \frac{1}{\mathrm{Im}\,\Omega_m}\frac{1}{\pi g_{\max}} \simeq \frac{(\overline{\Delta\omega_m^2})^{1/2}}{\mathrm{Im}\,\Omega_m}, \tag{76}$$

where the quantity

$$\overline{\Delta\omega_m^2} = m_z^2 \int_{-\infty}^{\infty} \mathrm{d}\omega(\omega - \bar{\omega}_m)^2 g(\omega)$$

determines the betatron oscillation spread.

Substituting the excitational increments obtained in the previous sections, one can get formulas for the threshold current for the beam interacting with matched plates. Then the expression for the maximal increment should be substituted in (76). The value of it can be estimated for the arbitrary smooth distribution (see sec. III).

For example, if the machine's chromaticity is not too large:

$$\varphi_0 \left| m_z \frac{\mathrm{d}\omega_z}{\mathrm{d}\omega_s} \right| \ll 1$$

then for the axial-longitudinal excitations ($m_c \neq 0$) N_{th} can be written in the form

a) $l_B \ll |m_c| l$;

$$N_{\mathrm{th}} = \frac{\gamma}{\delta_0} \frac{(\overline{\Delta\omega_m^2})^{1/2} L_m^s}{l\left(-m_z \dfrac{\mathrm{d}\omega_z}{\mathrm{d}\omega_s}\right)}, \tag{77}$$

where the quantity L_m^s depends on the form of the distribution (for the δ-like distribution $L_m^s = 4m_c^2 - 1$).

b) $l_B \gg |m_c| l$;

$$N_{\mathrm{th}} = \frac{\pi^2}{m_z \delta_0} \frac{\gamma l_B (\overline{\Delta\omega_m^2})^{1/2}}{\left(-2L_s^{(l)} \dfrac{\mathrm{d}\ln \nu_z}{\mathrm{d}\ln R_0} - 1\right)} \tag{78}$$

For the δ-like distribution the factor $L_m^{(l)}$ equals $\ln(l_B/l|m_c|)$; for a smooth distribution it is of the order of $(|m_c| + 1)^{-1}$. Note, that the last formula is in a good qualitative agreement with the experimental results obtained on the installations ADONE (Ref. 3), ACO (Ref. 9), and CEA (Ref. 10).

If the machine's chromaticity is large

$$\varphi_0 \left| m_z \frac{\mathrm{d}\omega_z}{\mathrm{d}\omega_s} \right| \gg 1$$

then the formula (22a) must be used for the estimate of the threshold current. Then N_{th} has a form

$$N_{\mathrm{th}} = \frac{\gamma\pi}{\delta_0} \frac{m_z l_B}{R_0} \frac{\mathrm{d}\omega_z}{\mathrm{d}\omega_s} L_m (\overline{\Delta\omega_m^2})^{1/2}, \tag{79}$$

where the factor L_m depends on the form of the oscillation amplitude distribution function in the stationary state.

It is seen from the formulas (77)–(79) that functional dependencies of the threshold current are varying strongly depending on the relation between characteristic parameters of the problem (in particular, the beam's length and the chromaticity of the machine).

If the coherent frequency shift is large $|\mathrm{Re}\,\Omega_m| \gg |\mathrm{Im}\,\Omega_m|$ then $P(\omega_{\mathrm{th}}) \gg g(\omega_{\mathrm{th}})$, which is only possible if $|\omega_{\mathrm{th}} - \bar{\omega}_m| \gg (\overline{\Delta\omega_m^2})^{1/2}$.

In this case $P(\omega_{\mathrm{th}}) \simeq 1/\omega_{\mathrm{th}}$, and, according to (69), $\omega_{\mathrm{th}} = \mathrm{Re}\,\Omega_m$. Therefore, we have a transcendental equation for N_{th}

$$\mathrm{Im}\,\Omega_m = \pi N_{\mathrm{th}} (\mathrm{Re}\,\bar{\Omega}_m)^2 g(N_{\mathrm{th}}\,\mathrm{Re}\,\bar{\Omega}_m). \tag{80}$$

This work was performed in 1972 (see Ref. 11).

