ε ε

INS Report

INS-Rep.-1171
Nov. 1996

EQUIVALENT CIRCUIT STUDY OF BEAM-LOADING USING A MOMENT METHOD

Tai-Sen F. Wang*

November 5, 1996

* On leave of absence from Los Alamos National Laboratory, Los Alamos, New Mexico, USA

Institute for Nuclear Study
University of Tokyo
Tanashi, Tokyo 188, Japan

EQUIVALENT CIRCUIT STUDY OF BEAM-LOADING USING A MOMENT METHOD

Tai-Sen F. Wang*

November 5, 1996

Abstract

In this work, we present a formalism for the equivalent circuit model to include all harmonics of the synchrotron oscillation in a beam-cavity interaction system by considering the perturbations in the moments of a bunched beam. The dispersion relation obtained by this new method was compared with that derived from the linearized Vlasov equation up to the second harmonic of the synchrotron frequency. We found good agreements between these two approaches. We also discuss about the possibility of extending the moment method to the cases with nonlinear longitudinal focusing forces and to the time domain tracking of the beam-cavity interaction with a feedback control.

1. Introduction

In designing the radio frequency (RF) system of an accelerator or storage ring, the interaction between the charged particle beam and the RF cavity is often modeled by an equivalent circuit. The wide application of this kind of modeling can range from a simple estimation of power requirement to complicated studies of system stability and control designs. As an example, for an RF system without external control, K. Robinson derived a set of well-known criteria for a stable beam-cavity interaction system.[1,2] These stability conditions were originally derived by using an equivalent circuit model and the equation of synchrotron motion before the more elaborate kinetic theory of bunched-beam stability was formulated.[3,4] The equivalent circuit analysis can show that the Robinson instability is due to the frequency-dependent cavity impedance and the coupling of the upper and lower synchrotron sidebands next to the RF frequency. In the kinetic theory, the frequencies of these sidebands correspond to the frequency of the coherent dipole mode perturbation in the charge density of the bunch. For a beam with more than one bunch, the content of Robinson's stability criteria covers only the coherent motion of dipole modes among bunches.

Because of its mathematical simplicity and its practical importance, the dipole mode in the beam-cavity system has been extensively studied by using the equivalent circuit model and by using the Vlasov equation of the kinetic description.[5] One of the advantages of using the Vlasov equation over the equivalent circuit model is that the coupling among perturbation modes of all synchrotron harmonics, including the dipole mode, are covered in the formalism in a natural way. A widely used formalism in applying the Vlasov equation was developed by Lebedev and Sacherer.[3,4] In that formalism, the perturbations of the beam-particle distribution in phase space are categorized according to the harmonics of the synchrotron frequency. Since the synchrotron frequency is usually much lower than the fundamental frequency of an RF cavity, synchrotron harmonics may appear in the beam signal as sidebands around the resonant frequency of the cavity. For narrow-band resonators, only those synchrotron sidebands near the resonant frequencies of the cavity contribute significantly to the beam-cavity interaction. Yet, depend on situations, the higher order modes could also be important in some cases. It has been

* On leave of absence from Los Alamos National Laboratory, Los Alamos, New Mexico, USA

discussed in an earlier work, that for a tightly bunched beam interacting with a highly or moderately detuned narrow-band resonator, the neglect of higher synchrotron harmonics is a good approximation. However, for long beam bunches or small cavity detuning, higher synchrotron harmonics may affect the stability appreciably; therefore, at least a few of the higher synchrotron harmonics should be considered.[6] When more synchrotron harmonics are included, the mathematics involved in solving the Vlasov equation becomes difficult except for a few special equilibrium phase-space distributions.

The equivalent circuit model used so far can not describe the dynamics involved in the collective motion of particles within the bunch. However, it is simpler and easier to be incorporated with engineering designs than the kinetic theory model. The simplicity of the equivalent circuit model can be appreciated more when we want to track the beam-cavity interaction in the time domain. The reason that the equivalent circuit model used so far can cover only the dipole mode is because only the single particle motion is considered. In such a model, a bunch of particles is envisioned as a single macro-particle with no internal degree of freedom.

The purpose of this study is to formulate, at least in the linear longitudinal focusing regime, an approach in the equivalent circuit model that can take any number of synchrotron sidebands into account. As will be discussed in the followings, that using the moment method, one can incorporate any number of synchrotron sidebands in the equivalent circuit model. A more general moment method has been previously used to study the space charge effect in the dynamics of transverse motion of beam particles.[7,8] The "moment" we will study at here is the moment in the configuration space of the bunch beam, it does not include all the moments in the phase space as some previous studies did. In the following sections, we will derive the linearized equations of motion for the longitudinal moments of a bunched beam and relate the variations in moments to the perturbed beam current that drives the perturbed field in an RF cavity. We will also discuss about the possibility of extending the moment method to include the effects of nonlinear focusing and to study the nonlinear evolution of the beam-cavity interaction. An example will be given for the time domain tracking of a beam-cavity system with feedback control. For simplicity, we limit our study here to the scope of coherent mode, or the "0" mode stability of a multibunch system, i.e., the coupled-bunch modes are not considered here. We shall concentrate on the case below transition. Above transition, the analysis is similar.

2. Linearized Equations of Particle Motion

The equation of synchrotron motion of a charged particle is

$$\frac{d^2\varphi}{dt^2} = \frac{\omega_{s0}^2}{V_s \cos\psi_s}\left[V\sin(\psi_s + \phi_v - \varphi) - V_s \sin\psi_s\right] , \tag{2.1}$$

where φ is the phase deviation of the particle's position with respect to the "synchronous phase" ψ_s, t is the time,

$$\omega_{s0}^2 = -\frac{q\eta h V_s \cos\psi_s}{2\pi m_0 \gamma R^2} , \tag{2.2}$$

ω_{s0} is the synchrotron frequency at equilibrium, q and m_0 are the charge and the rest mass of a beam particle, respectively, V_s is the maximum RF voltage on the cavity when the system is in steady state, V is the voltage on the cavity, ϕ_v is the deviation of the voltage phase from its equilibrium, γ is the ratio between the total energy and the rest mass of the synchronized beam particle, h is the RF harmonic number, R is the effective machine radius,

$$\eta = \frac{1}{\gamma_t^2} - \frac{1}{\gamma^2} \ , \tag{2.3}$$

is the momentum slip factor for a machine with transition gamma γ_t. Integrating the equation of motion, yields

$$\begin{aligned} \left(\frac{d\varphi}{dt}\right)^2 - \frac{2\omega_{s0}^2}{V_s \cos\psi_s}\left[V\cos(\psi_s + \phi_v - \varphi) - (V_s \sin\psi_s)\varphi\right] \\ = K_0 + \frac{2\omega_{s0}^2}{V_s \cos\psi_s}\int\left[V\sin(\psi_s + \phi_v - \varphi)\frac{d\phi_v}{d\varphi} - \cos(\psi_s + \phi_v - \varphi)\frac{dV}{d\varphi}\right]d\varphi \end{aligned} \ , \tag{2.4}$$

where K_0 is the integration constant to be determined. We notice that $\varphi = \varphi'_{\max}$ when $\phi_v = 0$, $V = V_S$, and $d\varphi/dt = 0$, where $\varphi'_{\max}$ is the maximal excursion of the particle in the unperturbed RF potential well. We therefore find that

$$K_0 = -\frac{2\omega_{s0}^2}{\cos\psi_s}\left[\cos(\psi_s - \varphi'_{\max}) - (\sin\psi_s)\varphi'_{\max}\right] \ . \tag{2.5}$$

Assuming that φ and ϕ_v are small quantities, $V = V_s + \hat{V}$ with $\hat{V} << V_s$, Eqs. (2.1) and (2.4) can be linearized to obtain

$$\frac{d^2\varphi}{dt^2} + \omega_{s0}^2\varphi = \omega_{s0}^2\left(\phi_v + \frac{\hat{V}}{V_s}\tan\psi_s\right) \ , \tag{2.6}$$

and

$$\begin{aligned} \left(\frac{d\varphi}{dt}\right)^2 &\approx -\omega_{s0}^2\varphi^2\left[1 + (\hat{V}/V_s) - \phi_v\tan\psi_s\right] \\ &\quad + \omega_{s0}^2\left\{H + 2\left[(\hat{V}/V_s) - \phi_v\tan\psi_s\right]\right\} - 2\omega_{s0}^2\int\left[d(\hat{V}/V_s) - \tan\psi_s d\phi_v\right] \\ &= -\omega_{s0}^2\varphi^2 - \omega_{s0}^2\varphi^2\left(\frac{\hat{V}}{V_s} - \phi_v\tan\psi_s\right) + \omega_{s0}^2 H , \end{aligned} \tag{2.7}$$

where $H = \varphi'^2_{\max}$ is a constant depends on the total energy of the particle.

