Renormalization in words

① Calculate the rad. corr. to the definition of the bare parameters in terms of "measurements":

$$m_Z^{0\,2} = m_Z^2 + \delta m_Z^2 \qquad \delta m_Z^2 = \mathrm{Re}\,\Sigma_{ZZ}(m_Z^2)$$

$$m_W^{0\,2} = m_W^2 + \delta m_W^2 \qquad \delta m_W^2 = \mathrm{Re}\,\Sigma_{WW}(m_W^2)$$

$$e_0 = e + \delta e \qquad \delta e = \frac{1}{2}\Pi_{\gamma\gamma}(0) - \frac{s_W}{c_W}\frac{\Sigma_{\gamma Z}(0)}{m_Z^2}$$

② Calculate the prediction for something (cross section, width, asymmetry ....) using the bare parameters

③ Express the bare parameters in the physical ones, using the counterterms

④ All infinities cancel !!

Only physical parameters appear !!

Order by order in perturbation theory !!

Only 3⊛ counterterms needed !!

IF the theory is renormalizable

⊛ plus mass counterterms for fermions, Higgs...

Extra renormalization ?

The physical quantities in the theory are masses and charges.
Therefore physical predictions ($\sigma, \Gamma, A_{FB} \dots$) are made finite by mass and charge renormalization.

Some people also want to have unphysical quantities (external lines in diagrams, numerators of propagators) finite: to do this, counterterms for unphysical quantities have to be added: wave function renormalization

PRO w.f.R.

It makes "building blocks" of diagrams and cross sections finite (by cancelling infinities that would cancel between building blocks)
$\Rightarrow$ easier for careful bookkeeping, especially in complicated processes or higher orders

CONTRA w.f.R.

It is irrelevant for physical quantities
$\Rightarrow$ conceptually simpler to disregard it!

Propagators are finite (rather than "finite") after w.f.R.

Calculating the renormalized photon self-energy $\overline{\Pi}(s)$

Dispersion integral for unrenormalized $\Pi_\gamma(s)$:

$$\mathrm{Re}\,\Pi^f(s) = \frac{\alpha}{3\pi}\int_{4m^2}^{\infty} dt\, \frac{R_f(t)}{t-s}$$

log. divergence if $\lim_{t\to\infty} R_f(t) = Q_f^2$.

Put $\mathrm{Re}\,\overline{\Pi}(s) = \mathrm{Re}\,\Pi(s) - \mathrm{Re}\,\Pi(0)$:

$$\mathrm{Re}\,\overline{\Pi}^f(s) = \frac{\alpha}{3\pi}\int_{4m^2}^{\infty} dt\, R_f(t)\left(\frac{1}{t-s} - \frac{1}{t}\right)$$

$$= \frac{\alpha s}{3\pi}\int_{4m^2}^{\infty} dt\, \frac{R_f(t)}{t(t-s)}$$

better convergence!

This is a once-subtracted dispersion integral

$$R_f(t) = \frac{\sigma(ee \to \gamma \to f\bar{f})}{\sigma(ee \to \gamma \to \mu\mu)}$$

Using the general results of p. 3 and p. 4:

$$\sigma(ee \to \gamma \to f\bar{f}) = \frac{2\pi\alpha^2}{3s} Q_f^2\, \beta(3-\beta^2) \qquad \beta = \sqrt{1-4m^2/s}$$

$$R_f(t) = \frac{1}{2} Q_f^2\, \beta(t)\left(3-\beta(t)^2\right) \qquad \beta(t) = \sqrt{1-4m^2/t}$$

$$Re\, \overline{\Pi}^f(s) = \frac{\alpha Q_f^2 s}{6\pi} \int_{4m^2}^{\infty} dt\, \frac{\beta(t)(3-\beta(t)^2)}{t(t-s)}$$

$\underline{s \geq 4m^2:}$

$$Re\, \overline{\Pi}^f(s) = -\frac{\alpha Q_f^2}{3\pi} \left\{ \frac{\beta(3-\beta^2)}{2} \ln\left(\frac{1+\beta}{1-\beta}\right) - \frac{8}{3} + \beta^2 \right\} \qquad \beta = \sqrt{1-4m^2/s}$$

$$Im\, \Pi^f(s) = Im\, \overline{\Pi}^f(s) = \frac{\alpha}{6}\beta(3-\beta^2)$$

$\underline{s \gg 4m^2:}$

$$\underline{\underline{Re\, \overline{\Pi}^f(s) = -\frac{\alpha Q_f^2}{3\pi}\left[\ln\frac{s}{m^2} - \frac{5}{3}\right]}}$$

$\underline{s \leq 4m^2:}$

$$Re\, \overline{\Pi}^f(s) = \frac{\alpha Q_f^2}{3\pi}\left[-\gamma(3+\gamma^2)\arctan\left(\frac{1}{\gamma}\right) + \frac{8}{3} + \gamma^2\right] \qquad \gamma = \sqrt{\frac{4m^2}{s} - 1}$$

$$Im\, \Pi^f(s) = Im\, \overline{\Pi}^f(s) = 0$$

$\underline{0 < s \ll 4m^2:}$

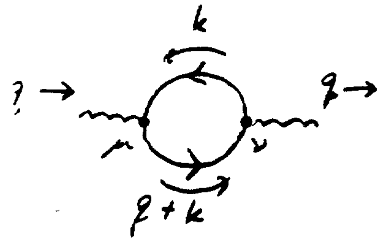
$$\underline{\underline{Re\, \overline{\Pi}^f(s) = \frac{\alpha Q_f^2}{15\pi}\frac{s}{m^2}}}$$

An important consequence:

If $m_f^2 \gg s$, the contribution to $\overline{\Pi}^f(s)$ goes away!
(de-coupling theorem)

Direct diagrammatic calculation of $\Pi_\gamma(s)$

We can also calculate $\Pi_\gamma(s)$ from the Feynman diagrams:



$$\equiv -i\,\Sigma^{\mu\nu}_\gamma(s)$$

A technical (but interesting ?) excercise!

Implement Feynman rules:

Fermions!

$$-i\,\Sigma^{\mu\nu}_\gamma(s) = (-1)\int \frac{d^4k}{(2\pi)^4}\,\mathrm{Tr}\left\{ i\,\frac{\not q+\not k+m}{(q+k)^2-m^2}\,(-ie)\gamma^\mu\, i\,\frac{\not k+m}{k^2-m^2}\,(-ie)\gamma^\nu \right\}$$

$$\Sigma^{\mu\nu}_\gamma = -ie^2\int\frac{d^4k}{(2\pi)^4}\,\frac{\mathrm{Tr}\left((\not q+\not k+m)\gamma^\mu(\not k+m)\gamma^\nu\right)}{\left((q+k)^2-m^2+i\varepsilon\right)\left(k^2-m^2+i\varepsilon\right)} = \infty$$

work out the trace:

$$\mathrm{Tr}\left((\not q+\not k+m)\gamma^\mu(\not k+m)\gamma^\nu\right) =$$

$$= 4\left(q^\mu k^\nu + q^\nu k^\mu + 2k^\mu k^\nu - (qk)g^{\mu\nu} - (k^2-m^2)g^{\mu\nu}\right)$$

we have to compute

$$I(\text{☺}) = \int\frac{d^4k}{(2\pi)^4}\,\frac{\text{☺}}{(k^2-m^2)((q+k)^2-m^2)}$$

with $\text{☺} = 1,\ k^\mu$ or $k^\mu k^\nu$

Regularization

In the loop calculations we will meet divergencies. To handle them, use regularization:

$$\begin{pmatrix}\text{infinite}\\ \text{quantity}\end{pmatrix} = \lim_{\begin{pmatrix}\text{some parameter}\\ \to \text{some value}\end{pmatrix}} \begin{pmatrix}\text{finite, parameter-}\\ \text{dependent quantity}\end{pmatrix}$$

Most popular nowadays:

DIMENSIONAL REGULARIZATION

(some parameter) = D, the number of dimensions of space time

(some value) = 4 of course!

The loop integration element:

$$\int \frac{d^4k}{(2\pi)^4} \longrightarrow \mu^{4-D} \int \frac{d^Dk}{(2\pi)^D}$$

"engineering dimension"
to keep the right power of (GeV) in the cross section.
It is not the renormalization scale!

Dim. Reg. is nice since it does not influence the gauge symmetry.
On the other hand, some technical problems:

$$\text{4-dim } \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \implies D\text{-dim } \gamma^5 = \;???$$

Define standard integrals : <u>SCALAR INTEGRALS</u>

$$\mu^{4-D}\int\frac{d^Dk}{(2\pi)^D}\,\frac{1}{k^2-m^2} \equiv \frac{i}{16\pi^2}A(m)$$

$$\mu^{4-D}\int\frac{d^Dk}{(2\pi)^D}\,\frac{1}{\underbrace{(k^2-m^2)}_{[k]}\underbrace{((q+k)^2-m^2)}_{[q+k]}} \equiv \frac{i}{16\pi^2}B(q^2,m)$$

The other integrals can be reduced to these :

$$\int\frac{k^\mu}{[k][q+k]} = F_0\, q^\mu \quad : \text{ find } F_0 \,!$$

$$F_0\, q^2 = \int\frac{(k\cdot q)}{[k][k+q]} = \int\frac{\frac{1}{2}\left([k+q]-[k]-q^2\right)}{[k+q]\cdot[k]}$$

$$= \frac{1}{2}\left\{\int\frac{1}{[k]} - \int\frac{1}{[q+k]} - q^2\int\frac{1}{[q+k][k]}\right\}$$

$$= \frac{1}{2}\left(A(m) - A(m) - q^2 B(q^2,m)\right) \quad \Rightarrow F_0 = -\frac{1}{2}B(q^2,m)$$

Slightly more complicated:

$$X^{\mu\nu} \equiv \int \frac{k^\mu k^\nu}{[k][q+k]} = F_1 g^{\mu\nu} + F_2 q^\mu q^\nu \qquad \text{find } F_1, F_2\ !$$

$$X^\mu_{\ \mu} = \int \frac{k^2}{[k][q+k]} = \int \frac{[k]+m^2}{[k][q+k]} = A(m) + m^2 B(q^2,m) = DF_1 + q^2 F_2$$

$$X^{\mu\nu} q_\mu q_\nu = \int \frac{(qk)^2}{[k][q+k]} = \frac{1}{2}\left\{ \int \frac{(qk)}{[k]} - \int \frac{(qk)}{[q+k]} - q^2 \int \frac{(qk)}{[k][q+k]} \right\}$$

$$= \frac{1}{2}\left\{ 0 - \int \frac{(q\cdot k - q)}{[k]} - q^2\left(-\frac{1}{2} q^2 B(q^2,m)\right)\right\}$$

$$= \frac{1}{2} q^2 A(m) + \frac{1}{4} q^4 B(q^2,m) = q^2 F_1 + q^4 F_2$$

Solve:

$$\left\{ \begin{aligned} A(m) + m^2 B(q^2,m) &= DF_1 + q^2 F_2 \\ \frac{1}{2} A(m) + \frac{1}{4} q^2 B(q^2,m) &= F_1 + q^2 F_2 \end{aligned} \right\}$$

$$\Rightarrow F_1 = \frac{1}{D-1}\left[\frac{1}{2} A(m) + \left(m^2 - \frac{1}{4} q^2\right) B(q^2,m) \right]$$

$$F_2 = \dots$$

Combining all terms:

$$\boxed{\begin{aligned} \Sigma_\gamma^{\mu\nu}(s) &= g^{\mu\nu} \cdot \frac{\alpha}{\pi}\left\{ -\frac{2}{3} A(m) + \frac{1}{3}(s + 2m^2) B(s,m^2) \right\} \\ &\quad + (q^\mu q^\nu \text{ terms}) \leftarrow \text{drop!} \end{aligned}}$$

Calculation of $A(m)$ and $B(s,m^2)$

$$\frac{i}{16\pi^2} A(m) = \frac{\mu^{4-D}}{(2\pi)^D} \int d^D k \, \frac{1}{k^2 - m^2}$$

For $B(s,m^2)$ the Feynman trick:

$$\frac{1}{a \cdot b} = \int_0^1 dx \, \frac{1}{(ax + b(1-x))^2}$$

$$\frac{i}{16\pi^2} B(s,m^2) = \frac{\mu^{4-D}}{(2\pi)^D} \int_0^1 dx \int d^D k \, \frac{1}{[k^2 + (x - x^2) q^2 - m^2]^2}$$

$k \to k + xq$

A complication:

$k^2 = (k^0)^2 - \vec{k}^2$ not positive definite

$\Rightarrow$ rewrite in Euclidean coordinates:

$$\left.\begin{array}{l} k^0 \equiv i k_E^0 \\ \vec{k} \equiv \vec{k}_E \end{array}\right\} \left\{\begin{array}{l} d^D k = i \, d^D k_E \\ k^2 = -k_E^2 = -(k_E^{0\,2} + \vec{k}_E^{\,2}) \end{array}\right.$$

$$\Rightarrow \begin{cases} A(m) = -16\pi^2 \cdot \dfrac{\mu^{4-D}}{(2\pi)^D} \cdot \displaystyle\int \frac{d^D k_E}{k_E^2 + m^2} \\[2ex] B(s,m^2) = -16\pi^2 \cdot \dfrac{\mu^{4-D}}{(2\pi)^D} \displaystyle\int_0^1 dx \int \frac{1}{[k_E^2 + m^2 + (x^2 - x)s]^2} \end{cases}$$

$$\int \frac{d^D k_E}{(k_E^2 + Q)^n} = \int \frac{k_E^{D-1}\, dk_E\, d\Omega_D}{(k_E^2 + Q)^n}$$

$$= \frac{2\pi^{D/2}}{\Gamma(\frac{D}{2})} \int_0^\infty dk \, \frac{k^{D-1}}{(k^2+Q)^n} = \pi^{\frac{D}{2}} \frac{\Gamma(n-\frac{D}{2})}{\Gamma(n)} Q^{\frac{D}{2}-n}$$

Use $n=1$ for $A(m)$

$n=2$ for $B(s,m^2)$

and $D = 4-\varepsilon$, $\varepsilon \to 0$

Handy relations $\Gamma(-1+\frac{\varepsilon}{2}) = -\frac{2}{\varepsilon} + \gamma_E - 1 + \cdots$

$\Gamma(\frac{\varepsilon}{2}) = \frac{2}{\varepsilon} - \gamma_E + \cdots$

$\gamma_E = 0.577\ldots$ Euler's constante

Dropping all terms that go to 0 as $\varepsilon \downarrow 0$:

$$A(m) = m^2\left[\frac{2}{\varepsilon} - \gamma_E + \ln 4\pi - \ln\frac{m^2}{\mu^2} + 1\right] + \cdots$$

$$B(s,m^2) = \frac{2}{\varepsilon} - \gamma_E + \ln 4\pi - \ln\frac{m^2}{\mu^2} - \int_0^1 dx \log\left[\frac{s(x^2-x)+m^2}{m^2} - i0\right]$$

defines Im part for values <0

Result of the diagram calculation

Contribution from fermion loop

$$\mathrm{Re}\,\Pi^f(s) = \Pi^f_\infty + \mathrm{Re}\,\bar{\Pi}^f(s)$$

$$\Pi^f_\infty = \frac{\alpha}{3\pi} Q_f^2 \left[\frac{2}{\varepsilon} - \gamma_E + \ln(4\pi) - \ln\frac{m_f^2}{\mu^2} \right]$$

survives if $s \to 0$

$$\mathrm{Re}\,\bar{\Pi}^f(s) = -\frac{\alpha}{3\pi} Q_f^2 \left\{ \frac{\beta(3-\beta^2)}{2} \ln\left(\frac{1+\beta}{1-\beta}\right) - \frac{8}{3} + \beta^2 \right\} \qquad \beta = \sqrt{1-4m^2/s}$$

for $s \geq 4m^2$

$$= -\frac{\alpha}{3\pi} Q_f^2 \left\{ \frac{\gamma(3+\gamma^2)}{2} \arctan\left(\frac{1}{\gamma}\right) - \frac{8}{3} - \gamma^2 \right\} \qquad \gamma = \sqrt{4m^2/s-1}$$

for $s \leq 4m^2$

vanishes if $s \to 0$

This is precisely the result derived before!

- (OF COURSE!)

FOR
- unequal masses
- internal boson lines
- ≥ 2-point integrals

things become more complicated but the principles remain the same

In order to be able to present the forthcoming self energy expressions as compact as possible, without loosing any transparancy, we are led to introduce a few more shorthand notations (only to be used in this appendix) :

$$z = M_Z \quad , \quad w = M_W \quad , \quad h = M_H \quad , \quad \Delta_i = \Delta_{M_i} \quad , \tag{I.3}$$

where Δ_M has been defined in appendix G. By decomposing the scalar 2-point function in appendix G according to

$$B_0(s, M_1, M_2) = \frac{1}{2}(\Delta_1 + \Delta_2) + 1 - \frac{M_1^2 + M_2^2}{M_1^2 - M_2^2} \log\left(\frac{M_1}{M_2}\right) + F(s, M_1, M_2) \, , \tag{I.4}$$

where the function $F(s, M_1, M_2)$ is symmetric in the masses, the unrenormalized transverse gauge boson self energies $\Sigma(s)$ occuring in the above expressions can be brought in the form : (the fermion summation also extends over the quark colours)

$$\begin{aligned}
\Sigma^{\gamma}(s) &= \frac{\alpha}{4\pi}\left\{\frac{4}{3}\sum_f Q_f^2\left[s\Delta_f + (s + 2m_f^2)\, F(s, m_f, m_f) - \frac{s}{3}\right]\right. && \leftarrow f\bar{f} \\
&\qquad \left. - \left[3s\Delta_W + (3s + 4w^2)\, F(s, w, w)\right]\right\} && \leftarrow W^+W^- \\
\Sigma^{\gamma Z}(s) &= \frac{\alpha}{4\pi}\left\{-\frac{4}{3}\sum_f Q_f v_f\left[s\Delta_f + (s + 2m_f^2)\, F(s, m_f, m_f) - \frac{s}{3}\right]\right. && \leftarrow f\bar{f} \\
&\qquad + \frac{1}{c_w s_w}\left[\left((3c_w^2 + \frac{1}{6})\, s + 2w^2\right)\Delta_W\right. \\
&\qquad\quad \left.\left. + \left((3c_w^2 + \frac{1}{6})\, s + (4c_w^2 + \frac{4}{3})\, w^2\right) F(s, w, w) + \frac{s}{9}\right]\right\} && \leftarrow W^+W^-
\end{aligned}$$

from W. Beenakker
Ph. D. thesis , 1990

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$$
\begin{aligned}
\Sigma^{Z}(s) = {}& \frac{\alpha}{4\pi}\Bigg\{\frac{4}{3}\sum_{l=e,\mu,\tau} 2a_l^2 s\left[\Delta_l + \frac{5}{3} - \log\left(-\frac{s+i\epsilon}{m_l^2}\right)\right] && \leftarrow \nu\bar{\nu} \\
& - \frac{4}{3}\sum_{f\neq\nu}\Big[(v_f^2+a_f^2)\left(s\Delta_f + (s+2m_f^2)F(s,m_f,m_f) - \frac{s}{3}\right) && \leftarrow f\bar{f} \\
& \qquad - 6a_f^2 m_f^2\,(\,\Delta_f + F(s,m_f,m_f))\,\Big] \\
& - \left[(3 - \frac{19}{6s_w^2} + \frac{1}{6c_w^2})\,s + (4 + \frac{1}{c_w^2} - \frac{1}{s_w^2})\,z^2\right]\Delta w && \leftarrow W^+W^- \\
& + \Big[\left(-c_w^4\,(40s + 80w^2) + (c_w^2 - s_w^2)^2(8w^2+s) + 12w^2\right)\,F(s,w,w)\Big) \\
& + \left(10z^2 - 2h^2 + s + \frac{(h^2-z^2)^2}{s}\right)F(s,h,z) && \leftarrow ZH \\
& - 2h^2\log\left(\frac{h^2}{w^2}\right) - 2z^2\log\left(\frac{z^2}{w^2}\right) \\
& + (10z^2 - 2h^2 + s)\left(1 - \frac{h^2+z^2}{h^2-z^2}\log\left(\frac{h}{z}\right) - \log\left(\frac{hz}{w^2}\right)\right) \\
& + \frac{2}{3}s\,(\,1 + (c_w^2 - s_w^2)^2 - 4c_w^4\,)\Big]\ \frac{1}{12c_w^2 s_w^2}\Bigg\} \\
\Sigma^{W}(s) = {}& \frac{\alpha}{4\pi}\frac{1}{3s_w^2}\Bigg\{\sum_{l=e,\mu,\tau}\left[(s - \frac{3}{2}m_l^2)\,\Delta_l + (s - \frac{m_l^2}{2} - \frac{m_l^4}{2s})\,F(s,0,m_l) + \frac{2}{3}s - \frac{1}{2}m_l^2\right] && \leftarrow l\bar{\nu} \\
& + \sum_{j\,(\mathrm{quark})}\Bigg[\frac{\Delta_{j+}}{2}(s - \frac{5}{2}m_{j+}^2 - \frac{1}{2}m_{j-}^2) + \frac{\Delta_{j-}}{2}(s - \frac{5}{2}m_{j-}^2 - \frac{1}{2}m_{j+}^2) && \leftarrow q\bar{q}' \\
& \qquad + \left(s - \frac{m_{j+}^2 + m_{j-}^2}{2} - \frac{(m_{j+}^2 - m_{j-}^2)^2}{2s}\right)F(s,m_{j+},m_{j-}) \\
& \qquad + \left(s - \frac{m_{j+}^2 + m_{j-}^2}{2}\right)\left(1 - \frac{m_{j+}^2 + m_{j-}^2}{m_{j+}^2 - m_{j-}^2}\log\left(\frac{m_{j+}}{m_{j-}}\right)\right) - \frac{s}{3}\Bigg] \\
& - \left[\frac{19}{2}s + 3w^2\left(1 - \frac{s_w^2}{c_w^2}\right)\right]\Delta w && \leftarrow WZ \\
& + \Bigg[3s_w^4 z^2 - c_w^2\left(7z^2 + 7w^2 + 10s - 2\frac{(z^2-w^2)^2}{s}\right) \\
& \qquad - \frac{1}{2}\left(w^2 + z^2 - \frac{s}{2} - \frac{(z^2-w^2)^2}{2s}\right)\Bigg]\,F(s,z,w) \\
& + s_w^2\left[-4w^2 - 10s + \frac{2w^4}{s}\right]F(s,0,w) && \leftarrow W\gamma \\
& + \frac{1}{2}\left[5w^2 - h^2 + \frac{s}{2} + \frac{(h^2-w^2)^2}{2s}\right]F(s,h,w) && \leftarrow WH \\
& + \left[c_w^2\,(\,3z^2 + 11w^2 + 10s) - 3s_w^4 z^2 + \frac{1}{2}(2w^2 - \frac{s}{2})\right]\frac{z^2}{z^2-w^2}\log\left(\frac{z^2}{w^2}\right) \\
& - (2w^2 + \frac{s}{4})\,\frac{h^2}{h^2-w^2}\log\left(\frac{h^2}{w^2}\right) - c_w^2\,(7z^2 + 7w^2 + \frac{32}{3}s) \\
& + 3s_w^4 z^2 + \frac{1}{2}(\frac{5}{3}s + 4w^2 - z^2 - h^2) - s_w^2\,(4w^2 + \frac{32}{3}s)\Bigg\}\,.
\end{aligned}
\qquad (\mathrm{I.5})
$$

