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# STABILITY OF A BUNCHED BEAM INTERACTING WITH MATCHED LINES

Ya.S. Derbenev, N.S. Dikanskij and D.V. Pestrikov

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ABSTRACT

The report investigates the stability of a bunched beam interacting with a matched transmission line. The functional dependences of the decrements (increments) of betatron and synchrobetatron oscillations of arbitrary multipolarity on the typical parameters of the problem (length of the plate, length of the bunch, chromatism) are determined.

An experimental investigation of coherent beam stability in storage rings has shown that there are coherent effects, whose decrements and increments do not depend on the selection of the operating point in terms of the particles' oscillation frequencies. The latter indicates that these effects are conditioned by the interaction of the beam with low-Q systems where fields excited by the beam damp out in a shorter time than the particle revolution period (i.e. "single-turn" effects).

The first such effect discovered on the VEPP-2 device, was the so-called 'fast damping' of vertical beam oscillations /1/. It was characterized by the fact that oscillation decrements were determined by the total charge of the bunch (Ne) and did not depend on the bunch length $\ell_b$

$$\delta \sim \frac{N}{E} \tag{1.1}$$

where E is the particle energy. This phenomenon was explained by the interaction of a coherently oscillating beam with the principal wave field (TEM) in matched transmission lines /2/.

Slightly later, instabilities in transverse oscillations, which may also be related to the single-turn effects, were discovered on the ACO and ADONE devices. The increments of these instabilities are inversely proportional to the bunch length and depend on the machine's chromatism $(d\ln\nu \;/ d\ln R)$ and on the number of particles in a given bunch. The empirical dependence of the threshold current on these parameters has the form /3/:

$$I_{th} \sim \frac{E\,\ell_b\,\Delta\nu}{1-\frac{d\ln\nu}{d\ln R}} \tag{a}$$

where $\nu$ is the dimensionless betatron frequency, $\Delta\nu$ the frequency spread and $2\pi R$ the perimeter of the orbit.

An explanation of the instability mechanism related to the dependence $\nu(R)$ was given by C. Pellegrini and M. Sands. It was called the head-tail effect/4/; however, the oscillation decrements obtained by him in specific cases vanish as the bunch length or the machine's chromatism

$$\delta \sim \frac{N B}{\ell}\,\frac{Q'\omega}{c\,\ell R} \tag{1.2}$$

approaches zero. In particular, in the case of matched lines, there is no term in the decrement that corresponds to "fast damping" (which does not vanish as $\ell_b \to 0$). This result is due to the fact that the interaction of the beam with a low-Q element was not taken fully into account.

In earlier papers (/5/, /6/, /7/) a theoretical study was made of the interaction of the bunch with matched lines and a low-Q resonator. It was shown that, by introducing matched lines, it is possible to ensure the damping of the basic type (one-dimensional betatron or synchrotron) of coherent beam oscillations whilst the interaction with the low-Q resonator may lead to instability. Expressions for the decrements of betatron oscillations were obtained in the limit of a short bunch length and therefore did not contain terms of type (1.2).

This report studies the stability of a bunch of arbitrary length (but not shorter than the chamber's cross-section), interacting with a matched line.

The first part gives a general integral equation defining the spectrum of normal collective beam excitations near a certain stationary state in the presence of an interaction with an exterior system. Using the kinetic equation in canonical variables (previously proposed in /5/), a wide range of problems relating to collective beam motion may be examined by one and the same method. This method is particularly efficient when the collective interaction affects only slightly the motion of the particles. The non-stationary part of the distribution of particles in phase space, describing the normal collective excitation, takes the form

$$F = F_m(I)\exp(i m_k \psi^k - i\omega t), \quad \omega = m_k\omega_k \qquad \textbf{(b)}$$

where $I$, $\Psi$ are the action-phase variables of the **stationary** state (in which, by definition, the distribution is uniform over the phases); $\omega_k$ are the partial frequencies of the **unperturbed particle motion,** and $m_k$ integers defining the excitation's multipolarity. In particular, the dipole excitation corresponds to the case $|m| = 1$.

In the second part we investigate the decrement expressions **for the $\delta$-type function distribution of** synchrotron oscillation amplitudes in the stationary state. It is shown that the excitation decrement of arbitrary **multipolarity** may, generally speaking, be represented by the sum of two terms, one of which corresponds to fast damping /5/ and the other depends on the machine's chromatism ($d\ln\nu \,/\, d\ln R$).

The first term is related to the excitation of the principal(**TEM**) wave by the transverse betatron motion and is therefore always positive. The second is linked to the excitation of the principal wave by the longitudinal motion at the ends of the plates.

**The** betatron phase shift along the bunch, necessary for instability, is produced by the energy dependence of the betatron frequency. Depending on the ratio of the beam and plate length and also on the type of excitation, the decrement's value is determined **either by the first or the second term.**

Thus, in the case of vertical betatron excitations ($\omega = n_z\omega_z + \Delta\omega$), the functional dependence of the decrement on the parameters of the machine and bunch has the form

[illegible]

a) $l_b < l_p$; [illegible] long plate, ultra-relativistic case ($\gamma = E/m_0c^2$, $m_0$ is the particle's mass).

$$\delta \sim \frac{N}{\gamma}\,\frac{\ell}{2\pi R}\left(1-\frac{8\ell_b}{\pi\ell}\,\frac{1}{\nu_z}\,\frac{d\omega_z}{d\omega_0}\right) \qquad (1.3)$$

where $\ell$ is the plate's length.

It can be seen from (1.3) that as $d\omega_z / d\omega_0$ approaches zero, the decrement is determined by the first term. Instability may occur when the following inequality is fulfilled

$$\frac{d\omega_z}{d\omega_0} > \frac{\nu_z \pi \ell}{8\ell_b}$$

b) $\ell_b > \ell,\ \gamma^2 \ell_b >> \ell$ "short plate"

$$\delta \sim \frac{N}{\gamma}\cdot\frac{\ell}{\ell_b}\cdot\frac{\ell}{2\pi R}\left(1+2\frac{d\omega_z}{d\omega_0}\cdot C_1\frac{\ell_b}{\ell}\right) \qquad (1.4)$$

It is clear that in the presence of a non-vanishing chromatism, the decrement's sign may be determined by the second term.

The paper also investigates two-dimensional synchro-**betatron oscillations** ($\omega = m_k\omega_k + m_c\omega_c + \Delta\omega\ ;\ k = r, z$ ). This type of **oscillation** is characterized by the fact that the beam performs coherent oscillations both in the transverse and in the longitudinal directions with the multipolarities $m_k$ and $m_c$ respectively. In this case, if the betatron oscillations are monitored with a pick-up electrode of corresponding multipolarity, the frequency of the signal obtained will be modulated by the frequency $m_c\,\omega_c$ .

If the plate is longer than the bunch ($\ell_b < \ell(|m_c| + 1$ ) then the decrement of synchrobetatron excitation is proportional to the quantity

$$\delta \sim \frac{N}{\gamma} \cdot \frac{\ell_b}{R \nu_z} \cdot \frac{d\omega_z}{d\omega_o} \frac{1}{4m_c^2 - 1} \tag{1.5}$$

and the contribution of "fast damping" is negligible.

If the plate is short, then the expression for the decrement practically coincides with (1.4).

For strong-focusing machines it may be interesting to study a case where the chromatism is large ( $\frac{\ell_b}{R} \left|\frac{d\omega_z}{d\omega_o}\right| \gg 1$ ). Under this condition the decrement is proportional with logarithmic accuracy to the quantity

$$\delta \sim - \frac{N}{\gamma} \frac{R}{\ell_b \frac{d\omega_z}{d\omega_o}} \cdot \ln\left(\frac{\ell_b}{R(|m_c+1)}\left| m_z \frac{d\omega_z}{d\omega_o} \right|\right) \tag{1.6}$$

and the contribution of "fast damping" may be ignored.

When examining the radial and longitudinal excitations, it is essential to take into account the accelerator's inherent coupling of these degrees of freedom. Without allowing for this coupling the expressions for the decrements of radial betatron and synchrotron excitations are analogous to (1.3), (1.6), with a substitution of indexes $z \rightarrow r$, provided the decrements of synchrotron excitations are small.

The coupling of radial and longitudinal motions, due to the dependence of coherent energy losses on the radial position of particles when a line is excited (by the beam), leads to a redistribution of the decrements, as a result of which, generally speaking, the radial betatron or synchrotron excitations may become unstable.

This mechanism may be used to damp the beam's synchrotron oscillations. If the decrements are redistributed by means of matched lines, then for all excitations with multipolarity $|m_c| < m_{max.} = \ell_b / \ell_\perp$ , where $\ell_\perp$ is the chamber's cross-section, the damping decrements do not depend on $m_c$. (In this case, the oscillations of the separate bunches are damped independently). The maximum value of a decrement is limited by the stability condition of the radial betatron and **synchro-betatron oscillations.**

In the third and fourth parts of the paper, qualitative methods are used to investigate the stability of a beam having a smooth equilibrium distribution of synchrotron oscillation amplitudes.

For limiting cases (short or long plate) the integral equation investigated is converted into an integral equation with a symmetrical positive kernel. This property enables the stability of coherent excitations to be examined in a general form of arbitrary smooth distributions of synchrotron amplitude oscillation in a stationary state. In the last part of the paper we investigate the solution of the dispersion equation taking into account the frequency spread of betatron oscillations.

The results of this paper show that the use of matched plates may be particularly effective for damping the basic types of oscillations (single-dimensional betatron and synchrotron). Simultaneous stability **of the radial and axial synchro-betatron oscillations is already guaranteed at low levels of machine chromatism.**

## 1. METHOD

The state of a beam interacting with an external system may be described by the equations:

$$\frac{\partial F}{\partial t} + \left\{ \mathcal{H}; F \right\} = 0 \tag{1}$$

$$\Delta \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = -\frac{4\pi e}{c} \int d^3_p \, \vec{v} F + \frac{1}{c} \vec{\nabla} \frac{\partial \varphi}{\partial t} \; ; \tag{2}$$

$$\Delta \varphi = -4\pi e \int d^3_p F \; ; \quad div \vec{A} = 0.$$

Here $\vec{A}$ and $\varphi$ are the vector and scalar potentials of the fields induced by the beam which satisfy the boundary conditions $\vec{A}_t = 0$, $\varphi = 0$ at the electrodes; $\{\,;\,\}$ are the Poisson brackets:

$$\{\mathcal{H};\ \} = \frac{\partial \mathcal{H}}{\partial \vec{\mathcal{P}}} \cdot \frac{\partial}{\partial \vec{z}} - \frac{\partial \mathcal{H}}{\partial \vec{z}} \cdot \frac{\partial}{\partial \vec{\mathcal{P}}}$$

$\vec{\mathcal{P}} = \vec{p} + \frac{e}{c}(\vec{A}_\varphi + \vec{A})$ is the canonical, and $\vec{p} = m_0 \vec{v}(1 - v^2/c^2)^{-1/2}$ is the kinetic momentum, $A_\varphi$ is the potential of the focusing fields; $\mathcal{H}$ is the Hamiltonian describing the motion of an individual particle in the focusing and beam-induced fields;

$$\mathcal{H} = c\sqrt{(\vec{\mathcal{P}} - \frac{e}{c}(\vec{A}_\varphi + \vec{A}))^2 + m_0^2 c^2} \tag{3}$$

$m_0$, $e$ are the mass and charge of a particle, and c is the speed of light; $F = F(\vec{\mathcal{P}}, \vec{z}, t)$ is the particle distribution function normalized to the total number of particles in the bunch N:

$$\int d^3 z\, d^3 p\, F = \int d\Gamma\, F = N$$

In this paper we shall examine the stability of stationary beam states with respect to small coherent excitations. In the absence of coherent oscillations, the fields acting on the particle are periodically dependent on time (where the frequency equals the rotation frequency $\omega_s$ ). In this case the particles perform oscillations around a particular equilibrium trajectory. It is convenient to describe these oscillations in action-phase variables $(I, \Psi)$, which are related to $\vec{\mathcal{P}}$ and $\vec{z}$ by a canonical transformation. In the stationary state $I^i$ and $\Psi_i$ are integrals of motion, and the Hamiltonian in these variables depends only on $I: \mathcal{H}_{st} \rightarrow \mathcal{H}_0(I)$:

$$I^i(\vec{\mathcal{P}}, \vec{z}, \theta) = const,$$

$$\dot{\psi}_i = \omega_i(I) = \frac{\partial \mathcal{H}_{st}}{\partial I^i}, \qquad i = 1,2,3$$

Therefore, the distribution function in the stationary state which satisfies the system of equations (1) and (2) will depend only on the $I$ action variables.

In the excited state

$$F = F_{st}(I) + \tilde{F}$$

and

$$(\vec{A}, \varphi) = (\vec{A}, \varphi)_{st} + (\tilde{\vec{A}}, \tilde{\varphi})$$

To investigate the stability of **small oscillations the system** of equations (1) and (2) may be linearized in terms of the deviations from the stationary state ($\tilde{F}, \tilde{\vec{A}}, \tilde{\varphi}$ ). In the linear approximation

$$\mathcal{H} = \mathcal{H}_{st} + \tilde{V} = \mathcal{H}_{st} - \frac{e}{c}(\vec{v}\tilde{\vec{A}}) + e\tilde{\varphi}$$

In the variables ( $I$ , $\psi$ ) the linearized equation for $\tilde{F}$ takes the form:

$$\frac{\partial \tilde{F}}{\partial t} + \omega_k \frac{\partial \tilde{F}}{\partial \psi_k} - \frac{\partial \tilde{V}}{\partial \psi_k} \cdot \frac{\partial F_{st}}{\partial I^k} = 0 \tag{4}$$

and the potentials $\tilde{\vec{A}}$ and $\tilde{\varphi}$ satisfy the equations (2), where in the right hand part $F$ is replaced by $\tilde{F}$

The normal solution of system (4), (2) has the form

$$X_\omega (I, \psi, \theta_s)\, e^{-i\omega t}$$

where

$$X_\omega(I, \psi, \theta_s + 2\pi) = X_\omega(I, \psi + 2\pi, \theta_s) = X_\omega(I, \psi, \theta_s)$$

(here symbol X denotes any of the quantities $\tilde{F}$ , $\tilde{\vec{A}}$ , $\tilde{\varphi}$ ).