3. The Moments and Their Equations of Motion

We assume that the particle density within the bunch $\rho(z,t)$ is a steady part $\rho_0(z)$ plus a perturbation part $\xi(z,t)$, i.e.,

$$\begin{aligned} \rho(z,t) &= \rho_0(z) + \xi(z,t) \\ &= \int_{-\infty}^{\infty} f_0(z,v_z)dv_z + \int_{-\infty}^{\infty} f_1(z,v_z,t)dv_z \ , \end{aligned} \tag{3.1}$$

where $f_0(z,v_z)$ and $f_1(z,v_z)$ are the equilibrium and the perturbed distribution functions of beam particles in the (z,v_z) phase space, respectively; z is the coordinate along the bunch length, and v_z is the speed of a beam particle. The origin of the coordinate is chosen to coincide with the bunch center in the steady state which is assumed to be "synchronized" with the RF phase. Assuming there are M identical bunches in the ring, the hth harmonic of the beam current I_h is then given by

$$\begin{aligned} I_h &= \frac{qMc\beta}{2\pi R}\int_{-\pi R}^{\pi R} e^{ihz/R}\rho(z)dz \\ &= \frac{qMc\beta}{2\pi R}\int_{-L/2}^{L/2} e^{ihz/R}\left[\rho_0(z)+\xi(z,t)\right]dz \\ &= I_h^{(0)} + \frac{qMc\beta}{2\pi R}\int_{-L/2}^{L/2}\sum_{n=1}^{\infty}\left(\frac{ihz}{R}\right)^n \frac{\xi(z,t)}{n!}dz \\ &= I_h^{(0)} + \frac{qMc\beta}{2\pi R}\int_{-\varphi_{\max}}^{\varphi_{\max}}\sum_{n=1}^{\infty}(i\varphi)^n \frac{R\xi(z,t)}{n!h}d\varphi \\ &= I_h^{(0)} + I_{dc}\sum_{n=1}^{\infty}\frac{i^n}{n!}\left\langle\varphi^n\right\rangle_p \\ &= I_h^{(0)} + I_{dc}\sum_{j=1}^{\infty}\frac{(-1)^j}{(2j)!}\left\langle\varphi^{2j}\right\rangle_p + iI_{dc}\sum_{j=0}^{\infty}\frac{(-1)^j}{(2j+1)!}\left\langle\varphi^{2j+1}\right\rangle_p \quad , \end{aligned} \tag{3.2}$$

where

$$I_{dc} = \frac{qMc\beta N}{2\pi R} \quad , \tag{3.3}$$

is the averaged (dc) beam current, c is the speed of light, β is equal to the averaged particle speed divided by c, L is the full length of the bunch, $i=\sqrt{-1}$, N is the total number of particles in one bunch, $\varphi_{\max}$ is one half of the bunch length in the RF phase,

$$I_h^{(0)} = \frac{I_{dc}}{N}\int_{-\pi R}^{\pi R} e^{ihz/R}\rho_0(z)dz \quad , \tag{3.4}$$

and the nth moment $\left\langle\varphi^n\right\rangle$ is defined according to

$$\left\langle\varphi^n\right\rangle = \frac{1}{N}\int_{-\varphi_{\max}}^{\varphi_{\max}}\varphi^n\frac{R\rho(z,t)}{h}d\varphi = \frac{1}{N}\int_{-\varphi_{\max}}^{\varphi_{\max}}\varphi^n\hat{\rho}(\varphi,t)d\varphi \quad , \tag{3.5}$$

where $\hat{\rho}(\varphi,t) = R\rho(z,t)/h$ follows from the change of variable from z to φ, and the moment due to the perturbation is given by

$$\left\langle\varphi^n\right\rangle_p = \frac{1}{N}\int_{-\varphi_{\max}}^{\varphi_{\max}}\varphi^n\hat{\xi}(\varphi,t)d\varphi \ . \tag{3.6}$$

Following the definition of $\left\langle\varphi^n\right\rangle$ in Eq. (3.5), we can derive that

$$\langle \varphi^0 \rangle = 1 \ , \tag{3.7}$$

$$\frac{d\langle \varphi^m \rangle}{dt} = \frac{m}{N} \int_{-\varphi_{max}}^{\varphi_{max}} \varphi^{m-1} \frac{d\varphi}{dt} \hat{\rho}(\varphi,t) d\varphi \ , \tag{3.8}$$

and

$$\begin{aligned}
\frac{d^2\langle \varphi^m \rangle}{dt^2} &= \frac{m(m-1)}{N} \int_{-\varphi_{max}}^{\varphi_{max}} \varphi^{m-2} \left(\frac{d\varphi}{dt}\right)^2 \hat{\rho}(\varphi,t) d\varphi + \frac{m}{N} \int_{-\varphi_{max}}^{\varphi_{max}} \varphi^{m-1} \frac{d^2\varphi}{dt^2} \hat{\rho}(\varphi,t) d\varphi \\
&= -m^2\omega_{s0}^2 \langle \varphi^m \rangle + m\omega_{s0}^2 \left(\phi_v + \frac{\hat{V}}{V_s} \tan\psi_s \right) \langle \varphi^{m-1} \rangle \\
&\quad + m(m-1)\omega_{s0}^2 \left[\langle H\varphi^{m-2} \rangle - \left(\frac{\hat{V}}{V_s} - \phi_v \tan\psi_s \right) \langle \varphi^m \rangle \right] \ ,
\end{aligned} \tag{3.9}$$

where we have applied Eqs.(2.6) and (2.7) to obtain Eq. (3.9). Thus, the moments obey the following equations

$$\frac{d^2\langle \varphi \rangle}{dt^2} + \omega_{s0}^2 \langle \varphi \rangle = \omega_{s0}^2 \left(\phi_v + \frac{\hat{V}}{V_s} \tan\psi_s \right) \ , \tag{3.10}$$

and

$$\begin{aligned}
\frac{d^2\langle \varphi^m \rangle}{dt^2} + m^2\omega_{s0}^2 \langle \varphi^m \rangle &= m\omega_{s0}^2 \left(\phi_v + \frac{\hat{V}}{V_s} \tan\psi_s \right) \langle \varphi^{m-1} \rangle \\
&\quad + m(m-1)\omega_{s0}^2 \left[\langle H\varphi^{m-2} \rangle - \left(\frac{\hat{V}}{V_s} - \phi_v \tan\psi_s \right) \langle \varphi^m \rangle \right] \ ,
\end{aligned} \tag{3.11}$$

for $m > 1$. Using a similarly procedure in deriving Eq. (3.11), one can show that

$$\begin{aligned}
\frac{d^2\langle H^n\varphi^m \rangle}{dt^2} &= -m^2\omega_{s0}^2 \langle H^n\varphi^m \rangle + m\omega_{s0}^2 \left(\phi_v + \frac{\hat{V}}{V_s} \tan\psi_s \right) \langle H^n\varphi^{m-1} \rangle \\
&\quad + m(m-1)\omega_{s0}^2 \left[\langle H^{n+1}\varphi^{m-2} \rangle - \left(\frac{\hat{V}}{V_s} - \phi_v \tan\psi_s \right) \langle H^n\varphi^m \rangle \right] \ .
\end{aligned} \tag{3.12}$$

