UCLA/96/TEP/27

Non–Critical Strings At High Energy

Kenichiro Aoki[1] and Eric D'Hoker[2]

[1] *Department of Physics, Hiyoshi Campus*
Keio University
Hiyoshi, Kouhoku-ku
Yokohama 223, JAPAN

[2] *Theory Division, CERN*
1211 Geneva 23, Switzerland
and
Department of Physics
University of California Los Angeles
Los Angeles, California 90024–1547*, USA*

Abstract

We consider scattering amplitudes in non-critical string theory of N external states in the limit where the energy of all external states is large compared to the string tension. We argue that the amplitudes are naturally complex analytic in the matter central charge c and we propose to define the amplitudes for arbitrary value of c by analytic continuation. We show that the high energy limit is dominated by a saddle point that can be mapped onto an equilibrium electro-static energy configuration of an assembly of N pointlike (Minkowskian) charges, together with a density of charges arising from the Liouville field. We argue that the Liouville charges accumulate on segments of curves, and produce quadratic branch cuts on the worldsheet. The electro-statics problem is solved for string tree level in terms of hyper-elliptic integrals and is given explicitly for 3- and 4-point functions. We show that the high energy limit should behave in a string-like fashion with exponential dependence on the energy scale for generic values of c.

[1] Internet electronic mail address: `ken@hc.keio.ac.jp`

[2] Internet electronic mail address: `dhoker@physics.ucla.edu`

1. Introduction

Ever since string theory was reformulated in terms of a summation over random surfaces, [1], there has been a renewed and persistent interest in the construction of consistent string theories away from the critical dimensions (26 for the bosonic string, 10 for superstrings), [2], [3], [4], [5], [6], [7] [8], [9], [10]. Reviews on and extensive references to work on non-critical string theory may be found in [11] and [12]. Several motivations are driving these studies, of which we shall just mention the most immediate ones. There is the uncovering of more general or new consistent string theories : for example a six dimensional tensionless string theory has recently been considered [13]. There is the improvement of our understanding – perhaps at the non-perturbative level – of critical string theory via models with fewer physical degrees of freedom. There is the reinterpretation of the worldsheet properties of the string in terms of two-dimensional quantum gravity, providing simple examples of quantized gravity [6]. There is the possibility of mapping the scaling behavior of two-dimensional statistical mechanics models on random lattices onto 2D quantum gravity [7], [9]. There is the proposed reformulation of the critical behavior of the three-dimensional Ising model in terms of fermionic strings [14][15][16]. There are proposals to view non-critical string theory as a conformally invariant off-shell realization of critical string theory [17], and as a critical string theory in a flat space-time background metric with a dilaton field that grows linearly with time [18].

Our understanding of non-critical string theory to date is very advanced for bosonic models with rational matter central charge $c < 1$. The mapping between discretized random surfaces and random matrices, combined with the double scaling limit, produces exact results for correlation functions, to all orders of perturbation theory [7], [9], see [12] for a review. Surprisingly, it has turned out to be very difficult to reproduce, with the help of the Liouville model of Polyakov's proposal, even the simplest results obtained via matrix models. Also, a direct reformulation of fermionic non-critical strings in terms of matrix models seems problematic, if not impossible [19]. Finally, it has proven to be quite difficult to cross the barrier at $c = 1$ and analyze the region of perhaps most pronounced physical interest $c > 1$, for either the bosonic or fermionic strings.

One of the most basic obstacles to reaching beyond the $c = 1$ barrier (within the Liouville field theory approach), is the appearance of conformal primary fields with complex weights, and thus of string states with complex masses. In a series of ingenious papers [3] and [4], it was proposed that the string spectrum may be restricted to a subset of "physical states", that have real conformal weights only. This restriction appears to be possible only provided space-time dimension assumes certain special values : 1, 7, 13 and 19 for the bosonic string, and 1, 3, 5 and 7 for the fermionic string. (The truncation of the spectrum in [3]and [4]is analogous to the truncation of the Kac table of conformal primary fields used in [6] for rational conformal field theories with $c < 1$.) [1]

A truncation of the spectrum at the free string level will be consistent at the interacting level, only if interactions between physical states produce only physical states. The simplest direct check would be on the factorization of the four point function at tree level; unfortunately, this amplitude is not available. Instead, it was verified in a series of papers [21], and [22], that the algebra of operators corresponding to "physical states" closes under operator product expansion, again provided the dimension of space-time belongs to the list given above. This somewhat less direct check on the truncation of string states provides

[1] See also [20], where a truncation, analogous to the GSO projection, was proposed for the non-critical fermionic string.

strong evidence that consistent string theories indeed exist in these special dimensions. Clearly, however, it would be very valuable to have access to the four point function for a more direct check of factorization.

The primary goal of this paper is to develop calculational methods that allow us to evaluate scattering amplitudes in non-critical string theory. We shall show that this can indeed be achieved, in the limit where the energies of the incoming and outgoing strings is large.