In the absence of interaction ( $N \to 0$), the spectrum of normal beam oscillations ($\tilde{F} \sim \exp.(-i\omega t + i m_k \psi^k$ )is $\omega = m_k \omega_k$, $m_k$ are integers).

If the interaction of the beam with induced fields is weak, **i.e. the particle motion is only slightly distorted during one oscillation** period ($\sim 2\pi/\omega_K$), then the spectrum and form of the excitations must be close to the unperturbed values. This means that the main contribution to the normal excitation $F\omega$ close to $m_K\omega_K(\omega = m_K\omega_K + \Delta\omega$: $|\Delta\omega| \ll \min.\{\omega_K\})$ is given by the harmonic (5)

$$F_{\omega,m} \sim \exp.(i m_K \Psi_K)$$

and the effect of the others may be neglected.

Thus, in order to determine the spectrum of excitations due to a first order interaction, the approximated equations may be used:

$$(\omega - m_k\omega_k) F_{\omega,m} = -m_k \cdot \frac{\partial F_{st}}{\partial I^k} \cdot \tilde{V}_{\omega,m}(I) ; \tag{6}$$

$$\Delta \tilde{\tilde{A}} - \frac{1}{c^2} \cdot \frac{\partial^2 \tilde{\tilde{A}}}{\partial t^2} = -\frac{4\pi e}{c} \int d^3p\, \vec{v} F_{\omega,m} e^{i m_k \psi_k - i\omega t} + \frac{1}{c} \vec{\nabla} \frac{\partial \tilde{\varphi}}{\partial t} ;$$

$$\Delta \tilde{\varphi} = -4\pi e \int d^3p\, F_{\omega,m} \cdot e^{i m_k \psi_k - i\omega t} ; \quad \mathrm{div} \tilde{\tilde{A}} = 0, \tag{7}$$

where

$$X_{\omega,m} = \int_0^{2\pi} \frac{d\theta_s}{2\pi} \int_0^{2\pi} \frac{d^3\psi}{(2\pi)^3} X_\omega(I, \psi, \theta_s)\, e^{-i m_k \psi_k}$$

The relative error occuring in the determination of the shift $\Delta\omega = \omega - m_K\omega_K$ will be of the order $|\Delta\omega| / |\ell\omega_s + p_K\omega_K|$.

We shall be interested in effects caused by low-frequency interactions of the beam excitations with the principal-wave field of an ideally matched **double-connected*** wave guide. The remaining part of the fields in the system where the beam is at rest, is of a quasi-static nature, and therefore will not be taken into account in what follows.

*) **i.e. the boundary of its transverse cross-section is a double-connected contour.**

The potential of the "principal (TEM) wave" in an infinite doubly-connected wave guide has the form:

$$\vec{A}(\vec{r},t) = c\,\frac{\vec{A}_0(\vec{r}_\perp)}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\, Q_k(t)\, e^{iky} \qquad (8)$$

where $\vec{A}_0(\vec{r}_\perp)$ is proportional to the electric field set up by the potential difference $U_0$ between the electrodes;

$$\vec{A}_0(\vec{r}_\perp) = (c Z_0)^{1/2}\cdot\frac{\vec{\mathcal{E}}(\vec{r}_\perp)}{U_0} \quad ; \quad \Delta_\perp \vec{\mathcal{E}}(\vec{r}_\perp) = 0$$

$Z_0$ is the wave impedance and c is the speed of light.

Under real conditions the beam interacts with a wave guide segment, the ends of which are terminated with the characteristic resistance ($Z_0$). For low frequency oscillations ($\omega \ell_\perp / c \ll 1$) the wave guide's potential may be represented by expression (8) where $\vec{A}_0(\vec{r})$ is the real electrostatic field exponentially decreasing on a length of the order of the transverse dimensions of the chamber ($\ell_\perp$),as one moves along the wave-guide chamber, away from the terminating sections. It is significant that in the boundary domains the electric field of the"principal wave" has a longitudinal component; therefore it may be excited by the longitudinal motion of the particles.

The value $Q_k(t)$ from (8) satisfies the equation

$$\ddot{Q}_k + c^2k^2 Q_k = e\int d\Gamma\, \vec{v}\vec{A}_0(\vec{r})\, e^{-iky}\cdot f_{\omega,m}\, \overline{e^{i m_k \psi_k - i\omega t}}\,, \qquad (9)$$

the solution of which may be written in the form

$$\overline{Q_k(t)} = \sum_{n=-\infty}^{\infty} Q_{k,n} \exp(-i\omega t - i n \omega_s t)\,, \qquad (10)$$

and

$$Q_{k,n} = e\,\frac{\int d\Gamma (vA_0 e^{iky})^*_{m,n}\cdot F_{\omega,m}}{c^2K^2-(\omega+n\omega_s)^2}\ . \qquad (11)$$

Substituting (10) into (8) we obtain from (6) the integral equation for $F_{\omega,m}$ :

$$(\omega - m_k\omega_k)\,F_{\omega,m} = e^2 m_k\cdot\frac{\partial F_{st}}{\partial I^k}\sum_n{}' \int_{-\infty}^{\infty}\frac{dK(\vec{v}\vec{A}_0 e^{iKy})_{m,n}}{c^2K^2-(\omega+n\omega_s)^2}\,\cdot \qquad (12)$$
$$\cdot\int d\Gamma (vA_0 e^{iky})^*_{m,n}\,F_{\omega,m}$$

For further calculations we shall require formulae for the transition from the co-ordinates and momenta ($\vec{r}$ , $\vec{\mathcal{P}}$) of the particle to the action-phase variables ($I$ , $\Psi$). The effect of stationary induced fields on particle motion may be ignored when the operating point is far from the machine's resonances. Then the transition formulae have the usual form:

$$r = r_b + r_c\ ,\quad r_c = R_0\,\Psi(\theta)\frac{\Delta p_{\parallel}}{p_s}\ ,\quad \Delta p_{\parallel} = p - p_s\ ,$$
$$(r_b, z) = \frac{a_{r,z}}{2|f_{r,z}|_{max}}\left[f_{r,z}(\theta)\exp\left(i\psi_{r,z} + i\varphi_c\frac{d\omega_z}{d\omega_0}\Big|_s\right) + \text{K.C.}\right], \qquad (13)$$
$$\vec{p}_{\perp} = \frac{p_s}{R_0}\cdot\frac{d\vec{z}}{d\theta}\ ;\ \Delta p_{\parallel} = \mu_c\dot{\varphi}_c\ ;\ \theta = \theta_s + \varphi_c + \vartheta_b\ ;\ \theta_s = \omega_s t\ ;$$
$$\vartheta_b = \frac{1}{R_0}\left(\Psi(\theta)\cdot\frac{dr_b}{d\theta} - r_b\frac{d\Psi}{d\theta}\right);\ \varphi_c = \varphi\cdot\sin\psi_c\ ;\ \dot{\psi}_i = \omega_i(p_s),\ i = r, z, c$$
$$\frac{d\omega_{r,z}}{d\omega_0}\Big|_s = \left(\frac{d\omega_{r,z}}{dp}\Big/\frac{d\omega_0}{dp}\right)_s\ ;\quad a_{r,z} = 2|f_{r,z}|_{max}\sqrt{2I_{r,z}\frac{R_0}{p_s}}\ .$$

Here the subscript $S$ denotes the values relating to the equilibrium particle; $2\pi R_0$ is the machine's perimeter; $\omega_0(p)$ is the revolution frequency of the particles; $\mu_c = (d\omega_0/dp)^{-1}_s$ is the mass of synchrotron motion; $f_{r,z}$ are the Floquet functions fulfilling the normalization conditions:

$$f_K\cdot(f_K' + i\nu_K f_K)^* - \text{K.C.} = -2i \qquad (K = r, z)$$

$\nu_{r,z} = (\omega_{r,z}/\omega_0)_s$ is the number of betatron oscillations per turn; $|f_{r,z}|_{max}$ is the greatest value of the modulus of the Floquet function over one machine period; ; $\Psi(\theta)$ is the inhomogeneous periodic solution of the equation:

$$\frac{d^2}{d\theta^2}\psi + \frac{R_0^2}{R^2(\theta)}\left(1 - n(\theta)\right)\psi(\theta) = \frac{R_0}{R(\theta)}$$

where $n(\theta)$ is the guide field **index** and $R(\theta)$ is the orbit's radius of curvature.

**The phase modulation of the transverse** oscillation is related to the **energy dependence of the frequencies** $\omega_z$ **and** $\omega_r$ ;

$$\omega_k(E) = \omega_0(E)\,\nu_k(E) \quad ; \; k = r, z$$

The calculation of the harmonics entering (12) may be simplified by using the field potentiality (field derived from a potential) $\vec{\mathcal{E}}(\vec{r})$

$$\vec{\mathcal{E}}(\vec{r}) = -\frac{U_0}{(c\,Z_0)^{1/2}}\,\vec{\nabla}U(\vec{r})$$

By definition of the Fourier harmonic, we have:

$$\left(\vec{v}\vec{A}_0 e^{iky}\right)_{m,n} = \overline{\left(\frac{dU(\vec{r}(t))}{dt} e^{ikR_0\theta(t) - i m_k \psi_k(t) - in\theta_s}\right)}, \tag{14}$$

where the line denotes the time averaging along the particle trajectory. **By performing the time integration by parts in (14), we can** rewrite this expression in the form

$$\overline{\left(\vec{v}\vec{A}_0 e^{iky}\right)}_{m,n} = -i\omega_s(R_0 k - n - m_k \nu_k)\cdot \overline{V^k_{m,n}}(I), \tag{15}$$

$$V^k_{m,n} = \int_0^{2\pi}\frac{d^3\psi}{(2\pi)^3}\int_{-\pi}^{\pi}\frac{d\theta_s}{2\pi}\, e^{ikR_0\theta - in\theta_s - im_k\psi_k}\, U(\vec{r}_\perp(\theta), \theta) \tag{15a}$$

For low-frequency field excitations ($k\ell_\perp \ll 1$) the azimuthal dependence $U$ may be approximated by the expression

$$\overline{U(\vec{r}_\perp, \theta)} = \begin{cases} U(\vec{r}_\perp) & , \; |\theta| \le \ell/2R_0 \\ 0 & , \; |\theta| > \ell/2R_0 \end{cases} \tag{16}$$

where $\ell$ is the plate length and it is assumed that $\theta$ is measured from the middle of the wave guide.

The harmonic $V^{K}_{m,n}(I)$ may then be calculated by means of a Taylor expansion of (15a) in terms of power of the amplitudes of the transverse oscillations ( $I_{\perp}$ ). The resulting expression is extremely cumbersome and therefore we shall give it only later for particular values of $m$ , where necessary.

In formula (15) the term proportional to ( $R_o K - n$ ), corresponds to the interaction of a beam with the edge fields and the term proportional to $m_K \nu_K$ describes the interaction over the plate length.

We note that when the beam interacts with a system "without a memory", the sum over $n$ in (12) depends weakly on the accurate value of $\omega$ and therefore for a first order accuracy $|\Delta\omega|/\omega_i$ the frequency $\omega$ in the right hand part of (12) may be replaced by $m_i \omega_i$ . When calculating sums over $n$ we shall use the summation formula

$$\sum_{n=-\infty}^{\infty} b_n = \sum_{q=-\infty}^{\infty} \int_{-\infty}^{\infty} dn\, b(n)\, e^{2\pi i q n} \tag{17}$$

By directly substituting (17) into (12), it can be seen that all the terms of the sum with $q \neq 0$ vanish i.e. the integrands in the integrals over $n$ have no singularities in the plane of the complex variable $n$ . In physical terms this corresponds to a total damping of the induced fields during the period corresponding to one revolution of the beam.

Taking into account the above, we rewrite the integral equation for $F_{\omega,m}$ in the form

$$\Delta\omega_m F'_{\omega,m} = e^{\prime} m_K \frac{\partial F_{st}}{\partial I^K} \int_{-\infty}^{\infty} dn \int_{-\infty}^{\infty} \frac{dk (R_o k - n - m_i \nu_i)^2 V^{K}_{m,n}}{c^2 K^2 - (m_i \omega_i + n\omega_s + i\varepsilon)^2} \int d\Gamma\, V^{K*}_{m,n} \cdot F_{\omega,m} \tag{18}$$

where $\Delta\omega_m = \omega - m_i \omega_i$ ; $\varepsilon \to +0$ defines the integration contour. The function $V^{K}_{m,n}$ may be represented in the form

$$V^{K}_{m,n}(a_r, a_z, \varphi) = \frac{a_r^{|m_r|} \cdot a_z^{|m_z|}}{|m_r|! \cdot |m_z|!} \cdot g^{K}_{m,n}(\varphi) + O(a_{\perp}^{|m_r|+|m_z|+1})$$

where

$$g^{\kappa}_{m,n}(\varphi) = \left.\frac{\partial^{|m_z|+|m_z|} V^{\kappa}_{m,n}}{\partial a_z^{|m_z|}\,\partial a_z^{|m_z|}}\right|_{a_z = a_z = 0}$$

depends only on the amplitude of the synchrotron oscillations $\varphi$;

$a_\perp$ is the maximum beam transverse dimension.

In this paper we shall examine only those cases where only the betatron oscillations are non-linear in the stationary state and the non-linearity of the synchrotron oscillations may be ignored.