Now, assuming that the moment $\langle \varphi^n \rangle$ is composed of a steady part $\langle \varphi^n \rangle_0$ and a small perturbation part $\langle \varphi^n \rangle_1$, i.e.

$$\langle \varphi^n \rangle = \langle \varphi^n \rangle_0 + \langle \varphi^n \rangle_1 \ ; \tag{3.13}$$

then using the conditions of $d^2\langle \varphi^n \rangle_0 / dt^2 = 0$ when $\hat{V} = \phi_v = \langle \varphi^n \rangle_1 = 0$, the steady part can be shown is given by

$$\langle\varphi^2\rangle_0 = \frac{\langle H\rangle_0}{2} = \varphi_{\max}^2 \ , \tag{3.14}$$

$$\langle H^n\varphi^{2j+1}\rangle_0 = 0 \ , \ \text{for } n,j = 0,1,2,3........; \tag{3.15}$$

and

$$\langle H^n\varphi^{2(j+1)}\rangle_0 = \frac{2j+1}{2(j+1)}\langle H^{n+1}\varphi^{2j}\rangle_0 \ , \ \text{for } n,j = 1,2,3........ \tag{3.16}$$

Applying these relations, we can derive the following linearized equations of motion for moments:

$$\frac{d^2\langle\varphi\rangle_1}{dt^2} + \omega_{s0}^2\langle\varphi\rangle_1 = \omega_{s0}^2\left(\phi_v + \frac{\hat{V}}{V_s}\tan\psi_s\right) \ , \tag{3.17}$$

$$\frac{d^2\langle\varphi^2\rangle_1}{dt^2} + 4\omega_{s0}^2\langle\varphi^2\rangle_1 = -2\omega_{s0}^2\varphi_{\max}^2\left(\frac{\hat{V}}{V_s} - \phi_v\tan\psi_s\right) \ , \tag{3.18}$$

....

$$\frac{d^2\langle\varphi^m\rangle_1}{dt^2} + m^2\omega_{s0}^2\langle\varphi^m\rangle_1 = m\omega_{s0}^2\left(\phi_v + \frac{\hat{V}}{V_s}\tan\psi_s\right)\langle\varphi^{m-1}\rangle_0$$
$$+ m(m-1)\omega_{s0}^2\left\{\langle H\varphi^{m-2}\rangle_1 - \left(\frac{\hat{V}}{V_s} - \phi_v\tan\psi_s\right)\langle\varphi^m\rangle_0\right\} \ . \tag{3.19}$$

We can also derive the following equations for the perturbation $\langle H^n\varphi^m\rangle_1$ in $\langle H^n\varphi^m\rangle$ and the perturbation $\langle H\varphi^2\rangle_1$ in $\langle H\varphi^2\rangle$:

$$\frac{d^2\langle H^n\varphi^m\rangle_1}{dt^2} + m^2\omega_{s0}^2\langle H^n\varphi^m\rangle_1 = m\omega_{s0}^2\left(\phi_v + \frac{\hat{V}}{V_s}\tan\psi_s\right)\langle H^n\varphi^{m-1}\rangle_0$$
$$+ m(m-1)\omega_{s0}^2\left[\langle H^{n+1}\varphi^{m-2}\rangle_1 - \left(\frac{\hat{V}}{V_s} - \phi_v\tan\psi_s\right)\langle H^n\varphi^m\rangle_0\right] \ , \tag{3.20}$$

and

$$\frac{d^2\langle H\varphi^2\rangle_1}{dt^2} + 4\omega_{s0}^2\langle H\varphi^2\rangle_1 = -4\omega_{s0}^2\varphi_{\max}^4\left(\frac{\hat{V}}{V_s} - \phi_v\tan\psi_s\right) \ . \tag{3.21}$$

Eqs. (3.18) and (3.21) indicate that $\langle H\varphi^2\rangle_1/(2\varphi_{\max}^2)$ and $\langle\varphi^2\rangle_1$ obey the same equation of motion. Therefore we can infer that

$$\langle H\varphi^2\rangle_1 = \langle H\rangle_0\langle\varphi^2\rangle_1 = 2\varphi_{\max}^2\langle\varphi^2\rangle_1 \ . \tag{3.22}$$

Similarly, we can infer that

$$\langle H^n\varphi\rangle_1 = \langle H^n\rangle_0\langle\varphi\rangle_1 , \tag{3.23}$$

$$\langle H^n\varphi^2\rangle_1 = \langle H^n\rangle_0\langle\varphi^2\rangle_1 , \tag{3.24}$$

and

$$\langle H\varphi^n\rangle_1 = \frac{\langle H^2\rangle_0}{\langle H\rangle_0}\langle\varphi^n\rangle_1 , \text{ for } n = 3,4,5........ \tag{3.25}$$

....

Making the use of the relations in Eqs.(3.22)-(3.25), one can rewrite the linearized equations of motion for moments as

$$\frac{d^2\langle\varphi\rangle_1}{dt^2} + \omega_{s0}^2\langle\varphi\rangle_1 = \omega_{s0}^2\left(\phi_v + \frac{\hat{V}}{V_s}\tan\psi_s\right) , \tag{3.26}$$

$$\frac{d^2\langle\varphi^2\rangle_1}{dt^2} + 4\omega_{s0}^2\langle\varphi^2\rangle_1 = -2\omega_{s0}^2\varphi_{\max}^2\left(\frac{\hat{V}}{V_s} - \phi_v\tan\psi_s\right) , \tag{3.27}$$

$$\frac{d^2\langle\varphi^3\rangle_1}{dt^2} + 9\omega_{s0}^2\langle\varphi^3\rangle_1 = 3\omega_{s0}^2\varphi_{\max}^2\left(4\langle\varphi\rangle_1 + \phi_v + \frac{\hat{V}}{V_s}\tan\psi_s\right) , \tag{3.28}$$

$$\frac{d^2\langle\varphi^4\rangle_1}{dt^2} + 16\omega_{s0}^2\langle\varphi^4\rangle_1 = 12\omega_{s0}^2\varphi_{\max}^2\left[2\langle\varphi^2\rangle_1 - \varphi_{\max}^2\left(\frac{\hat{V}}{\hat{V}} - \phi_v\tan\psi_s\right)\right] , \tag{3.29}$$

and

$$\begin{aligned} \frac{d^2\langle\varphi^m\rangle_1}{dt^2} + m^2\omega_{s0}^2\langle\varphi^m\rangle_1 &= m(m-1)\omega_{s0}^2\langle H\varphi^{m-2}\rangle_1 \\ &+ m\omega_{s0}^2\left(\phi_v + \frac{\hat{V}}{V_s}\tan\psi_s\right)\langle\varphi^{m-1}\rangle_0 - m(m-1)\omega_{s0}^2\varphi_{\max}^2\left(\frac{\hat{V}}{V_s} - \phi_v\tan\psi_s\right)\langle\varphi^{m-2}\rangle_0 , \end{aligned} \tag{3.30}$$

for $m = 2,3........$

To proceed the analysis further, we identify the perturbed moment $\langle\varphi^n\rangle_1$ here with $\langle\varphi^n\rangle_p$ in Eq. (3.2). The hth harmonic of the beam current then can be expressed as

$$\begin{aligned} I_h &= I_h^{(0)} + I_{dc}\sum_{j=1}^{\infty}\frac{(-1)^j}{(2j)!}\langle\varphi^{2j}\rangle_1 + iI_{dc}\sum_{j=0}^{\infty}\frac{(-1)^j}{(2j+1)!}\langle\varphi^{2j+1}\rangle_1 \\ &= I_h^{(0)} + \sum_{j=1}^{\infty} I_{h,2j}^{(1)} + i\sum_{j=0}^{\infty} I_{h,2j+1}^{(1)} , \end{aligned} \tag{3.31}$$

where

$$I_{h,2j}^{(1)} = \frac{(-1)^j}{(2j)!}I_{dc}\langle\varphi^{2j}\rangle_1 , \tag{3.32}$$

and

$$I^{(1)}_{h,2j+1} = \frac{(-1)^j}{(2j+1)!} I_{dc} \langle \varphi^{2j+1} \rangle_1 \ . \tag{3.33}$$