We propose to define non-critical string amplitudes in the region $1 < c < 25$ by analytic continuation in c throughout the complex c plane, [2] starting from the line $c < 1$. As was already shown in [23], the integral representation of the non-critical scattering amplitudes in the Liouville formulation, is complex analytic in the central charge, and thus naturally lends itself to such a definition. Also, this type of analytic continuation is very similar to that required to defining scattering amplitudes for all values of external momenta, already in the critical string. (See *e.g.* [25].)

The definition of non-critical string amplitudes by analytic continuation in the central charge that we use as a starting point is a priori different from the one used in [21] and [22]. There, substantial modifications occur in the structure of the Liouville field dynamics as one crosses from the weak coupling phase (for $c < 1$) into the strong coupling phase (for $c > 1$). In particular, the two chiralities of the Liouville field become uncoupled, a new cosmological constant appears and the vertex operators are modified. Whether the definition of the amplitudes by analytic continuation in the central charge c takes us to the non-critical strings of [3] and [4] is an open and exciting question.

The exact evaluation of the scattering amplitudes, even to tree level, for general c and general external momenta, would require that we can carry out a set of multiple integrals that are more general than those available from [26], or from [27]. Evaluating these integrals remains an open problem.

In this paper, we propose an evaluation of non-critical string amplitudes for any complex c, in the limit where the energies of incoming and outgoing string states are all large compared to the (square root of the) string tension.[3] We shall show that the Liouville approach lends itself naturally to taking the high energy limit, where the integral representations for the amplitudes become tractable, for any complex value of c. To string tree level, we succeed in producing explicit formulas for the limit in terms of hyper-elliptic integrals. We shall not, at this stage, perform any truncation on the spectrum of states in the non-critical string theory. Thus, our results are applicable to non-critical string theories in general, including those in which the Liouville field is reinterpreted as an extra dimension of space-time, as in [18].

For string theory in the critical dimension, the high energy limit of scattering amplitudes is dominated by a saddle point in the positions of the vertex operators for external string states, as well as in the moduli of the surface. This problem is equivalent to finding the equilibrium configuration of an array of electro-static Minkowskian charges (attached to the vertex operators) on a surface of variable shape. In a series of beautiful papers [29], it was shown how the saddle point can be constructed by symmetry arguments, for the four point function, to any order in perturbation theory.

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[2] Analytic continuations in the central charge were used in [23] as technical tools to prove the validity of the continuation procedure of [24]. Here, we go one step further, and take the analytic continuation as a definition of the amplitudes.

[3] The high energy limit of non-critical string theories with $c = 1$ was considered in [28].

For non-critical string theory, the high energy limit is still dominated by a saddle point, which is equivalent to the equilibrium configuration of an array of (complex) charges on a surface of variable shape. In addition to the charges from the external vertex operators, we now also have charges from the Liouville exponential operator. In fact, the number of Liouville charges on the surface increases linearly with energy and, in the limit of large energy, accumulate onto a continuous charge density. We shall show that this Liouville charge density consists of line segments, producing quadratic branch cuts on the worldsheet.

We shall solve explicitly the equivalent electro-statics problem for a worldsheet with the topology of a sphere (tree level) in terms of hyper-elliptic functions, and use it to deduce the high energy limit of tree level scattering amplitudes. The solution in this limit is valid for any complex value of the matter central charge c, and we use analytic continuation to define the non-critical string amplitudes throughout the complex c plane. For higher genus topologies, the solution involves quadratic branch cuts of higher genus surfaces, but we shall postpone a full derivation of this case to a later publication.

The main result is that, at least for generic values of the matter central charge c, the non-critical amplitudes behave in a string like fashion, with exponential dependence on the energy scale, in the limit of high energy. While it is logically possible that this generic exponential behavior could be absent (and replaced by power-like behavior) at isolated points in the complex c plane, we believe that this is unlikely to occur in the region $1 < c < 25$. It is thus unlikely that the non-critical string theories in this region ever become "quantum field theories".

The remainder of this paper is organized as follows. In Sect. 2, we establish the equivalence between the high energy limit saddle point and the electro-static equilibrium configuration of an array of charges, including Liouville charges, on a Riemann surface with variable moduli. In Sect. 3, we solve the electro-statics problem at string tree level (i.e. on the complex plane), find the configuration of charges and determine their electro-static energy in terms of hyper-elliptic integrals. In Sect. 4, we work out the cases of the string tree-level 3- and 4-point functions in detail, and use the symmetric scattering amplitude as a simple explicit example. In Sect. 5, we present a brief discussion of open problems, in particular of the higher loop case, of the possibility of power law behavior and of some practical applications. Some useful formulas are derived in Appendix A.

2. High Energy Limit and Equivalent Electro-statics

We begin by reviewing some basic results in the Liouville field theory formulation of bosonic non-critical string theory. The starting point is a "matter" conformal field theory, describing Poincaré invariant string dynamics in a d-dimensional space-time. In addition to the string coordinates $x^\mu(z)$ with $\mu = 1, \cdots d$, there may be further "internal degrees of freedom", collectively denoted by $\psi(z)$ in what follows. The worldsheet metric is denoted by g_{mn} and the associated Laplacian on scalar functions by Δ_g. The action for the string coordinate x is given by free field theory[4]

$$S_M = \frac{T}{4\pi} \int d^2z \sqrt{g}\ x^\mu \Delta_g x_\mu \tag{2.1}$$

[4] The string tension T can naturally be absorbed into x, and will be set equal to 1 in the remainder.