Then, by means of the substitution

$$F_{\omega,m}(a_z, a_z, \varphi) = \frac{a_z^{|m_z|}\cdot a_z^{|m_z|}}{\omega - m_k\omega_k(I)}\cdot \chi_m(\varphi)$$

**leaving only the lowest powers of the amplitude of transverse oscillations in (18), we obtain the equation for $\chi_m(\varphi)$;**

$$\Omega_m \chi_m(\varphi) = \xi(\varphi)\int\limits_{-\infty}^{\infty} dn \int\limits_{-\infty}^{\infty} du \frac{(u-n-m_i\nu_i)^2 g^{\kappa}_{m,n}(\varphi)}{u^2-\beta^2(m_i\nu_i+n+i\varepsilon)^2}\cdot \int\limits_0^{\infty} d\varphi'\varphi' g^{\kappa\,n}_{m,n}(\varphi')\chi_m(\varphi'). \qquad (18a)$$

Here we introduced the following notation:

$$\frac{1}{\Omega_m} = \frac{1}{A_{m_\perp}}\left\langle m_\kappa \cdot \frac{\partial}{\partial I^k}\,\frac{a_z^{|m_z|}\,a_z^{|m_z|}}{\omega - m_i\omega_i(I)}\right\rangle ,$$

$$A_{m_\perp} = \left\langle m_\kappa \cdot \frac{\partial}{\partial I^k}\left[a_z^{2|m_z|}\cdot a_z^{2|m_z|}\right]\right\rangle ; \qquad (18b)$$

$$\xi(\varphi) = \begin{cases} -\dfrac{Ne^2 m_c\,\omega_c^2 R_0}{c^2}\cdot\dfrac{\partial\rho}{\partial I_c}, & m_z\cdot m_z = 0 \\[2ex] -\dfrac{Ne^2\beta^2\rho(\varphi)}{R_0}\cdot\dfrac{A_{m_\perp}}{(|m_z|!\cdot|m_z|!)^2}, & |m_z|, |m_z| > 0 \end{cases}$$

$\beta = v/c$, the brackets $\langle\ \ \rangle$ denote the mean of $F_0(I_\perp)$ (we assume here that $F_{st}(I_\perp, I_c)$ can be **factorized**);

$$F_{st}(I_\perp, I_c) = F_0(I_\perp)\,\rho(I_c).$$

With no frequency spread (the oscillations in the stationary state are linear) the spectrum of normal collective excitations coincides with the spectrum of eigenvalues of equation (18a):

$$\omega - m_K\,\omega_\perp = \Omega_m .$$

If the oscillations in the stationary state are non-linear, the frequencies of normal oscillations $\omega$ may be found from the dispersion equation (18b), having found first the eigenvalues of equation (18a).

Generally speaking, the solution of the integral equation (18a) with some smooth distributions $\rho(\varphi)$ requires the application of numerical methods. We shall investigate below a series of cases where the qualitative dependence of the spectrum of normal oscillations on the characteristic parameters of the problem (length of plate, length of bunch, $d\omega_\perp/d\omega_s$ etc.) can be **obtained analytically.**

## II. MODEL SOLUTIONS

For simplicity's sake, we shall examine only those excitations for which the collective betatron oscillations in the beam are one-dimensional, i.e. $m_z \cdot m_{\bar z} = 0$, $|m_z| + |m_{\bar z}| > 0$.

### 1. Axial and axial-longitudinal excitations

First let $m_z = 0$ (axial-longitudinal excitations). The formula for $\Delta\omega$ is easily obtained for a model distribution

$$\rho(\varphi) = \delta(\varphi^2 - \Delta^2) \tag{19}$$

where $\Delta$ is linked with the beam's "length" $l_b = 2R_0\Delta$.
In this case $g^u_{m,n}(\varphi)$ is equal to

$$g^u_{m,n}(\varphi) = \frac{1}{(2\pi)^{3/2}} J_{m_c}\left(\varphi\left[n + m_z \frac{d\omega_z}{d\omega_s}\Big|_s\right]\right)\frac{B^{(z)}_{u,n}}{2^{|m_z|}} \cdot \frac{\partial^{|m_z|} U}{\partial z^{|m_z|}}\Big|_{z=0} \tag{20}$$

$$B^{(z)}_{u,n} = \int\limits_{-\ell/2R_0}^{\ell/2R_0} d\theta\, \bar{f}_z^{\,m_z}(\theta)\, e^{i(u\cdot n)\theta} \tag{20a}$$

where the notation $\bar{f}_z(\theta) = f_z(\theta)/|f_z|_{max}$. is introduced.

By substituting (19) and (20) into (18a), in the ultrarelativistic case ($\gamma^2 \ell_b \gg \ell$ , $\gamma = E_s/m_0 c^2$ ), we obtain the decrements ($\delta = -\mathrm{Im}\,\omega$) of the axial-longitudinal excitations;

$$\delta_m = N\delta_0 \int\limits_{-\infty}^{\infty} dn\, J^2_{m_c}\left(\Delta\left[n + m_z \frac{\omega_z}{\omega_s}\right]\right) \left|B^{(z)}_{-n-m_z\nu_z,\, n}\right|^2 (n + m_z \nu_z) \tag{21}$$

Here $J_m(x)$ is the Bessel function of order $m$ ;

$$\delta_0 = \frac{r_0 c\, |f_z|^2_{max}}{2\pi^2 \gamma}\, \mathrm{sign}(m_z) \left(\frac{\nu_z}{2}\right)^{2|m_z|-2} \sum \left[\frac{1}{(|m_z|-1)!} \cdot \frac{\partial^{|m_z|} U}{\partial z^{|m_z|}}\bigg|_{z=0}\right]^2$$

$r_0 = \dfrac{e^2}{m_0 c^2}$ is the classical radius of a particle.

Formula (21) simplifies in two limiting cases:

a) $\ell_b \ll (|m_c| + 1)\ell$ short bunch. After integration over $n$, we obtain

$$\delta_m = N\delta_0 m_z \left(\frac{\pi \ell}{2R_0} \Phi_1 + \frac{4\ell_b}{\pi R_0}\left[\Phi_2 - \frac{d\nu_z}{d\ell_b R_0} \cdot \Phi_3\right] \cdot \frac{1}{4m_c^2 - 1}\right), \tag{22}$$

The factors $\Phi_1$ , $\Phi_2$ and $\Phi_3$ are equal to

$$|f_z|^2_{max} \Phi_1 = J^2_{m_c}\left(\frac{m_z \ell_b}{2} \frac{d\nu_z}{dR_0}\right) \frac{R_0}{\ell} \int\limits_{\theta_1}^{\theta_2} d\theta\, \bar{f}_z^{\,2|m_z|-2}(\theta) \;;$$

$$|f_z|^2_{max} \Phi_2 = \begin{cases} \dfrac{3|m_z|-2}{2|m_z|-2} \left|\bar{f}_z(\theta_2)\right|^{2|m_z|-2} + \dfrac{|m_z|-2}{2|m_z|-2} \left|\bar{f}_z(\theta_1)\right|^{2|m_z|-2}, & |m_z| > 1\,; \\ 2 - \ln\left(\dfrac{|f_z(\theta_1)|}{|f_z(\theta_2)|}\right), & |m_z| = 1\,; \end{cases}$$

$$\Phi_3 = |\bar{f}_z(\theta_2)|^{2|m_z|} + |\bar{f}_z(\theta_1)|^{2|m_z|} ,$$

where $\theta_2 = \ell/2R_o$, $\theta_1 = -\ell/2R_o$ are the co-ordinates of the ends of the plate.

Formula (22) differs from the corresponding formula from paper (4), in the terms proportional to $\Phi_1$ and $\Phi_2$.

The first term in (22), proportional to the length of the plate, represents the damping decrement of the coherent motion of the beam due to the energy loss into the matched line due to the transverse motion of the bunch. We should point out that this decrement is always positive and corresponds to the "fast damping" previously obtained in (5). The second term, which does not depend on the length of the plate, corresponds to the so-called "head-tail" effect. This term's sign depends on the type of excitation and the sign and value of $d\nu_z / d\ln R_o$. Its occurrence is physically linked to the fact that at the edges of the plate, where the electric field of the "main wave" has a longitudinal component, a (TEM) wave is excited by longitudinal motion.

For $m_c \neq 0$ the factor $\Phi_1$ is proportional to the value $\left(\frac{m_1 \ell_b}{2} \cdot \frac{d\nu_z}{dR_o}\right)^{2|m_c|}$ and therefore the synchrobetatron excitations may become unstable. However, excitations with $m_c = 0$ may always be made stable by selecting an adequate plate length.

We should point out in particular that the value of the "boundary" terms is proportional to the $2|m_z|$ -order of the Floquet function modulus at the edges of the line. This factor may be decisive in the selection of the position of the plates in machines with large beats of the Floquet function (machines with low $\beta$ - function values).

In the azimuthally-symmetrical case, if the guide field is linear, then

$$\frac{d\nu_z}{d\ln R_o} = \frac{n(1+n)}{2\nu_z}; \quad f_z = \frac{1}{\sqrt{\nu_z}}; \quad 0 < n < 1$$

$n$ is the guide field exponent. The stability condition has the form $\delta > 0$

$$
\begin{aligned}
&1) \qquad m_z = 0 \qquad \frac{8\ell_b}{\pi^2 \ell} \cdot \frac{1}{\nu_z} \cdot \frac{d\omega_z}{d\omega_s} = \frac{4\ell_b}{\pi^2 \ell}(1-n) < 1 \\
&2) \qquad m_z \neq 0 \qquad \frac{1}{\nu_z} \cdot \frac{d\omega_z}{d\omega_s} = \frac{1-n}{2} > 0,
\end{aligned}
\tag{23}
$$

that is, in this case, the axial-longitudinal excitations are stable. A more general stability condition (for arbitrary focusing) may be obtained directly from (22).

We shall give also the formula for the decrement for the most important type of oscillation when $|m_z| = 1$ :

$$
\delta_m = \frac{N e_0 c}{2p} \cdot \frac{e}{2\pi R_0} \cdot \left(\frac{\partial U}{\partial z}\Big|_{z=0}\right)^2 \left[ J_{m_z}^2 \left( 2 \frac{d\nu_z}{dn} \right) + \frac{4\ell_b}{\pi^2 \ell} \left\{ 2 + \ln\left(\frac{|f_z(\theta_2)|}{|f_z(\theta_1)|}\right) - \frac{d\nu_z}{d\ln R_0} \left( |\bar{f}_z(\theta_2)|^2 + |\bar{f}_z(\theta_1)|^2 \right) \right\} \frac{1}{4m_z^2 - 1} \right\}
\tag{24}
$$

If the machine's chromatism is large

$$
\frac{\ell_b}{R_0} \left| m_z \frac{d\omega_z}{d\omega_s} \right| \gg 1
$$

(as is the case in strong-focusing machines), then the decrement of a short bunch may be estimated with logarithmic accuracy by means of the formula

$$
\delta_m \simeq - \frac{N \delta_0 R_0}{\pi m_z \ell_b \frac{d\omega_z}{d\omega_s}} \ln \left( \frac{\ell_b}{R_0(|m_z|+1)} \left| m_z \frac{d\omega_z}{d\omega_s} \right| \right)
\tag{22a}
$$

In obtaining (22a), it was assumed that the guide field was azimuthal by uniform.

We note that the decrement $\delta$ is inversely proportional to the length of the bunch $\ell_b$ and the quantity $d\omega_z / d\omega_s$ related to the machine's chromatism (the dependence of the numerator in (22a) on $\ell_b$ is weak, because it is logarithmic).

b) $\ell_b \gg \ell(|m_c|+1)$ – a long bunch. For simplicity's sake we shall assume that the guide field is a azimuthal-uniform. We then obtain from (21)

$$\delta_m = N\delta_0 \frac{\ell}{\ell_b R_0}\cdot\frac{m_z \nu_z c}{\ell_0}\left(1+2\frac{d\ln\nu_z}{d\ln R_0}\ln\left[\frac{\ell_b}{\ell(|m_z|+1)}\right]\right) \tag{25}$$

It is clear that instability may occur when the machine's chromatism ($d\ln\nu_z / d\ln R_0$ ) is negative. We should draw attention to the fact that the decrement in (25) is inversely proportional to the bunch length $\ell_b$ . This is related to the fact that at $\ell_b \gg \ell(|m_c|+1)$ , the main contribution to the integral (21) is made by the harmonic interval $\Delta n \sim R_0/\ell_b$ ; the contribution of each harmonic in this interval is of the order of ( $(\ell/2R_0)^2$ ), so that

$$\delta \sim \frac{\ell}{R_0}\cdot\frac{\ell}{\ell_b}$$

2. Radial-longitudinal excitations ( $m_z = 0$)

Collective excitations of this type may differ substantially from axial-longitudinal excitations due to the coupling of radial and longitudinal particle motion in the storage ring. Therefore, when calculating $g^u_{m,n}(\varphi)(m=\{m_r, m_c, 0\}$ modulation of the azimuth by betatron motion needs to be taken into account. The function $g^u_{m,n}(\varphi)$ is represented by the sum

$$g^{u(1)}_{m,n}(\varphi) + g^{u(2)}_{m,n}(\varphi) \tag{26}$$

where $g^{u(1)}_{m,n}(\varphi)$ is obtained from (20) by substituting the index $z$ for $r$ and $g^{u(2)}_{m,n}(\varphi)$, which describes the effect of radial-longitudinal coupling, is given by

$$g^{u(2)}_{m,n}(\varphi) = \frac{in}{R_0(2\pi)^{3/2}} J_{m_c}\left(\varphi\left[n+m_z\frac{d\omega_z}{d\omega_s}\right]\right)\frac{g^{(z)}_{u,n}}{2^{|m_z|}}\cdot \left(\left.\frac{\partial^{|m_z|-1}U}{\partial z^{|m_z|-1}}\right|_{z=0}\right) \tag{27}$$

where
$$b^{(2)}_{u,n} = \int_{\theta_1}^{\theta_2} d\theta \, \bar{f}_2^{\,m_2-1}(\theta) \left\{ \psi(\theta)\left[\bar{f}_2' + i\nu_2 \bar{f}_2\right] - \bar{f}_2(\theta)\,\psi'(\theta) \right\} e^{i(u-n)\theta}$$

After substituting (27) and (19) into (18a), we obtain the formula for the decrements

$$\delta_m = \delta_m^{(0)} - \delta_m^{(1)} ,$$

in which $\delta_m^{(0)}$ is obtained from (21) by substituting the index $z$ for $r$ , $\delta_m^{(1)}$ equals

$$\delta_m^{(1)} = N \delta_{01} \int_{-\infty}^{\infty} dn\, n(n+m_2\nu_2)\, J^2_{m_2}\left(\Delta\left[n + m_2 \frac{d\omega_2}{dJ_2}\right]\right) \cdot \tag{28}$$

$$\cdot \, \mathrm{Im} \left\{ b^{(2)}_{u,n} \cdot b^{(2)*}_{u,n} \right\}_{u = n - m_2\nu_2}$$

where

$$\delta_{01} = \frac{|m_2| z_0 c\, |f_2|^2_{max}}{2\pi^2 p R_0} \left\langle \left(\frac{a_2}{2}\right)^{2|m_2|-2} \right\rangle \frac{1}{(|m_2|-1)!^2} \cdot \frac{\partial}{\partial z} \left( \frac{\partial^{|m_2|-1} U}{\partial z^{|m_2|-1}} \right)^2 \Bigg|_{z=0}$$

The integral over $n$ in formula (28) diverges logarithmically. However, in a real case, where the "length" of decrease of the edge field is finite, the integrand in (28) can be multiplied by a factor which cuts off the integral over $n$ at a value of the order $n_0 \sim R_0/\ell_\perp$. In order to estimate the decrement in terms of its order of magnitude, the specific form of this factor is not important. In particular, the infinite limits of integration in (28) may simply be replaced by finite limits with $|n_{max.}| = n_0$ .