In the polar coordinate notation, the hth harmonic of the beam current can be written as

$$I_h = I_b(t) \mathrm{e}^{-i\phi_b(t)} \ , \tag{3.34}$$

where

$$I_b = \sqrt{\left(I_h^{(0)} + \sum_{j=1}^{\infty} I^{(1)}_{h,2j} \right)^2 + \left(\sum_{j=0}^{\infty} I^{(1)}_{h,2j+1} \right)^2} \approx I_h^{(0)} + \sum_{j=1}^{\infty} I^{(1)}_{h,2j} \ , \tag{3.35}$$

and

$$\phi_b = -\tan^{-1}\left[\frac{\sum_{j=0}^{\infty} I^{(1)}_{h,2j+1}}{I_h^{(0)} + \sum_{j=1}^{\infty} I^{(1)}_{h,2j}} \right] \approx -\tan^{-1}\left[\sum_{j=0}^{\infty} I^{(1)}_{h,2j+1} \Big/ I_h^{(0)} \right] \approx -\sum_{j=0}^{\infty} I^{(1)}_{h,2j+1} \Big/ I_h^{(0)} \ . \tag{3.36}$$

If only the first moment or the dipole mode is considered, then

$$I_b \approx I_h^{(0)} \ , \tag{3.37}$$

and

$$\phi_b = -\frac{I^{(1)}_{h(1)}}{I_h^{(0)}} \approx -\frac{I_{dc} \langle \varphi \rangle_1}{I_h^{(0)}} \ . \tag{3.38}$$

To the second moment, the quadrupole mode, we have

$$I_b \approx I_h^{(0)} + I^{(1)}_{h(2)} = I_h^{(0)} - \frac{I_{dc}}{2} \langle \varphi^2 \rangle_1 \ , \tag{3.39}$$

and

$$\phi_b \approx -\frac{I^{(1)}_{h(1)}}{I_h^{(0)}} \approx -\frac{I_{dc}}{I_h^{(0)}} \langle \varphi \rangle_1 \ . \tag{3.40}$$

To the third moment, the sextuple mode, we have

$$I_b \approx I_h^{(0)} + I^{(1)}_{h(2)} = I_h^{(0)} - \frac{I_{dc}}{2} \langle \varphi^2 \rangle_1 \ , \tag{3.41}$$

and

$$\phi_b \approx -\frac{I^{(1)}_{h(1)}}{I_h^{(0)}} - \frac{I^{(1)}_{h(3)}}{I_h^{(0)}} \approx -\frac{I_{dc} \langle \varphi \rangle_1}{I_h^{(0)}} + \frac{I_{dc} \langle \varphi^3 \rangle_1}{6 I_h^{(0)}} \ . \tag{3.42}$$

4. Equivalent Circuit Model for an RF Cavity

In the equivalent circuit model, an RF cavity is envisioned as a parallel *RLC* circuit; the applied RF power source and the circulating beam current are envisioned as currents i_g and i_b, respectively. The schematic is shown in Fig. 1.

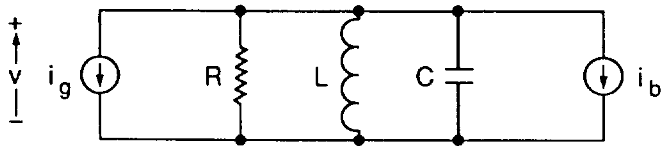

Fig. 1. The equivalent circuit model of the beam-cavity interaction system.

Using Kirchhoff's law, one can derive that the total voltage on the cavity satisfies the differential equation

$$\frac{d^2v}{dt^2} + 2\alpha\frac{dv}{dt} + \omega_r^2 v = 2\alpha R_s \frac{di_t}{dt} , \tag{4.1}$$

where v is the total voltage, $\alpha = \omega_r/(2Q)$, $Q = R_s/(L\omega_r)$ is the quality factor of the cavity, $\omega_r = 1/\sqrt{LC}$ is the resonant frequency of the cavity, R_s, L, C are the shunt resistance, the inductance, and the capacitance of the cavity, respectively;

$$i_t = i_g + i_b . \tag{4.2}$$

Making the substitutions of

$$v = \tilde{V}(t)e^{-i\omega_g t} , \tag{4.3}$$

$$i_g = \tilde{I}(t)e^{-i\omega_g t} , \tag{4.4}$$

and

$$i_b = \tilde{I}_b(t)e^{-i\omega_g t} , \tag{4.5}$$

in Eq. (4.1), yields

$$\frac{d^2\tilde{V}}{dt^2} + 2(\alpha - i\omega_g)\frac{d\tilde{V}}{dt} + (\omega_r^2 - \omega_g^2 - 2i\alpha\omega_g)\tilde{V} = 2\alpha R_s\left(\frac{d\tilde{I}}{dt} - i\omega_g\tilde{I}\right) , \tag{4.6}$$

where ω_g is the frequency of the driving RF power and

$$\tilde{I} = \tilde{I}_g - \tilde{I}_b . \tag{4.7}$$

For synchronization, the RF frequency is chosen according to the relation

$$\omega_g = \omega_{rf} = h\Omega_0 , \tag{4.8}$$

where Ω_0 is the averaged revolution frequency of beam particles and h is the harmonic number. For high-Q and high frequency resonators, $\alpha << \omega_g$, and $d^2\tilde{V}/dt^2 << \omega_g d\tilde{V}/dt$. If we also assume that $d\tilde{I}/dt << \omega_g\tilde{I}$, which is true in most cases, Eq. (4.6) can be approximated by

$$\frac{d\tilde{V}}{dt} + \left[\alpha - \frac{i(\omega_g^2 - \omega_r^2)}{2\omega_g}\right]\tilde{V} = \alpha R_s \tilde{I} \ . \tag{4.9}$$

The relations among these phasors are shown in Fig. 2, where we have chosen a rotating polar coordinate system such that the steady state $\tilde{I}_b$ is on the real axis.