The conformal primary fields are vertex operators of the type

$$V_\delta = \mathcal{P}_\Delta(\partial x^\mu, \psi) e^{ik\cdot x} \qquad\qquad \delta = \Delta + \frac{1}{2}k^2 \tag{2.2}$$

where $\mathcal{P}_\Delta$ depends on ψ and derivatives of x^μ only. The conformal dimensions of V_δ and $\mathcal{P}_\Delta$ are (δ,δ) and (Δ,Δ) respectively. Without loss of generality, we may consider vertex operators associated to external string states of definite mass and spin; the factor $\mathcal{P}_\Delta$ then grows with momenta (and energy) no faster than polynomially.

The above conformal field theory is coupled to a quantized worldsheet metric g, which, in conformal gauge, decomposes into the Liouville field $\phi(z)$ and a fiducial metric $\hat{g}(m_j)$ that only depends on the moduli m_j of the Riemann surface Σ, with $g = \hat{g}\exp\{2\phi\}$. The action for the Liouville field is

$$S_L = \frac{1}{4\pi}\int \sqrt{\hat{g}}\left[\frac{1}{2}\phi\Delta_{\hat{g}}\phi - \kappa R_{\hat{g}}\phi + \mu e^{\alpha\phi}\right] \tag{2.3}$$

Here $R_{\hat{g}}$ is the Gaussian curvature of the metric $\hat{g}$, and the coupling constants κ and α are given in terms of the matter central charge c as follows

$$3\kappa^2 = 25 - c \qquad\qquad \alpha^2 + \kappa\alpha + 2 = 0 \tag{2.4}$$

Each conformal primary field V_δ may be coupled to the worldsheet metric in a diffeomorphism invariant way. In conformal gauge this is achieved by multiplying V_δ by a Liouville exponential $\exp\{\beta(\delta)\phi\}$, and adjusting $\beta(\delta)$ in such a way that the resulting operator has conformal weight $(1,1)$. This gravitationally dressed operator may be integrated in a diffeomorphism invariant way and we obtain

$$\mathcal{V}_\delta \equiv \int d^2z V_\delta(z) e^{\beta\phi(z)}, \qquad \beta(\delta) = \frac{-\sqrt{25-c} + \sqrt{1-c+24\delta}}{2\sqrt{3}} \tag{2.5}$$

The analytic continuation in c and the external momentum will dictate which branches of the square roots should be chosen. When $\delta = 0$, the operator $\mathcal{V}_0$ is just the Liouville exponential interaction, present in the Liouville action.[5]

Correlation functions of the operators $\mathcal{V}_\delta$ are obtained in standard fashion by combining matter and Liouville correlation functions[6]

$$\langle\prod_{i=1}^N \mathcal{V}_i\rangle = \sum_{h=0}^\infty \int_{\mathcal{M}_h} dm Z_{gh}(m) \int_{\Sigma_h}\prod_i d^2z_i \langle\prod_{i=1}^N e^{\beta_i\phi(z_i)}\rangle_L \langle\prod_{i=1}^N V_i(z_i)\rangle_M \tag{2.6}$$

[5] For special values of c, it is possible to have primary fields other than just exponentials of the Liouville field, so that combinations other than (2.5) may occur. This happens at $c = 25$ and $c = -2$ for example. Thus, the operators we are considering in (2.5) may not be the most general physical operators possible in non–critical string theory. Also, we do not know in general which operators correspond to the complete set of physical states in a particular theory. A careful analysis of the tree level four point function which we compute below should shed light on these questions.

[6] To simplify notation, we denote V_{δ_i}, $\mathcal{V}_{\delta_i}$, and $\beta(\delta_i)$ by V_i, $\mathcal{V}_i$ and β_i respectively.

Here, $\mathcal{M}_h$ is the moduli space of compact Riemann surfaces Σ_h with h handles, dm stands for the measure on moduli space, and $Z_{gh}(m)$ consists of Fadeev-Popov ghost determinants, including their zero mode normalization factors (see e.g. [30]). The conformal field theory correlation functions for the matter integrals and Liouville integrals are respectively defined by

$$\begin{aligned}\langle\prod_{i=1}^{N} V_i(z_i)\rangle_{_M} &\equiv \int D_{\hat{g}}x \int D_{\hat{g}}\psi \; e^{-S_M - S_\psi} \prod_i V_i(z_i) \\ \langle\prod_{i=1}^{N} e^{\beta_i\phi(z_i)}\rangle_{_L} &\equiv \int D_{\hat{g}}\phi \; e^{-S_L} \prod_i e^{\beta_i\phi(z_i)}\end{aligned} \tag{2.7}$$

S_ψ denotes the action for the additional internal degrees of freedom, but we shall not need it here. The functional measures $D_{\hat{g}}x \; D_{\hat{g}}\psi$ and $D_{\hat{g}}\phi$ are built from the $\mathbf{L}^2$ norms on the functions with respect to the fiducial metric $\hat{g}$ (see [8], [31]).