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- $M_1 = 0$

$$F(s, 0, M_2) = 1 + \left[\frac{M_2^2}{s} - 1\right] \log\left(1 - \frac{s + i\epsilon}{M_2^2}\right) \qquad (I.6)$$

- $s \ll M_1^2, M_2^2$ and $M_1 \neq M_2$:

$$F(s, M_1, M_2) \approx \frac{s}{(M_1^2 - M_2^2)^2} \left[\frac{M_1^2 + M_2^2}{2} - \frac{M_1^2 M_2^2}{M_1^2 - M_2^2} \log\left(\frac{M_1^2}{M_2^2}\right)\right] \qquad (I.7)$$

$s \ll M_1^2 = M_2^2$:

$$F(s, M_1, M_1) \approx \frac{s}{6M_1^2}\left[1 + \frac{s}{10M_1^2}\right] \qquad (I.8)$$

$s \ll M_1^2 \ll M_2^2$:

$$F(s, M_1, M_2) \approx \frac{s}{2M_2^2} \qquad (I.9)$$

and hence $F(0, M_1, M_2) = 0$ provided M_1 and M_2 are not both equal to zero.

- $s \gg M_1^2, M_2^2$:

$$F(s, M_1, M_2) \approx 1 - \log\left(-\frac{s + i\epsilon}{M_1 M_2}\right) + \frac{M_1^2 + M_2^2}{M_1^2 - M_2^2} \log\left(\frac{M_1}{M_2}\right) \qquad (I.10)$$

- $M_1^2 \ll s \ll M_2^2$:

$$F(s, M_1, M_2) \approx \frac{s}{2M_2^2}\left[1 + \frac{s}{3M_2^2}\right] . \qquad (I.11)$$

70.3

What happened to θ_W?

The weak mixing angle θ_W is not a fundamental parameter of the theory.
At tree level we had:

$$1 - \frac{M_W^2}{M_Z^2} \underset{\rightarrow}{=} s_W^2 \underset{\leftarrow}{=} \frac{e^2}{8g_W^2}$$

– two consistent definitions of $s_W^2 = \sin^2\theta_W$

After loop corrections these two definitions are no longer (necessarily) consistent: we have to take (at most) one of them (note that we can write everything without ever using $\sin^2\theta_W$ anyway!)

– The most aesthetic choice: (Veltman, Sirlin, ...)

$$\sin^2\theta_W \equiv 1 - \frac{M_W^2}{M_Z^2}$$

before and after radiative corr.s

- Therefore we no longer have $s_W^2 = e^2/8g_W^2$ beyond tree level.
- "Wherever you see θ_W, read $\arccos\left(\frac{M_W}{M_Z}\right)$"

Alternative renormalization schemes

So far we have considered the on-shell scheme: the renormalized mass is defined to be the physical mass:

$$\delta m_W^2 \equiv \mathrm{Re}\, \Sigma_W(m_W^2)$$

so that

$$\mathrm{Re}\, \overline{\Sigma}_W(m_W^2) = 0$$

One might consider alternatives, for instance with

$\delta m_W^2 \equiv$ the infinite part only of $\mathrm{Re}\,\Sigma_W(m_W^2)$, proportional to

$$\frac{2}{\varepsilon} - \gamma_E + \ln(4\pi)$$

so that

$$\mathrm{Re}\, \overline{\Sigma}_W(m_W^2) \neq 0 \quad (\text{but of course } < \infty)$$

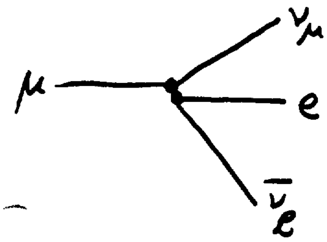
This is the $\overline{MS}$ scheme (customary in QCD): now the pole in the propagator is not necessarily at the value of m_W

- The on-shell is conceptually simpler
- The $\overline{MS}$ scheme is mathematically simpler

A fundamental radiative correction calculation:

Δr

Remember the starting point of our discussion: μ decay, leading to the introduction of the W



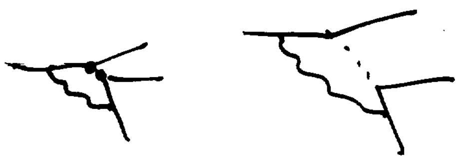
$$\Gamma_\mu = \frac{1}{\tau_\mu} = \frac{G^2 m_\mu^5}{192\pi^3} + \mathcal{O}\left(\frac{m_e^2}{m_\mu^2}\right)$$

One can include QED corrections

$$\Gamma_\mu = \frac{G^2 m_\mu^5}{192\pi^3}\Big[1 + \underbrace{\frac{\alpha}{2\pi}\left(\frac{25}{4} - \pi^2\right)}_{-0.0042}\Big]$$

- Use this formula to define G
- At tree level, G is "predicted" to be

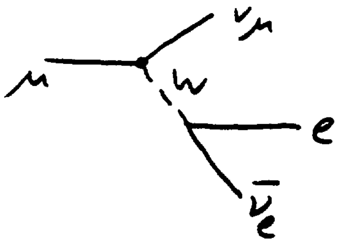
$$\frac{G}{\sqrt{2}} = \frac{e^2}{8 s_W^2 m_W^2}$$



- What is the prediction including corrections?

The strategy of "calculating" $G/\sqrt{2}$

(1) Calculate weak corrections to the amplitude



in terms of the bare parameters e_0, m_W^0, m_Z^0
$\rightarrow$ infinities in result

(2) Express the bare parameters e_0, m_W^0, m_Z^0 in the physical ones e, m_W, m_Z and the counterterms δe, δm_W^2, δm_Z^2
$\rightarrow$ extra infinities

(3) Truncate to fixed (1^{st}) order

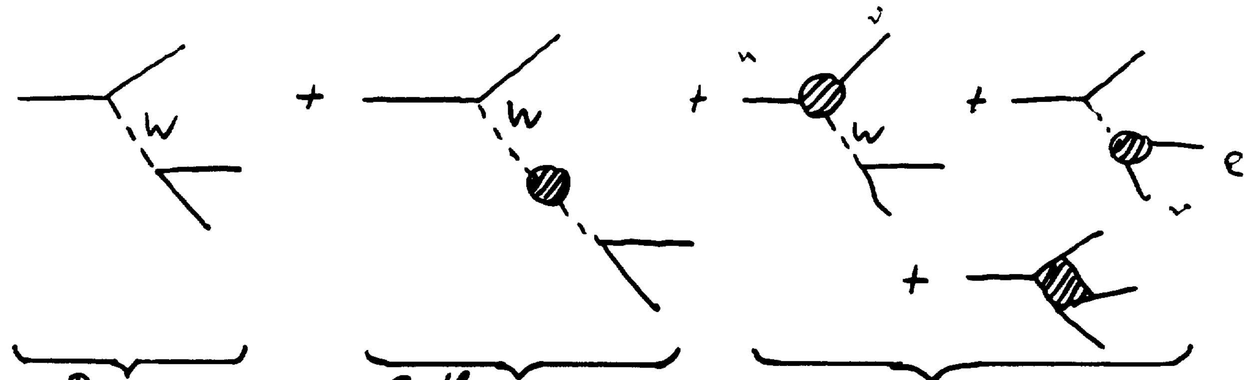
$\rightarrow$ infinities should drop

(4) Hopefully the resulting expression for the amplitude is analytically and numerically close to the tree-level one

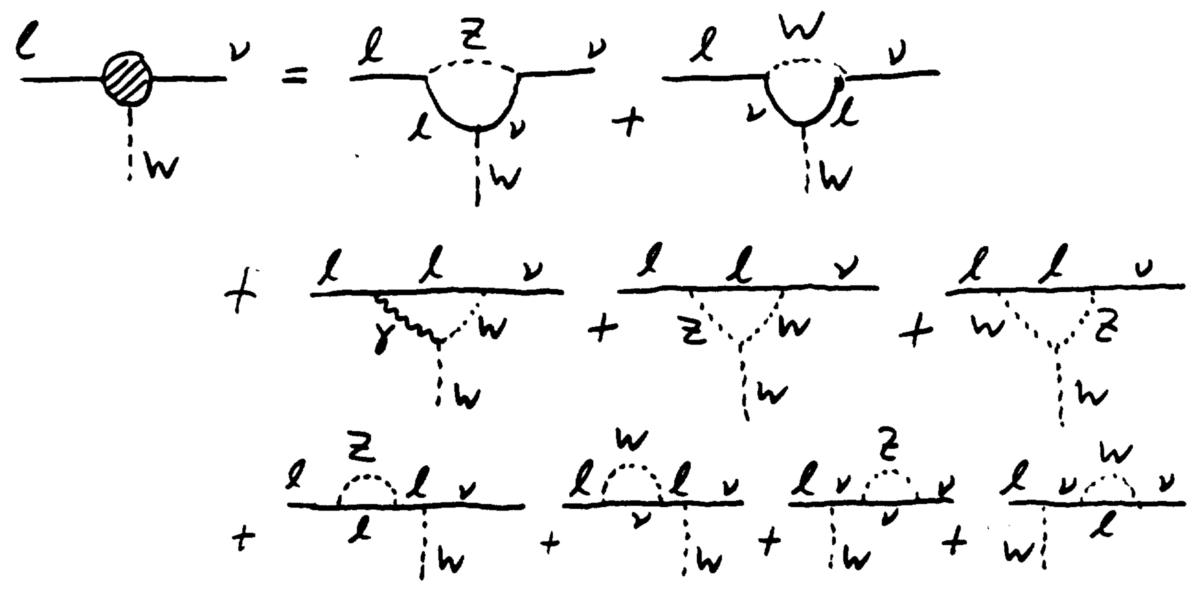
N.B. Renormalizability guarantees that the radiative corrections are finite, but not that they are small !

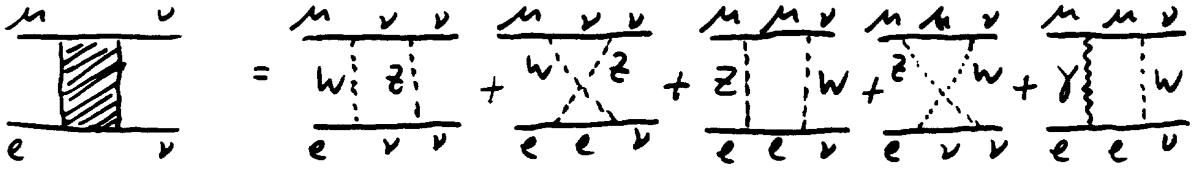
Weak correction diagrams

N.B. The photonic ("QED") corrections are already included in the definition of G! Avoid double counting!



Born — Self-energy — Vertex, Box





⇐ this one was added already in the QED corrections the same at Born and corr.

The Born (tree level) relation was (leaving out the Dirac matrices and spinors)

$$\frac{G}{\sqrt{2}} = \frac{e^2}{8 m_W^2 s_W^2} \qquad s_W^2 \equiv 1 - \frac{m_W^2}{m_Z^2}$$

After calculation of the loop corrections

$$\frac{G}{\sqrt{2}} = \frac{e_0^2}{8 m_W^{o2} s_W^{o2}} \left[1 + \frac{\Sigma_W(0)}{m_W^2} + \delta_{VB} \right] \qquad s_W^{o2} \equiv 1 - \frac{m_W^{o2}}{m_Z^{o2}}$$

1) self-energy correction:

$$\frac{-i}{s - m_W^2} \left(-i\Sigma_W(s)\right) \frac{-i}{s - m_W^2} \Big|_{s \simeq 0} = \frac{-i}{s - m_W^2} \cdot \frac{\Sigma_W(0)}{m_W^2}$$

2) vertex/box corrections:

$$\delta_{VB} = \frac{\alpha}{\pi s_W^2}\left[\Delta - \ln\frac{m_W^2}{\mu^2}\right] + \frac{\alpha}{4\pi s_W^2}\left(6 + \frac{7 - 4s_W^2}{2 s_W^2}\ln c_W^2\right)$$

note the correspondence with the nonabelian charge counterterm:

$$\frac{\alpha}{\pi s_W^2}\left[\Delta - \ln\frac{m_W^2}{\mu^2}\right] = \frac{2}{c_W s_W} \frac{\Sigma_{\gamma Z}(0)}{m_Z^2}$$

$$\Delta \equiv \frac{2}{\varepsilon} - \gamma_E + \ln(4\pi)$$

Express the bare parameters in the physical ones:

$$e_0^2 = (e + \delta e)^2 = e^2\left(1 + \frac{2\delta e}{e}\right)$$

$$= e^2\left(1 + \Pi_\gamma(0) - 2\frac{s_W}{c_W}\cdot\frac{\Sigma_{\gamma Z}(0)}{m_Z^2}\right)$$

$$\frac{1}{m_W^{0\,2}} = \frac{1}{(m_W^2 + \delta m_W^2)} = \frac{1}{m_W^2 + Re\,\Sigma_W(m_W^2)} \sim \frac{1}{m_W^2}\left(1 - \frac{Re\,\Sigma_W(m_W^2)}{m_W^2}\right)$$

$$S_W^{0\,2} = 1 - \frac{m_W^{0\,2}}{m_Z^{0\,2}} = 1 - \frac{m_W^2 + Re\,\Sigma_W(m_W^2)}{m_Z^2 + Re\,\Sigma_Z(m_Z^2)} = 1 - \frac{m_W^2}{m_Z^2}\,\frac{(1 + Re\,\Sigma_W(m_W^2)/m_W^2)}{(1 + Re\,\Sigma_Z(m_Z^2)/m_Z^2)}$$

$$\sim 1 - c_W^2\left[1 + \frac{Re\,\Sigma_W(m_W^2)}{m_W^2} - \frac{Re\,\Sigma_Z(m_Z^2)}{m_Z^2}\right]$$

$$= S_W^2\left(1 - \frac{c_W^2}{S_W^2}\left[\frac{Re\,\Sigma_W(m_W^2)}{m_W^2} - \frac{Re\,\Sigma_Z(m_Z^2)}{m_Z^2}\right]\right)$$

Adding everything:

$$\frac{G}{\sqrt{2}} = \frac{e^2}{8\, m_W^2 S_W^2}\left(1 + \Delta r\right)$$

$$\Delta r = \Pi_\gamma(0) - 2\frac{S_W}{c_W}\frac{\Sigma_{\gamma Z}(0)}{m_Z^2} + \frac{2}{c_W S_W}\frac{\Sigma_{\gamma Z}(0)}{m_Z^2}$$

$$- \frac{c_W^2}{S_W^2}\left[\frac{Re\,\Sigma_Z(m_Z^2)}{m_Z^2} - \frac{Re\,\Sigma_W(m_W^2)}{m_W^2}\right] + \frac{\Sigma_W(0)}{m_W^2} - \frac{Re\,\Sigma_W(m_W^2)}{m_W^2}$$

$$+ \frac{\alpha}{4\pi S_W^2}\left\{6 + \frac{7 - 4S_W^2}{2S_W^2}\ln c_W^2\right\}$$

And this should be finite!

Using

$$\Pi_\gamma(0) \doteq \mathrm{Re}\,\Pi_\gamma(m_Z^2) - \mathrm{Re}\,\overline{\Pi}_\gamma(m_Z^2)$$

we can write

$$\begin{aligned}
\Delta r = &-\mathrm{Re}\,\overline{\Pi}_\gamma(m_Z^2) \\
&+ \mathrm{Re}\,\Pi_\gamma(m_Z^2) + 2\frac{c_W}{s_W}\frac{\Sigma_{\gamma Z}(0)}{m_Z^2} \\
&- \frac{c_W^2}{s_W^2}\left[\frac{\mathrm{Re}\,\Sigma_Z(m_Z^2)}{m_Z^2} - \frac{\mathrm{Re}\,\Sigma_W(m_W^2)}{m_W^2}\right] \\
&+ \frac{\Sigma_W(0)}{m_W^2} - \frac{\mathrm{Re}\,\Sigma_W(m_W^2)}{m_W^2} \\
&+ \frac{\alpha}{4\pi s_W^2}\left\{6 + \frac{7-4s_W^2}{2s_W^2}\ln c_W^2\right\}
\end{aligned}$$

We can distinguish several contributions:

1) from light fermions (i.e. $m_f \to 0$ when possible)

2) from heavy fermions ($2m_f > m_Z$ etcetera)

3) from the Higgs

4) from the gauge bosons

3) and 4): <u>not</u> separately gauge-invariant

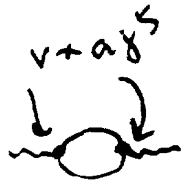
The light fermion contribution

Consider an "up", "down" pair $((\nu, \ell), (u,d), \dots)$ with $m_{u,d} \ll m_W, m_Z$ and look at the significant terms in the Δr contribution

$$\frac{\Sigma_W(0)}{m_W^2} \sim \mathcal{O}\left(\frac{m_f^2}{m_W^2}\right) \quad : \text{ negligible in leading order}$$

$$\frac{\Sigma_{\gamma Z}(0)}{m_Z^2} \sim \mathcal{O}\left(\frac{m_f^2}{m_Z^2}\right) \quad : \qquad \text{idem}$$

$$-\ \frac{\mathrm{Re}\,\Sigma_Z(m_Z^2)}{m_Z^2} \sim \frac{\alpha}{3\pi}\,\frac{v_u^2 + a_u^2 + v_d^2 + a_d^2}{e^2}\left[\Delta - \ln\frac{m_Z^2}{\mu^2} + \frac{5}{3}\right]$$



$$\frac{\mathrm{Re}\,\Sigma_W(m_W^2)}{m_W^2} \sim \frac{\alpha}{3\pi}\,\frac{2g_W^2}{e^2}\left[\Delta - \ln\frac{m_W^2}{\mu^2} + \frac{5}{3}\right]$$

$$= \frac{\alpha}{3\pi}\cdot\frac{2g_W^2}{e^2}\cdot\left[\Delta - \ln\frac{m_Z^2}{\mu^2} + \frac{5}{3}\right] - \frac{2\alpha g_W^2}{3\pi e^2}\ln c_W^2$$

shift in scale from m_W to m_Z

$$-\ \mathrm{Re}\,\Pi_\gamma(m_Z^2) \sim \frac{\alpha}{3\pi}\cdot\frac{Q_u^2 + Q_d^2}{e^2}\left[\Delta - \ln\frac{m_Z^2}{\mu^2} + \frac{5}{3}\right]$$