We shall give the formulae of the decrements in two limiting cases:

a)
$$\ell_b \ll (|m_2|+1)\,\ell$$
$$\delta_m^{(1)} = \frac{N \delta_{01}}{\pi\, \ell_b} \cdot \Phi_4 \cdot \ln\left[\frac{\ell_b}{\ell_\perp (|m_2|+1)}\right] \tag{29}$$

where

$$2|f_2|^2_{max}\, \Phi_4 = \psi(\theta_2)\,|\bar{f}_2(\theta_2)|^{2|m_2|-2} + \psi(\theta_1)\,|f_2(\theta_1)|^{2|m_2|-2}$$

We note that $\delta_m^{(1)}$ does not depend on $d\omega_z/d\omega_s$ and, at

$$\Phi_4 \cdot \frac{\partial}{\partial z}\left(\frac{\partial^{|m_z|-1} U}{\partial z^{|m_z|-1}}\right)^2\Bigg|_{z=0} > 0 \tag{30}$$

it reduces the decrement of the radial oscillations.

If $\delta_m^{(0)}$ is positive, then, by comparing (29) and (22), we see that the radial oscillations ($m_c = 0$) may become unstable when

$$\overline{\psi} \cdot \frac{\ell_\perp R_0}{\ell_b \cdot \max\{\ell, \ell_s \left|\frac{d\omega_z}{d\omega_s}\right|\}} > 1$$

and the radial-phase oscillations when

$$\overline{\psi} \cdot \frac{\ell_\perp R_0}{\ell_b^2 \left|\frac{d\omega_z}{d\omega_s}\right|} > 1$$

In this case, the increments of normal oscillations are inversely proportional to the length of the bunch $\ell_b$. We shall give the decrement expression for the most important type of oscillation $|m_z| = 1$.

$$\delta_m^{(1)} = \frac{N z_0 c}{4\pi^3 \gamma} \cdot \frac{\psi(\theta_1) + \psi(\theta_2)}{\ell_b} \cdot \ln\left[\frac{\ell_b}{\ell_\perp(|m_c|+1)}\right] \cdot \frac{\partial U^2}{\partial z}\Bigg|_{z=0} \tag{31}$$

b) $\ell_b \gg (|m_c|+1)\,\ell$. For an azimuthally-uniform machine ($\psi = \overline{\psi} = const$, $f_z = 1/\sqrt{\nu_z}$) $\delta_m^{(1)}$ has the form

$$\delta_m^{(1)} = \frac{N z_0 c\,|m_z|}{2\pi^3 \gamma} \frac{\overline{\psi}}{\ell_b} \ln\left(\frac{\ell}{\ell_\perp}\right) \left\langle \left(\frac{a_z}{2}\right)^{2|m_z|-2} \right\rangle \cdot \tag{32}$$

$$\cdot \frac{1}{(|m_z|-1)!^2} \cdot \frac{\partial}{\partial z}\left(\frac{\partial^{|m_z|-1} U}{\partial z^{|m_z|-1}}\right)^2\Bigg|_{z=0} .$$

It must be pointed out that the appearance of large logarithms in formulae (25), (29) and (32) is a specific peculiarity of distribution (19). This is so because the decrements in formulae (25), (29) and (32) for an arbitrary smooth distribution $\rho(\varphi)$, determine the sums of the decrements of normal beam excitations.

At large excitation(mode)numbers the separate decrements decrease slowly as the mode number $k\,(\delta_k \sim 1/k)$ increases, so that the sum turns out to be logarithmically large.

## 3 Synchrotron excitations

For excitations with $\omega \simeq m_c\omega_c$, $m_r = m_z = 0$ with an accuracy up to the order $(a_\perp / \ell_\perp)$ ($a_\perp$ is the tranverse beam's width), equation (18a) may be written in the form

$$\lambda \chi_m(y) = \int_{-\infty}^{\infty} dx\, \mathcal{L}\left(\frac{x}{\Delta}\right) \left[ U^2_{z=0} + \frac{R_0 \mu_c \omega_c m_c \Delta}{\beta_s x} \mp \frac{\partial U^2}{\partial z}\Big|_{z=0} \right] \cdot \qquad (33)$$

$$\cdot J_{m_c}(xy) \int_0^{\infty} dy' y' \frac{dq}{dy'} J_{m_c}(xy') \chi_m(y'),$$

where

$$\lambda = \frac{2\pi^2 R_0^2 \mu_c \omega_c}{N e^2 m_c} \Delta^3 (\omega - m_c \omega_c), \quad x = n\Delta,$$

$$\rho(\varphi) = \frac{1}{\Delta^2} q(\varphi/\Delta)\ ; \quad \int_0^{\infty} dy\, y\, q(y) = 1.$$

In this case, we shall use the assumptions that the guide field is azimuthal by uniform and that the bunch moves parallel to the axis of the wave guide. In view of the low value of the synchrotron frequency ( $\omega_c \ll \omega_s$ ), the function $\mathcal{L}$ mav be represented as:

$$\mathcal{L}(x) = \frac{\Delta}{4} \cdot \frac{\sin\left(\frac{4\ell}{\ell_s} x\right)}{x} + \frac{i\Delta}{2} \cdot \frac{\sin^2\left(\frac{2\ell x}{\ell_s}\right)}{x}\ ; \quad \ell_s = 2R_0\Delta$$

In this section, we obtain a solution for (33) for a case where $q(y)$ is a "step",

$$q(y) = \begin{cases} 2, & 0 \le y \le 1 \\ 0, & y > 1 \end{cases} \qquad (34)$$

From (33), we immediately obtain the dispersion equation

$$\lambda = -2 \int_{-\infty}^{\infty} dx\, \mathcal{L}\left(\frac{x}{\Delta}\right) \left[ U^2_{z=0} + \frac{\mu_c \omega_c m_c R_0 \Delta}{x \beta_s} \mp \frac{\partial U^2}{\partial z}\Big|_{z=0} \right] J^2_{m_c}(x) \qquad (35)$$

As $\Im m\, \mathcal{L}(x)$ is an odd function of $x$ , the first term in (35), which is directly due to the total energy losses, contributes only to the real part of the frequency shift. The second term , which is proportional to the loss gradient, determines the excitation decrements, equalling:

$$\delta_m = \frac{2 N z_0 c}{\pi^2 \gamma} \cdot \frac{m_c^2 \overline{\Psi}}{\ell_b} \left( \frac{\partial U^2}{\partial z} \Big|_{z=0} \right) \int_0^\infty dx \frac{\sin^2\left(\text{[illegible]}\right)}{x^4} J_{m_c}^2(x) \qquad (36)$$

The above expression simplifies substantially in two limiting cases:

a) $\ell_b \ll |m_c| \ell$ "long" plate. When this condition is fulfilled, the square of the sine in (36) oscillates rapidly and it may be replaced by the mean value:

$$\delta_m \simeq \frac{N z_0 c}{\gamma \pi^3} \frac{\overline{\Psi}}{\ell_b} \left( \frac{\partial U^2}{\partial z} \Big|_{z=0} \right) \qquad (37)$$

The condition for the damping of coherent synchrotron oscillations is

$$\overline{\Psi} \left( \frac{\partial U^2}{\partial z} \Big|_{z=0} \right) > 0 \qquad (38)$$

This signifies that, for $\overline{\Psi} > 0$, the plates must be situated on the outside of the equilibrium orbit. The physical reasons for this are obvious: the modulation of the coherent (energy) losses must be such that when the energy increases these losses increase. We wish to draw particular attention to the outstanding characteristic of this method of damping: for all excitations $c|m_c| < \ell_b / \ell_\perp$ the decrement's value does not depend ont the multipolarity number (the constraint on $m_c$ is related to the fact that at $|m_c| > \ell_b / \ell_\perp$ the excitation of other types of wave has to be taken into acoount). This is because the radiation formation length equals the "length" of the edge.

$$\ell_p \sim \ell_\perp \ll \frac{\ell_b}{|m_c|}$$

that is, separate "bunches" radiate independently (6).

b) $\ell_b \gg |m_c|\,\ell$ short plate. In this case, the main contribution to the integral in (36) is made by the region $|m_c| < x < \ell_b/\ell$. We then obtain with logarithmic accuracy:

$$\tilde{\delta}_m \approx \frac{8 N r_0 c}{\gamma \pi^3} \cdot \left(\frac{2\pi c}{\ell_b}\right)^2 \frac{\overline{\Psi}}{\omega_s} \left(\frac{\partial U^2}{\partial z}\Big|_{z=0}\right) \ln\left(\frac{\ell_b}{\ell |m_c|}\right) . \tag{39}$$

The condition for the stability of excitations, as in the case of a short bunch, is also given by inequality (38).

## III. QUALITATIVE PICTURE OF THE EXCITATION SPECTRUM FOR SMOOTH $\rho(\varphi)$

In this section, we shall examine a serie of limiting cases where equation (18a) may be converted into an equation with a real symmetrical kernel. The spectra of the eigen values of such equations have been well-studied (viz. for example (8)), so that it is possible to study the stability of excitations for a comparatively wide range of smooth $\rho(\varphi)$. For simplicity's sake we consider an azimuthally-symmetrical guide field and assume that $\rho(\varphi)$ depends on one parameter: the "width" $\Delta$. That is,

$$\rho(\varphi) = \frac{1}{\Delta^2}\, q(\varphi/\Delta) \; ; \quad \int_0^\infty dy\, y\, q(y) = 1 \tag{40}$$

For the time being we shall assume that $\overline{\Psi} = 0$, and then equation (18a) may be written in the form

$$\lambda_m \chi_m(y) = \int_0^\infty dy'\, y'\, q(y')\, \mathcal{K}_\perp(y|y')\, \chi_m(y') \tag{41}$$

where

$$\lambda_m = -\frac{\Delta}{N\delta_1}\left(\omega - m_i \omega_i\right),$$

$$\delta_1 = \frac{r_0 c}{2\pi^2 \gamma} \left\langle \left[\frac{\operatorname{sign}(m_r)}{\nu_r}\left(\frac{a_r}{m_r}\right)^2 + \frac{\operatorname{sign}(m_z)}{\nu_z}\left(\frac{a_z}{m_z}\right)^2\right] \cdot \left(\frac{a_r}{2}\right)^{2|m_r|-2} \cdot \left(\frac{a_z}{2}\right)^{2|m_z|-2} \sum \left[\frac{1}{(|m_r|-1)!\,(|m_z|-1)!} \cdot \frac{\partial^{|m_r|+|m_z|} U}{\partial x^{|m_r|}\, \partial z^{|m_z|}}\Big|_{\vec{r}_\perp = 0}\right]^2\right.$$

$$\mathcal{K}_\perp(y|y') = \int_{-\infty}^{\infty} dx\, \varphi_\perp\!\left(\frac{x}{\Delta} - m_\kappa \frac{d\omega_\kappa}{d\omega_s}\right) J_{m_c}(xy)\, J_{m_c}(xy') \;, \quad \kappa = r, z$$

The function $\mathcal{L}_\perp(X)$ equals:

$$\mathcal{L}_\perp(x) = -\frac{\ell}{2R_0}\frac{m_\varkappa \nu_\varkappa}{2x+m_\varkappa \nu_\varkappa} - \frac{x\cdot \sin\frac{\ell}{R_0}(2x+m_\varkappa\nu_\varkappa)}{(2x+m_\varkappa\nu_\varkappa)^2} + \tag{42}$$

$$+ 2i(x+m_\varkappa \nu_\varkappa)\cdot\frac{\sin^2\frac{\ell}{2R_0}(2x+m_\varkappa\nu_\varkappa)}{(2x+m_\varkappa\nu_\varkappa)^2}, \quad \varkappa = r, z$$

The kernel $\mathcal{K}_\perp(y/y')$ is clearly symmetrical:

$$\mathcal{K}_\perp(y/y') = \mathcal{K}_\perp(y'/y)$$

1. Betatron excitations ($m_c = 0$)

Let us first examine one-dimensional, say axial excitations of a short bunch $\ell_b < \ell$ . Ignoring quantities of the order

$$\frac{\ell_b}{\ell}\cdot\frac{d\nu_z}{d\ln R_0} \ll 1 \; ; \; \frac{\ell_b \nu_z}{\ell} \ll 1,$$

we rewrite $\mathcal{K}_\perp(y/y')$ in the form

$$\mathcal{K}_\perp(y/y') = i\,\frac{\Delta\cdot m_z\cdot\nu_z\ell}{R_0}\,\overline{\mathcal{K}}_\perp(y/y'),$$

$$\overline{\mathcal{K}}_\perp(y/y') = \int_0^\infty dx\,\frac{\sin^2 x}{x^2}\,J_0\!\left(\frac{xy\ell_b}{2\ell}\right)J_0\!\left(\frac{xy'\ell_b}{2\ell}\right)$$

(41) is transformed into the equation

$$\overline{\lambda}_m \chi_m(y) = \int_0^\infty dy'y'q(y')\,\overline{\mathcal{K}}_\perp(y/y')\,\chi_m(y'), \quad \overline{\lambda}_m = -\frac{i\lambda R_0}{\Delta m_z \nu_z \ell}$$

in which the function $q(y)$ is positive by definition. It is shown in the annex that all the characteristic roots of such an equation are positive numbers.