2.1. Correlation functions as multiple integrals

In what follows, we shall compute Liouville and matter correlation functions in the high energy limit, which will allow us to evaluate correlation functions in non–critical string theory in the same approximation.

To evaluate the Liouville correlation functions, we follow the procedure of [24], [23][32] (see also [33]). We split the Liouville field ϕ as follows $\phi = \phi_0 + \varphi$, where ϕ_0 is constant on the worldsheet and φ is orthogonal to constants. The integration splits accordingly, and the integral over ϕ_0 may be carried out explicitly, as follows

$$\int D_{\hat{g}}\phi \, e^{-S_L} \prod_{j=1}^{N} e^{\beta_j\phi(z_j)} = \frac{\Gamma(-s)\mu^s}{\alpha(4\pi)^s} \int D_{\hat{g}}\varphi \; e^{-S'_L} \left(\int\sqrt{\hat{g}} \; e^{\alpha\varphi}\right)^s \prod_{j=1}^{N} e^{\beta_j\varphi(z_j)} \tag{2.8}$$

Here $D_{\hat{g}}\varphi$ denotes the integration over the field φ, which is orthogonal to constants, by definition. The new Liouville action S'_L is a free action now, given by

$$S'_L = \frac{1}{4\pi}\int d^2z\sqrt{\hat{g}}\left[\frac{1}{2}\varphi\Delta_{\hat{g}}\varphi - \kappa R_{\hat{g}}\varphi\right] \tag{2.9}$$

The variable s is a scaling dimension, given in terms of α, κ, the genus h of the surface and the energies β_j as follows

$$\alpha s = -\kappa(1-h) - \sum_{j=1}^{N}\beta_j \tag{2.10}$$

In general, s does not have to be integer, or does not even have to be rational. The prescription of [24] is to proceed and carry out the functional integration over φ as if s were an integer, and then later on continue in s. We are confident that this procedure is reliable in view of the semi-classical analysis carried out in [23].

The integration over φ is completely analogous to the Coulomb gas problem, but here with generalized complex charges.

$$\langle\prod_{j=1}^{N} e^{\beta_j\phi(z_j)}\rangle_{_L} = \frac{\Gamma(-s)\mu^s}{\alpha(4\pi)^s} Z_s(m)^{-1/2}\int\prod_{p=1}^{s} d^2w_a \exp\Big\{\sum_{\substack{i,j=1\\ i<j}}^{N}\beta_i\beta_j G(z_i,z_j) + \sum_{a=1}^{s}\sum_{j=1}^{N}\alpha\beta_j G(z_j,w_a) + \sum_{\substack{a,b=1\\ a<b}}^{s}\alpha^2 G(w_a,w_b) + \mathcal{R}\Big\} \tag{2.11}$$

Here, $Z_s(m)$ represents the functional determinant of the scalar Laplace operator on a surface with metric $\hat{g}(m)$ and takes the form

$$Z_s(m) = \frac{\mathrm{Det}'\Delta_{\hat{g}}}{\int d^2z\sqrt{\hat{g}}} \tag{2.12}$$

The additional term $\mathcal{R}$ in (2.11) involves all the integrals over the Gaussian curvature $R_{\hat{g}}$. Clearly, this term is absent at string tree level where all curvature can be concentrated at ∞. It is also absent at one loop level where we can take $R_{\hat{g}} = 0$. Later on, we shall establish that this term is subdominant in the high energy limit and may be omitted.

The integration over the matter fields will depend upon the precise nature of the matter conformal field theory. Fortunately, in anticipation of the high energy limit of the amplitudes, we shall only need to exhibit the part of the matter amplitudes that involves the exponential factors.

$$\langle\prod_{j=1}^{N} V(z_j)\rangle_{_M} = Z_s(m)^{-d/2}\mathcal{P}(z_i)\exp\Big\{-\sum_{\substack{i,j=1\\ i<j}}^{N} k_ik_jG(z_i,z_j)\Big\} \tag{2.13}$$

Here, the function $\mathcal{P}(z_i)$ involves all matter Green function factors other than exponential, and $Z_s(m)$ represents the functional determinant of the scalar Laplace operator of (2.12).