Use the coupling constants:

$$g_W = \frac{e}{s_W \sqrt{8}} \qquad Q_u = Q_d + e$$

$$a_u = \frac{e}{4 s_W c_W} \qquad v_u = a_u\left(1 - \frac{Q_u}{e} \cdot 4 s_W^2\right)$$

$$a_d = \frac{-e}{4 s_W c_W} \qquad v_d = a_d\left(1 + \frac{Q_d}{e} \cdot 4 s_W^2\right)$$

Then for the light fermion contribution:

$$\Delta r^{(l.f.)} =$$

$$- \mathrm{Re}\, \overline{\Pi}_\gamma (m_Z^2)$$

$$+ \frac{\alpha}{3\pi}\left(\Delta - \ln\frac{m_Z^2}{\mu^2} + \frac{5}{3}\right)\underbrace{\left[Q_u^2 + Q_d^2 - \frac{c_W^2}{s_W^2}\left(v_u^2 + a_u^2 + v_d^2 + a_d^2 - 2g_W^2\right) - 2g_W^2\right]\frac{1}{e^2}}_{\text{cancels !}}$$

$$- \frac{\alpha}{3\pi}\frac{2g_W^2}{e^2}\left(\frac{c_W^2}{s_W^2} - 1\right)\ln c_W^2$$

$$= \underbrace{- \mathrm{Re}\, \overline{\Pi}_\gamma (m_Z^2)}_{\shortparallel} \; \underbrace{- \frac{\alpha}{3\pi}\frac{c_W^2 - s_W^2}{4 s_W^4}\ln c_W^2}_{0.00055 \text{ per } (u,d) \text{ doublet}}$$

$$\frac{\alpha}{3\pi} N_c\left[Q_u^2\left(\ln\frac{m_Z^2}{m_u^2} - \frac{5}{3}\right) + Q_d^2\left(\ln\frac{m_Z^2}{m_d^2} - \frac{5}{3}\right)\right]$$

Contributions to Re $\overline{\Pi}_\gamma(m_Z^2)$

(1) Leptons:

ℓ	m_ℓ (GeV)	Result (%)
e	0.000511	1.77
μ	0.106	0.92
τ	1.87	0.47

(2) Lightest quarks : u, d, s

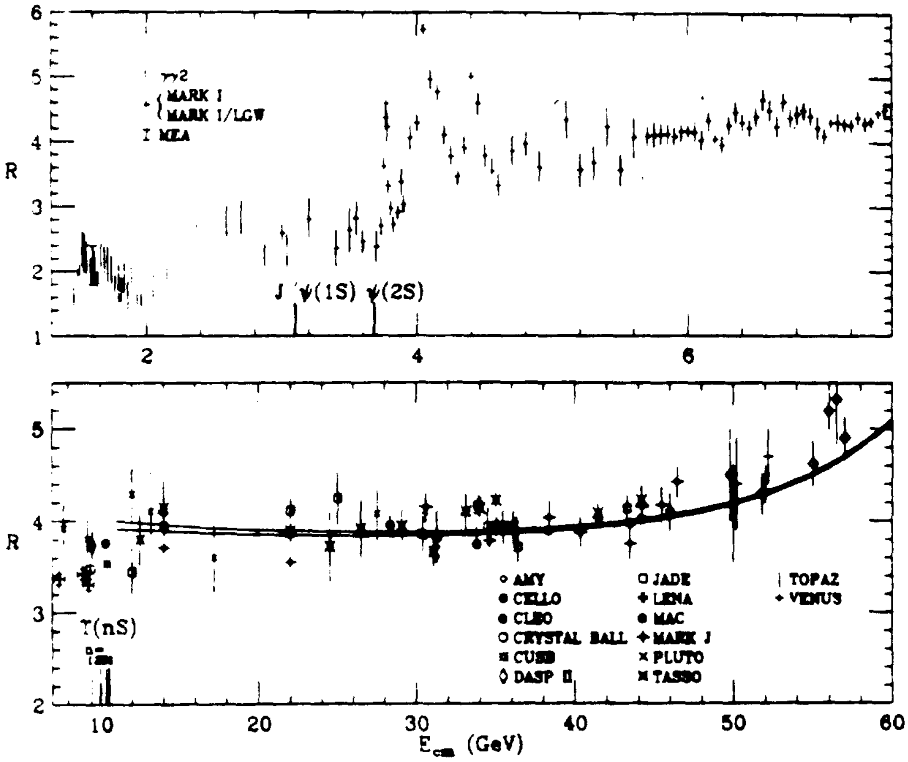
q	m_q (GeV)	Result (%)	
u	0.005	1.85	"current quark" masses
d	0.007	0.45	
S	0.150	0.29	
u	0.3	1.01	"constituent quark" masses
d	0.3	0.25	
s	0.5	0.22	

(3) Semi-heavy quarks: c, b

q	m_q (GeV)	Result (%)
c	1.5	0.68
b	5.0	0.11

PLOTS OF CROSS SECTIONS AND RELATED QUANTITIES (Cont'd)

R in e^+e^- Collisions



Selected measurements of $R \equiv \sigma(e^+e^- \to \text{hadrons})/\sigma(e^+e^- \to \mu^+\mu^-)$, where the annihilation in the numerator proceeds via one photon or via the Z^0. Measurements in the vicinity of the Z^0 mass are shown in the following figure. The denominator is the calculated QED single-photon process; see the section on Cross-Section Formulae for Specific Processes. Radiative corrections and, where important, corrections for two-photon processes and τ production have been made. Note that the ADONE data ($\gamma\gamma 2$ and MEA) is for ≥ 3 hadrons. The points in the $\psi(3770)$ region are from the MARK I – Lead Glass Wall experiment. To preserve clarity only a representative subset of the available measurements is shown – references to additional data are included below. Also for clarity, some points have been combined or shifted slightly ($< 4\%$) in E_{cm}, and some points with low statistical significance have been omitted. Systematic normalization errors are not included; they range from ~5–20%, depending on experiment. We caution that especially the older experiments tend to have large normalization uncertainties. Note the suppressed zero. The horizontal extent of the plot symbols has no significance. The positions of the $J/\psi(1S)$, $\psi(2S)$, and the four lowest Υ vector-meson resonances are indicated. Two curves are overlaid for $E_{cm} > 11$ GeV, showing the theoretical prediction for R, including higher order QCD [M. Dine and J. Sapirstein, Phys. Rev. Lett. **43**, 668 (1979)] and electroweak corrections. The Λ values are for 5 flavors in the $\overline{\rm MS}$ scheme and are $\Lambda^{(5)}_{\overline{\rm MS}} = 60$ MeV (lower curve) and $\Lambda^{(5)}_{\overline{\rm MS}} = 250$ MeV (upper curve). References (including several references to data not appearing in the figure and some references to preliminary data):

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Particle Data Book
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Hadronic vacuum polarization

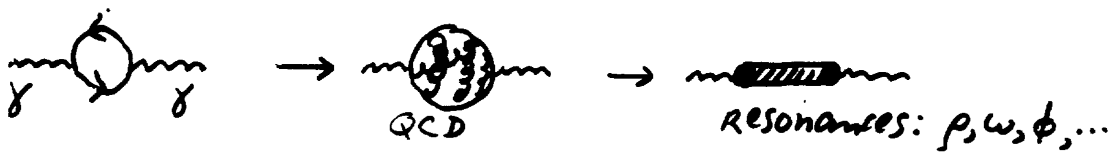
Problematic:

1) what are the quark masses ?

current masses: $u+d+s \rightarrow 2.59\,\%$ — an unacceptable difference!

constituent masses: $1.48\,\%$

2) QCD corrections are expected to be large !



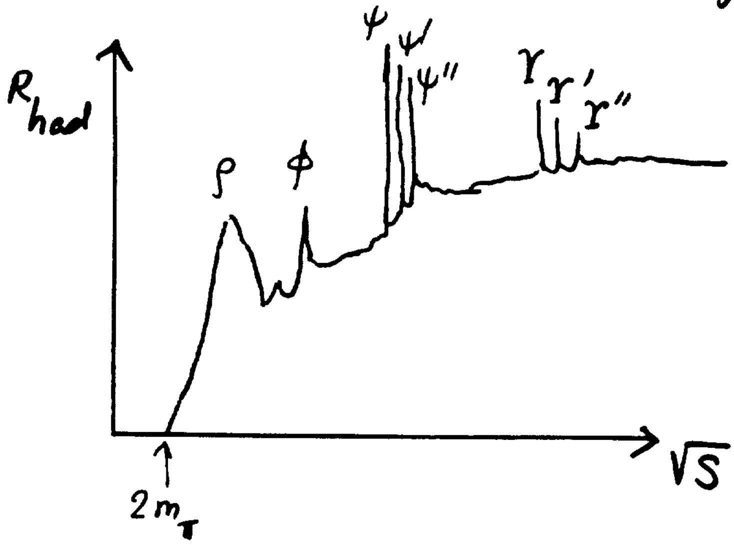
Solution: Back to the dispersion relation !

$$\mathrm{Re}\,\overline{\Pi}^{had}(m_Z^2) = \frac{\alpha m_Z^2}{3\pi} \int_{4m_\pi^2}^{\infty} ds \, \frac{R_{had}(s)}{s(s-m_Z^2)}$$

$R_{had}(s)$ is built up out of:

(1) hadronic resonances $\Rightarrow$ integrate analytically

(2) smooth background $\Rightarrow$ integrate numerically



A careful integration: (H. Burkhardt et al.)

$$-\mathrm{Re}\,\overline{\Pi}^{had}(s) = 0.00165 + 0.0030 \ln\frac{s}{\mathrm{GeV}^2} \pm 0.0009 \qquad s \sim m_Z^2$$

$$-\mathrm{Re}\,\overline{\Pi}^{had}(t) = 0.008 \ln\frac{|t|}{\mathrm{GeV}^2} \qquad t \lesssim 0$$

(1) At $s \sim m_Z^2$:

$$-\mathrm{Re}\,\overline{\Pi}^{had}(m_Z^2) \sim 0.0288 \pm 0.0009$$

the error comes from the exp. error in R_{had}

(2) Can be mimicked for $s \sim m_Z^2$ by

$$-\mathrm{Re}\,\overline{\Pi}^{had}(s) = \sum_q N_c Q_q^2 \left[\ln\frac{s}{m_q^2} - \frac{5}{3}\right]\left(1 + \frac{\alpha_s}{\pi}\right)$$

with

$m_u = 0.053$ $\quad m_c = 1.5$ $\quad m_b = 4.5$

$m_d = 0.071$ $\quad m_s = 0.174$

$\alpha_s = 0.124$

but these values do not work for $-\infty < s < (\sim 20\,\mathrm{GeV})^2$

The total contribution from leptons and light quarks:

$$\boxed{-\mathrm{Re}\,\overline{\Pi}_\gamma(m_Z^2) = -0.0602 \pm 0.0009}$$

... but don't forget the $\ln c_W^2$ term!

Heavy fermion contribution

In the MSM there is one heavy fermion: **TOP**

with unknown mass m_t. Unambiguous limits:

$m_t > 45$ GeV from LEP (assuming "nothing")

$m_t > 89$ GeV from Tevatron (assuming MSM)

Again, collect the relevant terms in the (b,t) contribution to Δr

(keep terms $\sim \ln m_t^2/m_Z^2$ and $\sim m_t^2/m_Z^2$)

$$\Pi_\gamma^t(m_Z^2) = \frac{\alpha N_c}{3\pi}\left(\frac{Q_t}{e}\right)^2 \Delta_t + \cdots \qquad \Delta_t = \underbrace{\frac{2}{\varepsilon} - \gamma_E + \ln 4\pi}_{} - \ln\frac{m_t^2}{\mu^2}$$

$$\Pi_\gamma^b(m_Z^2) = \frac{\alpha N_c}{3\pi}\left(\frac{Q_b}{e}\right)^2\left[\Delta_b - \ln\frac{m_Z^2}{m_b^2} + \frac{5}{3}\right] + \cdots$$

$$\frac{1}{m_Z^2}\mathrm{Re}\,\Sigma_Z^t(m_Z^2) = \frac{\alpha N_c}{3\pi}\frac{1}{e^2}\left[\left(v_t^2 + a_t^2 - 6a_t^2\frac{m_t^2}{m_Z^2}\right)\Delta_t + \cdots\right]$$

$$\frac{1}{m_Z^2}\mathrm{Re}\,\Sigma_Z^b(m_Z^2) = \frac{\alpha N_c}{3\pi}\frac{1}{e^2}\left[(v_b^2 + a_b^2)\left(\Delta_b - \ln\frac{m_Z^2}{m_b^2} + \frac{5}{3}\right) + \cdots\right]$$

$$\frac{1}{m_W^2}\mathrm{Re}\,\Sigma_W^{tb}(m_W^2) = \frac{\alpha N_c}{3\pi}\frac{1}{4e^2s_W^2}\left[\frac{\Delta_t}{2}\left(1 - \frac{5m_t^2}{2m_W^2}\right) + \frac{\Delta_b}{2}\left(1 - \frac{m_t^2}{2m_W^2}\right) - \frac{m_t^2}{4m_W^2} + \left(1 - \frac{m_t^2}{2m_W^2}\right)\left(1 - \ln\frac{m_t}{m_b}\right) + \cdots\right]$$

$$\frac{1}{m_W^2}\Sigma_W^{tb}(0) = \frac{\alpha N_c}{3\pi}\frac{1}{4e^2s_W^2}\left[\frac{\Delta_t}{2}\left(-\frac{5m_t^2}{2m_W^2}\right) + \frac{\Delta_b}{2}\left(-\frac{m_t^2}{2m_W^2}\right) - \frac{m_t^2}{2m_W^2}\left(1 - \ln\frac{m_t}{m_b}\right) + \cdots\right]$$

$$\frac{1}{m_Z^2}\Sigma_{\gamma Z}^{t,b}(0) = 0$$

Check on the part $\propto \frac{2}{4-D} - \gamma_E + \ln(4\pi)$:

$$\frac{\alpha N_c}{3\pi}\cdot\frac{1}{e^2}\left\{Q_t^2+Q_b^2-\frac{c_w^2}{s_w^2}\left(v_t^2+a_t^2+v_b^2+a_b^2-6a_t^2\frac{m_t^2}{m_Z^2}-\frac{1}{4s_w^2}\left[1-\frac{3}{2}\frac{m_t^2}{m_W^2}\right]\right)-\frac{1}{4s_w^2}\right\}$$

$= 0$: finite again!

- the terms without m_t cancel as before, due to $a_t = -a_b = \frac{e}{4s_w c_w}$, $v_t = a_t\left(1-\frac{Q_t}{e}\cdot 4s_w^2\right)$, $v_b = a_b\left(1+\frac{Q_b}{e}\cdot 4s_w^2\right)$
- to make the terms with m_t^2 cancel, we also need $m_W = c_w m_Z$ $\Rightarrow$ nontrivial influence of the Higgs sector!

For $m_t \gg m_W$ the finite part is dominated by m_t^2

$$\Delta r^{(t,b)} \sim -\frac{c_w^2}{s_w^2}\Delta\rho$$

$$\Delta\rho = \frac{\alpha N_c}{16\pi s_w^2 c_w^2}\cdot\frac{m_t^2}{m_Z^2}$$

Some remarks on $\Delta\rho$

① We also have

$$\Delta\rho^{t,b} = \frac{\Sigma_Z^{t,b}(0)}{m_Z^2} - \frac{\Sigma_W^{t,b}(0)}{m_W^2}$$

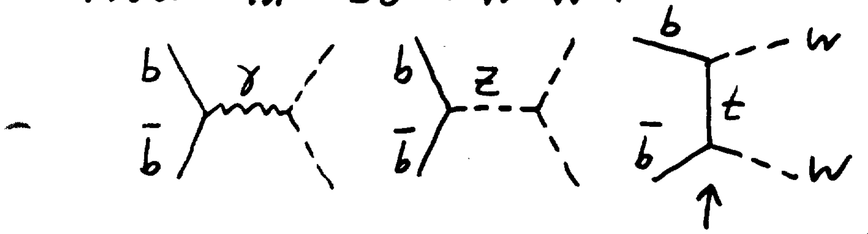
which is precisely the (t,b) contribution to the corrections to the NC/CC ratio of νN scattering

○ $\lim_{m_t \to \infty} \Delta\rho \sim m_t^2 \to \infty$:

For $m_t \to \infty$ the rad. corr. diverges!

This is a reminder of the fact that the MSM without t is not renormalizeable!

Note: in $b\bar{b} \to W^+W^-$:



unitarity is destroyed if we remove this diagram $\Rightarrow$ quadratic divergences

This is nowadays called the Non-decoupling theorem although this is a crazy name!

③ $\lim\limits_{m_H \to \infty} \Delta\rho \sim \ln m_H^2/m_W^2$

This is a much milder divergence. Veltman screening

(Related to the fact that there is only one, neutral Higgs) "accidental SU(2)"

At 2-loop level, one indeed gets terms $\sim m_H^2 \cdot \alpha^2$

④ $\Delta\rho$ can be considered as a measure for the amount of isospin symmetry breaking

(significant $\Delta\rho \Rightarrow m_W$ and m_Z renormalized in significantly different way

$m_t \to \infty \Rightarrow$ "no t quark" $\Rightarrow$ b quark all alone! $\Rightarrow \Delta\rho = \infty$)

⑤ New physics can also contribute to $\Delta\rho$ ("oblique corrections") for instance

(1) new generations with large mass splittings

(2) susy partners

(3) techicolor, compositeness

(4) non-minimal Higgs structure

Note: in this case we already have $\rho \neq 1$ at tree level: ρ is an additional free parameter here!

$\rho = 1 + \Delta\rho + \Delta\rho_0$

$\Rightarrow$ influence of large m_t much more complicated....

Summary on Δr in $O(\alpha)$

use $m_Z = 91.16$
$m_W = 80.6$
$s_W^2 = 0.218$
$m_t = 137 \leftarrow$
$m_H = 200$

In the minimal standard model

$$\Delta r = \Delta\alpha - \frac{c_W^2}{s_W^2}\Delta\rho + \Delta r_{Rest}$$

$$\Delta\alpha = -\mathrm{Re}\,\overline{\Pi}_\gamma(m_Z^2) \qquad = 0.0602 \pm 0.0009$$

$$\Delta\rho = \frac{\alpha N_c}{16\pi s_W^2 c_W^2}\frac{m_t^2}{m_Z^2} = 0.00255\frac{m_t^2}{m_Z^2} \qquad = 0.00576$$

$$\Delta r_{Rest} = \Delta r_{Rest}^{(t)} + \Delta r_{Rest}^{(H)} + \Delta r_{Rest}^{Rest}$$

$$\Delta r_{Rest}^{(t)} = \frac{\alpha}{4\pi s_W^2}\left(\frac{c_W^2}{s_W^2} - 1\right)\ln\frac{m_t}{m_Z} \qquad = 0.0028$$

$$\Delta r_{Rest}^{(H)} = \frac{\alpha}{16\pi s_W^2}\cdot\frac{11}{3}\left(\ln\frac{m_H^2}{m_W^2} - \frac{5}{6}\right) \qquad = 0.0018$$

$$\Delta r_{Rest}^{Rest} = \text{very tiny}$$

$$\Delta r = \Delta r(m_Z, m_W, m_t, m_H, \text{(known masses)}, \alpha, \alpha_s, \dots)$$

But: there is a constraint!

Higher orders in Δr

In $\mathcal{O}(\alpha)$:

$$\Delta r = \Delta\alpha - \frac{c_w^2}{s_w^2}\Delta\rho + \Delta r_{rest}$$

entering in the relation for G as

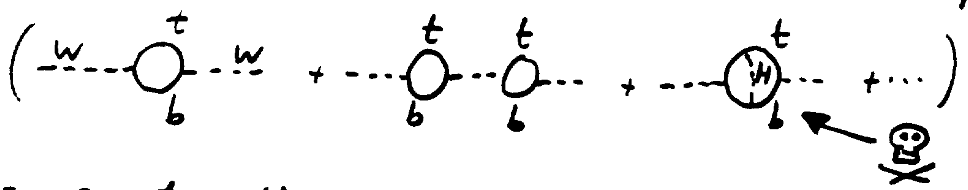
$$(1+\Delta r) \Longrightarrow \frac{1}{1-\Delta r}$$

How to include leading higher orders?

- The renormalization group tells us how to include large logarithms to all orders by

$$1+\Delta\alpha \longrightarrow \frac{1}{1-\Delta\alpha}$$

- ... but the m_t^2 terms are not large logs! An explicit calculation of the two-loop terms



indicates that

$$(1+\Delta r) \longrightarrow \frac{1}{1-\Delta\alpha}\cdot\frac{1}{1+\frac{c_w^2}{s_w^2}\Delta\bar{\rho}} + \Delta r_{rest}$$

$$\Delta\bar{\rho} = N_c \frac{G m_t^2}{8\pi^2\sqrt{2}}\left[1 + \frac{G m_t^2}{8\pi^2\sqrt{2}}(19-2\pi^2)\right]$$

gives all large terms to second order at least.