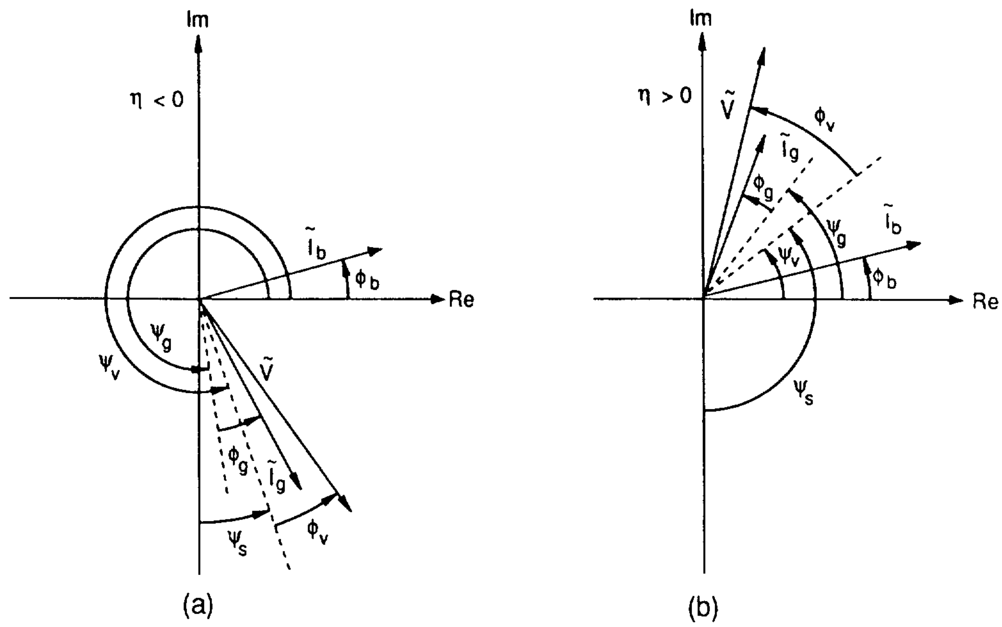

Fig.2. Phasor diagram showing the relations among the beam current, $\tilde{I}_b$, the generator current, $\tilde{I}_g$, and the total cavity voltage, $\tilde{V}$, for (a) $\gamma < \gamma_t$, (b) $\gamma > \gamma_t$. ψ_s represents the synchronous angle and dashed lines designate steady state angles.

For most systems, the phasors will oscillate about their steady states, so we denote ϕ_v and ϕ_b as the angular deviations of $\tilde{V}$, and $\tilde{I}_b$ from their steady states, respectively. We also introduce the synchronous angle between the total voltage and the beam current, ψ_s, as shown in Fig. 2.

Using the notations defined above, we can write the phasors in polar form as:

$$\tilde{I}_b = I_b(t)e^{-i\phi_b(t)} \ , \tag{4.10}$$

$$\tilde{I}_g = I_g(t)e^{-i\psi_g} \ , \tag{4.11}$$

and

$$\tilde{V} = V(t)e^{-i\psi_v - i\phi_v(t)} \ . \tag{4.12}$$

Substituting the above polar representations into Eq. (4.9) and equating the real and imaginary parts on both sides of the equality, we have

$$\frac{dV}{dt} + \alpha V = \alpha R_s\left[-I_b\cos(\phi_b - \phi_v - \psi_v) + I_g\cos(\phi_v - \psi_g + \psi_v)\right] , \tag{4.13}$$

and

$$\frac{d\phi_v}{dt} = \omega_r - \omega_g - \frac{\alpha R_s}{V}\left[I_b \sin(\phi_b - \phi_v - \psi_v) + I_g \sin(\phi_v - \psi_g + \psi_v)\right], \qquad (4.14)$$

where use has been made of the approximation $\omega - \omega_r \approx 2\omega$. One can prove that when $\psi_g = \psi_v$ the system will be in tune, that is, the RF source will see a *real* impedance. In this case, Eqs. (4.13) and (4.14) are simplified to

$$\frac{dV}{dt} + \alpha V = \alpha R_s\left[-I_b \cos(\phi_b - \phi_v - \psi_v) + I_g \cos\phi_v\right], \qquad (4.15)$$

and

$$\frac{d\phi_v}{dt} = \omega_r - \omega_g - \frac{\alpha R_s}{V}\left[I_b \sin(\phi_b - \phi_v - \psi_v) + I_g \sin\phi_v\right]. \qquad (4.16)$$

The subsequent analysis in this and the next sections will always assume that the system is in tune. It should be noted here that the total voltage in the steady state V_s, is given by

$$V_s = R_s\left[I_g - I_b \cos\phi_v\right]. \qquad (4.17)$$

Also note that in the steady state,

$$\frac{R_s I_b \sin\psi_v}{V_s} = \frac{\omega_r - \omega_g}{\alpha} = -\tan\phi_y . \qquad (4.18)$$

Assuming that $I_b = I_{b0} + I_{b1}$ with $I_{b1} << I_{b0}$, and a small perturbation $\hat{V}$ is introduced to the voltage, one can linearize Eqs. (4.15) and (4.16) to yield

$$\frac{d\hat{V}}{dt} + \alpha\hat{V} = -\alpha R_s\left[I_{b1}\cos\psi_v + I_{b0}(\phi_b - \phi_v)\sin\psi_v\right], \qquad (4.19)$$

and

$$\frac{d\phi_v}{dt} + \alpha\phi_v = \frac{-\alpha R_s I_{b0}\cos\psi_v}{V_s}\left[\phi_b - \left(\frac{I_{b1}}{I_{b0}}\right)\tan\psi_v + \left(\frac{\hat{V}}{V_s}\right)\tan\psi_v\right]. \qquad (4.20)$$

The above two equations and Eqs. (3.26)-(3.36) are the basic equations for studying the beam-cavity interaction in the linear regime.

5. Dispersion Relations and Comparison with Kinetic Theory

The purpose of this section is to test our formalism established so far by comparing the characteristic equation, or the generalized dispersion relation, derived from the moment method with that obtained from the kinetic theory. We shall study the dispersion relation up to the second harmonic of the synchrotron oscillation, i.e. to the second moment.

5.1 Dispersion Relations

The complete set of equations, up to the second moment, are

$$\frac{d\hat{V}}{dt}+\alpha\hat{V}=-\alpha R_s\left[I_{b1}\cos\psi_v+I_{b0}(\phi_b-\phi_v)\sin\psi_v\right], \tag{5.1}$$

$$\frac{d\phi_v}{dt}+\alpha\phi_v=\frac{-\alpha R_s I_{b0}\cos\psi_v}{V_s}\left[\phi_b-\left(\frac{I_{b1}}{I_{b0}}\right)\tan\psi_v+\left(\frac{\hat{V}}{V_s}\right)\tan\psi_v\right], \tag{5.2}$$

$$I_b=I_{b0}+I_{b1}\approx I_h^{(0)}-\frac{I_{dc}}{2}\langle\varphi^2\rangle_1, \tag{5.3}$$

$$\phi_b\approx-\frac{I_{dc}\langle\varphi\rangle_1}{I_h^{(0)}}, \tag{5.4}$$

$$\frac{d^2\langle\varphi\rangle_1}{dt^2}+\omega_{s0}^2\langle\varphi\rangle_1=\omega_{s0}^2\left(\phi_v+\frac{\hat{V}}{V_s}\tan\psi_s\right), \tag{5.5}$$

and

$$\frac{d^2\langle\varphi^2\rangle_1}{dt^2}+4\omega_{s0}^2\langle\varphi^2\rangle_1=-2\omega_{s0}^2\varphi_{\max}^2\left(\frac{\hat{V}}{V_s}-\phi_v\tan\psi_s\right), \tag{5.6}$$

where $I_{b0}=I_h^{(0)}$, and

$$I_{b1}=-\frac{I_{dc}}{2}\langle\varphi^2\rangle_1. \tag{5.7}$$

The Laplace transformation of Eqs. (5.1), (5.2), (5.5) and (5.6) are

$$(s+\alpha)(\hat{V}/V_s)=-\lambda\left[(\phi_b-\phi_v)\sin\psi_v+(I_{b1}/I_{b0})\cos\psi_v\right], \tag{5.8}$$

$$(s+\alpha)\phi_v=-\lambda\left[(\hat{V}/V_s)\sin\psi_v+\phi_b\cos\psi_v-(I_{b1}/I_{b0})\sin\psi_v\right], \tag{5.9}$$

$$(s^2+\omega_{s0}^2)\phi_b=\omega_{s0}^2\mu_1\left[\phi_v+(\hat{V}/V_s)\tan\psi_s\right], \tag{5.10}$$

and

$$(s^2+4\omega_{s0}^2)(I_{b1}/I_{b0})=4\omega_{s0}^2\mu_2\left[(\hat{V}/V_s)-\phi_v\tan\psi_s\right], \tag{5.11}$$