Combining the Liouville and matter parts of the correlation functions, we obtain our final expression for non-critical string correlation functions, given by the following expression

$$\langle\prod_{i=1}^{N}\mathcal{V}_i\rangle = \frac{\Gamma(-s)\mu^s}{\alpha(4\pi)^s}\sum_{h=0}^{\infty}\int_{\mathcal{M}_h} dm\, Z_s(m)^{-(d-1)/2}\int_{\Sigma_h}\prod_{i=1}^{N} d^2z_i\mathcal{P}(z_i)\int_{\Sigma_h}\prod_{p=1}^{s} d^2w_p\exp\{-\mathcal{E}_h(z_i,w_p)\} \tag{2.14}$$

The total partition function $Z(m)$ is given by

$$Z(m) = Z_{gh}(m)Z_s^{-(d+1)/2}(m) \tag{2.15}$$

The argument of the exponential is given by

$$\mathcal{E}_h(z_i,w_p) = \sum_{\substack{i,j=1\\ i<j}}^{N} u_{ij}G(z_i,z_j) - \sum_{j=1}^{N}\sum_{p=1}^{s}\alpha\beta_jG(z_j,w_p) - \sum_{\substack{p,q=1\\ p<q}}^{s}\alpha^2G(w_p,w_q) - \mathcal{R} \tag{2.16}$$

The u_{ij} are analogues of the Mandelstam variables extended to the case of non-critical string theory and are given by

$$u_{ij} \equiv -\beta_i\beta_j + k_i \cdot k_j \tag{2.17}$$

Momentum conservation $\sum_j k_j = 0$, Eq. (2.10) together with the defining equations for α and β in (2.4) and (2.5), guarantee diffeomorphism invariance of the correlation function.

It is instructive to view the function $\mathcal{E}_h(z_i, w_p)$ as the electro-static energy of an array of electric charge vectors $K_0 = (\alpha; 0)$ placed at points w_p and electric charge vectors $K_i = (\beta_i; k_i)$ placed at points z_j. A natural inner product may be defined on these $d+1$-dimensional vectors in the following way

$$K_i \cdot K_j \equiv -(\beta_i + \frac{\kappa}{2})(\beta_j + \frac{\kappa}{2}) + k_i \cdot k_j \tag{2.18}$$

The on-shell condition for external states characterized by conformal dimension Δ_i in (2.2) is simply reads

$$K_i \cdot K_i = -m_i^2 \qquad\qquad m_i^2 = \frac{1-c}{12} + 2\Delta_i \tag{2.19}$$

Thus, the computation of correlation functions in non–critical string theory has been reduced to that of the free energy of an array of electric charge vectors in two dimensional electrostatics. In addition to the charges associated with external vertex operators, which are familiar from critical string amplitudes, the Liouville operator introduces additional, internal, charges.

The problem of evaluating correlation functions in non-critical string theory is seemingly reduced to the problem of computing a finite dimensional multiple integral with respect to z_i, w_a over the Riemann surface. Since s is not, in general, an integer however, the correlation function is not well defined as it stands. From the arguments presented in [23], it is clear that the original expression for the amplitudes in (2.6) is complex analytic in the external momenta and in the central charge c, even though the intermediate expressions (2.8), (2.11) and (2.14) only make sense for integer s. Thus, the results obtained by evaluating (2.14) for integer s will have to be analytically continued in s. This is achieved through a combination of analytic continuation in the external momenta (just as in the critical string, [25]) and in the central charge c.

For rational $c < 1$, and to string tree level, it was proposed in [24] to analytically continue in the variable s, using certain rearrangement formulas for ratios of Euler Γ-functions (that are specific to tree level). The validity of this procedure is justified, after the fact, since it produces agreement with results from matrix models. More importantly, agreement can be established from first principles, as was shown in [23], using a saddle point approximation in the limit when $\alpha \to 0$, i.e. when $c \to \infty$. We shall take these analyticity properties as a definition for the amplitudes away from $c < 1$ and rational.

2.2. Correlation functions in the high energy limit

For tree level amplitudes, the above multiple integrals are of the same type as those discussed by Selberg and in [26]. The 3-point function was obtained in their work for arbitrary parameters, but results on the 4-point function are limited to $c < 1$ conformal matter. In general, even to string tree level, the integrals of (2.14) and (2.16) are not available in explicit form.

We propose to evaluate the non-critical string correlation functions in the limit where the energies and momenta of the external string states all become large compared to the square root of the string tension. This limit is physically interesting and was extensively explored in the case of the critical string [29]. Also, the limit of the amplitudes is calculable and could be used as a starting point for a more systematic expansion in high energy. In fact, we shall establish that the amplitudes, in the high energy limit, are given by a saddle point, which corresponds to the electro-static equilibrium configuration of the associated electro-statics problem. The saddle point configuration may be evaluated and the associated electro-static energy — the quantity $\mathcal{E}_h$ entering (2.14) — may be obtained explicitly, at least to tree level.

We shall define the high energy limit by rescaling all momenta k_i by a common factor $\lambda \to \infty$. We have the following asymptotic behavior

$$k_i \to \lambda k_i \qquad\qquad \beta_i \to \pm\lambda|k_i| + \mathcal{O}(1) \tag{2.20}$$

In the high energy limit, $K \equiv (\beta; k)$ naturally corresponds to the momentum of a massless particle as expected. The scaling properties of other quantities are easily deduced from the above : u_{ij} scales like λ^2, s scales like λ^1 while c and α scale like λ^0. Notice that external vertex operators always remain conformally invariant under this scaling.