Therefore, in the given approximation, the decrements of one-dimensional excitations are positive and equal to

$$\delta_K = \frac{\lambda_K N r_o c}{\pi \gamma} \cdot \frac{e\,|m_z|}{2\pi R_o} \cdot \left\langle \left(\frac{a_z}{2}\right)^{2|m_z|-2} \right\rangle \cdot$$

$$\cdot \left( \frac{1}{(|m_z|-1)!} \cdot \frac{\partial^{|m_z|} \gamma}{\partial z^{|m_z|}} \bigg|_{z=0} \right)^2 ; \quad K = 1, 2, 3, \ldots ,$$

As the eigenvalues $\bar{\lambda}_K$ satisfy the inequality $\bar{\lambda}_K \lesssim \pi/2$ (viz. (A.6)), the decrements fulfil $\delta_K \lesssim \delta_0$ where $\delta_0$ is the decrement of one-dimensional betatron excitations obtained in (5).

When $\frac{l_b}{l} \left| \frac{d\omega_z}{d\omega_s} \right|$ is of the order of unity, the numerical solution (41) is required in order to investigate the stability of the excitations.

In the case of two-dimensional betatron excitations ($|m_r \cdot m_z| > 0$), the coherent oscillations may become unstable. The excitation decrements equal:

$$\delta_K = \bar{\lambda}'_K \cdot \frac{N S_1 m_r \nu_r l}{R_o} ; \quad \bar{\lambda}'_K \geqslant 0 , \quad K = 1, 2, 3, \ldots$$

Hence we immediately obtain the oscillations stability condition

$$\left\langle \left[ \frac{\mathrm{sign}(m_r)}{\nu_r} \left(\frac{a_r}{m_r}\right)^2 + \frac{\mathrm{sign}(m_z)}{\nu_z} \cdot \left(\frac{a_z}{m_z}\right)^2 \right] \cdot \right.$$

$$\left. \cdot \left(\frac{a_z}{2}\right)^{2|m_z|-2} \cdot \left(\frac{a_r}{2}\right)^{2|m_r|-2} \right\rangle \cdot (m_r \nu_z + m_z \nu_r) > 0$$

coinciding with that obtained in /5/.

## 2. Axial-longitudinal excitations ( $m_r = 0$)

a) Short bunch $l_b \ll (|m_c|+1) l$. In this case, with accuracy up to the order of $(l_b \cdot d\nu_z / dR_o)^2 / 4 \ll 1$, the kernel $\mathcal{K}_1(y/y')$ equals:

$$\mathcal{K}_1(y/y') = \left\{ \frac{2l}{R_o} \left( \frac{d\nu_z}{da_z R_o} - \frac{\nu_z}{2} \right) + \frac{i}{2} \cdot \frac{d\omega_z}{d\omega_s} \right\} m_z \Delta^2 \int_0^\infty \frac{dx}{x^2} J_{m_c}(xy) J_{m_c}(xy')$$

Therefore, the excitation decrements may be written in the form

$$\delta_k = \frac{\lambda_k N r_0 c |m_c|}{4\pi^2 \gamma} \Delta \cdot \frac{d\omega_z}{d\omega_s} \left\langle \left(\frac{c}{2}\right)^{2|m_c|-2} \right\rangle \left( \frac{1}{(|m_c|-1)!} \cdot \frac{\partial^{|m_c|} U}{\partial z^{|m_c|}} \Big|_{z=0} \right)^2 \qquad (43)$$

where $\lambda_k$ are the characteristic roots of the equation

$$\lambda_k \chi_m(y) = \int_0^\infty \frac{dx}{x^2} J_{m_c}(xy) \int_0^\infty dy' y' g(y') J_{m_c}(xy') \chi_m(y') \qquad (44)$$

which are clearly real positive numbers (viz. annex). Consequently, the condition for the stability of excitations takes the form:

$$\frac{d\omega_z}{d\omega_s} > 0 \qquad (45)$$

as for a $\delta$ -type distribution.

If the excitation is unstable $(d\omega_z / d\omega_s < 0)$, then it is interesting to know the region containing the numerical value of the maximum increment. According to the formula (43) $\delta_{max.}$ is determined by the maximum eigenvalue of equation (44) $\lambda_{max}$ for which the inequalities (A.6) of the annex are correct.

This estimate depends both on the type of distribution and also on the type of "test" function $\rho_m(y)$ (viz.(A.5)). For example, for the distribution $g = 2\exp$ and the test function $(-y^2)$

$$g = 2\exp(-y^2)$$
$$\rho_m(y) = \left(\frac{2}{|m_c|!}\right)^{1/2} y^{|m_c|+1/2} e^{-y^2/2}$$

we obtain inequalities for $\lambda_{max}$

$$\sqrt{\frac{\pi}{2}} \cdot \frac{2^{-2|m_c|}(2|m_c|-1)!!}{|m_c|! \cdot (2|m_c|-1)} \leqslant \lambda_{max} \leqslant \frac{2}{\sqrt{\pi}} \cdot \frac{1}{4m_c^2-1}$$

b) Lastly, we shall examine the axial-longitudinal excitation of a long bunch $\ell_b \gg \ell(|m_c|+1)$. In this case, the main contribution to the interaction of the bunch with the line is made by the region of the harmonics of the revolution frequency ( $n = X/\Delta$ )

$$|n| < R_0/\ell \qquad (46)$$

In this case the kernel $\mathcal{K}_\perp(y/y')$ may be written in the form

$$\mathcal{K}_\perp(y/y') = \frac{i}{2}\left(\frac{\ell}{R_0}\right)^2 \frac{d\nu_z}{d\ln R_0} \int_0^{x_{max}} dx\, J_{m_z}(xy)\, J_{m_z}(xy') \tag{47}$$

where $X_{max.} = \frac{\ell}{\ell_b}$.

Using the standard method (viz. above), equation (47) leads to an equation with a positive symmetrical kernel. Therefore, the stability of the excitations is determined only by the sign of the chromatism ( $d\nu_z/d\ln R_0$ ).

The excitation decrements, equal to

$$\delta_K = \frac{\lambda_K N e_0 c}{\pi \gamma}\left(\frac{\ell}{2\pi R_0}\right)\cdot\frac{|m_z|\ell}{\ell_b}\cdot\frac{d\nu_z}{d\ln R_0}\left\langle\left(\frac{a_z}{2}\right)^{2|m_z|-2}\right\rangle\cdot \left(\frac{1}{(|m_z|-1)!}\cdot\frac{\partial^{|m_z|}U}{\partial z^{|m_z|}}\bigg|_{z=0}\right)^2 \tag{48}$$

are expressed by means of the eigenvalues of the equation

$$\lambda_K \chi_m(y) = \int_0^{x_{max}} dx\, J_{m_z}(xy) \int_0^{\infty} dy' y' q(y')\, J_{m_z}(xy')\, \chi_m(y')$$

which is easily obtained from (41), taking into account (47). The corresponding estimate for the maximum increment will be given below (viz. point 4).

### 3. Radial-longitudinal excitations ( $m_z = 0$)

As mentionned above, this type of excitation differs from axial-longitudinal excitations in storage rings by the presence of coupling between radial and longitudinal collective bunch motion.

Taking into account the modulation of the azimuth $\theta$ by the radial oscillations, we rewrite (18a) in the form

$$\lambda_m \chi_m(y) = \int_0^{\infty} dy' y' q(y')\left(\mathcal{K}_\perp(y/y') - \mathcal{K}_\parallel(y/y')\right)\chi_m(y') \tag{49}$$

where $\lambda_m$ and $\mathcal{K}_\perp(y/y')$ are obtained from the corresponding values in (33) by replacing the index $z$ by $\lambda$, and $\mathcal{K}_{\parallel}$ equals

$$\mathcal{K}_{\parallel}(y/y') = \zeta_{\parallel} \int_0^{x_0} dx \sin^2 \frac{2\ell x}{\ell_s} \cdot J_{m_z}(xy)\, J_{m_z}(xy'),$$

$$\zeta_{\parallel} = \frac{2i \mp \nu |m_z|}{R_0} \left[ \frac{\partial}{\partial z} \left( \frac{\partial^{|m_z|-1} U}{\partial z^{|m_z|-1}} \right)^2 \Bigg|_{z=0} \Bigg/ \left( \frac{\partial^{|m_z|} U}{\partial z^{|m_z|}} \Bigg|_{z=0} \right)^2 \right] \tag{50}$$

The equality (50) is obtained by taking into account the fact that $\mathcal{K}_{\parallel}(y/y')$ describes the interaction of the bunch with the boundary fields, when harmonics of the revolution frequency with $n > k_0$, $\min\{\ell_b, \ell\}$ are significant. The nature of the excitation spectrum is determined by the value

$$\xi_m = \mp\nu \frac{\ell_s R_0}{\ell_b \ell^{(m)}}$$

where

$$\ell^{(m)} = \begin{cases} \max\left(\ell, \ell_s \dfrac{d\omega_z}{d\omega_s}\right), & m_z = 0 \\ \ell_s \cdot \dfrac{d\omega_z}{d\omega_s}, & m_z \neq 0 \end{cases}$$

If the value $\xi_m$ is low ($|\xi_m| \ll 1$), then it is clear that the influence of $\mathcal{K}_{\parallel}$ on the solution of (49) is weak and it can be taken into account in terms of perturbation theory. In this case the stability of excitations is determined by the properties of the kernel $\mathcal{K}_\perp(y/y')$ so that the stability conditions have the same form as in the above section.

We shall calculate the distortion of the spectrum $\mathcal{K}_\perp$ due to $\mathcal{K}_{\parallel}$. For simplicity's sake, we shall assume that the eigenvalues of $\mathcal{K}_\perp(y/y')$ are non-degenerate. We then have in the first order of the perturbation theory

$$\lambda^k_m - \lambda^k_{m_0} = \zeta_{\parallel} \int_0^{x_0} dx \sin^2 \frac{2\ell x}{\ell_b} \left( \int_0^{\infty} dy\, y\, g(y)\, J_{m_z}(xy)\, \chi^{(k)}_{m_0}(y) \right)^2 \tag{51}$$

where $\lambda^{K}_{m,o}$ and $\chi^{(K)}_{m,o}$ are determined from (49) at $\mathcal{K}_{\parallel}(y/y')=0$, and $\chi^{(K)}_{m,o}$ satisfy the normalization conditions

$$\int_0^\infty dy\, y\, g(y)\, \chi^{(K)}_{m,o}\, \chi^{(K')*}_{m,o} = \delta_{K,K'} .$$

By comparing (51) with (42), (43) or (48) we find that the ratio $|\lambda^{K}_{m}-\lambda^{K}_{m,o}|/\lambda^{K}_{m,o}$ equals $|\xi_m|$ in the order of magnitude.

In the inverse limiting case ($|\xi_m| \gg 1$) solutions of (49) are mainly determined by $\mathcal{K}_{\parallel}(y/y')$, and $\mathcal{K}_{\perp}$ may be considered as a small perturbation.

Without perturbation ($\mathcal{K}_{\perp} = 0$) the excitation spectrum is determined by the equation

$$\lambda \chi_m(y) = \int_0^{x_0} dx \sin^2\frac{2\ell x}{\ell_s}\, J_{m_\varepsilon}(xy) \int_0^\infty dy'\, y'\, g(y')\, J_{m_\varepsilon}(xy')\, \chi_m(y') \qquad (52)$$

where $\lambda = \lambda_m / \zeta_{11}$. By means of the substitution

$$\chi_m(y) = \sqrt{\frac{g(y)}{y}}\, \bar{\chi}_m(y)$$

the above equation is transformed into an equation with a positive symmetrical kernel. Therefore, all the eigenvalues of (52) are positive. The excitation decrements equal:

$$\delta_K = - \frac{\lambda_K N e_0 c |m_z|}{2\pi^2 \gamma}\, \frac{\bar{\psi}}{B_0} \left\langle \left(\frac{a_z}{2}\right)^{2|m_z|-2} \right\rangle \frac{1}{(|m_z|-1)!^2} \cdot \frac{\partial}{\partial z}\left(\frac{\partial^{|m_z|-1} U}{\partial z^{|m_z|-1}}\right)^2 \Bigg|_{z=0} , \qquad (53)$$

where $\lambda_K$ is the eigenvalue of (47) with the number $K$.

The stability condition ($\delta_K > 0$) has the form

$$\bar{\psi} \cdot \frac{\partial}{\partial z}\left(\frac{\partial^{|m_z|-1} U}{\partial z^{|m_z|-1}}\right)^2 < 0 \qquad (54)$$

Generally speaking, the eigenvalues of equation (52) depend on the parameter $\ell_b/\ell$. However, it is clear from (52) that this dependence is extremely weak and in the limit $\ell_b/\ell \to 0$ drops out completely.

## 4. Synchrotron excitations

In this case the integral equation has the form (33). We should point out that if $\rho(\Psi)$ depends only on one parameter, then, for a convergence of the normalized integral for $q$, the following is generally required

$$\frac{dq}{dy} < 0 , \qquad 0 \leq y \leq \infty$$

For such $q(y)$ distributions, in the case of extremely short $(\ell_b \ll |m_c|\ell)$ or extremely long $(\ell_b \gg |m_c|\ell)$ bunches, the stability condition of the synchrotron excitations coincides with (38) and does not depend on the form of the distribution in terms of the amplitudes of the synchrotron oscillations.

First let $\ell_b \ll |m_c|\ell$. Then equation (33) may be rewritten in the form:

$$\lambda_1 \chi_m(y) = \int_0^\infty \frac{dx}{x^2} J_{m_c}(xy) \int_0^\infty dy' y' \left|\frac{dq}{dy'}\right| J_{m_c}(xy')\, \chi_m(y') \tag{55}$$

where

$$\lambda_1 = i\,\frac{(\omega - m_c\omega)\,\bar{\pi}^2 \gamma\, \ell_b}{N r_0 c\, m_c^2} \left(\overline{\Psi} \cdot \frac{\partial U^2}{\partial \tau}\Big|_{\tau=0}\right)^{-1} \tag{56}$$

Equation (55) leads easily* to the integral equation with a symmetrical positive kernel. Therefore all its characteristic roots are positive numbers.