where

$$\lambda=\frac{\alpha R_s I_{b0}}{V_s}, \tag{5.12}$$

$$\mu_1=\frac{\phi_b}{\langle\varphi\rangle_1}=-\frac{I_{dc}}{I_h^{(0)}}=-\frac{I_{dc}}{I_{b0}}, \tag{5.13}$$

$$\mu_2=\frac{-I_{b1}\varphi_{\max}^2}{2I_{b0}\langle\varphi^2\rangle_1}=\frac{I_{dc}\varphi_{\max}^2}{2I_{b0}}, \tag{5.14}$$

and we made no distinguish between the notations for the time domain quantities and the notations for the Laplace transformed quantities. In the matrix form, Eqs. (5.8)-(5.11) can be written as

$$\begin{bmatrix} (s+\alpha) & -\lambda\sin\psi_v & \lambda\sin\psi_v & \lambda\cos\psi_v \\ \lambda\sin\psi_v & (s+\alpha) & \lambda\cos\psi_v & -\lambda\sin\psi_v \\ \omega_{s0}^2\mu_1\tan\psi_s & \omega_{s0}^2\mu_1 & -(s^2+\omega_{s0}^2) & 0 \\ -4\omega_{s0}^2\mu_2 & 4\omega_{s0}^2\mu_2\tan\psi_s & 0 & (s^2+4\omega_{s0}^2) \end{bmatrix} \begin{bmatrix} (\hat{V}/V_s) \\ \phi_v \\ \phi_b \\ (I_{b1}/I_{b0}) \end{bmatrix} = 0 . \quad (5.15)$$

Equating the determinant of the matrix in Eq. (5.15) to zero, one arrives the following dispersion relation:

$$s^6 + p_5 s^5 + p_4 s^4 + p_3 s^3 + p_2 s^2 + p_1 s + p_0 = 0 , \quad (5.16)$$

where

$$p_0 = 4\omega_{s0}^4\left[\alpha^2\sec^2\phi_y - \lambda^2(\mu_1+\mu_2) + \lambda^2\mu_1\mu_2\sec^2\psi_s\right] , \quad (5.17)$$

$$p_1 = 8\alpha\omega_{s0}^4 , \quad (5.18)$$

$$p_2 = 4\omega_{s0}^4 + 5\alpha^2\omega_{s0}^2\sec^2\phi_y - \omega_{s0}^2\lambda^2(\mu_1+4\mu_2) , \quad (5.19)$$

$$p_3 = 10\alpha\omega_{s0}^2 , \quad (5.20)$$

$$p_4 = 5\omega_{s0}^2 + \alpha^2\sec^2\phi_y , \quad (5.21)$$

and

$$p_5 = 2\alpha . \quad (5.22)$$

5.2 Comparison with Kinetic Theory

The dispersion relation including the second harmonic of the synchrotron frequency has been derived from linearized Vlasov equation in a previous work.[6] Therefore only a summary is given in the following. The starting point here is the linearized Vlasov equation

$$(\omega - l\omega_{s0})R_l(r) = \frac{q^2 M\eta\Omega_0 l}{2\pi m_0\gamma r}\left(\frac{df_0}{dr}\right) \times \sum_{n,m}\frac{Z_n(\omega+n\Omega_0)}{n} i^{m-l-1} J_l\left(\frac{nr}{R}\right)\int_0^\infty R_m(r')J_m\left(\frac{nr'}{R}\right)r'dr' , \quad (5.23)$$

where n is the index for the harmonic number around the ring, $R_l(r)$ is the lth radial mode of the perturbation in the phase space of $(z, v_z/\omega_{s0})$, $r = \left[z^2 + (v_z/\omega_{s0})^2\right]^{1/2}$, $J_l(x)$ is the Bessel Function of the lth order, and

$$Z_n(\omega \pm n\Omega_0) = \frac{R_s}{1 + i\left[(\omega_R \mp n\Omega_0 \pm \omega)/\alpha\right]} , \quad (5.24)$$

is the impedance of the RF cavity.

Considering the cases of $n = \pm h$, $l = \pm1, \pm2$, and $m = \pm1, \pm2$, we have

$$(\omega - 2\omega_{s0})\Gamma_2 = -2i\vartheta_2[\Xi_-(\Gamma_2 + \Gamma_{-2}) - i\Xi_+(\Gamma_1 - \Gamma_{-1})] , \tag{5.25}$$

$$(\omega - \omega_{s0})\Gamma_1 = -i\vartheta_1[i\Xi_+(\Gamma_2 + \Gamma_{-2}) + \Xi_-(\Gamma_1 - \Gamma_{-1})] , \tag{5.26}$$

$$(\omega + \omega_{s0})\Gamma_{-1} = -i\vartheta_1[i\Xi_+(\Gamma_2 + \Gamma_{-2}) + \Xi_-(\Gamma_1 - \Gamma_{-1})] , \tag{5.27}$$

and

$$(\omega + 2\omega_{s0})\Gamma_{-2} = -2i\vartheta_2[-\Xi_-(\Gamma_2 + \Gamma_{-2}) + i\Xi_+(\Gamma_1 - \Gamma_{-1})] , \tag{5.28}$$

where

$$\vartheta_m = \frac{q^2 M\eta\Omega_0}{2\pi h m_0 \gamma}\int_0^\infty \left[J_m\left(\frac{hr}{R}\right)\right]^2 \frac{df_0}{dr} dr , \tag{5.29}$$

$$\Gamma_m = \int_0^\infty R_m(r) J_m\left(\frac{hr}{R}\right) r dr , \tag{5.30}$$

$$\Xi_- = Z_h(\omega + h\Omega_0) - Z_h(\omega - h\Omega_0) , \tag{5.31}$$

and

$$\Xi_+ = Z_h(\omega + h\Omega_0) + Z_h(\omega - h\Omega_0) . \tag{5.32}$$

Equating the determinant of the coefficients of Γ_2, Γ_1, Γ_{-1}, and Γ_{-2} to zero and making a substitution of $\omega = is$ yield the dispersion relation

$$s^6 + b_5 s^5 + b_4 s^4 + b_3 s^3 + b_2 s^2 + b_1 s + b_0 = 0 , \tag{5.33}$$

where

$$b_0 = 4\alpha^2\omega_{s0}^2[\omega_{s0}^2 \sec^2\phi_y - 4R_s\omega_{s0}(\vartheta_1 + \vartheta_2)\tan\phi_y + 16R_s^2\vartheta_1\vartheta_2] , \tag{5.34}$$

$$b_1 = 8\alpha\omega_{s0}^4 , \tag{5.35}$$

$$b_2 = 4\omega_{s0}^4 + 5\alpha^2\omega_{s0}^2\sec^2\psi_v - 4R_s\omega_{s0}\alpha^2(\vartheta_1 + 4\vartheta_2)\tan\phi_y , \tag{5.36}$$

$$b_3 = 10\alpha\omega_{s0}^2 , \tag{5.37}$$

$$b_4 = 2\omega_{s0}^2 + \alpha^2\sec^2\phi_y , \tag{5.38}$$

and

$$b_5 = 2\alpha . \tag{5.39}$$

Comparing Eqs.(5.16)-(5.22) with Eqs.(5.33)-(5.39), we find that the differences are in $\mu_1\omega_{s0}^2\lambda^2$ versus $4R_s\omega_{s0}\alpha^2\tan\phi_y\vartheta_1$, and $\mu_2\omega_{s0}^2\lambda^2$ versus $4R_s\omega_{s0}\alpha^2\tan\phi_y\vartheta_2$. Using the definitions of α, λ, μ_1, μ_2, ϑ_m, and ω_{s0}, one can show that

$$\mu_1 = \frac{4R_s\omega_{s0}\alpha^2\tan\phi_y\vartheta_1}{F_1\omega_{s0}^2\lambda^2} , \tag{5.40}$$

and

$$\mu_2 = -\frac{2R_s\omega_{s0}\varphi_{\max}^2\alpha^2\tan\phi_y\vartheta_2}{F_2\omega_{s0}^2\lambda^2} \quad . \tag{5.41}$$

where

$$F_m = \frac{4\int_0^\infty \left[J_m\left(\frac{hr}{R}\right)\right]^2 \frac{df_0}{dr}dr}{\left(\frac{h}{R}\right)^2 \int_0^\infty f_0(r)rdr} \quad , \tag{5.42}$$

is the *reduced form factor.* The values of F_1 and F_2 have been calculated and charted for some different phase space distributions.[6] In general, the value of F_1 (F_2) decreases (increases) when the bunch length increases. For very short bunch lengths, F_1 and F_2 have values near 2.0 and 0, respectively. At the bunching factor of 0.5, most distributions have the values of F_1 near 1.3 and the values of F_2 near 0.17.