To determine the high energy limit of the non-critical scattering amplitude in (2.14), we begin by analyzing the high energy behavior of the electro-static energy function $\mathcal{E}_0$. From the expression for $\mathcal{E}_0$ in (2.16), it can be readily show that $\mathcal{E}_h$ scales like λ^2 for large λ. This is manifest for the first term in $\mathcal{E}_h$ in (2.16). Actually, the next two terms in (2.16) also scale like λ^2 for large λ. Although the couplings in the second term only scale linearly in λ, the number of Liouville insertion points, s, also grows like λ. In the third term, the coupling α scales like $\lambda^0 = 1$, but there are now s^2 Liouville insertion points, so again this term scales like λ^2. The last term, $\mathcal{R}$, can be neglected in what we do. Firstly, to tree and one loop levels, the curvature term is irrelevant. Also, its scaling with λ is at most linear in λ, as can be seen from combining (2.8) and (2.9).

The next ingredient needed in the determination of the high energy limit of the non-critical scattering amplitudes is the degree of dependence of this limit on any specific matter conformal field theory. The most important simplification in this respect comes from the observation that the conformal primary fields $\mathcal{P}_\Delta$ involve and produce only polynomial dependence on the space-time momenta k_i. Thus, in the high energy limit, where the contributions from the saddle point will be generically exponential (as we shall establish below), we may neglect the polynomial contributions from the vertex functions $\mathcal{P}_\Delta$. Thus, only the exponential vertex operator parts contribute to the high energy limit.

The fact that the entire function $\mathcal{E}_h$ scales in a homogeneous way for large λ, allows us to use routine saddle point methods to calculate the limiting behavior.[7] The leading

[7] In [23], a saddle point approximation was applied to the Liouville functional integral directly to the Liouville field in the limit semi-classical limit where $\alpha \to 0$. In the high energy limit however, the Liouville action does not scale homogeneously and a similar procedure does not yield a good valid saddle point formulation. To summarize, the limits taken are different; in [23], the external states were fixed and c was taken to $-\infty$ and here, c is fixed to be of order one while the external momenta are taken to be large.

order contribution will be given by the value of $\mathcal{E}_h$ at the saddle point, specified by the vertex positions z_i^0, Liouville insertion points w_p^0 and moduli m^0 at the saddle.

$$\langle \prod_i \mathcal{V}_i \rangle = \exp\{-\mathcal{E}_h\left(z_i^0, w_p^0; m^0\right)\} \tag{2.21}$$

The values of z_i^0, w_p^0 and m^0 are determined from the saddle-point equations

$$\left.\frac{\partial \mathcal{E}_h}{\partial z_i}\right|_{z_i^0, w_p^0, m^0} = \left.\frac{\partial \mathcal{E}_h}{\partial w_p}\right|_{z_i^0, w_p^0, m^0} = \left.\frac{\partial \mathcal{E}_h}{\partial m}\right|_{z_i^0, w_p^0, m^0} = 0 \tag{2.22}$$

Thus, the overall dependence on the momenta at high energy will be exponential. This saddle point equation is just the expression for electro-static equilibrium for an array of vector charges. In addition to the vertex operators present for the critical string, we now also have a number s of Liouville charges.

The number of Liouville charges grows with increasing energy, and tends to ∞ in the infinite energy limit. If we assume that the saddle point Riemann surface remains compact in the high energy limit then the Liouville charges must accumulate somewhere on the surface. What could be the limiting distribution of the Liouville charges ? A priori, this distribution could be two-dimensional and fill regions of the surface; or it could lie along line segments on the surface; or it might be arranged in more exotic configurations, like Cantor sets. To find out which one of these distributions is physically realized, we shall examine the case of tree level amplitudes first, where explicit formulas are readily obtained.

3. Tree Level : Electro-statics on the Plane

To tree level, the worldsheet topology is that of the sphere (or by stereographic projection, of the complex plane), there are no moduli, and all determinant factors $Z(m)$ are constants. The Green function is the electro-static potential on the two dimensional plane, given by[8]

$$G(z, z') = -\ln\{|z - z'|^2 + \epsilon^2\} \tag{3.1}$$

Tree-level non-critical amplitudes, – evaluated for vertex operators that are exponentials only) then reduce to a simple multiple integral expression[9]

$$\begin{aligned}\langle \prod_{i=1}^N \mathcal{V}_i \rangle = \frac{\Gamma(-s)\mu^s}{\alpha(4\pi)^s} \int \prod_{i=1}^N d^2 z_i \prod_{\substack{i,j=1 \\ i<j}}^N |z_i - z_j|^{2u_{ij}} \\ \times \int \prod_{p=1}^s d^2 w_p \prod_{j=1}^N \prod_{p=1}^s |z_j - w_p|^{-2\alpha\beta_j} \prod_{\substack{p,q=1 \\ p<q}}^s |w_p - w_q|^{-2\alpha^2}\end{aligned} \tag{3.2}$$

[8] We have made explicit in the definition of the Green function a short distance regulator ϵ, which will not be exhibited in the sequel, but will always be subsumed.

[9] Notice that these integrals resemble those evaluated in [26] and [27], but they involve more general exponents. As a result, no explicit formulas appear to be available for their evaluation.