Self-consistent solutions

$$\Delta r = \Delta r(m_Z, m_W, m_t, m_H)$$

$$\frac{G}{\sqrt{2}} = \frac{\pi\alpha}{2}\,\frac{1}{m_Z^2 s_W^2 c_W^2}\cdot(1+\Delta r) \qquad , \; s_W^2 \equiv 1 - \frac{m_W^2}{m_Z^2}$$

Therefore, m_W must satisfy the equation

$$m_W^2 = \frac{m_Z^2}{2}\left\{1 + \left[1 - 4\left(\frac{\pi\alpha}{G\sqrt{2}}\right)\frac{1}{m_Z^2}(1+\Delta r(m_Z, m_W, m_t, m_H))\right]^{1/2}\right\}$$

Can easily be solved numerically

input parameters

$$\alpha = (137.0359895(61))^{-1}$$

$$G = 1.16637(2)\, 10^{-5}\ \text{GeV}^{-2}$$

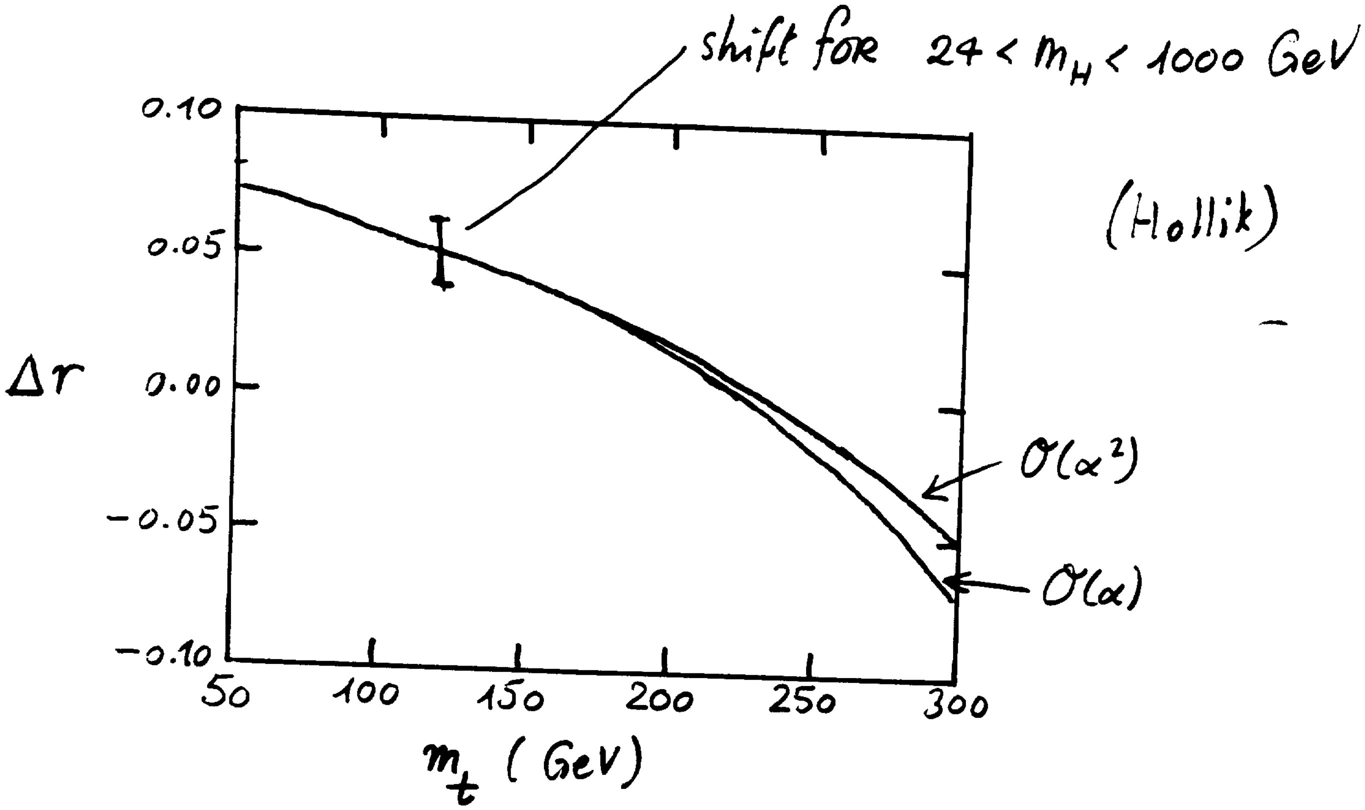
$$m_Z = 91.16 \pm 0.03\ \text{GeV}$$

output parameters

$$m_W,\ \Delta r,\ s_W^2$$

Result for Δr

$$m_Z = 91.15 \text{ GeV}$$
$$m_H = 100 \text{ GeV}$$



Δr can be experimentally defined as

$$\Delta r = 1 - \frac{\pi\alpha}{\sqrt{2}\,G}\,\frac{1}{m_W^2\left(1 - m_W^2/m_Z^2\right)}$$

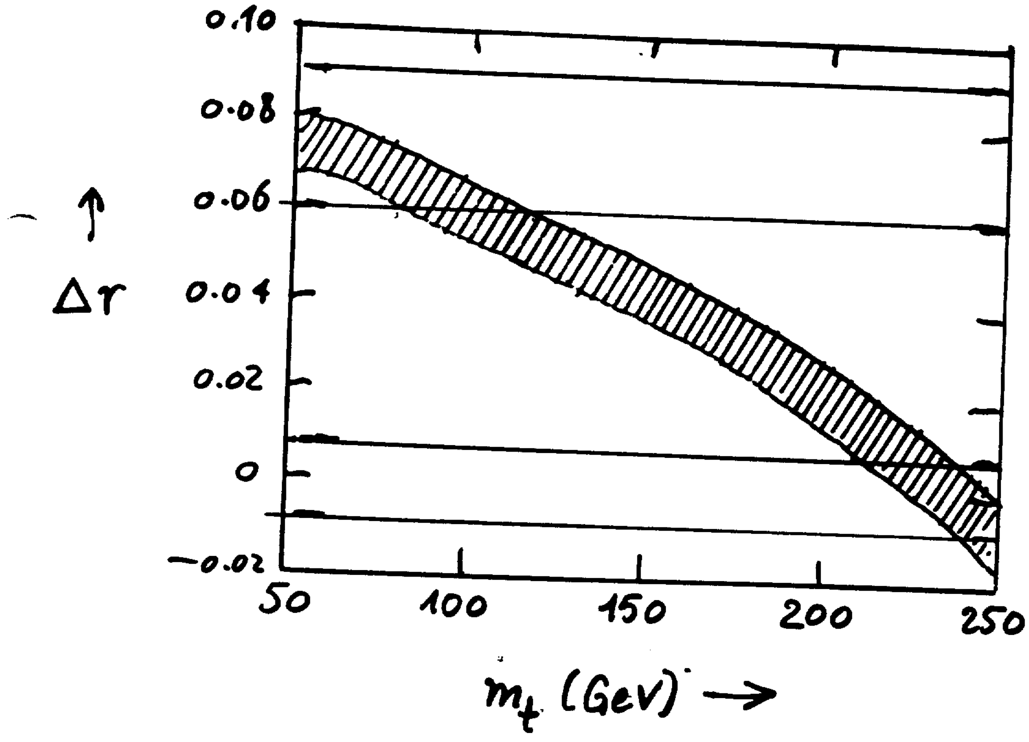
$\Rightarrow$ measurements of α, G, m_W and m_Z tell you the value of $\Delta r(\alpha, G, m_Z, m_W, m_H, m_t \ldots)$
$\Rightarrow$ obtain info on m_t, m_H!

Rough comparison with experiment

$m_Z = 91.15 \pm 0.03$ GeV

24 GeV $< m_H <$ 1000 GeV

50 GeV $< m_t <$ 250 GeV



1σ bounds from

$m_W = 80.0 \pm 0.62$ GeV

1σ bounds from

$\frac{m_W}{m_Z} = 0.8829 \pm 0.0055$

Any conclusions are up to you!

A note on the input parameters

The original Lagrangian (starting from the gauge symmetry principle) has parameters:

$$g_{U(1)_Y}\ ,\ g_{SU(2)_L}\ ,\quad \lambda, \mu^2\quad ,\quad g_Y$$

gauge couplings — Higgs potential — Yukawa couplings

Equivalent to the more physical set:

$$e\ ,\quad m_W\ ,\ m_Z\ ,\ m_H\quad ,\quad m_f$$

the on-shell scheme

Of these are known:

- e, m_f (except m_t) and m_Z "very precisely"
- m_W poorly
- m_t, m_H not at all

Actually we can trade m_W for G_μ:

$$e\ ,\ m_Z\ ,\ G\ ,\ m_H\ ,\ m_f$$

which is the <u>practical on-shell scheme</u>.

The Z° propagator revisited

A slightly simplified treatment:

$$\frac{1}{s - m_Z^{0\,2} + \mathrm{Re}\,\Sigma_Z(s)} = \frac{1}{s - m_Z^2 + \mathrm{Re}\,\Sigma_Z(s) - \mathrm{Re}\,\Sigma_Z(m_Z^2) + i\,\mathrm{Im}\,\Sigma_Z(s)}$$

$$\Downarrow$$

$$\mathrm{Re}\,\Sigma_Z(s) - \mathrm{Re}\,\Sigma_Z(m_Z^2) \sim (s - m_Z^2)\underbrace{\left[\frac{\partial}{\partial s}\mathrm{Re}\,\Sigma_Z(s)\right]_{s=m_Z^2}}_{X} + \ldots$$

$$\left.\begin{array}{l} \mathrm{Im}\,\Sigma_Z(s) \propto s \\ \mathrm{Im}\,\Sigma_Z(m_Z^2) = m_Z \Gamma_Z^{(0)} \end{array}\right\} \Rightarrow \mathrm{Im}\,\Sigma_Z(s) \sim \frac{s}{m_Z}\Gamma_Z^{(0)}$$

$$= \frac{1}{(s - m_Z^2)(1 + X) + i\frac{s}{m_Z}\Gamma_Z^0}$$

$$= \frac{1}{1 + X} \cdot \frac{1}{s - m_Z^2 + i\frac{s}{m_Z}\Gamma_Z^{phys}}$$

① X becomes finite when combined with vertices (or: after wave function renormalization!)

$$X \to \overline{\Pi}_Z(s) = \frac{\mathrm{Re}\,\Sigma_Z(s) - \mathrm{Re}\,\Sigma_Z(m_Z^2)}{s - m_Z^2}$$

$$- \Pi_\gamma(0) + \frac{c_W^2 - s_W^2}{s_W^2}\left(\frac{\mathrm{Re}\,\Sigma_Z(m_Z^2)}{m_Z^2} - \frac{\mathrm{Re}\,\Sigma_W(m_W^2)}{m_W^2} - 2\frac{s_W}{c_W}\frac{\Sigma_{\gamma Z}(0)}{m_Z^2}\right)$$

② $\Gamma_Z^{phys} = \dfrac{\Gamma_Z^{(0)} + \Gamma_Z^{(1)}}{1 + \overline{\Pi}_Z(m_Z^2)}$ non-negligible higher orders!

Corrections to the process $ee \to f\bar{f}$ at LEP 1

Typical e.w. corrections are of order

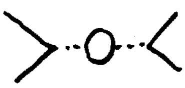
$$\frac{\alpha}{\pi} \times \begin{cases} \ln \frac{m_Z^2}{m_f^2} & \text{for light fermions} \\ m_t^2/m_Z^2 & \text{for heavy top quark} \\ 1 & \text{otherwise } \left(\frac{m_W^2}{m_Z^2},\ \ln m_H^2/m_Z^2, \ldots \right) \end{cases}$$

easily comparable to the experimental accuracy!

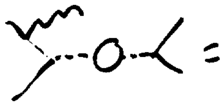
(of course one of the main reasons behind LEP....)

The corrections fall into two "classes":

① QED corrections  (next lecture)

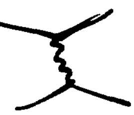
② "purely weak" corrections  (this lecture)

Remarks

(1) This split-up is gauge-invariant, but only unambiguous at 1 loop :  = ?

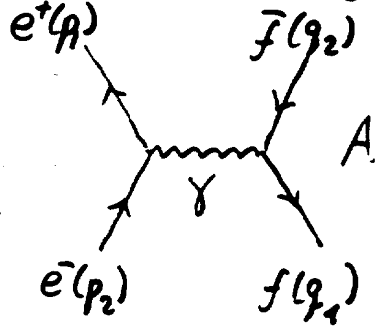
(2) not gauge invariant for charged-current processes (cf. μ decay!)

(3) not so easy e.g. for $ee \to W^+W^-$ (LEP 200!)

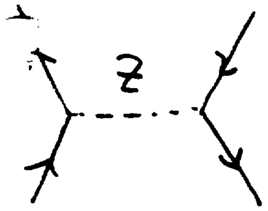
(4) similar results, but quantitatively different, for Bhabha scattering : $e^+e^- \to e^+e^-$ 

The tree level result

Two diagrams (for $f = \mu, \tau$, quarks)



$$A^0_\gamma \equiv i \cdot \frac{1}{s} Q_e Q_f \times \left[\bar{v}(p_1) \gamma^\mu u(p_2) \otimes \bar{u}(q_1) \gamma_\mu v(q_2) \right]$$



$$A^0_Z \equiv i \frac{1}{s - m_Z^2 + i m_Z \Gamma_Z}$$

$$\times \left[\bar{v}(p_1)(v_e + a_e \gamma^5) \gamma^\mu u(p_2) \right.$$

$$\left. \otimes \bar{u}(q_1)(v_f + a_f \gamma^5) \gamma_\mu v(q_2) \right]$$

Kinematics:

$$p_1^\mu = E(1, 0, 0, 1) \qquad e^+$$

$$p_2^\mu = E(1, 0, 0, -1) \qquad e^-$$

$$q_1^\mu = E(1, -\sin\theta \sin\varphi, -\sin\theta\cos\varphi, -\cos\theta) \qquad \mu^-$$

$$q_2^\mu = E(1, \sin\theta\sin\varphi, \sin\theta\cos\varphi, \cos\theta) \qquad \mu^+$$

Assume a degree of R.h. polarization P for the e^+:

$$\sum_{spins} v(p_1)\bar{v}(p_1) = \not{p}_1 (1 + P\gamma^5) \qquad \sum_{spins} u(p_2)\bar{u}(p_2) = \not{p}_2$$

Very explicitly:

$$\langle |M|^2 \rangle = \frac{1}{4} \sum_{spins} |M|^2 = \frac{A_{\gamma\gamma}}{s^2} + \frac{A_{ZZ}}{(s-m_Z^2)^2 + m_Z^2 \Gamma_Z^2} + \frac{(s-m_Z^2) A_{\gamma Z}}{s((s-m_Z^2)^2 + m_Z^2 \Gamma_Z^2)}$$

$$A_{\gamma\gamma} = s^2 Q_e^2 Q_f^2 (1 + \cos\theta^2)$$

$$A_{ZZ} = s^2 \left[(v_e^2 + a_e^2)(v_f^2 + a_f^2)(1 + \cos^2\theta) + 8 v_e v_f a_e a_f \cos\theta \right]$$
$$+ 2\mathcal{P} s^2 \left[v_e a_e (v_f^2 + a_f^2)(1 + \cos^2\theta) + 2 v_f a_f (v_e^2 + a_e^2) \cos\theta \right]$$

$$A_{\gamma Z} = 2 s^2 Q_e Q_f \left[v_e v_f (1 + \cos^2\theta) + 2 a_e a_f \cos\theta \right]$$
$$+ 2 \mathcal{P} s^2 Q_e Q_f \left[a_e v_f (1 + \cos^2\theta) + 2 v_e a_f \cos\theta \right]$$

$$\frac{d\sigma}{d\cos\theta} = \frac{1}{32 \pi s} \langle |M|^2 \rangle$$

$$\sigma_{tot} = \int_{-1}^{1} \frac{d\sigma}{d\cos\theta} \, d\cos\theta \Big|_{\mathcal{P}=0}$$

$$A_{FB} = \left[\int_0^1 \frac{d\sigma}{d\cos\theta} \, d\cos\theta - \int_{-1}^{0} \frac{d\sigma}{d\cos\theta} \, d\cos\theta \right]_{\mathcal{P}=0} \Big/ \sigma_{tot}$$

$$A_{LR} = \left[\int_{-1}^{1} \frac{d\sigma}{d\cos\theta} \, d\cos\theta \Big|_{\mathcal{P}=+1} - \int_{-1}^{1} \frac{d\sigma}{d\cos\theta} \, d\cos\theta \Big|_{\mathcal{P}=-1} \right] \Big/ \sigma_{tot}$$

$$\sigma_{tot} = \frac{1}{12\pi s}\left\{ \frac{s^2}{(s-m_Z^2)^2+m_Z^2\Gamma_Z^2}(v_e^2+a_e^2)(v_f^2+a_f^2) \right. \qquad |Z|^2$$

$$+ \quad Q_e^2 Q_f^2 \qquad |\gamma|^2$$

$$\left. + \frac{(s-m_Z^2)}{(s-m_Z^2)^2+m_Z^2\Gamma_Z^2} 2 Q_e Q_f v_e v_f \right\} \qquad \gamma \otimes Z$$

for $s \sim m_Z^2$ neglect the γ channel:

$$A_{FB} = 3\frac{v_e a_e v_f a_f}{(v_e^2+a_e^2)(v_f^2+a_f^2)} \equiv \frac{3}{4} A_e A_f$$

$$A_{LR} = \frac{2 v_e a_e}{(v_e^2+a_e^2)} \equiv A_e$$

other possibilities:

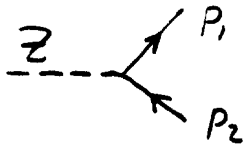
A^{τ}_{POL} : $\sim A_\tau \equiv A_{LR}$

A^{POL}_{FB} : polarized forward-backward asymmetry (fancy!)

Note: These, the usual expressions for the asymmetries, are only approximate, with $m_f = 0$ and no photon. In actual practice one of course uses the full thing!

Unitarity notation for σ_{tot}

Anologous to the W case: $Z \to f\bar{f}$ decay



$$\mathcal{M} = i\,\epsilon_\mu\,\bar{u}(p_1)(v_f + a_f\gamma^5)\gamma^\mu u(p_2)$$

$$\frac{1}{3}\sum|\mathcal{M}|^2 = \frac{4}{3}(v_f^2 + a_f^2)\,m_Z^2$$

$$\Gamma(Z \to f\bar{f}) = \frac{1}{12\pi}(v_f^2 + a_f^2)\,m_Z \quad \Rightarrow \quad (v_f^2 + a_f^2) = 12\pi\frac{\Gamma_{f\bar{f}}}{m_Z}$$

The total cross section (neglecting the γ) can therefore be written as

$$\sigma_{tot}(s) = 12\pi\left(\frac{\Gamma_{ee}}{m_Z}\right)\left(\frac{\Gamma_{ff}}{m_Z}\right)\frac{s}{(s-m_Z^2)^2 + m_Z^2\Gamma_Z^2}$$

at the peak:

$$\sigma_{tot}(m_Z^2) = \underbrace{\frac{12\pi}{m_Z^2}}_{\text{unitarity limit for } J=1 \text{ channel}}\cdot\underbrace{\left(\frac{\Gamma_{ee}}{\Gamma_Z}\right)}_{\text{branching ratio for initial state}}\underbrace{\left(\frac{\Gamma_{ff}}{\Gamma_Z}\right)}_{\text{branching ratio for final state}}$$

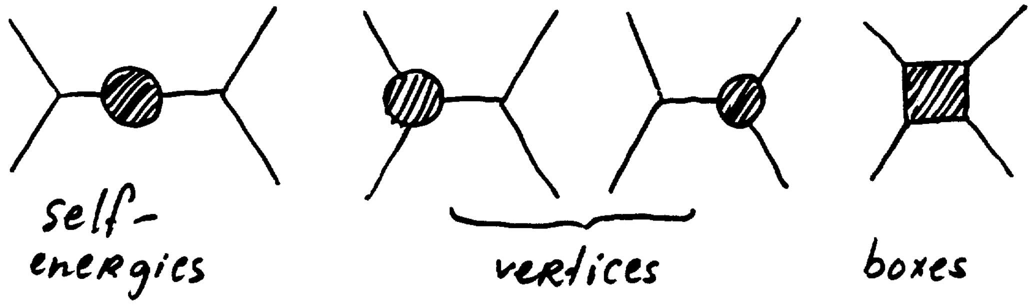
- At the peak the cross section is essentially the maximum that is physically possible

- $\Rightarrow$ At the peak, higher order effects will be either <u>very small</u> or <u>negative</u>! (cf. $t\bar{t}$–Z interference)

Applying Radiative (weak) corrections in $e^+e^- \to f\bar{f}$

The usual strategy:

(1) Calculate the one-loop corrections using the bare parameters



no virtual γ loops!