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* viz. Annex

The excitation decrements are expressed by means of the eigen-values of equation (55) according to the formula

$$\delta_\varkappa = \frac{\lambda_\varkappa N r_0 c\, m_c^2}{\pi^2 \gamma \ell_b} \overline{\Psi_\varkappa} \cdot \frac{\partial U}{\partial \tau}\Big|_{\tau=0}^{2}, \quad \varkappa = 1, 2, 3, \ldots, \tag{57}$$

$K$ is the solution number (53). Hence it is clear that $\delta_K$ will be positive if (38) is fulfilled.

The numbers $\lambda_{1K}$ are bounded from above by the value (viz.(A.6))

$$\frac{1}{\pi} \int_0^\infty dy\, q(y) / (m_c^2 - 1/4), \tag{58}$$

and the integral in the numerator of (58) depends fairly weakly on the specific form of the function $q(y)$. Therefore, the decrement $\delta$ given by (37) may be considered to be the upper limit for (55). If the bunch is longer than the plate $\ell_b \gg \ell\,|m_c|$, then equation (33) may also be transformed (with accuracy up to terms of the order $|m_c|\ell/\ell_b$) to an equation with a real positive kernel. Therefore, the stability condition in this case also has the form (38).

## IV. SPECTRUM OF SHORT-WAVE OSCILLATIONS

In this section we obtain the spectrum of excitations caused by the interaction of a bunch with the high-frequency part of the induced fields ($n \gg (|m_c|+1)/\Delta$).

Let us first examine the axial-longitudinal excitations. By introducing the new unknown function

$$C_m(x) = \int_0^\infty dy\, y\, q(y)\, \mathcal{X}_m(y)\, \mathcal{I}_{m_c}(xy)$$

we rewrite equation (41) in the form:

$$\lambda_m C_m(x) = \int_0^\infty dx'\, \Phi(x')\, g(x/x')\, C_m(x'), \tag{41a}$$

where $$g(x|x') = \int_0^\infty dy\, y\, q(y)\, J_{m_c}(xy)\, J_{m_c}(x'y)\,;$$

$$\Phi(x) = \mathcal{L}_1\left(\frac{x}{\Delta} - m_2 \cdot \frac{d\omega_2}{d\omega_s}\right) + \mathcal{L}_1\left(-\frac{x}{\Delta} - m_2 \cdot \frac{d\omega_2}{d\omega_s}\right) \tag{59}$$

and $\mathcal{L}_1(x)$ is determined by formula (42).

In the high-frequency region ($x \gg |m_c| + 1$), the function $g(x|x')$ has a sharp maximum (with a width of the order of one unit) at $x' \simeq x$, and $g(x|x')$ decreases quickly as $x$ moves away from $x$ *.

It is possible to obtain the short-wavelength part of the spectrum in two limiting cases:

a) $\ell_b \ll \ell$ $\Delta x \sim 1$ Fast oscillations of $\Phi(x)$ may be ignored in the range which corresponds to the substitution

$$\Phi(x) \simeq \overline{\Phi(x)} = \int_{\Delta x' \simeq 1} dx'\, \Phi(x + \Delta x')$$

b) In the opposite limiting case ($\ell_b \gg \ell$), as can be seen from (42), the variation of $\Phi(x)$ may be ignored in the range $\Delta x' \sim 1$ :

$$\Phi(x') \longrightarrow \Phi(x)$$

Therefore, in the case under investigation, we nay write approximately

$$g(x|x') \simeq \frac{1}{x}\,\delta(x - x') \;;\quad x \gg |m_c| + 1 \tag{60}$$

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*For example for

$$q(y) = 2\exp(-y^2)\,;$$
$$g(x|x') \sim \frac{1}{x}\exp\left(-(x-x')^2/4\right), \quad x \gg |m_c| + 1\,;$$

for $$q(y) = \frac{1}{1+y^2}\,,\; g(x|x') \sim \exp(-|x-x'|)/x,\; x \gg |m_c| + 1.$$

By substituting (60) into (41a) we obtain the spectrum of excitations in the short-wave region

$$\lambda_m(x) = \begin{cases} \dfrac{\overline{\Phi(x)}}{x}, & \ell_b \ll \ell \\ \dfrac{\Phi(x)}{x}, & \ell_b \gg \ell \end{cases} \tag{61}$$

The decrease of the correlation of the revolution frequency harmonics $C_m(X)$ when $X \gg |m_c|+1$ means in physical terms that the normal excitations due to the interaction of the bunch with the high-frequency part of the induced fields are close to "plane waves":

$$F_{\omega,m}(I_1, I_c) \sim \left(e^{in\varphi_c - i\omega t}\right)_{m_c}, \quad \left(n \gg \frac{|m_c|+1}{\Delta}\right)$$

In this case the "distance" between the separate modes is of the order of the width $g(X|X')$ $(\Delta X' \simeq 1)$ which corresponds to $\Delta n \simeq 1/\Delta$ $(X = n\Delta)$.

The calculation of the decrements is simplified in two limiting cases:

a) $\ell_b \ll \ell$. The function $\overline{\Phi(X)}$ may be written in the form

$$\overline{\Phi(x)} = \begin{cases} \dfrac{i\Delta^2}{2} \cdot \dfrac{m_z}{X^2} \cdot \dfrac{d\omega_z}{d\omega_s}, & X \gg |m_z|/\nu_z\Delta;\ \Delta\left|m_z \cdot \dfrac{d\omega_z}{d\omega_s}\right| \\ -\dfrac{i}{2} \cdot \dfrac{1}{m_z \cdot \dfrac{d\omega_z}{d\omega_s}}, & X \ll \Delta\left|m_z \cdot \dfrac{d\omega_z}{d\omega_s}\right| \end{cases}$$

By substituting $\overline{\Phi(X)}$ into (61), we obtain the **expressions for the decrements** $(\delta = \Im m\,\omega)$

$$\delta(X) = \begin{cases} \dfrac{N\delta_0 m_z}{2X^3} \Delta \dfrac{d\omega_z}{d\omega_s}, & X \gg |m_c| > \Delta\left|m_z \cdot \dfrac{d\omega_z}{d\omega_s}\right| \\ -\dfrac{N\delta_0}{m_z \cdot \dfrac{d\omega_z}{d\omega_s}\Delta} \cdot \dfrac{1}{2X}, & \Delta\left|m_z \cdot \dfrac{d\omega_z}{d\omega_s}\right| \gg X \gg |m_c|+1 \end{cases} \tag{62}$$

By integrating the decrements (62) over $X$ it is easy to obtain the corresponding sums of the decrements ($|m_c| \neq 0$):

$$\sum_{X \geq |m_c|}^{\infty} \delta(X) \simeq \int_{|m_c|}^{\infty} dX\,\delta(X) = \begin{cases} N\delta_0 m_z \cdot \dfrac{\Delta}{4m_c^2} \cdot \dfrac{d\omega_z}{d\omega_s}, & |m_c| > \Delta \left| m_z \cdot \dfrac{d\omega_z}{d\omega_s} \right| \\[2ex] -\dfrac{N\delta_0 \ln\left(\Delta \left| \dfrac{m_z}{m_c} \cdot \dfrac{d\omega_z}{d\omega_s} \right|\right)}{2\Delta m_z \cdot \dfrac{d\omega_z}{d\omega_s}}, & \Delta \left| m_z \cdot \dfrac{d\omega_z}{d\omega_s} \right| \gg |m_c| \end{cases}$$

The above formulae coincide well (at $m_c \neq 0$) with (22) and (22a) respectively. This coincidence is conditioned by the fact that the main contribution to the synchrobetatron excitation of a short bunch is made by the interaction with boundary fields which are substantially non-uniform in $\theta$. Therefore, the spectrum of the high-frequency excitations is expected to join directly the spectrum of the low-frequency excitations.

Hence it follows that formulae (62), extrapolated into the low-frequency region $X \simeq |m_c| > 0$, should give (in order of magnitude) the maximum decrement (increment) of the axial-longitudinal excitations of a short bunch:*

$$\delta_{max} \simeq \begin{cases} \dfrac{N\delta_0 m_z}{2|m_c|^3} \Delta \dfrac{d\omega_z}{d\omega_s}, & |m_c| > \Delta \left| m_z \dfrac{d\omega_z}{d\omega_s} \right| \\[2ex] -\dfrac{N\delta_0}{m_z \Delta \dfrac{d\omega_z}{d\omega_s}} \cdot \dfrac{1}{2|m_c|}, & \Delta \left| m_z \dfrac{d\omega_z}{d\omega_s} \right| > |m_c| \end{cases}$$

b) $\ell_b \gg \ell$. In this case, the most interesting region from the point of view of the estimate of the oscillation's maximum increment, is $\ell_b/\ell \gg X \gg |m_c|$. Using (59), we may write $\Phi(X)$ in this

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* For excitations with $m_c = 0$ this extrapolation is not valid as the determining contribution to the decrements of long-wave excitations is made by the harmonics $n \lesssim R_0/\ell \ll \dfrac{1}{\Delta}$.

region in the form:

$$\Phi(x) \simeq i\left(\frac{e}{R_0}\right)^2 m_z \cdot \frac{d\nu_z}{d\ln R_0} \quad , \quad \frac{\ell_b}{\ell} \gg x \gg |m_z|$$

In this case, the maximum increment may be estimated by formula:

$$\delta_{max} \simeq 2N\delta_0 m_z \cdot \frac{e}{2R_0} \cdot \frac{e}{\ell_b} \frac{d\nu_z}{d\ln R_0} \frac{1}{|m_z|+1}$$

For radial-longitudinal excitations the function $\Phi(x)$ in equation (41a) is replaced by

$$\Phi(x) - \tau_{\parallel} \sin^2 \frac{2\ell x}{\ell_b} \cdot$$

In this case the decrements of high-frequency excitations may be written in the form:

$$\delta_m(x) = \delta_{\perp}'(x) - \delta_{\parallel}(x)$$

where $\delta_{\perp}(x)$ is obtained from the decrement of axial-longitudinal excitations by substituting the index $z$ for $r$, $\delta_{\parallel}(x)$ describes the effect of radial-longitudinal coupling.

For a short bunch ($\ell_b \ll \ell$) the quantity $\delta_{\parallel}(x)$ may be calculated using the formula

$$\delta_{\parallel}(x) \simeq \frac{\delta_{\parallel}^{(0)}}{2x} \quad , \quad \frac{\ell_b}{\ell_{\perp}} \gg x \gg |m_z| , \tag{63}$$

where

$$\delta_{\parallel}^{(0)} = \frac{N r_0 c |m_z|}{\pi^2 \gamma \ell_b} \left\langle \left(\frac{a_r}{2}\right)^{2|m_z|-2} \right\rangle \cdot \frac{\overline{\psi}}{(|m_z|-1)!^2} \cdot \frac{\partial}{\partial z}\left(\frac{\partial^{|m_z|-1} U}{\partial z^{|m_z|-1}}\right)^2 \Bigg|_{z=0}$$

The maximum decrement in order of magnitude equals

$$\delta_{\parallel max} \simeq \frac{\delta_{\parallel}^{(0)}}{2(|m_z|+1)}$$

If the bunch is longer than the plate $\ell_b \gg |m_z|\,\ell$ , then

$$\delta_\parallel(x) \simeq \begin{cases} \delta_\parallel^{(0)} \cdot \dfrac{2\ell x}{\ell_b} , & \dfrac{\ell_b}{\ell} \gg x \gg |m_z| \\ \dfrac{\delta_\parallel^{(0)}}{2x} , & \dfrac{\ell_b}{\ell_\perp} \gg x \gg \dfrac{\ell_b}{\ell} \end{cases} \tag{63a}$$

In this case the maximum decrement in order of magnitude equals:

$$\delta_{\parallel\,max} \simeq \frac{N r_0 c |m_z|}{\pi^2 \gamma \ell_b}\left(\frac{\ell}{\ell_b}\right)\left\langle\left(\frac{a_z}{2}\right)^{2|m_z|-2}\right\rangle \frac{\overline{\Psi}}{(|m_z|-1)!^2}\cdot\frac{\partial}{\partial z}\left(\frac{\partial^{|m_z|-1}U}{\partial z^{|m_z|-1}}\right)^2\Bigg|_{z=0}$$

## V. ON THE EFFECT OF FREQUENCY SPREAD

In this paper we shall investigate only those cases where the oscillation's frequency spread in the stationary state is determined only by the non-linearity of transverse motion, and the non-linearity of longitudinal motion may be ignored. For an excitation of arbitrary multipolarity, the dispersion equation (viz. (18b)) has the form

$$1 = -\frac{\Omega_k}{A_m}\int dI_z dI_\tau \frac{I_z^{|m_z|}\, I_\tau^{|m_\tau|}\left(m_z \cdot \frac{\partial F_0}{\partial I_z} + m_\tau \cdot \frac{\partial F_0}{\partial I_\tau}\right)}{\omega - m_z \omega_z(I_z, I_\tau) - m_\tau \omega_\tau(I_z, I_\tau)} , \tag{64}$$

where $A_m$ is the normalizing constant equal to

$$\left\langle\left(m_\tau |m_z| I_z + m_z |m_\tau| I_\tau\right) I_\tau^{|m_\tau|-1} \cdot I_z^{|m_z|-1}\right\rangle,$$

$\Omega_k$ is the eigenvalue of equation (18a) with the number $k$ (which is the solution of equation (64) without spread).