The associated electro-statics problem is that of an array of charges at points z_i and w_p, characterized by the following electrostatic energy function

$$\begin{aligned}\mathcal{E}_0(z_i, w_p) = &- \sum_{\substack{i,j=1\\ i<j}}^{N} u_{ij} \ln|z_i - z_j|^2 + \sum_{j=1}^{N}\sum_{p=1}^{s} \alpha\beta_j \ln|z_j - w_p|^2 \\ &+ \sum_{\substack{p,q=1\\ p<q}}^{s} \alpha^2 \ln|w_p - w_q|^2\end{aligned} \tag{3.3}$$

where α and β_i were defined in (2.4) and (2.5) and $u_{ij} = -\beta_i\beta_j + k_i \cdot k_j$. The saddle point equations for the integral (3.2) are just the equations for electro-static equilibrium of the associated electro-statics problem. They are given by

$$\begin{aligned}-\sum_{\substack{j=1\\ j\neq i}}^{N} \frac{2b_{ij}}{z_i - z_j} + \frac{1}{s}\sum_{p=1}^{s} \frac{a_i}{z_i - w_p} &= 0 \\ \frac{1}{s}\sum_{\substack{q\neq p\\ q=1}}^{s} \frac{2}{w_p - w_q} + \sum_{j=1}^{N} \frac{a_j}{w_p - z_j} &= 0\end{aligned} \tag{3.4}$$

Here,[10] we have defined parameters $a_i \equiv 2\beta_i/(\alpha s)$ and $b_{ij} \equiv u_{ij}/(\alpha s)^2$ both of which scale like λ^0 in the limit of large λ. Also, each summation over the number of Liouville charges at w_p, $p = 1, \cdots s$ has been divided by a factor of s, so that the entire equations (3.4) scale like λ^0 in the limit of large λ.

The integral (3.2) and the saddle point equations (3.4) are invariant under simultaneous conformal transformations of z_i and w_p. Using these transformations, we may fix three points $(z_{N-2}, z_{N-1}, z_N) = (0, 1, \infty)$, but we shall continue to denote 0 and 1 by z_{N-2} and z_{N-1} respectively.

We wish to solve the equations (3.4) in the limit where $s \to \infty$, while keeping a_i, b_{ij} and the number of vertex charges N fixed. The most difficult part of this problem is the solution of the second equation in (3.4), for the following reasons. Since the number of Liouville charges at w_p tends to ∞, they must accumulate somewhere on the Riemann sphere. (Viewing the sphere as the complex plane, the charges might accumulate at infinity.) We do not know how they accumulate and what the limiting distribution of Liouville charges at w_p will look like.

A priori, the limiting distribution might correspond to two-dimensional regions of charge, to one-dimensional line segments, to isolated points, or even to more exotic arrangements such as Cantor sets. We shall start by providing an answer to this question first, by carefully keeping the Liouville charges at w_p isolated, and taking the limit only when completely safe. We shall find that the distribution of Liouville charges at w_p is always of the form of a collection of curve segments, whose number is $N - 2$.

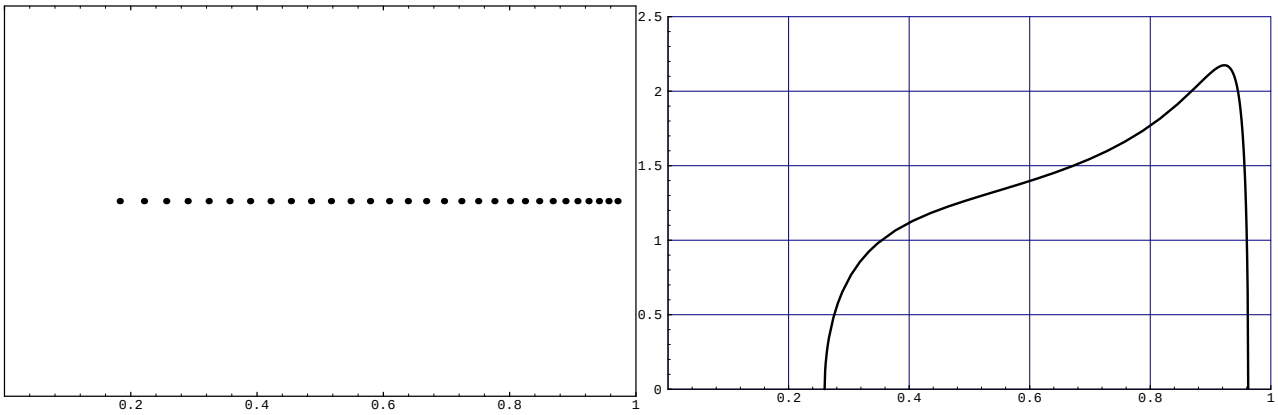
[10] The variables $\bar{z}_i$ and $\bar{w}_p$ satisfy (3.4) with z_i and w_p replaced by $\bar{z}_i$ and $\bar{w}_p$ respectively. When the charges a_i and b_{ij} are real, those respective equations are just the complex conjugates of one another. But when the charges a_i and b_{ij} are taken to be complex, the equations are no longer complex conjugates of one another, and $\bar{z}_i$ and $\bar{w}_p$ at the saddle point are no longer the complex conjugates of z_i and w_p respectively.