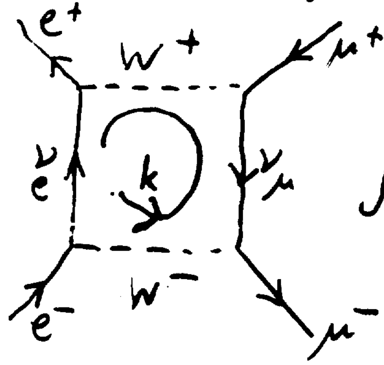
(2) Express the bare param^{s} in the physical ones, using the counterterms

(3) Truncate to desired (1^{st}) order

(4) Publish the (now finite) result.

A note on the box diagrams

The box diagrams are UV finite!



For large k^μ the loop integral is like

$$\int d^4k\, \frac{k}{k^2}\cdot\frac{1}{k^2}\,\frac{k}{k^2}\,\frac{1}{k^2} \sim \mathcal{O}\left(\int d^4k \frac{1}{k^6}\right) \quad \text{o.k.!}$$

$\Rightarrow$ boxes do not enter in our renormalization considerations

in fact, at the resonance the box diagrams give very small contributions

BUT

This only holds if indeed the W propagator goes like $\frac{1}{k^2}$: gauge dependent!

Feynman gauge:

$$\mu \overset{W}{\underset{\vec{k}}{- - - - -}} \nu \;=\; -i\,\frac{g^{\mu\nu}}{k^2-m^2} \sim \frac{1}{k^2} \;:\; \text{o.k.}$$

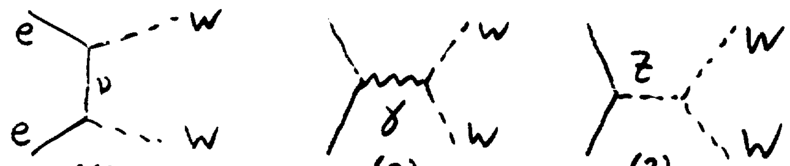
Unitary gauge:

$$\mu \overset{W}{\underset{\vec{k}}{- - - - -}} \nu \;=\; -i\,\frac{g^{\mu\nu} - \frac{k^\mu k^\nu}{m^2}}{k^2-m^2} \sim \frac{k^\mu k^\nu}{k^2} \;:\; \text{NOT o.k.}$$

In the unitary gauge the boxes are QUADRATICALLY DIVERGENT !?!?

The optical theorem again!

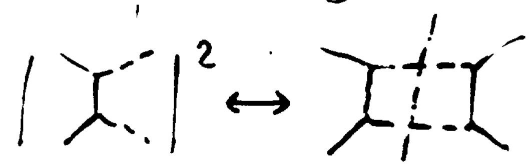
Consider $e^+e^- \to W^+W^-$: we needed 3 diagrams :

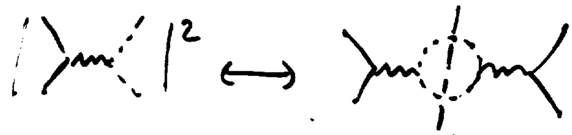


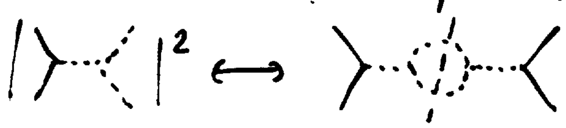
(1) (2) (3)

unitary gauge
longitudinal W's

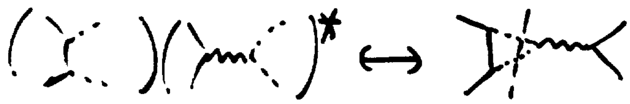
If we take only the first one, the cross section diverges!
Now remember that the cross section is related
to the (imaginary part of) the higher-order diagram:

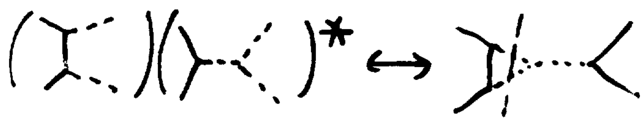
$|1|$  the box diagram!

$|2|^2$  γ self energy

$|3|^2$  Z self energy

$2 \times \bar{3}$  γZ mixing

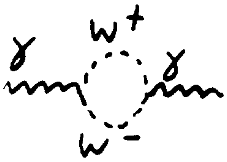
$1 \times \bar{2}$  γ vertex correction

$1 \times \bar{3}$  Z vertex correction

- Only the complete set is gauge-invariant!
- In unitary gauge, quadratic divergences from the box are cancelled by self energies / vertices
  $\Rightarrow$ no need for additional renormalization
- In the Feynman gauge the separate contributions are finite (but $\exists$ extra ghost diagrams)

A reflection on the self energies

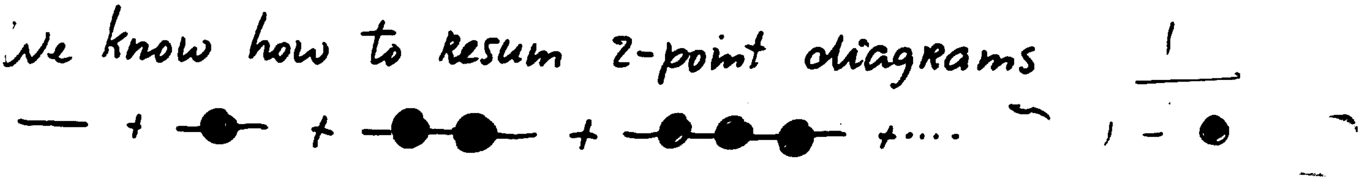
Apparently the W^+W^- loop contributions to the γ self energy are not gauge invariant.



BUT

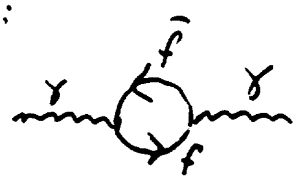
what about the Dyson summation?

We know how to resum 2-point diagrams



but not how to do this for 3-point / 4-point diagrams!

Solution adopted in practice:

1) Resum only fermionic loops


←

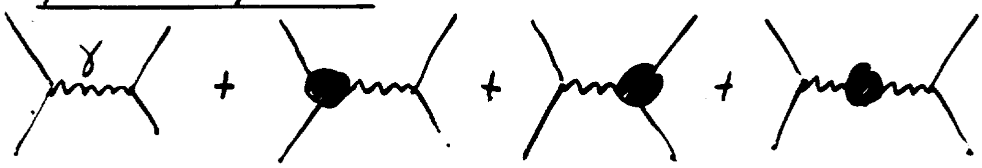
but keep the bosonic ones only to $\mathcal{O}(\alpha)$

OR

2) don't worry: since in the Feynman gauge the W^+W^- loops are small anyway, resumming them or not is practically irrelevant

Dressed amplitudes

If we can neglect the boxes the corrected amplitudes can be written in a form much like the Born one for the photon:



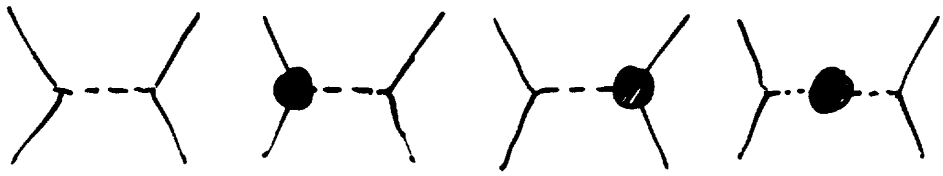
$$= i\,\frac{Q_e Q_f}{s}\cdot\frac{1}{1+\overline{\Pi}_\gamma(s)}\cdot\bar v(p_1)\left[(1+F_V^{\gamma e})+F_A^{\gamma e}\gamma^5\right]\gamma^\mu u(p_2)$$
$$\cdot\,\bar u(q_1)\left[(1+F_V^{\gamma f})+F_A^{\gamma f}\gamma^5\right]\gamma_\mu v(q_2)$$

- $\overline{\Pi}_\gamma(s)$: renormalised vacuum polarization $\sim -6\%$
- $F_{V,A}^{\gamma f}(s)$: "form factors" $=0$ for $s=0$
  $\mathrm{Re}\,F(m_Z^2)\sim 10^{-3}$
  $\mathrm{Im}\,F(m_Z^2)$ even smaller

The renormalization group tells us that we can sensibly use a running QED coupling

$$e^2(s)=\frac{e^2}{1+\overline{\Pi}_\gamma(s)} > e^2 \quad \text{since } \overline{\Pi}<0$$

For the Z:



$$A_Z = i \frac{1}{s - m_Z^2 + i\frac{s}{m_Z}\Gamma_Z} \cdot \bar{v}(p_1)\left(\hat{v}_e + \hat{a}_e \gamma^5\right)\gamma^\mu u(p_2)$$

$$\cdot\ \bar{u}(q_1)\left(\hat{v}_f + \hat{a}_f \gamma^5\right)\gamma_\mu v(q_2)$$

$$\hat{a}_f^2 = \sqrt{2}\, G\, m_Z^2\, \rho_f \qquad (\text{was } \sqrt{2}\, G\, m_Z^2)$$

$$\frac{\hat{v}_f}{\hat{a}_f} = 1 - \left|\frac{Q_f}{e}\right| \cdot 4 s_W^2\, \kappa_f$$

Again ρ_f and κ_f are form factors.
They contain universal (f-independent) and non-universal (f-dependent) contributions:

$$\rho_f = 1 + \Delta\rho_U + \Delta\rho_{NU}^f$$

$$\kappa_f = 1 + \Delta\kappa_U + \Delta\kappa_{NU}^f$$

More explicitly:

$$\Delta\rho_u = -\Delta r - \Pi^Z(s)$$

$$\Delta\rho^f_{NU} = \frac{1}{a_f} F_A^{Zf}(s)$$

$$\Delta\kappa_u = -\frac{c_W}{s_W}\Pi^{\gamma Z}(s)$$

$$\Delta\kappa^f_{NU} = -\frac{1}{2s_W^2 Q_f}\left[F_V^{Zf}(s) - \frac{v_f}{a_f}F_A^{Zf}(s)\right]$$

Usually the non-universal parts are small.

as before:

$$\Pi^Z(s) = \frac{\mathrm{Re}\,\Sigma_Z(s) - \mathrm{Re}\,\Sigma_Z(m_Z^2)}{s-m_Z^2} - \Pi_\gamma(0) + \frac{c_W^2 - s_W^2}{s_W^2}\left[\frac{\mathrm{Re}\,\Sigma_Z(m_Z^2)}{m_Z^2} - \frac{\mathrm{Re}\,\Sigma_W(m_W^2)}{m_W^2} - 2\frac{s_W}{c_W}\frac{\Sigma_{\gamma Z}(0)}{m_Z^2}\right]$$

now also:

$$\Pi^{\gamma Z}(s) = \frac{\Sigma^{\gamma Z}(s) - \Sigma^{\gamma Z}(0)}{s} - \frac{c_W}{s_W}\left(\frac{\mathrm{Re}\,\Sigma_Z(m_Z^2)}{m_Z^2} - \frac{\mathrm{Re}\,\Sigma_W(m_W^2)}{m_W^2}\right) + 2\frac{\Sigma^{\gamma Z}(0)}{m_Z^2}$$

these are both finite

Leading behaviour of $\Delta\rho_u$ and $\Delta\kappa_u$

The dominant behaviour of the universal parts:

$$\Delta\rho_u = \Delta\rho + \cdots$$

$$\Delta\kappa_u = \frac{c_W^2}{s_W^2}\,\Delta\rho + \cdots$$

And one can see why from the counterterms:

$$a_f^2 = \frac{e^2}{16\, s_W^2 c_W^2} \qquad \text{at Born}$$

$$\downarrow$$

$$= \frac{e_0^2}{16\, s_W^{0\,2} c_W^{0\,2}} = \frac{e^2}{16\, s_W^2 c_W^2}\left[1 + 2\frac{\delta e}{e} - \frac{c_W^2 - s_W^2}{s_W^2}\left(\frac{Re\,\Sigma_Z(m_Z^2)}{m_Z^2} - \frac{Re\,\Sigma_W(m_W^2)}{m_W^2}\right)\right]$$

$$= \sqrt{2}\, G\, m_Z^2 \left[1 + \underbrace{\left(\frac{Re\,\Sigma_Z(m_Z^2)}{m_Z^2} - \frac{Re\,\Sigma_W(m_W^2)}{m_W^2}\right)}_{\updownarrow} + \cdots\right]$$

$= \Delta\rho$ as far as the m_t^2 terms are concerned.

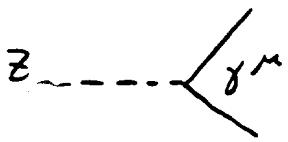
$$\updownarrow$$

$$s_W^2 \to s_W^{0\,2} = s_W^2\left[1 + \frac{c_W^2}{s_W^2}\overbrace{\left(\frac{Re\,\Sigma_Z(m_Z^2)}{m_Z^2} - \frac{Re\,\Sigma_W(m_W^2)}{m_W^2}\right)}\right] + \cdots$$

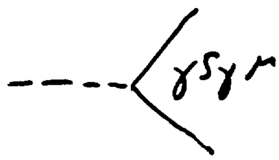
Effective $\sin^2\theta_W$: $\bar{s}_W^2$

After expressing the a_e, a_f in G, m_Z^2, the only place where $\sin^2\theta_W$ that appears is in v_f/a_f: corrected $s_W^2 \longrightarrow s_W^2 + c_W^2 \Delta\rho$

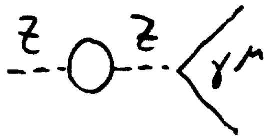
This can be understood from the diagrams

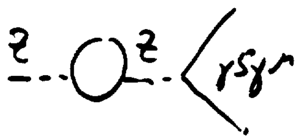


vs.

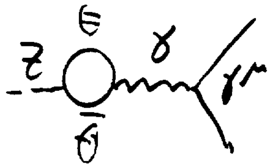


$\Rightarrow s_W^2$ at tree level.





overall correction $\Delta\rho$ but no change in v_f/a_f



no $\gamma^5\gamma^\mu$ diagram!

$\Rightarrow$ Ratio is changed!

Apparently we have an effective mixing angle

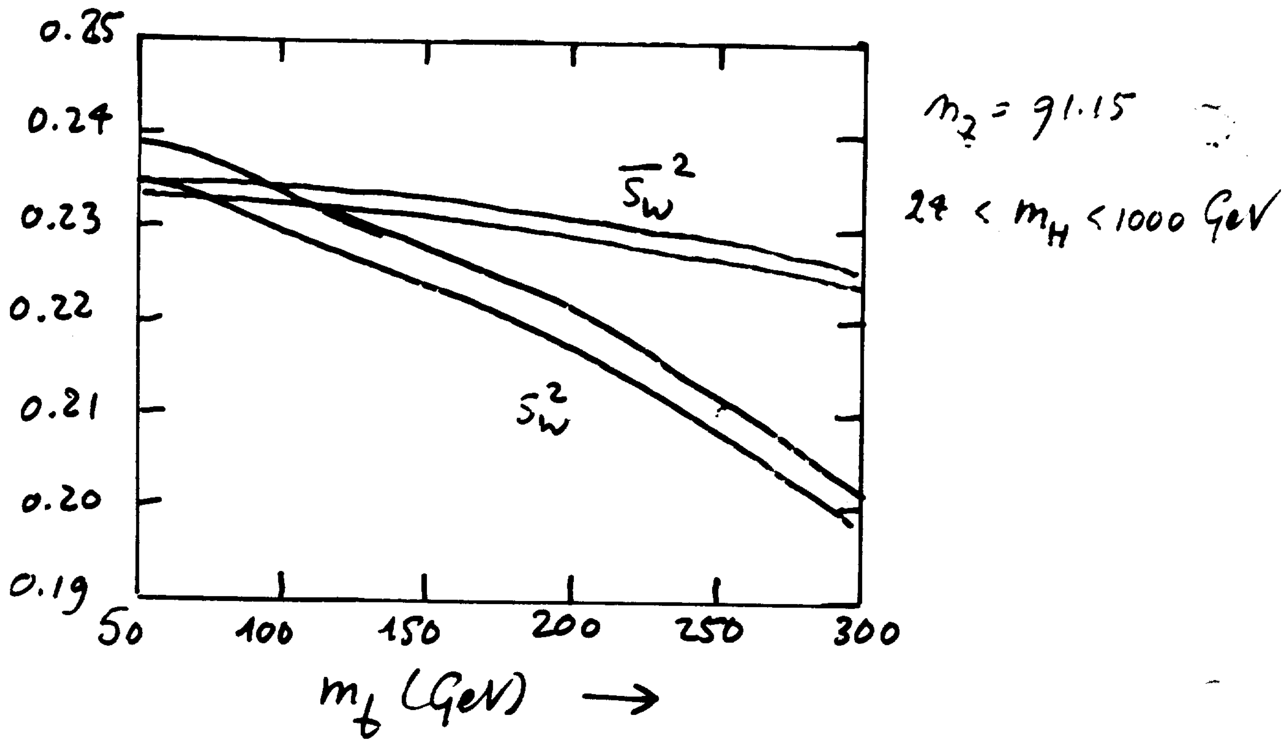
$$\bar{s}_W^2 \equiv s_W^2 + c_W^2 \Delta\rho$$

- s_W^2 depends on m_t
- $\bar{s}_W^2$ depends on m_t

$\Big\}$ but....

$\bar{s}_W^2$ depends much less on m_t than s_W^2 !

$\Rightarrow$ The major m_t dependence in the $Zf\bar{f}$ couplings is by way of $\rho_{e,f} \sim 1 + \Delta\rho$

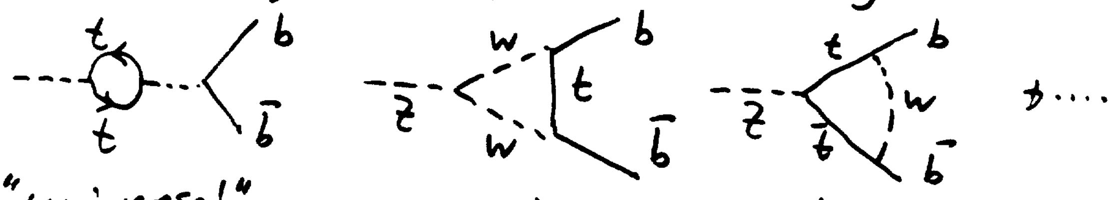


Also: if $\rho_0 \neq 1$ at tree level, $\Delta\rho_0 = \rho_0 - 1$ would act the same way everywhere!
Even if we measure $\Delta\rho_{(0)}$ we would not know the top mass.

Except in one case ....

$Z \to b\bar{b}$ decays

The top quark appears naturally in $Z \to b\bar{b}$:



"universal"

"non-universal"

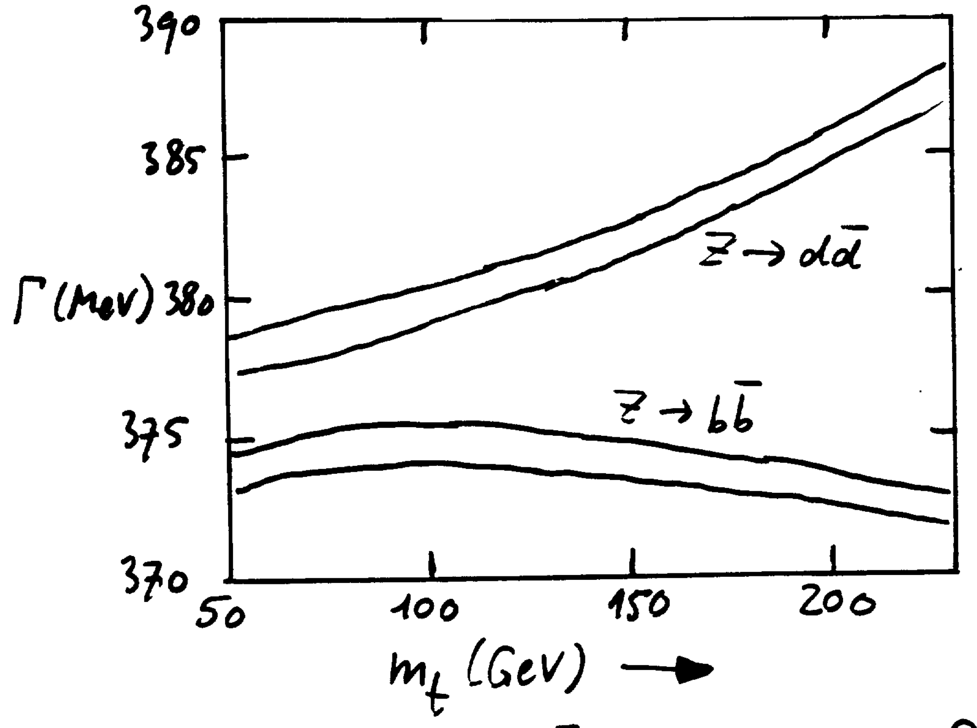
Remember that we argued non-renormalizability for $m_t \to \infty$ because of unitarity violation in $b\bar{b} \to WW$

$\Rightarrow$ quadratic divergence $\propto m_t^2 \Rightarrow \propto \Delta\rho$!

$$\Delta\rho^{b}_{N.U.} = -\frac{4}{3}\Delta\rho - \frac{\alpha}{4\pi s_W^2}\left(\frac{8}{3} + \frac{1}{6c_W^2}\right)\ln\frac{m_t^2}{m_W^2} + \dots$$

$$\Delta\kappa^{b}_{N.U.} = -\frac{1}{2}\Delta\rho^{b}_{N.U.}$$

This over compensates the universal part !