Thus, we have demonstrated that the dispersion relation derived from the equivalent circuit is the same as that inferred from the linearized Vlasov equation except for a factor that depends on the detail of the perturbation in the phase space.

6. Discussions

We have shown that in the regime of linear longitudinal focusing, the results of stability analysis using the equivalent circuit and the moment method agree very well with that using linearized Vlasov equation. In this section, we shall discuss the possibility of applying the moment method to nonlinear problems and give a simple example of taking the feedback control into consideration.

6.1 Nonlinear Longitudinal Focusing Forces

There are two main difficulties make it hard to apply the moment method to a general nonlinear problem. The first difficulty is that nonlinearities introduce coupling among all the equations of motion for moments. The second difficulty is the indefinite integral in Eq. (2.4) can not be carried out or expanded into the power series of φ unless the voltage response is given. Nonetheless, we shall discuss in the following, a possible approach by retaining only the zeroth order in the expansion of the integrand in Eq. (2.4). Under this approximation, one can derive a general equation of motion for the moments. These differential equations are all linear and coupled. One can truncate the coupling at any desirable order for numerical solutions. Readers are reminded that the approximation taken here could be a crude one.

Thus, using the approximation of

$$\int\left[V\sin(\psi_s+\phi_v-\varphi)\frac{d\phi_v}{d\varphi}-\cos(\psi_s+\phi_v-\varphi)\frac{dV}{d\varphi}\right]d\varphi \approx (V_s\sin\psi_s)\phi_v - \hat{V}\cos\psi_s \quad , \tag{6.1}$$

and expanding the trigonometric functions in Eqs. (2.1) and (2.4) we obtain

$$\frac{d^2\varphi}{dt^2} = \frac{\omega_{s0}^2}{V_s\cos\psi_s}\left[V\sin(\psi_s+\phi_v) - V_s\sin\psi_s\right]$$
$$+\frac{\omega_{s0}^2 V\sin(\psi_s+\phi_v)}{V_s\cos\psi_s}\sum_{n=1}^{\infty}\frac{(-1)^n\varphi^{2n}}{(2n)!} - \frac{\omega_{s0}^2 V\cos(\psi_s+\phi_v)}{V_s\cos\psi_s}\sum_{n=1}^{\infty}\frac{(-1)^n\varphi^{2n+1}}{(2n+1)!}\ , \tag{6.2}$$

and

$$\left(\frac{d\varphi}{dt}\right)^2 \approx \frac{\omega_{s0}^2}{V_s\cos\psi_s}\left[V\sin(\psi_s+\phi_v) - V_s\sin\psi_s\right]\varphi$$
$$+\frac{\omega_{s0}^2}{V_s\cos\psi_s}\left[V\cos(\psi_s+\phi_v) + K_0 + (V_s\sin\psi_s)\phi_v - \hat{V}\cos\psi_s\right] \tag{6.3}$$
$$+\frac{\omega_{s0}^2 V\cos(\psi_s+\phi_v)}{V_s\cos\psi_s}\sum_{n=1}^{\infty}\frac{(-1)^n\varphi^{2n}}{(2n)!} + \frac{\omega_{s0}^2 V\sin(\psi_s+\phi_v)}{V_s\cos\psi_s}\sum_{n=1}^{\infty}\frac{(-1)^n\varphi^{2n+1}}{(2n+1)!}\ .$$

Then, following the same procedures in deriving Eqs (3.10) and (3.11), we can derive the following equations:

$$\frac{d^2\langle\varphi\rangle}{dt^2} \approx \frac{\omega_{s0}^2}{V_s\cos\psi_s}\left[V\sin(\psi_s+\phi_v) - V_s\sin\psi_s\right]$$
$$+\frac{\omega_{s0}^2 V}{V_s\cos\psi_s}\sum_{n=1}^{\infty}(-1)^n\left[\frac{\langle\varphi^{2n}\rangle\sin(\psi_s+\phi_v)}{(2n)!} - \frac{\langle\varphi^{2n+1}\rangle\cos(\psi_s+\phi_v)}{(2n+1)!}\right]\ , \tag{6.4}$$

$$\frac{d^2\langle\varphi^2\rangle}{dt^2} \approx \frac{2\omega_{s0}^2}{V_s\cos\psi_s}\left[\frac{V}{2}\cos(\psi_s+\phi_v) + K_0 + (V_s\sin\psi_s)\phi_v - \hat{V}\cos\psi_s\right]$$
$$+\frac{4\omega_{s0}^2}{V_s\cos\psi_s}\left[V\sin(\psi_s+\phi_v) - V_s\sin\psi_s\right]\langle\varphi\rangle \tag{6.5}$$
$$-\frac{2\omega_{s0}^2 V}{V_s\cos\psi_s}\sum_{n=1}^{\infty}(-1)^n\left[\frac{(2n+1)\langle\varphi^{2n+2}\rangle\cos(\psi_s+\phi_v)}{(2n)!} - \frac{(2n+2)\langle\varphi^{2n+1}\rangle\sin(\psi_s+\phi_v)}{(2n+1)!}\right]\ ,$$

....

$$\frac{d^2\langle\varphi^m\rangle}{dt^2} \approx \frac{m(m-1)\omega_{s0}^2}{V_s\cos\psi_s}\left[V\cos(\psi_s+\phi_v) + K_0 + (V_s\sin\psi_s)\phi_v - \hat{V}\cos\psi_s\right]\langle\varphi^{m-2}\rangle$$
$$+\frac{m(m-1)\omega_{s0}^2}{V_s\cos\psi_s}\left[V\sin(\psi_s+\phi_v) - V_s\sin\psi_s\right]\langle\varphi^{m-1}\rangle - \frac{m(m-1)\omega_{s0}^2}{V_s\cos\psi_s}\cos(\psi_s+\phi_v)\langle\varphi^m\rangle$$
$$+\frac{m\omega_{s0}^2 V}{V_s\cos\psi_s}\sum_{n=1}^{\infty}(-1)^n\left[\frac{(2n+3-m)\langle\varphi^{2n+m}\rangle\cos(\psi_s+\phi_v)}{(2n+2)!} + \frac{(2n+m)\langle\varphi^{2n+m+1}\rangle\sin(\psi_s+\phi_v)}{(2n+1)!}\right]\ , \tag{6.6}$$

where $V = V_s + \hat{V}$. The variation of the moment $\langle\varphi^n\rangle_1$ can be calculated according to

$$\langle \varphi^n \rangle_1 = \langle \varphi^n \rangle - \langle \varphi^n \rangle_0 \, . \tag{6.7}$$

The variations of the moments are then related to the perturbed beam current through Eqs. (3.34) to (3.36). To complete the set of equations, we have to include Eqs. (4.15) and (4.16). These equations, in principle, can be solved numerically.