Appendix

We shall show here, that all of the eigenvalues of the integral equation

$$\lambda f(x) = \int_0^\infty dx' K\left(\frac{x}{x'}\right) q(x') f(x') \qquad (A.1)$$

have the same sign if $q(x)$ does not change its sign when $0 \leq x < \infty$, and the kernel K can be represented in the form

$$K\left(\frac{x}{x'}\right) = \int_0^\infty dt c^2(t) B(xt) B(x't), \qquad (A.2)$$

where $c(t)$, $B(x)$ are real functions.

To be specific, let us put $q(x) > 0[\int_0^\infty dx q(x) = 1]$. Multipling both sides of (A.I) by $q^{1/2}$ we will obtain the equation for

$$\varphi(x) = q^{1/2} f(x)$$

$$\lambda\varphi(x) = \int_0^\infty dx' K_1\left(\frac{x}{x'}\right)\varphi(x'). \qquad (A.3)$$

The spectrum of this equation coincides, clearly, with the spectrum of (A.1), and the kernel $K_1(x/x')$, which is related to the kernel $K(x/x')$ by the formula

$$K_1\left(\frac{x}{x'}\right) = \sqrt{q(x)q(x')}\, K\left(\frac{x}{x'}\right),$$

is real and symmetrical.

The usual requirement (see, for example, Ref. 8) is that K_1 should be a square integrable function, i.e., that the integral

$$\|K_1\|^2 = \int_0^\infty dx\, dx' K_1^2\left(\frac{x}{x'}\right)$$

is bounded by some number.

This requirement is fulfilled automatically if the sum of the eigenvalues of (A.1) is finite. Really, taking (A.2) into account and applying the Buniakovsky-Schwartz inequality we will find:

$$\|K_1\|^2 = \int_0^\infty dt\, dt' c^2(t) c^2(t') \times \left[\int_0^\infty dx q(x) B(xt) B(xt')\right]^2 \leq \left[\int_0^\infty dt\, c^2(t) \int_0^\infty dx\, q(x) B^2(xt)\right]^2 = \left[\int_0^\infty dx\, q(x) K\left(\frac{x}{x'}\right)\right]^2. \qquad (A.4)$$

The last quantity in (A.4) is exactly equal to the square of the sum of eigenvalues of (A.1). So

$$\|K_1\|^2 \leq \sigma^2 = \left[\int_0^\infty dx\, q(x) K_1\left(\frac{x}{x'}\right)\right]^2.$$

The eigenvalues of the equation (A.3) are real and positive. The latter follows from the fact that (see Ref. 8) $K_1(x/x')$ is a positive kernel, i.e., the integral

$$J[p] = \int_0^\infty dx\, dx' K_1\left(\frac{x}{x'}\right) p(x) p(x')$$

is positive. (Here $p(x)$ is such that it can be expanded by the eigenfunctions of (A.3).) Really:

$$J[p] = \int_0^\infty dt\, c^2(t) \left[\int_0^\infty dx\, q^{1/2}(x) B(xt) p(x)\right]^2 > 0.$$

For the estimates of the excitational increments the following formula can be useful: (Ref. 8)

$$\lambda_{\max} \geq J[p] \qquad (A.5)$$

when

$$\int_0^\infty dx\, p^2(x) = 1.$$

Since all λ are positive, it is clear that

$$J[p] \leq \lambda_{\max} \leq \sigma. \qquad (A.6)$$

The second inequality in (A.6) can be obtained by the direct application of the Buniakovsky-Schwartz inequality to the right side of (A.5).

If $q(x) < 0$, then substituting λ by $-\lambda$ and $q(x)$ by $-|q(x)|$ it is easy to reduce this equation to the form (A.3), which has a positive spectrum. Therefore for $q(x) < 0$, $0 \leq x < \infty$ all eigenvalues of (A.1) are negative.

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