Can we measure $\Gamma(Z \to b\bar{b})$ to 5 MeV ?

$\bar{s}_W^2$ values

$m_Z = 91.15$

"universal"

M_H(GeV)	m_t(GeV)	μ	u	d	b	$\bar{s}_W^2$
25	80	0.2335	0.2334	0.2332	0.2334	0.2327
	150	0.2316	0.2315	0.2313	0.2325	0.2308
	200	0.2297	0.2296	0.2295	0.2315	0.2289
	250	0.2273	0.2272	0.2271	0.2303	0.2266
100	80	0.2341	0.2340	0.2339	0.2340	0.2333
	150	0.2322	0.2321	0.2320	0.2331	0.2315
	200	0.2303	0.2302	0.2301	0.2321	0.2295
	250	0.2279	0.2278	0.2277	0.2309	0.2272
1000	80	0.2353	0.2352	0.2351	0.2353	0.2346
	150	0.2334	0.2333	0.2332	0.2343	0.2327
	200	0.2315	0.2314	0.2313	0.2333	0.2307
	250	0.2290	0.2290	0.2288	0.2321	0.2283

Table 1: *Effective mixing angles on-resonance for $M_Z = 91.15$ GeV*

Fighting about the "best" $\sin^2\theta_W$

- We started by adopting
$$s_W^2 = 1 - m_W^2/m_Z^2$$
to all orders.
This appears to be quite dependent on m_t (or new physics) $\Delta r = \Delta\alpha - \underbrace{\frac{c_W^2}{s_W^2}}_{\text{big!}}\Delta\rho$

- $ee \to f\bar{f}$ physics turns out not to be so dependent on m_t (or N.P.). $\bar{s}_W^2$, $\rho = 1 + \underbrace{1}_{\text{not big!}} \cdot \Delta\rho$

- Since s_W^2 is anyway only a bookkeeping device and not a fundamental parameter, why not change to another equivalent bookkeeping device that also is not too m_t dependent?

- Good idea!
$$\bar{s}_W^2 = s_W^2 + c_W^2 \Delta\rho$$
is such an effective thing, and it has much more to do with the ratio of couplings than with the ratio of masses!
$\Rightarrow$ back to the original introduction of s_W^2 in our derivation of the Lagrangian

A number of alternative $\bar{s}_W^2$ definitions exist

$\bar{s}_W^2$	"Hollik"	To avoid making enemies: this order is not chronological nor preferential
s_W^{*2}	"Lynn"	
s_W^{**2}	"Lynn"	
$\hat{s}_W^2(m_Z^2)$	"Sirlin"	
$\hat{s}_W^2(m_W^2)$	"Sirlin"	
s_{W*}^2	"Lynn"	

All these alternatives incorporate the IDENTICAL leading terms $(\Delta\rho)$.

None of them can incorporate all loop effects.

The question of which alternative is better is

(COMPLETELY) IRRELEVANT!

Since the left-over terms are truly small

The improved Born approximation

Since we understand well the dominant leading corrections in $e^+e^- \to f\bar{f}$ we can take them into account as follows:

(1) Take the Born amplitudes

(2) Replace

$e^2 \to e^2(s)$ in the photon graph A_γ^0

$s_W^2 \to \bar{s}_W^2$ in the ratio $v_{e,f}/a_{e,f}$

$a_{e,f} \to [\sqrt{2}\, G\, m_Z^2 \rho]^{1/2}$ in the Z graph A_Z^0

$\Gamma_Z \to \frac{s}{m_Z^2}\Gamma_Z$ in the Z propagator

(3) Publish the result.

It is typically good to a few ‰ around m_Z!

Improved Born approximations for $\sigma_{tot}(s)$

These typically read

$$\sigma(s) = \frac{12\pi}{m_Z^2} \frac{s\,\Gamma_{ee}\Gamma_{f\bar{f}}}{(s-m_Z^2)^2 + \frac{s^2}{m_Z^2}\Gamma_Z^2} + \frac{4\pi\alpha^2(s)}{3s} \cdot N_c(1+\delta_{QCD})$$

$$+ \text{(interference term)}$$

Such formulae are VERY NICE because

- compact, transparent, easy to understand
- incorporate all large radiative corrections if you stick to the MSM
- contain $\Gamma_{ee}, \Gamma_{tot}, \Gamma_{ff}, m_Z$ as independent free parameters:
  - good for fitting
  - good to go beyond the standard model!

As a function of time, the weak corrections have become smaller!

Because we have learned how to write the Born expression

"Mass shift" due to weak corrections

Neglecting the γ channel:

$$\sigma(s) \propto \frac{s}{(s-m_Z^2)^2 + \frac{s^2}{m_Z^2}\Gamma_Z^2}$$

The peak is no longer precisely at m_Z!

Two competing effects:

1) s in numerator pulls "mass" up:

$$\Rightarrow s_{peak} = m_Z^2 \longrightarrow m_Z^2\left[1+\frac{\Gamma_Z^2}{m_Z^2}\right]^{1/2} \sim m_Z^2 + 17\text{ MeV}$$

2) s-dependent width pulls "mass" down:

$$(s-m^2)^2 + \frac{s^2}{m^2}\Gamma^2 = s^2\left(1+\frac{\Gamma^2}{m^2}\right) - 2m^2 s + m^4$$

$$= \left(1+\frac{\Gamma^2}{m^2}\right)\left[s^2 - \frac{2s\,m^2}{1+\Gamma^2/m^2} + \left(\frac{m^2}{1+\frac{\Gamma^2}{m^2}}\right)^2\right] + m^4 - \frac{m^4}{1+\Gamma^2/m^2}$$

$$= \left(1+\frac{\Gamma^2}{m^2}\right)\left(s - \frac{m^2}{1+\Gamma^2/m^2}\right)^2 + \frac{m^2\Gamma^2}{1+\Gamma^2/m^2}$$

$$\Rightarrow s_{peak} = m_Z^2 \longrightarrow m_Z^2\left[1+\frac{\Gamma_Z^2}{m_Z^2}\right]^{-1} = m_Z^2 - 35\text{ MeV}$$

Net effect: a change of ~ -17 MeV

(without s-dependent width would be $+17$ MeV!)

Other observables

The most important classes:

(1) $$A_f \equiv \frac{2\hat{v}_f \hat{a}_f}{\hat{v}_f^2 + \hat{a}_f^2}$$

$$A_{LR} = A_e$$
$$A^{\tau}_{pol} = A_\tau$$
$$A^f_{FB} = \frac{3}{4} A_e A_f$$

essentially only dependent on $\bar{s}_W^2$

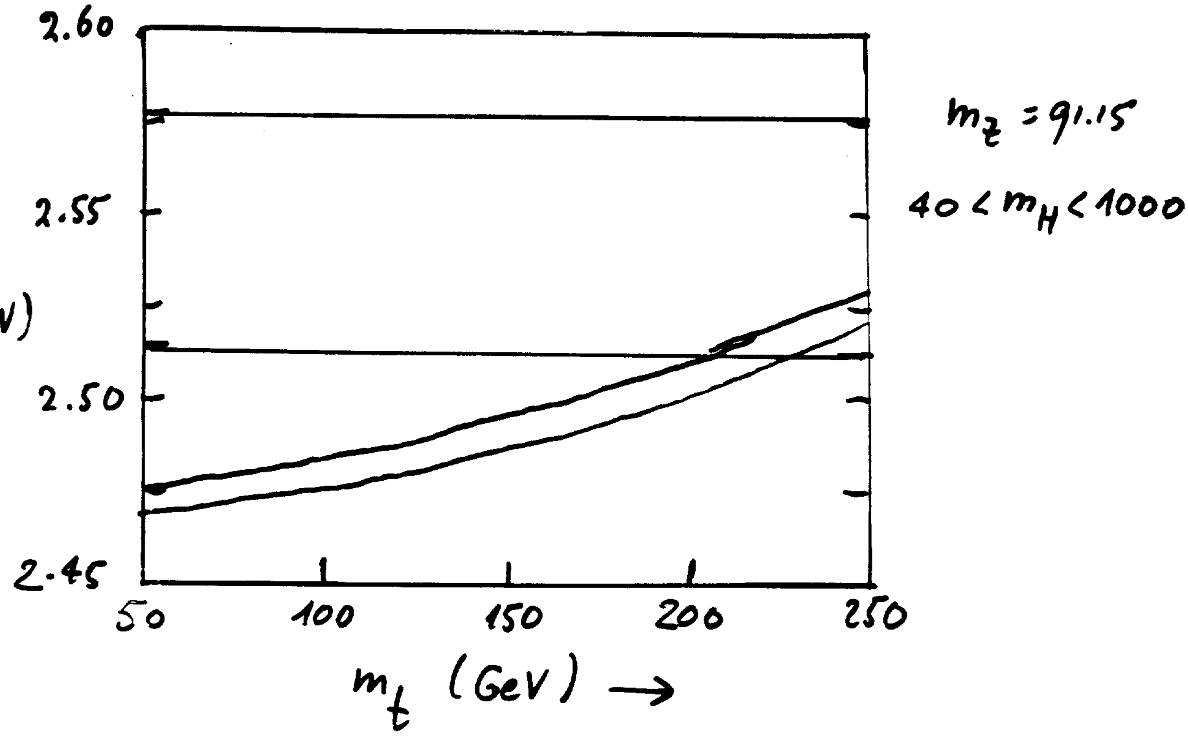
(2) $$\Gamma_f = \Gamma(Z \to f\bar{f}) = \frac{m_Z}{12\pi}(\hat{v}_f^2 + \hat{a}_f^2)(1+\delta_{QCD})(1+\delta_{QED})$$

$$m_f = 0: \begin{cases} \delta_{QED} = 1 + \frac{3}{4}\frac{\alpha}{\pi} & \sim 1.0017 \\ \delta_{QCD} \sim 1 + \frac{\alpha_s}{\pi} & \alpha_s \sim 0.11 \pm 0.01 \end{cases}$$

mainly dependent on $\Delta\rho_f$

$(\Delta\Gamma_{tot}) \sim 12$ MeV from the uncertainty in α_s

Total Z decay width



= MSM prediction

= 1σ bounds of measured Γ_Z

Do you find a limit on m_t?

Strategies of m_t searches at LEP1 (so far)

(1) Measure m_Z $\Rightarrow$ $\bar{s}_W^2$ as a function of m_t for fixed m_Z

(2) Measure Γ, A $\Rightarrow$ $\bar{s}_W^2$ directly

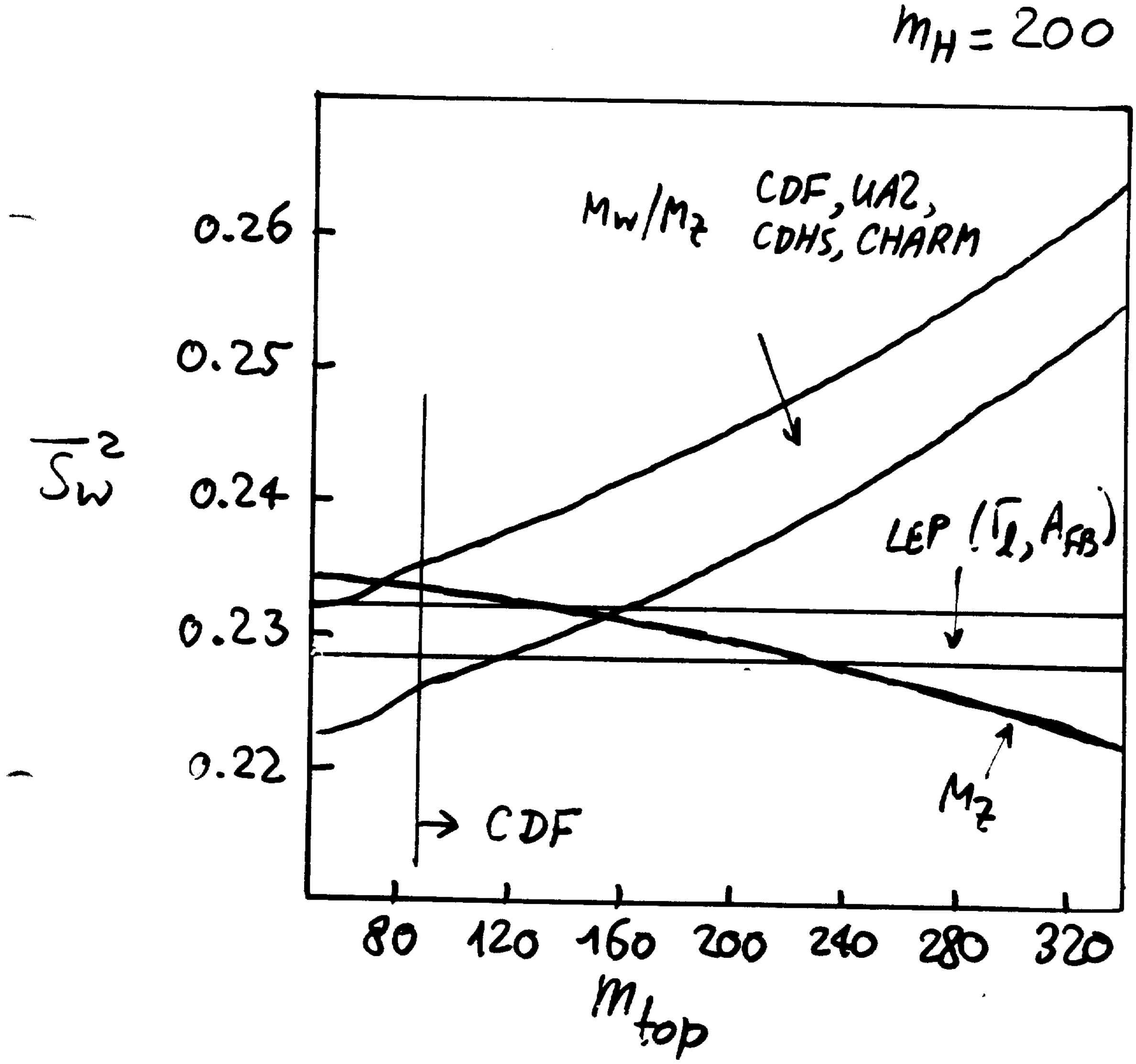
(3) Measure m_W $\Rightarrow$ s_W^2 directly $\Rightarrow$ $\bar{s}_W^2$ as a function of m_t

Find the overlap region!

Typical result (Dydak, Singapore '90):

Towards m_t and m_H

We can determine minimal-model allowed (m_t, m_H) ranges.



$$m_t = 137 \pm 33 \pm \underset{\hat{M_Z}}{3} \pm \underset{\uparrow M_H}{20} \text{ GeV}$$

$m_W = 80.15 \pm 0.25$ ← face up to LEP200!

"$\Delta\rho$- free physics"

Results that contain some $\Delta\rho$ can usually be fitted to the data by invoking large m_t or New Physics.

$\Rightarrow$ find some quantities in which $\Delta\rho$ is much suppressed or absent.

$\Rightarrow$ the Linear Combinations Game

- a poor man's way of doing a global fit
- leads to some understanding as well?

① $R = \frac{\Gamma_{had}}{\Gamma_{\ell\ell}}$ quite independent of m_t (partly fundamental partly coincidence)

② both A_{FB} and A_{LR} are m_t dependent, but...

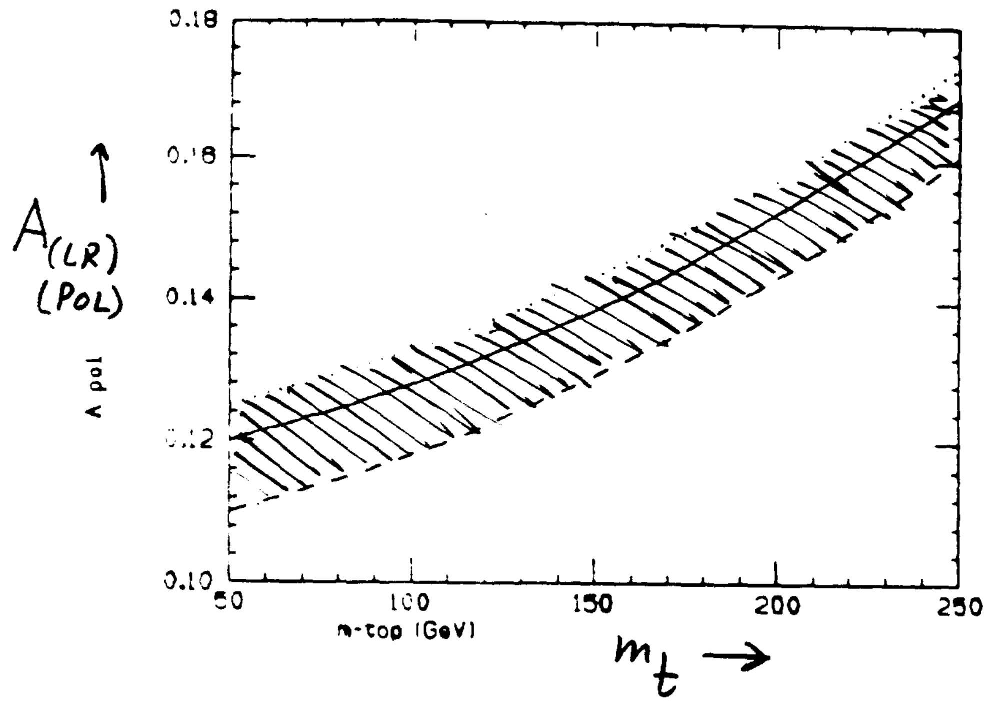


Fig. 13: *A_{pol} or A_{LR} in the minimal model. $M_Z = 91.15$ GeV; $M_H = 25$ (· · ·), 100 (——), 1000 (− − −) GeV*

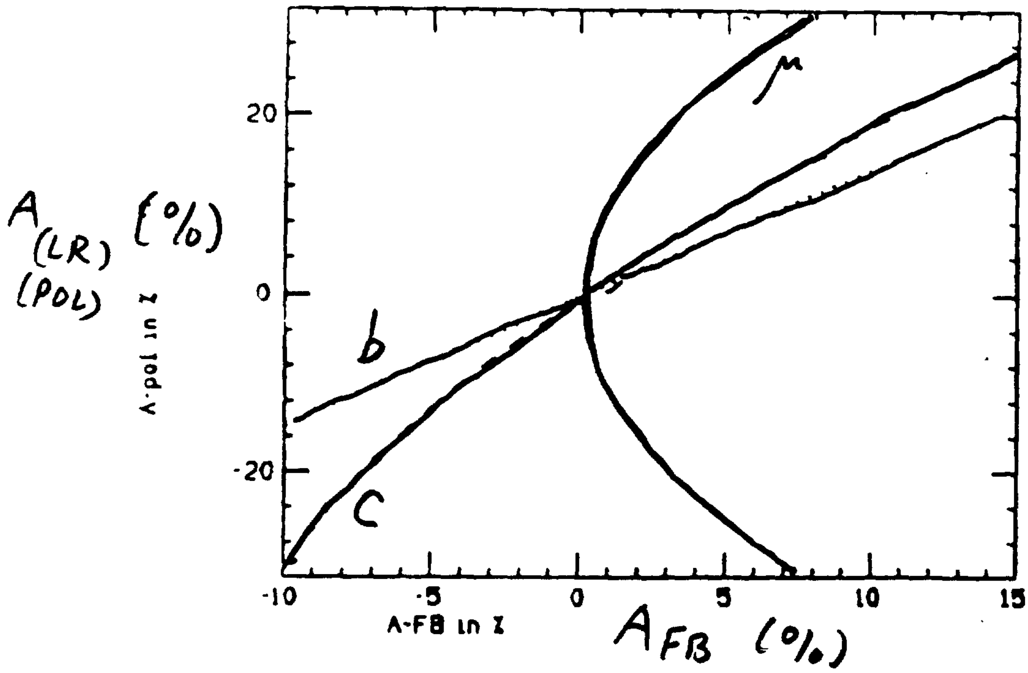
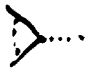
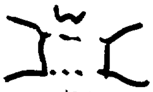
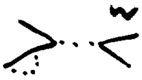


Fig. 14: *τ-polarization versus forward-backward asymmetries for muons (——), c-quarks (− − −), b-quarks (· · ·)*

QED corrections to $e^+e^- \to f\bar{f}$

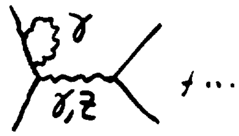
So far we have only considered:

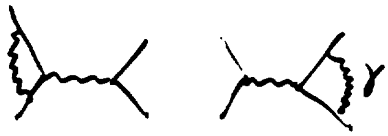
- γ and Z self-energy diagrams 
- vertex corrections with W or Z 
- boxes with W or Z 
- fermion self-energies 

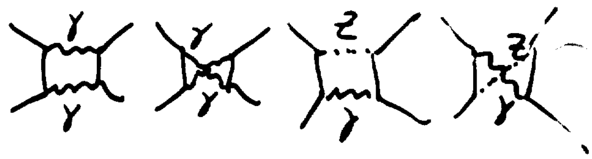
Now: QED corrections

= all diagrams obtained by adding 1 photon to either the γ (A^0_γ) or Z (A^0_Z) graph.