Despite its apparent complexity, equation (64) may be easily reduced to a standard form. To do so let us introduce a new variable into (64)

$$\mathcal{E} = \mathcal{E}_0 = \sum_{i=z,\tau} m_i\left[\omega_i(I_z, I_\tau) - \omega_i(0,0)\right] \; ; \; x = x_0(I_z, I_\tau) \tag{65}$$

where the variable $x_0(I_z, I_\tau)$ may be chosen such that the Jacobian of the transformation (65) is unity:

$$\frac{\partial(\varepsilon, x)}{\partial(I_z, I_z)} = 1$$

Equation (64) then changes to

$$1 = \Omega_K \int_{-\infty}^{\infty} d\varepsilon \frac{g(\varepsilon)}{\omega - \omega_o - \varepsilon} \tag{66}$$

where $\omega_o = m_z \omega_z(0,0) + m_z \omega_z(0,0)$ and the quantity $g(\varepsilon)$ corresponding to the "effective" frequency distribution density, is determined by the equality

$$g(\varepsilon) = -\frac{1}{A_m} \int dI_z dI_z \int_{-\infty}^{\infty} dx' \, \delta(\varepsilon - \varepsilon_o(I_z, I_z)) \cdot \delta(x' - x_o(I_z, I_z)) \cdot$$
$$\cdot I_z^{|m_z|} I_z^{|m_z|} \left( m_z \cdot \frac{\partial F_o}{\partial I_z} + m_z \cdot \frac{\partial F_o}{\partial I_z} \right) . \tag{67}$$

The function $g(\varepsilon)$, by definition, is normalized to unity:

$$\int_{-\infty}^{\infty} d\varepsilon \, g(\varepsilon) = 1$$

Equation (66) may be investigated by standard methods. sing Nyquist's criterion, it is easy to establish that, in order that all the roots of (66) lie on the lower half-plane, ($\omega$), it is sufficient to fulfil the inequality

$$1 - \pi \frac{|\Omega_K|^2}{Im\,\Omega_K} g(\varepsilon_i) > 0 \tag{68}$$

where $\varepsilon_i$ are real numbers determined from the equation

$$Im\,\Omega_K \fint_{-\infty}^{\infty} \frac{d\varepsilon \, g(\varepsilon)}{\varepsilon_i - \varepsilon} - \pi \, Re\,\Omega_K \, g(\varepsilon_i) = 0$$

Here $\fint$ signifies that the integral must be calculated as a principal value.

In particular, the inequality (68)

$$1 - \pi \frac{|\Omega_K|^2}{Im\,\Omega_K} g_{max} > 0 \tag{68a}$$

holds certainly,

where $g_{max.}$ is the largest val e of $g$ over the entire range of variation of $\varepsilon$ . Therefore, if the inequality(68a) is fulfilled, the coherent oscillations are damped.

This means that the stability of coherent motion may be guaranteed by selecting parameters of the external system so that the complex coherent frequency shift $\Omega_K$ falls within the circle:

$$\left| \Omega_K - \frac{i}{2\pi g_{max}} \right| < \frac{1}{2\pi g_{max}}$$

We should point out that conditions (68) and (68a), generally speaking, are not necessary. Therefore, if they are violated, it does not follow that the oscilla ions will be unstable.

Conditions for the stability of coherent oscillations when there is a spread, may also be obtained by investigating equation (68) close to the instability threshold $(\omega-\omega_0 \to \omega_{th} + i\delta, \delta \to 0)$ (5). Equation (66) then breaks up into two equations:

$$\pi g(\omega_{th}) = \frac{Im\,\Omega_K}{|\Omega_K|^2}$$
$$\mathcal{P}(\omega_{th}) = \int_{-\infty}^{\infty} \frac{d\varepsilon\, g(\varepsilon)}{\omega_{th} - \varepsilon} = \frac{Re\,\Omega_K}{|\Omega_K|^2} \qquad (69)$$

For a given distribution function $g(\varepsilon)$, the system of equation (69) defines in a parametric form the boundary of the stability region in the plane of the complex variable $\Omega_K$. The position of the stability region i. relation to the boundary curve is determined by the relationshi. /5/

$$\left( \frac{Im\,\Omega'}{|\Omega'|^2} - \pi g(\omega_{th}) \right) \cdot \frac{\partial \mathcal{P}(\omega_{th})}{\partial \omega_{th}} > 0 \qquad (70)$$

In the above relationship $\Omega'$ is a point in the plane of the complex variable $\Omega_K$ situated near the boundary of the stability region and $\omega_{th}$ corresponds to a point on the boundary curve.

For given parameters of the external systems, equations (69) may be used to calculate the threshold current and coherent frequency shift at the instability threshold. For this purpose it is more convenient to rewrite (69) in the form:

$$\tilde{\Pi}\,\frac{g(\omega_{th})}{\mathcal{P}(\omega_{th})} = \frac{\mathrm{Im}\,\bar{\Omega}_K}{\mathrm{Re}\,\bar{\Omega}_K}\,, \qquad N_{th} = \frac{\mathrm{Im}\,\bar{\Omega}_K}{|\bar{\Omega}_K|^2}\cdot\frac{1}{\tilde{\Pi}\,g(\omega_{th})} \tag{69a}$$

where the following notation is introduced $\Omega_K = N\bar{\Omega}_K$ .

For further analysis it is essential to specify the type of density $g(\varepsilon)$ . Let us examine, for example, two-dimensional synchrobetatron excitations ($\omega = m_z\omega_z + m_c\omega_c + \Delta\omega$ . Let us assume, moreover, that the non-linearity of motion in the stationary state is determined by the cubic non-linearity of the guide field. Then

$$\omega_z(I_z, I_z) = \omega_{z0} + \frac{\partial\omega_z}{\partial I_z} I_z + \frac{\partial\omega_z}{\partial I_z} I_z \tag{71}$$

The form of the distribution function $g(\varepsilon)$ depends substantially on the beam dimension determining the frequency spread (vertical or radial).

Let us say that the spread is "intrinsic" if

$$\langle I_z\rangle \left|\frac{\partial\omega_z}{\partial I_z}\right| \gg \langle I_z\rangle \left|\frac{\partial\omega_z}{\partial I_z}\right|$$

In this case the dispersion equation may be written in the form

$$1 = -\frac{\Omega_K}{m_z\,\alpha_{zz}\,A_{m_z}}\cdot\int_0^\infty \frac{d\varepsilon\,\varepsilon^{|m_z|}\,\frac{\partial F_0}{\partial\varepsilon}}{t - \varepsilon}\,, \tag{66a}$$

where

$$\alpha_{zz} = \frac{\partial\omega_z}{\partial I_z}\,, \quad t = (\omega - \omega_0)/\alpha_{zz}$$

In this case, $g(\varepsilon)$ equals:

$$g(\varepsilon) = -\frac{\varepsilon^{|m_z|}}{m_z\alpha_{zz}A_{m_z}}\,\theta(\varepsilon)\cdot\frac{\partial F_0}{\partial\varepsilon}\,, \tag{72}$$

where

$$\theta(x) = \begin{cases} 1\,, & x > 0 \\ 0\,, & x < 0 \end{cases}$$

In the inverse limiting case:

$$\langle I_z \rangle \cdot \left|\frac{\partial \omega_z}{\partial I_z}\right| << \langle I_z \rangle \cdot \left|\frac{\partial \omega_z}{\partial I_z}\right|$$

we shall call the spread extrinsic. In this case the function $g(\mathcal{E})$ is:

$$g(\varepsilon) = \frac{1}{m_z \alpha_{z,z}} \theta(\varepsilon) F_0(\varepsilon) \tag{73}$$

From (72) it is clear that the form of the frequency distribution is determined not only by the particle distribution in terms of the oscillation amplitudes but also, generally speaking, by the multipolarity number $m_z$ ($m_\eta$ in the case of radial-longitudinal excitations). We should point out in particular that the"effective" distribution width $g(\varepsilon)$ increases approximately like $\left(|m_z|\ ^1/_2\right)$. Moreover, $g(\mathcal{E})$ has at least one maximum which, for monotonic $F_0(I)$ occurs approximately at $\varepsilon \simeq |m_z| \langle I_z \rangle$ .

An important factor is the fact that $g(\varepsilon)$ is generally not symmetrical to its maximum. Therefore, the boundary of the stability region is given by

$$\begin{aligned} \frac{\mathcal{J}m\,\Omega_K}{|\Omega_K|^2} &= - \frac{\theta(\varepsilon)\,\varepsilon^{|m_z|}}{|m_z \alpha_{zz}|\,A_{m_z}} \cdot \frac{\partial F_0}{\partial \varepsilon} \\ \frac{Re\,\Omega_K}{|\Omega_K|^2} &= \frac{\mathcal{P}(\varepsilon)}{m_z \alpha_{zz}} \ , \end{aligned} \tag{74}$$

It is not symmetrical to the axis $\mathcal{J}m\,\Omega_K$ (viz. Fig. 2, 3). In particular, it coincides with the axis $Re\,\Omega\,K$ for

$$Re\,\Omega_K \leq Re\,\Omega_{K\,gp} = \frac{m_z \alpha_{zz}}{\mathcal{P}(0)} \tag{75}$$

Consequently, if the value of the coherent frequency shift introduced by the system is such that inequality (75) is fulfilled, it is impossible to stabilize coherent oscillations by means of a spread.

From the equations (74) it is clear that when the sign of $\alpha_{zz}$ is reversed, the stability region is reflected around the axis $\Im m\,\Omega_K$. Due to the asymmetry with respect to the $\Im m\,\Omega_K$ axis, the stability region does not transform into itself. This means that for external systems (in the case of "intrinsic" spread) with

$$|Re\,\Omega_K| > |Re\,\Omega_{K\,lim.}|$$

the coherent oscillations may become unstable after changing the sign of $\alpha_{zz}$.

In the case of extrinsic spread, the integral $\mathcal{P}(\varepsilon)$ diverges logarithmically when $\varepsilon \to 0$. However, this only affects the value of the coherent frequency shift limit $Re\,\Omega_{K\,lim.}$, which must not now be calculated by means of (75). Qualitatively, the results remain as before.

The value of the threshold instability current may be easily calculated for monotonic $F_0(I)$ in the case where $|\Im m\,\Omega_K| \gg |Re\,\Omega_K|$. Here, $g(\omega_{th}) \gg \mathcal{P}(\omega_{th})$, which means that the roots of the first equation (69a) must be close to $\bar{\omega}_m$ (corresponding to the maximum of $g(\omega)$), that is $|\omega_{th} - \bar{\omega}_m| \ll (\overline{\Delta\omega_m^2})^{1/2}$. The value of the threshold current may consequently be estimated by means of the formula:

$$N_{th} \simeq \frac{1}{\pi\, \Im m\, \bar{\Omega}_K\, g_{max}} \simeq \frac{(\overline{\Delta\omega_m^2})^{1/2}}{\Im m\, \bar{\Omega}_K}. \tag{76}$$

where the quantity $\overline{\Delta\omega_m^2} = m_z^2 \int_{-\infty}^{\infty} d\omega\,(\omega - \bar{\omega})^2 g(\omega)$ determines the frequency spread of betatron oscillations.

By substituting in formula (76) the excitation increments obtained in the previous sections, it is possible to obtain formulae for the threshold current of a beam interacting with matched plates. Clearly, the expression for the maximum increment, which may be estimated for an arbitrary smooth distribution (viz. section III) must be inserted into (76).

For example, if the machine's chromatism is not too great:

$$\frac{\ell_b}{R_0}\left|\frac{d\omega_z}{d\omega_s}\right| \ll 1$$

then, for axial longitudinal excitations ($m_z \neq 0$) $N_{th}$ may be written in the form

a) $\ell_b \ll |m_c|\ell$

$$N_{th} = \frac{\gamma}{\delta_0' m_z} \frac{R_0 (\overline{\Delta\omega_m^2})^{1/2}}{\ell_b\left(-\frac{d\omega_z}{d\omega_s}\right)} L_m^{(s)}, \quad \delta_0' = \gamma\delta_0 \tag{77}$$

where the quantity $L_m^{(s)}$ depends on the form of the distribution (for a $\delta$ -type distribution $L_m^{(s)} = 4m_c^2 - 1$).

b) $\ell_b \gg |m_c|\ell$

$$N_{th} = \frac{\tilde{\pi}}{\delta_0' m_z} \frac{\gamma R_0 \ell_b (\overline{\Delta\omega_m^2})^{1/2}}{\ell^2\left(-2L_m^{(\ell)} \frac{d\ln\nu_z}{d\ln R_0} - 1\right)} . \tag{78}$$

For a $\delta$ -type distribution the factor $L_m^{(\ell)}$ equals $\ln(\ell_b/\ell(|m_c|+1))$; for a smooth distribution it is of the order $1/(|m_c|+1)$. We should point out that the above formula is in good qualitative agreement with the experimental results obtained on ... /3/, ... /9/ and CEA /10/.

If the machine's chromatism is great

$$\frac{\ell_b}{R_0}\left|\frac{d\omega_z}{d\omega_s}\right| \gg 1$$

then, in order to estimate the threshold current, it is necessary to use formula (22a). In this case $N_{th}$ has the form:

$$N_{th} = \frac{\gamma\tilde{\pi}}{\delta_0'} \frac{m_z \ell_b}{R_0}\left(\frac{d\omega_z}{d\omega_s}\right)(\overline{\Delta\omega_m^2})^{1/2} L_m^{(3)}, \tag{79}$$

where the factor $L_m^{(3)}$ depends on the shape of the distribution function of the synchrotron oscillation amplitudes in the stationary state.

From formulae (77) - (79) it is clear that, depending on the relationship of the characteristic parameters of the problem, the functional dependences of the threshold current (in particular on beam length and on the machine's chromatism) may vary **considerably.**

If the value of the coherent frequency shift is great: $|Re\,\Omega_K| \gg |Im\,\Omega_K|$ , then $\mathcal{P}(\omega_{th}) \gg g(\omega_{th})$ , which is possible only for $|\omega_{th} - \bar{\omega}_m| \gg (\Delta\omega_m^2)^{1/2}$.