A special case that a full nonlinear effects can be studied is the one of zero bunch length, in which the particle density is given by a delta function, i.e.,

$$\rho(z,t) = N\delta(z - \langle z \rangle) \, . \tag{6.8}$$

In this case, it is easier to work on the moments with respect to the bunch center. Note that since the bunch center is always moving, the moments defined with respect to the bunch center are essentially defined in a noninertial coordinate system. The perturbed beam current is then given by

$$\begin{aligned} I_h &= \frac{qMc\beta}{2\pi R} \int_{-\pi R}^{\pi R} e^{ihz/R} \rho(z,t) dz \\ &= \frac{qMc\beta N}{2\pi R} \int_{-L/2}^{L/2} e^{ihz/R} \delta(z - \langle z \rangle) dz \\ &= I_{dc} e^{i\langle \varphi \rangle} \int_{-\varphi_{\max}}^{\varphi_{\max}} e^{i(\varphi - \langle \varphi \rangle)} \delta(\varphi - \langle \varphi \rangle) d\varphi \\ &= I_b (\cos\langle \varphi \rangle + i \sin\langle \varphi \rangle) \, . \end{aligned} \tag{6.9}$$

The equation of motion for the bunch center is

$$\frac{d^2\langle \varphi \rangle}{dt^2} = \frac{\omega_{s0}^2}{V_s \cos\psi_s} \left[V \sin(\psi_s + \phi_v - \langle \varphi \rangle) - V_s \sin\psi_s \right] . \tag{6.10}$$

All moments with respect to the bunch center vanish in this special case, i.e.,

$$\left\langle (\varphi - \langle \varphi \rangle)^n \right\rangle \propto \int_{-\varphi_{\max}}^{\varphi_{\max}} (\varphi - \langle \varphi \rangle)^n \delta(\varphi - \langle \varphi \rangle) d\varphi = 0 \, . \tag{6.11}$$

6.2 An Example of Including the Feedback Control

Since the control of the beam-cavity system is a well studied and documented subject, so we are not going to discuss it at here.[9] The purpose of this section is to present an example of the attempt to make the equivalent circuit equations complete for numerical solutions in the time domain study when the feedback control is included.

We consider the transfer matrix of a feedback control in the Laplace transformed domain like

$$\begin{pmatrix} \delta I_g / I_g \\ \phi_g \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} \hat{V}/V_s \\ \phi_v \end{pmatrix} , \tag{6.12}$$

where

$$A_{ij} = \frac{e^{-s\tau}\left(d_{ij}s^2 + c_{ij}s + e_{ij}\right)}{s\left(s^2 + g_i s + h_i\right)}, \qquad \text{for } i,j = 1 \text{ and } 2, \tag{6.13}$$

τ is the time delay in the feedback circuit, δI_g and ϕ_g are the corrections in the amplitude and phase of the generator current, respectively, d_{ij}, c_{ij}, e_{ij}, g_i, and h_i, are constants depend on the feedback circuit design. In the time domain, the corresponding differential equations are

$$\begin{aligned} &\frac{d^2}{dt^2}\left(\frac{\delta I_g}{I_g}\right) + g_1 \frac{d}{dt}\left(\frac{\delta I_g}{I_g}\right) + h_1\left(\frac{\delta I_g}{I_g}\right) = \\ &\quad c_{11}\left(\frac{\hat{V}}{V_s}\right) + d_{11}\frac{d}{dt}\left(\frac{\hat{V}}{V_s}\right) + e_{11}\int\left(\frac{\hat{V}}{V_s}\right)dt + c_{12}\phi_v + d_{12}\frac{d\phi_v}{dt} + e_{11}\int \phi_v dt , \end{aligned} \tag{6.14}$$

and

$$\begin{aligned} &\frac{d^2\phi_g}{dt^2} + g_2\frac{d\phi_g}{dt} + h_2\phi_g = \\ &\quad c_{21}\left(\frac{\hat{V}}{V_s}\right) + d_{21}\frac{d}{dt}\left(\frac{\hat{V}}{V_s}\right) + e_{21}\int\left(\frac{\hat{V}}{V_s}\right)dt + c_{22}\phi_v + d_{22}\frac{d\phi_v}{dt} + e_{21}\int \phi_v dt . \end{aligned} \tag{6.15}$$

The right hand sides of Eqs. (6.14) and (6.15) should be evaluated at the time $t-\tau$. The equations for cavity voltage and phase are

$$\frac{dV}{dt} + \alpha V = \alpha R_s\left[-I_b\cos(\phi_b - \phi_v - \psi_v) + \left(I_g + \delta I_g\right)\cos\left(\phi_v + \phi_g - \psi_g + \psi_v\right)\right], \tag{6.16}$$

and

$$\frac{d\phi_v}{dt} = \frac{\omega_r^2 - \omega_g^2}{2\omega_g} - \frac{\alpha R_s}{V}\left[I_b\sin(\phi_b - \phi_v - \psi_v) + \left(I_g + \delta I_g\right)\sin\left(\phi_v + \phi_g - \psi_g + \psi_v\right)\right]. \tag{6.17}$$

Eqs. (6.14)-(6.17) should be solved together with Eqs. (3.34)-(3.36), and the moment equations [Eqs. (3.26)-(3.30) or Eqs. (6.4)-(6.7)]. Note that in Eqs. (6.16) and (6.17), we have included the nonlinear part of the RF focusing force.

Conclusions

We have presented an approach to incorporate all harmonics of synchrotron oscillation in the equivalent circuit model of beam-cavity interaction by considering the perturbed moments of a bunched beam. In the regime of linear approximation, we found good qualitative agreements in comparing the dispersion relations obtained from this new approach with that derived from the linearized Vlasov equation up to the second synchrotron harmonic . We have also discussed about the possibility of extending the method to the cases of nonlinear longitudinal focusing forces and the time domain tracking with feedback control. It is found that the moment method can not be easily extended to the nonlinear focusing case without using some crude approximations.

Acknowledgments

The author would like to thank Japanese Hadron Project for the support and the hospitality he received during the course of this work. He would also like to thank Drs. Y. Mori, S. Machida, and C. Omori for their helpful discussions.

References

1. K. W. Robinson, "Radiofrequency Acceleration II," Report No. CEA-11, Cambridge Electron Accelerator, Cambridge, Mass. (Sept. 1956).
2. K. W. Robinson, "Stability of Beam in Radio-Frequency System," Report No. CEAL-1010, Cambridge Electron Accelerator, Cambridge, Mass. (Feb. 1964).
3. A. N. Lebedev, "Coherent Synchrotron Oscillations in the Presence of Space Charge," Atomnaya Energiya, **25** (2), pp. 100 (1968). (English translation p. 851).
4. F. J. Sacherer, "Bunch Lengthening and Microwave Instability," CERN Internal Report CERN/PS/BR 77-6 (1977).
5. See, for example, the refereces cited in Ref. 6.
6. T. F. Wang, "Bunched-Beam Longitudinal Mode-Coupling and Robinson Instabilities," *Particle Accelerators*, **54**, pp. 105-126 (1990).
7. P. J. Channell, "The Moment Approach to Charged Particle Beam Dynamics," IEEE Trans. Nucl. Sci., **30** (4), August 1983, p. 2607.
8. W. P Lysenko and M. Overley, "Moment Invariants for Particle Beams," in *Linear Accelerators and Beam Optics Codes*, AIP Conf. Proc. 177, ed. C. R. Eminhizer (AIP New York, 1988), p. 323.
9. For example see, D. Boussard, " Design of a Ring RF System," in CERN 92-03, *Proc. of CERN Accelerator School on RF Engineering for Particle Accelerators,* p.474, or F. Pedersen, "A novel RF Cavity Tuning Scheme for Heavy Beam Loading," IEEE Trans. Nucl. Sci. Vol. NS-32, No. 5, Oct 1985, p2138, or R. Garoby's lecture note in US/CERN/Japan Accelerator School on RF Engineering for Particle Accelerators, Hayama, Japan, September 1996.