Classes:

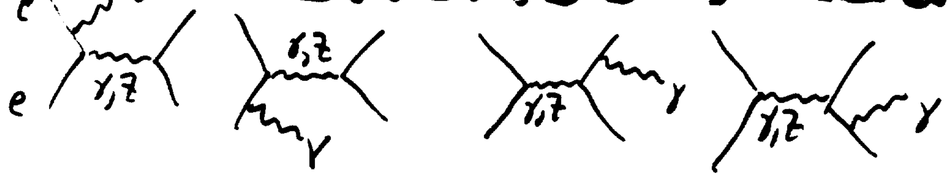
1) Fermionic self-energies 

2) Vertex corrections 

3) Box diagrams ($\gamma\gamma$ or γZ) 

4) Real photon emission:

BREMSSTRALUNG



General remarks on the QED corrections

(1) Adding 1 virtual photon in all possible ways is gauge invariant
$\Rightarrow$ also separately renormalizeable
(a theory with only fermions + Z^0 + γ is renormalizable!)

(2) – If W exchange would be involved instead of Z exchange (as in $u\bar{d} \to W \to \nu \bar{e}$, $\mu \to \nu_\mu e \bar{\nu}_e$ etc.) we would also have to include

$\Rightarrow$ affects renormalizability etc.

(3) The Bremsstrahlung diagrams describe a different final state: not $f\bar{f}$ but $f\bar{f}\gamma$

- virtual and real photon effects are connected by way of the

INFRARED PROBLEM

An introduction to Bremsstrahlung

(1) Single Bremsstrahlung phase space

$f(q_1)\bar{f}(q_2)\gamma(k)$

$$d\sigma = \frac{1}{2s\,(2\pi)^{3n-4}} \langle |M|^2 \rangle \, d(\mathrm{LIPS})$$

$$d(\mathrm{LIPS}) = d^4q_1\,\delta(q_1^2-m^2)\,d^4q_2\,\delta(q_2^2-m^2)\,d^4k\,\delta(k^2) \times \delta^4(p_1+p_2-q_1-q_2-k)$$

Several variable choices possible:

(1) k^0, Ω_k in lab frame, Ω_f^* in $f\bar{f}$ CM frame: ($f\bar{f}$ back-to-back!)

$$d(\mathrm{LIPS}) = d^4k\,\delta(k^2)\,\delta^4(p_1+p_2-Q) \times d^4q_1\,\delta(q_1^2-m^2)\,d^4q_2\,\delta(q_2^2-m^2)\,\delta^4(Q-q_1-q_2)$$

$$= \frac{1}{16}\,k^0\,dk^0\,d\Omega_k \cdot \beta_f\,d\Omega_f^* \qquad \beta_f=\sqrt{1-4m^2/s'} \qquad s'=Q^2=(q_1+q_2)^2$$

(2) $q_1^0, q_2^0, \Omega_1, \varphi_{12}$ (= azimuthal angle of $\vec{q}_2$ around $\vec{q}_1$) in lab:

$$d(\mathrm{LIPS}) = \frac{1}{8}\,dq_1^0\,dq_2^0\,d\Omega_1\,d\varphi_{12}$$

(3) k^0, Ω_k, Ω_1 ($\neq \Omega_2$ since no longer back-to-back) <u>in lab</u>

$$d(LIPS) = \frac{1}{16} \frac{q_1^{0\,2}}{E(E-k^0)} \, dk^0 \, d\Omega_k \, d\Omega_1 \qquad E = \tfrac{1}{2}\sqrt{s}$$

$$q_1^0 = \frac{2E(E-k^0)}{2E - k^0 + k^0 \cos\sphericalangle(\vec{q}_1, \vec{k})}$$

very simplified form! ($m=0$)

A dilemma?

- The simple phase space formulations have inadequate variables if you want to impose cuts
- The most "cut-friendly" formulation (3) has a complicated Jacobian and is still not good enough:

$$q_1^0(k^0, \Omega_k, \Omega_1) = \frac{2E(E-k^0)}{2E - k^0 + k^0\cos\sphericalangle(\vec{q}_1, \vec{k})}$$

$$q_2^0 = 2E - k^0 - q_1^0(k^0, \Omega_k, \Omega_1)$$

$$\cos\theta_2 = -\frac{k^0\cos\theta_k + q_1^0(k^0, \Omega_k, \Omega_1)}{2E - k^0 - q_1^0(k^0, \Omega_k, \Omega_1)}$$

$\Rightarrow$ cuts on q_2^0, Ω_2 are extremely difficult!

Simplification in limit $k^0 \to 0$:

Ω_f^* (in $f\bar{f}$ CM frame) $\to$ Ω_f (in lab frame)

(4) $$\left[d(LIPS)\right]_{k^0\ \text{small}} \sim \frac{1}{16} k^0 \, dk^0 \, d\Omega_f$$

Multileg amplitudes with soft bremsstrahlung

If we consider a process with >1 external legs:

- $e^+(p_1)\, e^-(p_2) \to f(q_1)\, \bar{f}(q_2)$

  amplitude $\mathcal{M}_0$

- $e^+(p_1)\, e^-(p_2) \to f(q_1)\, \bar{f}(q_2)\, \gamma(k) \qquad k^0 \to 0$

  amplitude $$\mathcal{M}_1 = \mathcal{M}_0 \left[-Q_e \frac{p_1\cdot\epsilon}{p_1\cdot k} + Q_e \frac{p_2\cdot\epsilon}{p_2\cdot k} - Q_f \frac{q_1\cdot\epsilon}{q_1\cdot k} + Q_f \frac{q_2\cdot\epsilon}{q_2\cdot k} \right]$$

- Check current conservation:

  $$\mathcal{M}_1\Big]_{\epsilon^\mu \to k^\mu} = \mathcal{M}_0 \left[-Q_e + Q_e - Q_f + Q_f \right] = 0$$

  (in general, current conserved if total e.m. charge conserved)

- Bremsstrahlung from internal lines does not contribute! At least, not in the leading $1/k^0$ terms – example: in $\mu \to \nu_\mu e \bar{\nu}_e$ diagram  is not dominant

- Form of soft-photon amplitude only depends on external charge flows, not on internal lines

  Physical picture: as $k^0 \to 0$, $\lambda_{photon} \to \infty$.

  Long-wavelength photons can not resolve the hard scattering, but only the long-distance charge distribution

The soft-photon cross section

Photon spin sum: $\sum_{spins} \epsilon^\mu \epsilon^{\nu *} = -g^{\mu\nu} \Rightarrow \sum \left(\frac{p_i\cdot\epsilon}{p_i\cdot k}\right)\left(\frac{p_j\cdot\epsilon}{p_j\cdot k}\right) = -\frac{p_i\cdot p_j}{(p_i k)(p_j k)}$

① $d\sigma = \frac{1}{2s}\frac{1}{(2\pi)^5}\langle|\mathcal{M}_1|^2\rangle\, d(LIPS)$

② $\langle|\mathcal{M}_1|^2\rangle \sim \langle|\mathcal{M}_0|^2\rangle \cdot -\left\{-Q_e\frac{p_1^\mu}{p_1\cdot k} + Q_e\frac{p_2^\mu}{p_2\cdot k} - Q_f\frac{q_1^\mu}{q_1\cdot k} + Q_f\frac{q_2^\mu}{q_2\cdot k}\right\}^2$

$= \langle|\mathcal{M}_0|^2\rangle \cdot e^2 R_{infra}$

$$R_{infra} \equiv \Big[\frac{s}{(p_1k)(p_2k)} - \frac{m_e^2}{(p_1k)^2} - \frac{m_e^2}{(p_2k)^2} + \frac{s}{(q_1k)(q_2k)} - \frac{m_f^2}{(q_1k)^2} - \frac{m_f^2}{(q_2k)^2} + \frac{2p_1\cdot q_1}{(p_1k)(q_1k)} - \frac{2p_1\cdot q_2}{(p_1k)(q_2k)} - \frac{2p_2\cdot q_1}{(p_2k)(q_1k)} + \frac{2p_2\cdot q_2}{(p_2k)(q_2k)}\Big]$$

③ $d(LIPS) \sim \frac{1}{2}k^0 dk^0 \cdot \frac{1}{8} d\Omega_f$

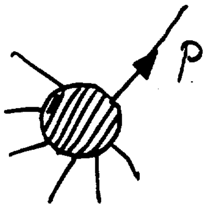
①+②+③ $\Rightarrow \frac{d\sigma}{d\Omega} = \frac{d\sigma_0}{d\Omega} \cdot \frac{\alpha}{4\pi^2} R_{infra}\, k^0 dk^0 d\Omega_k$

cross section factorizes in the soft-photon limit!

(ii) Soft-photon approximation

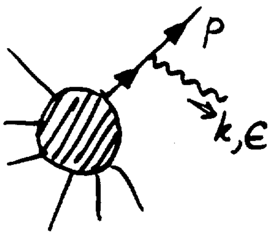
Not only phase space but also matrix elements become simple in the limit $k^0 \to 0$

Consider a process with an outgoing fermion:



$$\mathcal{M}_0 = \bar{u}(p) \underbrace{A(p)}_{\text{rest of diagram}}$$

Now attach Bremsstrahlung:



$$\mathcal{M}_1 = -Q_f \, \bar{u}(p) \not{\epsilon} \frac{\not{p} + m + \not{k}}{(p+k)^2 - m^2} A(p+k)$$

$\big\downarrow$ neglect k^0 where possible

$$\approx -Q_f \frac{1}{2p\cdot k} \bar{u}(p) \not{\epsilon} (\not{p} + m) A(p)$$

$\big\downarrow$ anticommute and use Dirac eqn.

$$= -Q_f \frac{p\cdot\epsilon}{p\cdot k} \underbrace{\bar{u}(p) A(p)}$$

$$= \left(-Q_f \frac{p\cdot\epsilon}{p\cdot k} \right) \mathcal{M}_0$$

- the lowest-order amplitude × a simple factor!
- amplitude scales as $\frac{1}{k^0}$ for $k^0 \to 0$

Integration of the soft-photon cross section

Concentrate on the terms $\frac{s}{(p_1k)(p_2k)} - \frac{m_e^2}{(p_1k)^2} - \frac{m_e^2}{(p_2k)^2}$ in R_{infra}

write

$p_1 \cdot k = E k^0 (1-\beta c)$ $\qquad \beta = |\vec{p}_1|/E = \sqrt{1-4m^2/s}$

$p_2 \cdot k = E k^0 (1+\beta c)$ $\qquad c = \cos \sphericalangle (\vec{p}_1, \vec{k})$

$$\Rightarrow \frac{d\sigma}{d\Omega} = \frac{d\sigma^0}{d\Omega} \cdot \frac{\alpha}{\pi^2} \left[\frac{1}{1-\beta^2 c^2} - \frac{m^2/s}{(1-\beta c)^2} - \frac{m^2/s}{(1+\beta c)^2} \right] \frac{1}{k^0} dk^0 \, dc \, d\varphi$$

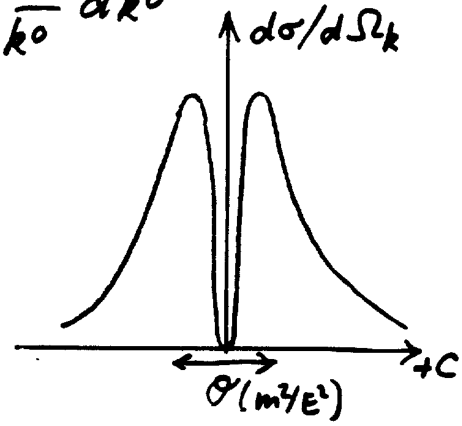
- Bremsstrahlung spectum: $\sim \frac{1}{k^0} dk^0$

- Tremendous angular peaks

$$\frac{1}{2} \leq \frac{1}{1 \pm \beta^2 c^2} \leq \frac{E^2}{m^2} \sim 10^{10}$$

at LEP1

for $\vec{k} \parallel \vec{p}_1, \vec{p}_2, \vec{q}_1, \vec{q}_2$



- Integrated over γ angles:

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma^0}{d\Omega} \cdot \underbrace{\frac{2\alpha}{\pi} \left[\ln \frac{s}{m_e^2} - 1 \right]}_{\text{"classical Radiator factor"} \sim 0.11 \text{ at LEP 1}} \frac{1}{k^0} dk^0$$

- Total Bremsstrahlung cross section:
  simply integrate over k^0!

BUT....

The infrared divergence

Soft bremsstrahlung spectrum : $\int \frac{dk^0}{k^0}$

- Upper limit provided by kinematics : $k^0 \leq E\left(1-\frac{4m^2}{s}\right) \sim E$
- Lower limit $= 0$!

* The total bremsstrahlung cross section is infinite

* The divergence comes from the region $k^0 \sim 0$.
i.e. zero-mass, zero-energy photons

* Are such photons real photons ?

* A physically sensible answer can only be expected when we combine contributions from real with those from virtual photons

This is <u>not</u> renormalization!
No redefinition of parameters involved

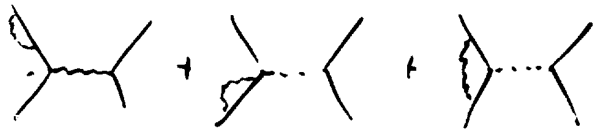
Regularization : give the photon a small finite mass m_γ
The spectrum integral becomes

$$\int_0^E \frac{dk^0}{k^0} \rightarrow \int_{m_\gamma}^E \frac{dk^0}{k^0} \sim \ln \frac{E}{m_\gamma}$$

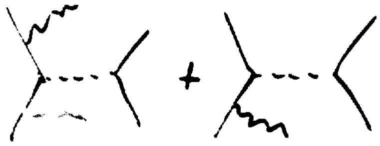
[The real calculation is a bit more complicated since if $m_\gamma \neq 0$, $|\vec{k}| \neq k^0$ in $(p_1 \cdot k)$, $(p_2 \cdot k)$]

Some results for real and virtual photon corrections

After renormalization to get rid of the UV divergencies:

 virtual γ correction

$$\delta_{virt} = \frac{2\alpha}{\pi}\left\{ \left(\ln\frac{s}{m_e^2} - 1\right)\ln\frac{m_\gamma}{m_e} - \frac{1}{4}\ln^2\frac{s}{m_e^2} + \frac{3}{4}\ln\frac{s}{m_e^2} + \frac{\pi^2}{3} - 1 \right\}$$

 soft γ correction

$$\delta_{soft} = \frac{2\alpha}{\pi}\left\{ \left(\ln\frac{s}{m_e^2} - 1\right)\ln\frac{2k^{max}}{m_\gamma} - \frac{1}{4}\ln^2\frac{s}{m_e^2} + \frac{1}{2}\ln\frac{s}{m_e^2} - \frac{\pi^2}{6} \right\}$$

where $k^{max} \ll E$

is the upper bound on what you still want to call soft bremsstrahlung (note that the soft-photon approximation assumes $k^0 \ll E, q_1^0, q_2^0 \ldots$)

$$\delta^{VS} = \delta_{virt} + \delta_{soft}$$

$$= \frac{2\alpha}{\pi}\left\{ \left(\ln\frac{s}{m_e^2} - 1\right)\ln\frac{k^{max}}{E} + \frac{3}{4}\ln\frac{s}{m_e^2} + \frac{\pi^2}{6} - 1 \right\}$$

- IR infinities have cancelled!
- no terms with $[\ln(s/m_e^2)]^2$ left
- if $k^{max} \downarrow 0$ then $\delta^{VS} \to -\infty$ again.

Additional Remarks on the IR cancellation

Up to now: initial-state Radiation

Similar Result for final-state Radiation

$$\delta^{VS}_{final} = \frac{2\alpha}{\pi}\left(\frac{Q_f}{e}\right)^2 \left\{ \left(\ln\frac{s}{m_f^2}-1\right)\ln\frac{k^{max}}{E} + \frac{3}{4}\ln\frac{s}{m_f^2} + \frac{\pi^2}{6} - 1 \right\}_{m_f \ll E}$$

Also similar (but more complicated) for the interference between initial- and final-state Radiation

Characteristic IR divergent terms:

initial: $\frac{2\alpha}{\pi}\left(\ln\frac{s}{m_e^2}-1\right)\frac{dk^0}{k^0}$ $\qquad s = (p_1+p_2)^2$

$\downarrow$

angle-independent

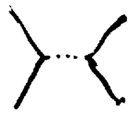
$\uparrow$

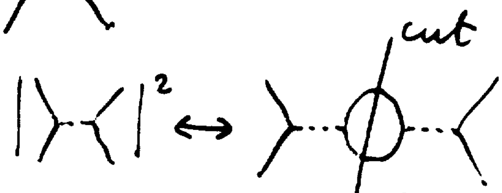
final: $\frac{2\alpha}{\pi}\left(\ln\frac{s}{m_f^2}-1\right)\frac{dk^0}{k^0}\cdot\left(\frac{Q_f}{e}\right)^2$ $\qquad s = (q_1+q_2)^2$

interference: $\frac{2\alpha}{\pi}\underbrace{\ln\left[\frac{(p_1-q_1)^2(p_2-q_2)^2}{(p_1-q_2)^2(p_2-q_1)^2}\right]}_{2\ln\tan\frac{\theta}{2}}\frac{dk^0}{k^0}\cdot\left(-\frac{Q_f}{e}\right)$

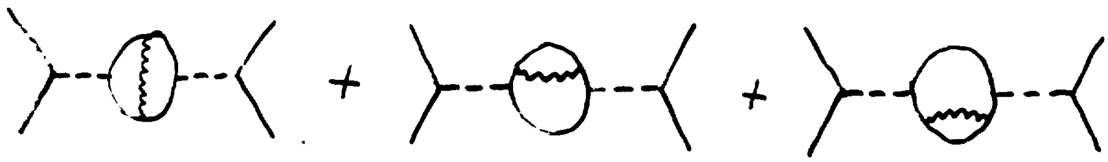
angle-dependent!
but not Large

Understanding the IR cancellation from the optical theorem

Lowest order diagram: 

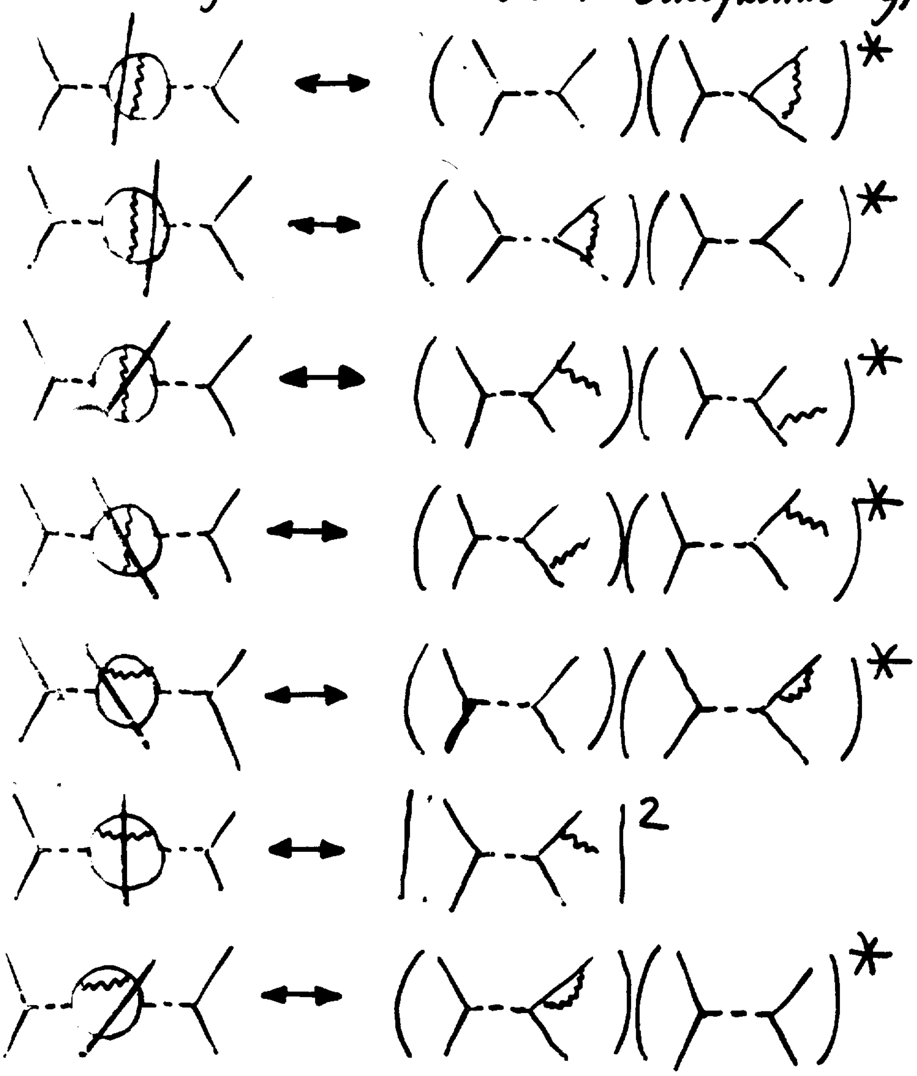
Lowest-order cross section: 

Add photonic (QED) corrections to this self-energy:



This sum is UV finite (after renormalization) and also IR finite

Cut again to see which diagrams give the cross section



$+ \cdots$

Only the sum is IR finite!