In this case $\mathcal{P}(\omega_{th}) \simeq 1/\omega_{th}$ and, according to (69), $\omega_{th} \simeq Re\,\Omega_K$ . In order to determine $N_{th}$ , we therefore have the transcendental equation

$$Im\,\bar{\Omega}_K = \pi N_{th} (Re\,\bar{\Omega}_K)^2 g(N_{th} \cdot Re\,\bar{\Omega}_K) \qquad (80)$$

Annex:

Here we shall show that all the eigenvalues of the integral equation

$$\lambda f(x) = \int_0^\infty dx' \mathcal{K}(x/x')\, q(x') f(x') \qquad (A.1)$$

have the same sign, where $q(x)$ does not change sign in $0 \leq X < \infty$ , and the kernel takes the form

$$\mathcal{K}(x/x') = \int_0^\infty dt\, c^2(t)\, b(xt)\, b(x't), \qquad (A.2)$$

where $c(t)$ and $b(x)$ are real functions.

To be specific, let us assume that $q(x) > 0 \left(\int_0^\infty dx\, q = 1\right)$ . After multiplying the right and left parts of (A.1) by $q^{1/2}(x)$, we obtain an equation for

$$\varphi(x) = q^{1/2} f(x)$$

$$\lambda \varphi(x) = \int_0^\infty dx' \mathcal{K}_1(x/x')\, \varphi(x') \qquad (A.3)$$

The spectrum for this clearly coincides with the spectrum of (A.1) and the kernel $\mathcal{K}_1(x/x')$, linked with $\mathcal{K}(x/x')$ by the relationship

$$\mathcal{K}_1(x/x') = \sqrt{q(x)q(x')}\,\mathcal{K}(x/x'),$$

is real and symmetrical.

Normally (viz. for example ([illegible])), **the square of $\mathcal{K}_1$ is integrable. This means that the integral**

$$\|\mathcal{K}_1\|^2 = \int_0^\infty dx \int_0^\infty dx'\, \mathcal{K}_1^2(x/x')$$

**has an upper limit. This requirement is certainly** fulfilled if the sum of the characteristic roots of (A.1) is finite. In fact, taking into account (A.2) and using the Bunjakowski/Schwarz inequality, we obtain

$$\|\mathcal{K}_1\|^2 = \int_0^\infty dt \int_0^\infty dt'\, c^2(t)\, c^2(t') \left( \int_0^\infty dx\, q(x)\, \ell(xt)\, \ell(xt') \right)^2 \leq$$

$$\leq \left( \int_0^\infty dt\, c^2(t) \int_0^\infty dx\, q(x)\, \ell^2(xt) \right)^2 = \left( \int_0^\infty dx\, q(x)\, \mathcal{K}(x/x) \right)^2$$

The last quantity in (A.4) exactly equals the square of the sum of the eigenvalues in (A.1). **Hence**

$$\|\mathcal{K}_1\|^2 \leq \sigma^2 = \left( \int_0^\infty dx\, q(x)\, \mathcal{K}(x/x) \right)^2$$

The characteristic numbers of equation (A.3) are real and positive. That is due to the fact that (viz. (3)) $\mathcal{K}_1(x/x')$ is a positive kernel, that is, there exists an integral

$$\mathcal{J}[\rho] = \int_0^\infty dx \int_0^\infty dx'\, \mathcal{K}_1(x/x')\, \rho(x)\, \rho(x'),$$

where $\rho(x)$ **is of such a type that it can be expanded in terms of the eigenfunctions of the integral equation (A.3).**

In fact:

$$\mathcal{J}[\rho] = \int_0^\infty dt\, c^2(t) \left( \int_0^\infty dx\, q^{1/2}(x)\, \ell(xt)\, \rho(x) \right)^2 > 0$$

In order to estimate the increment of the oscillations, the following formula may prove useful

$$\lambda_{max} \geqslant J[\rho], \tag{A.5}$$

when

$$\int_0^\infty dx \rho^2(x) = 1.$$

Since all $\lambda$ are positive, then clearly $\lambda_{max.}$ is within the limits

$$J[\rho] \leqslant \lambda_{max} \leqslant \sigma \tag{A.6}$$

The second inequality in (A.6) may, on the other hand, be obtained by the direct application of the Bunjakowski/Schwarz inequality to the right-hand part of (A.5).

If $q(x)<0$ then by substituting $\lambda$ for $-\lambda$ and $q(x)$ for $|q(x)|$, it is easy to reduce the equation to the form of (A.3), the spectrum of which is positive. Therefore, for $q(x)<0$, $0 \leqslant x < \infty$, all the eigenvalues of (A.1) are negative.

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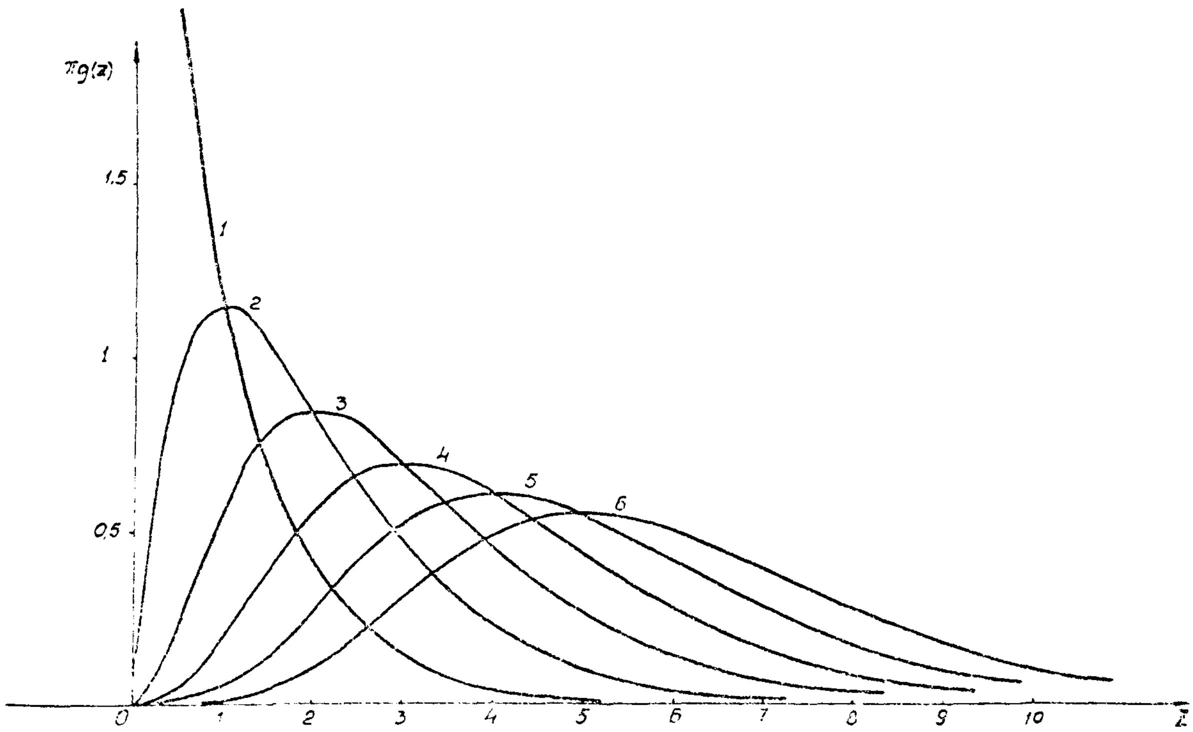


Fig.1 Effective frequency distributions for a beam with an exponential distribution in terms of the squares of betatron oscillations amplitudes in the stationary state (viz.(72)(73)). Curve 1 corresponds to the "**extrinsic**"spread. Curves (2), (3), (4), (5) and (6) correspond to the intrinsic spread and the multipolarities $m_z$ = 1, 2, 3, 4, 5 respectively.

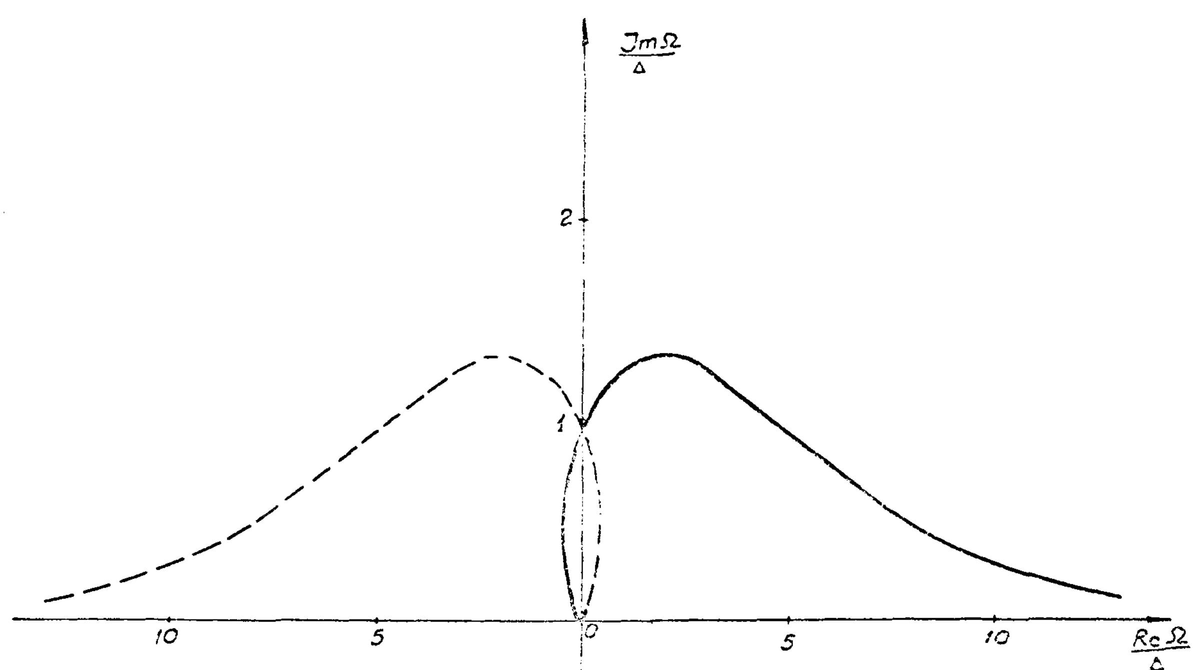


Fig.2 Boundary of the stability region for distribution (1) (Fig.1). The solid curve corresponds to $\alpha_{zz} > 0$, the hatched curve to $\alpha_{zz} < 0$.

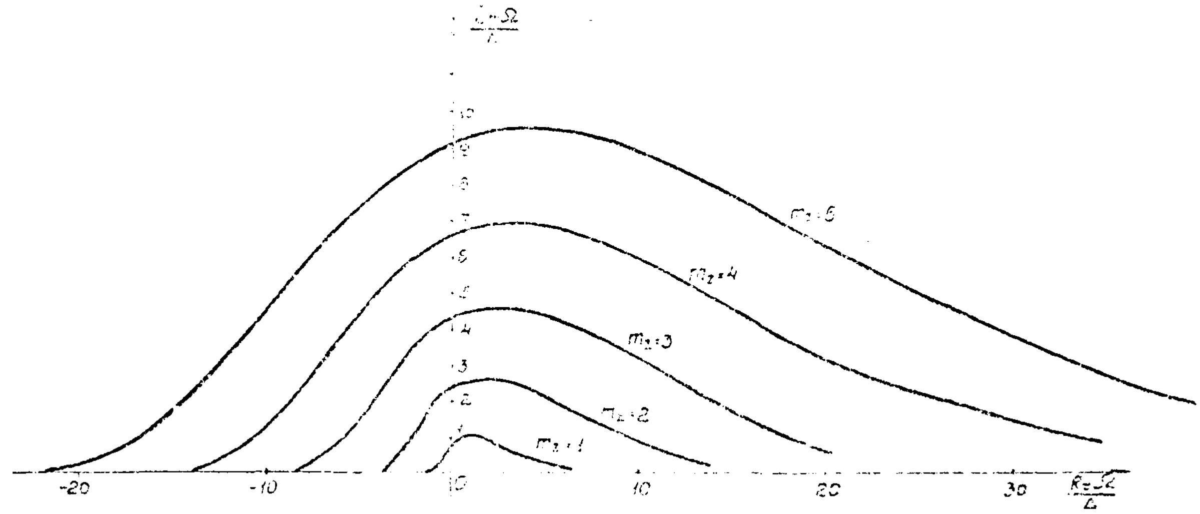


Fig.3 Boundary of the stability region for effective distributions (2), (3), (4), (5) and (6), (Fig.1) $\alpha_{zz} > 0$ .

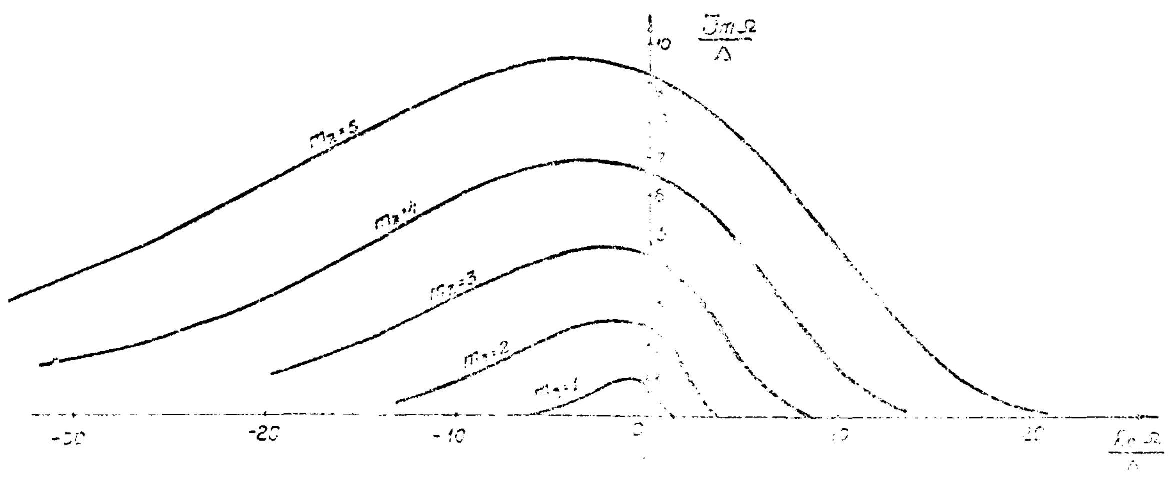


Fig.4 The same for $\alpha_{zz} < 0$.