(similar for initial-state rad. and interference) excercise!

Hard photon effects

We have cut the soft photons off at $k^0 = k^{max} << E$
but this is either

OR
- arbitrary $\Rightarrow$ have to add a piece with $k^0 \geq k^{max}$
- not a good model of an experimental set-up

$\Rightarrow$ We also have to account for the cross section from $k^0 > k^{max}$

This cross section is UV finite and IR finite but:

- matrix element becomes terrible! strong peaks curving through LIPS
- phase space formulation becomes awful!
- experimental constraints become horrible!

The only ways of attacking this are:

1. use extremely simplified (or no) cuts and work (semi) analytically
2. use **MONTE CARLO**

$\Downarrow$

This is worth an academic training course by itself

I ♥ MC

Here, we have to be content with just a few qualitative remarks.

For more details consult the "Z Physics at LEP1" Yellow Books

Remarks on initial-state radiation at LEP 1

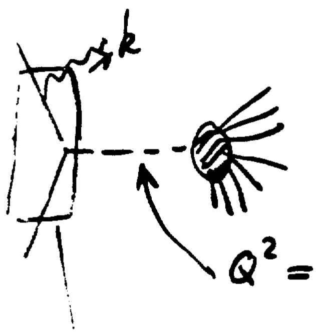
In the soft-photon approximation: for initial-state radiation:

$$\frac{\partial}{\partial k^0} \sigma(s) = \frac{2\alpha}{\pi}\left(\ln\frac{s}{m_e^2} - 1\right)\frac{1}{k^0}\,\sigma_0(s) \qquad k^0 < k^{max} \ll E$$

Including hard-photon effects:

$$\frac{\partial}{\partial k^0} \sigma(s) = \underbrace{\frac{2\alpha}{\pi}\left(\ln\frac{s}{m_e^2} - 1\right)\frac{1+(1-k^0/E)^2}{2k^0}}_{\text{Radiator}}\;\underbrace{\sigma_0\left(s(1-k^0/E)\right)}_{\substack{\text{cross section}\\ \text{at reduced}\\ \text{CM energy}}}$$

This can be understood diagrammatically:



$$Q^2 = (p_1+p_2-k)^2 = s\left(1-\frac{k^0}{E}\right)$$

Radiator

The shift in energy $s \to s\left(1-\frac{k^0}{E}\right)$ is important for resonant cross sections. Qualitatively:

① at resonance: resonance "disappears" for $s\left(1-\frac{k^0}{E}\right) \lesssim m_Z - \Gamma_Z$

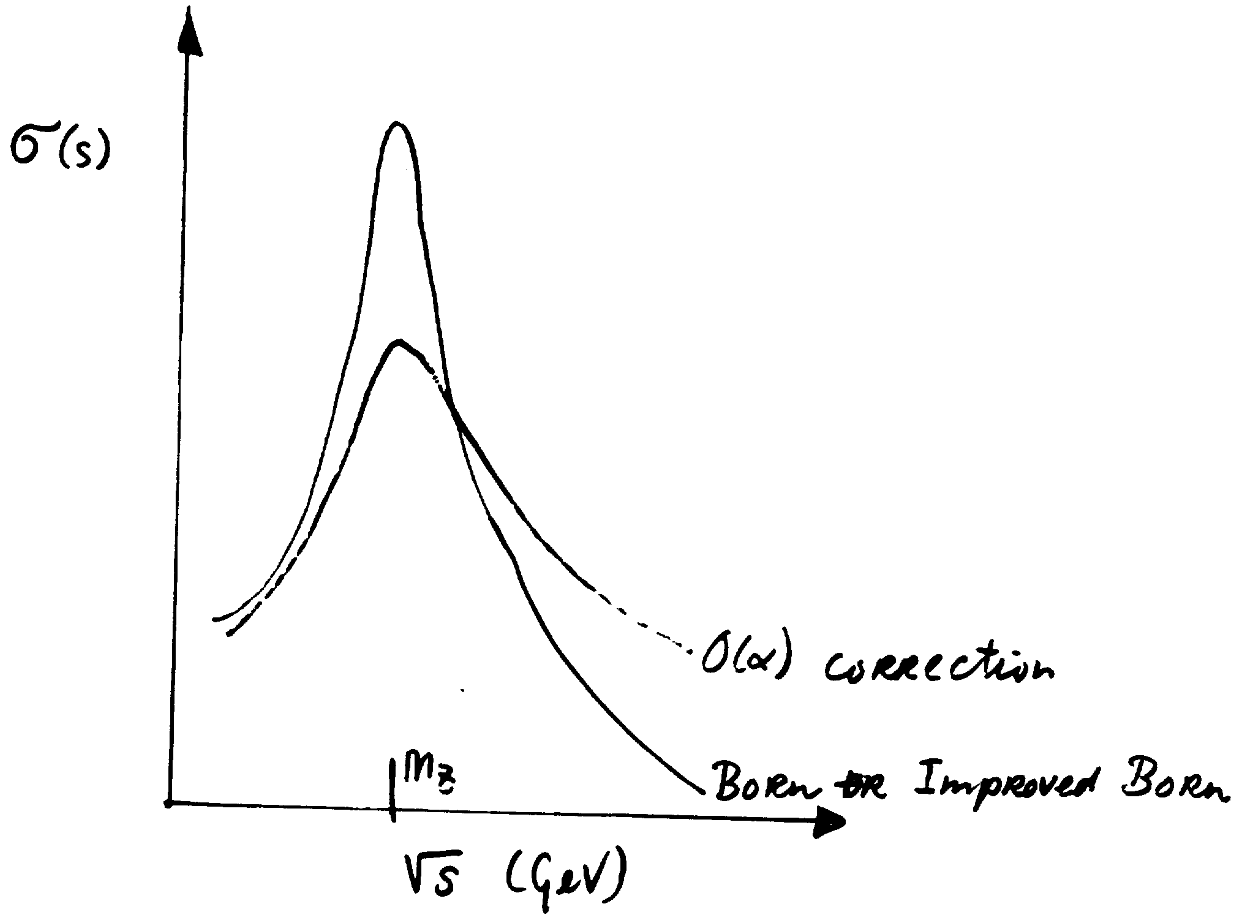
$\Rightarrow$ "natural" cutoff on k^0 of order $k^{max} \sim \Gamma_Z$

$\Rightarrow$ $\delta_{initial} \sim \frac{2\alpha}{\pi}\left(\ln\frac{s}{m_e^2} - 1\right)\ln(\Gamma_Z/M_Z) \quad \sim -30\,\%$

② above resonance: resonance "reappears" if k^0 is such that $s(1-k^0/E) \sim m_Z$

$\Rightarrow$ Large radiative tail!

The Z line shape



First-order correction $\sim -30\%$ is huge!

Final-state Radiation and interference

Final-state Radiation:

The KLN theorem says that the total correction is not singular as $m_f \to 0$ $\Rightarrow$ no terms $\sim \ln s/m_f^2$!

$$\delta_{final}^{tot} \sim \frac{3}{4}\frac{\alpha}{\pi} \sim 0.17\,\%$$

If you have strict cuts, can have $\delta_{final} = -$few %

Interference

At resonance, no cuts: very small! $\delta \sim 10^{-3}$

Physical: on resonance, the Z is produce with a non-negligible life-time $\Rightarrow$ 'wavefunctions' for production (with initial-state Rad) and decay (with final-state Rad) have small overlap.

- Away from the resonance, or with strict cuts, again have $\delta_{interference} \sim \pm$ few % like at PETRA/PEP/Tristan

Higher order effects: exponentiation

- If the $\mathcal{O}(\alpha)$ correction is -30%, we have to worry about higher orders !!!
- The correction is $-(\text{many})\%$ because k^{max} is small (about Γ_Z/m_Z): what if $\Gamma_Z \to 0$? $\delta \lesssim -100\%$???

Exponentiation (simplified form)

V = virtual + soft photon
H = hard photon

① $\mathcal{O}(\alpha)$ corrected cross section

$$\sigma^V \sim \sigma_0 (1 + \beta \ln \Delta + \cdots)$$

$$\sigma^H \sim \sigma_0 \beta \ln \frac{K}{\Delta} + \cdots$$

$$\Rightarrow \underline{\sigma^{(1)} = \sigma_0 (1 + \beta \ln K + \cdots)}$$

non-negligible but finite terms

$\beta = \frac{2\alpha}{\pi}\left(\ln \frac{s}{m_e^2} - 1\right)$

$\Delta = k^{max}/E$

$K = \text{max. value of } k^0/E$

② $\mathcal{O}(\alpha^2)$ corrected cross section

Bose symmetry of photons

$$\sigma^{V_1V_2} \sim \sigma_0 \left(1 + \beta \ln \Delta + \frac{1}{2}\beta^2 (\ln \Delta)^2 + \cdots\right)$$

$$\sigma^{V_1H_2 + V_2H_1} \sim \sigma_0 \beta \ln \frac{K}{\Delta} (1 + \beta \ln \Delta + \cdots)$$

$$\sigma^{H_1H_2} \sim \frac{1}{2} \sigma_0 \beta^2 \ln^2 \frac{K}{\Delta} + \cdots$$

$$\Rightarrow \underline{\sigma^{(2)} = \sigma_0 \left(1 + \beta \ln K + \frac{1}{2}\beta^2 \ln^2 K + \cdots\right)}$$

excercise: guess the $\mathcal{O}(\alpha^3)$ term
guess the $\mathcal{O}(\alpha^4)$ term
⋮
guess the $\mathcal{O}(\alpha^\infty)$ term

dominant behaviour summed to all orders:

$$\sigma^{(\infty)} \sim \sigma_0 [\exp(\beta \ln K) + \dots]$$

$$= \sigma_0 (K^\beta + \dots) \qquad \text{"EXPONENTIATION"}$$

obviously

① The real treatment is more complicated (Yennie - Frautschi - Suura)

If $K \to 0$: $K^\beta \to 0 \Rightarrow \sigma^{(\infty)} \sim 0$!

"There is no scattering without radiation" (Bloch - Nordsieck)

③ Modified Bremsstrahlung spectrum

$$E \frac{d\sigma^{(\infty)}}{dk^0} \equiv \left[\frac{\partial}{\partial K} \sigma^{(\infty)} \right]_{K = k^0/E}$$

$$= \underbrace{\beta \frac{E}{k^0} \sigma_0(s)}_{\text{the } O(\alpha) \text{ result}} \cdot \underbrace{\left(\frac{k^0}{E}\right)^\beta}_{}$$

regulating factor to soften the IR divergence

Rule of thumb:

if $\delta^{(1)} = -A \quad < 0$, A not small

then $\delta^{(2)} \sim +\frac{1}{2} A^2$

Structure functions for the line shape

Total initial-state corrected cross section to $\mathcal{O}(\alpha)$:

$$\sigma = \sigma_0 (1+\delta^V) + \int_{k^{max}}^{K} dk \, \frac{1+(1-k)^2}{2k} \cdot \beta \, \sigma_0(s')$$

$$s' = s(1-k/E)$$

Exponentiation takes virtual and hard photon effects together and allows to write

$$\sigma = \int_0^K dk \, \underbrace{F(s,k)}_{\text{Flux function}} \, \sigma_0(s') \qquad k \equiv 1 - s'/s \text{ now!}$$

or, when keeping separate contributions for radiation from the e^+ and from the e^-
(which is only possible in Leading Log Level):

$$\sigma = \iint dx_1 \, \underbrace{F_{e^+}(x_1) \, F_{e^-}(x_2)}_{\text{Structure functions}} \, \sigma(s x_1 x_2) \qquad x_i = \text{energy of } e^\pm \text{ after radiation}$$

This looks a lot like QCD (of course!)

historically: QED
$\downarrow$
QCD
$\downarrow$
QED

The flux function is known very precisely: to $\sim 0.1\,\%$

From the results of these calculations it becomes clear that certain terms can be resummed, and the division in $\delta(1-z)$ and $\theta(1-z-\epsilon)$ terms is not necessary anymore. A number of cases is listed below. The first case reads [22]

$$F(s,k) = G_A(z) = \beta\underbrace{(1-z)^{\beta-1}}_{\frac{\beta}{\epsilon}\,\epsilon^{\beta}\cdot\delta^{V+S}+\ldots}\delta^{V+S} + \delta^H , \qquad t = -\frac{\epsilon}{\ldots} \tag{3.12}$$

$L = \ln s/m_e^2$

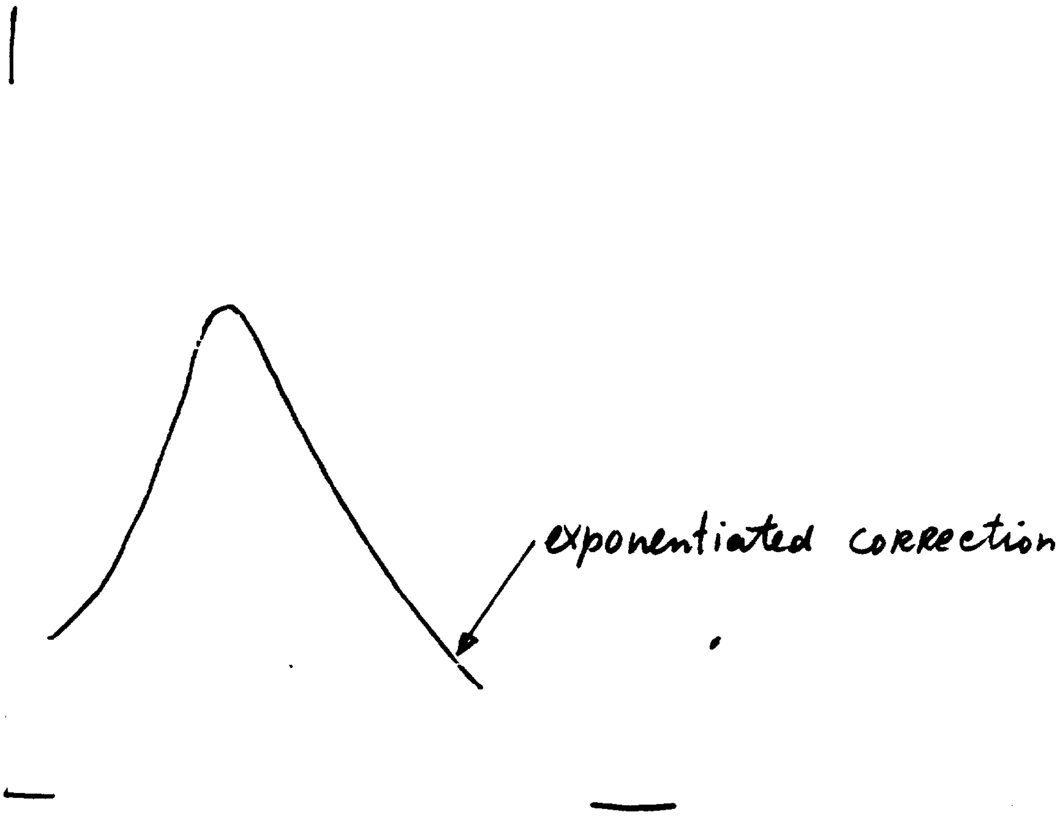
with

$$
\begin{aligned}
\beta &= \frac{2\alpha}{\pi}(L-1) , && (3.13)\\
\delta^{V+S} &= 1 + \delta_1^{V+S} + \delta_2^{V+S} , && (3.14)\\
\delta^H &= \delta_1^H + \delta_2^H , && (3.15)\\
\delta_1^{V+S} &= \frac{\alpha}{\pi}\left(\frac{3}{2}L + 2\zeta(2) - 2\right) , && (3.16)\\
\delta_2^{V+S} &= \left(\frac{\alpha}{\pi}\right)^2\left[\left(\frac{9}{8} - 2\zeta(2)\right)L^2 + \left(-\frac{45}{16} + \frac{11}{2}\zeta(2) + 3\zeta(3)\right)L\right. \\
&- \left.\frac{6}{5}\zeta(2)^2 - \frac{9}{2}\zeta(3) - 6\zeta(2)\ln 2 + \frac{3}{8}\zeta(2) + \frac{19}{4}\right] , && (3.17)\\
\delta_1^H &= -\frac{\alpha}{\pi}(1+z)(L-1) , && (3.18)\\
\delta_2^H &= \left(\frac{\alpha}{\pi}\right)^2\left\{X - (1+z)\left[2\ln(1-z)(L-1)^2\right.\right. \\
&+ \left.\left.(L-1)\left(\frac{3}{2}L + 2\zeta(2) - 2\right)\right]\right\} , && (3.19)
\end{aligned}
$$

$$
\begin{aligned}
X &= \left(-\frac{1+z^2}{1-z}\ln z + (1+z)\frac{1}{2}\ln z + z - 1\right)L^2 \\
&+ \left[\frac{1+z^2}{1-z}\left(Li_2(1-z) + \ln z\,\ln(1-z) + \frac{7}{2}\ln z - \frac{1}{2}\ln^2 z\right)\right. \\
&+ \left.(1+z)\frac{1}{4}\ln^2 z - \ln z + \frac{7}{2} - 3z\right]L \\
&+ \frac{1+z^2}{1-z}\left(-\frac{1}{6}\ln^3 z + \frac{1}{2}\ln z\,Li_2(1-z) + \frac{1}{2}\ln^2 z\ln(1-z)\right. \\
&- \left.\frac{3}{2}\,Li_2(1-z) - \frac{3}{2}\ln z\ln(1-z) + \zeta(2)\ln z - \frac{17}{6}\ln z - \ln^2 z\right) \\
&+ (1+z)\left(\frac{3}{2}Li_3(1-z) - 2S_{1,2}(1-z) - \ln(1-z)Li_2(1-z) - \frac{1}{2}\right) \\
&- \frac{1}{4}(1-5z)\ln^2(1-z) + \frac{1}{2}(1-7z)\ln z\,\ln(1-z) - \frac{25}{6}z\,Li_2(1-z) \\
&+ (-1 + \frac{13}{3}z)\zeta(2) + (\frac{3}{2} - z)\ln(1-z) + \frac{1}{6}(11 + 10z)\ln z \\
&+ \frac{2}{(1-z)^2}\ln^2 z - \frac{25}{11}z\ln^2 z - \frac{2}{3}\frac{z}{1-z}\left(1 + \frac{2}{1-z}\ln z + \frac{1}{(1-z)^2}\ln^2 z\right)
\end{aligned} \tag{3.20}
$$

In these definitions the polylogarithms $Li_n(z)$ and $S_{n,p}(z)$ have been introduced (cf. refs. [28] and [29]) and the Riemann zeta function $\zeta(2) = \pi^2/6$ and $\zeta(3) \approx 1.202$.

The terms $\delta_1^{V+S}, \delta_2^{V+S}$ originate from first and second order virtual and soft photon corrections. Similarly δ_1^H and δ_2^H originate from single and double hard bremsstrahlung.



CONCLUSION

- The QED correction ~ −28 % is Huge!
- But: it is known VERY PRECISELY!
- We can unfold them to isolate the Improved Born!
- We can do precision measurements!
- We will find the top! the Higgs! New Physics!

⇒ GOOD LUCK!