3.1. Solving the Electro-statics Problem for $N = 3$ by Jacobi Polynomials

The case of $N = 3$ may be solved for any finite integer s with the help of Jacobi polynomials, as is shown in Appendix A. For charges a_1 and a_2 located at -1 and 1 (with compensating charge at ∞), the positions of the Liouville charges at w_p are given by the zeros of the Jacobi polynomials

$$P_s^{(-1+a_1 s/2,\ -1+a_2 s/2)}(w) = 0 \tag{3.5}$$

For real positive a_1 and a_2, and integer s, it is clear that there are s solutions w_p, $p = 1, \cdots, s$, all located such that $-1 < w_p < 1$. In this case, one easily sees that the zeros accumulate onto a line segment between -1 and 1, as shown schematically in fig. 1. It is useful to keep this case in mind when generalizing to higher values of N.



(a) (b)

fig. 1(a),(b) *(a) Positions of Liouville charges w_p ($p = 1, 2, \ldots, 30$) for $a_1 = 3, a_2 = 1$. (b) The density of Liouville charges in the continuum limit for $a_1 = 3, a_2 = 1$.*

3.2. Solving the Associated Electro-statics Problem for General N

To study equation (3.4) for general N, we make use of a complex potential $W(z)$ for the charges z_i, defined by

$$W(z) \equiv -\sum_{j=1}^{N-1} a_j \ln(z_j - z) \tag{3.6}$$

in terms of which the equation (3.4) for the Liouville charges at w_p becomes

$$\frac{1}{s} \sum_{\substack{q=1 \\ q \neq p}}^{s} \frac{1}{w_p - w_q} = \frac{1}{2} W'(w_p) \, . \tag{3.7}$$

This equation, for general $W(z)$ is just the electro-statics condition for an assembly of s charges in the presence of an external potential $W(z)$ – given here by the potential

generated by the charges at z_i. We also introduce a complex analytic generating function $\omega(z)$, defined by

$$\omega(z) = \frac{1}{s}\sum_{p=1}^{s}\frac{1}{w_p - z} \tag{3.8}$$

which, physically, is just the electric field produced by the Liouville charges at w_p. Its divergence is obtained by applying the Cauchy-Riemann operator,

$$\rho^{(2)}(z) = -\frac{1}{2\pi i}\frac{\partial}{\partial \bar{z}}\omega(z) = \frac{1}{s}\sum_{p=1}^{s}\delta^{(2)}(z - w_p) \tag{3.9}$$

and yields the (two-dimensional) electric charge density $\rho^{(2)}(z)$, with unit integral over the plane.

One may re-express the set of s equations (3.7) in terms of the following Riccati equation[11] for $\omega(z)$

$$\omega^2(z) - \frac{1}{s}\omega'(z) + W'(z)\omega(z) + \frac{1}{4}R(z) = 0 \tag{3.10}$$

The auxiliary potential $R(z)$ is defined by

$$R(z) = \frac{4}{s}\sum_{p=1}^{s}\frac{W'(w_p) - W'(z)}{w_p - z} \tag{3.11}$$

For general $W(z)$, it would not be possible to carry out the sum in the definition of $R(z)$ in any simple way. When $W(z)$ is a rational function of z however, as is the case here, $R(z)$ is also rational, with poles at precisely the same locations as $W(z)$. It is easy to determine $R(z)$ explicitly :

$$R(z) = \sum_{i=1}^{N-1}\frac{R_i}{z - z_i}, \qquad \text{where} \qquad R_i = \frac{4a_i}{s}\sum_{p=1}^{s}\frac{1}{w_p - z_i} \tag{3.12}$$

The fact that we have been able to determine the functional form of $R(z)$ explicitly, in terms of a finite number of parameters is perhaps the most important ingredient in our solution of the associated electro-statics problem.[12]

All that precedes is still an exact transcription of the electro-statics equations (3.4), valid for any finite number of Liouville charges at w_p. We shall now bring about one further simplification by using the approximation in which the number of charges s is large. (Recall that, in the original non-critical string problem, this limit corresponds to high energy of all external string states.)

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[11] Similar methods involving generating functions were used in the study of matrix model spectral equations [12]. The potentials $W(z)$ were typically polynomials in that case.

[12] In the case of matrix model spectral equations, where $W(z)$ it typically polynomial, the analogous key ingredient is that $R(z)$ is polynomial.

The potential $W(z)$ is independent of s, while the electric field $\omega(z)$ and the auxiliary potential $R(z)$ converge to finite limits as $s \to \infty$. Thus, the electro-statics equation for $\omega(z)$ of (3.10) can be simplified in this limit, as the term in $\omega'(z)$ is suppressed by a factor of $1/s$ and may be dropped. Instead of the Riccati equation of (3.10), we obtain now an algebraic equation

$$\omega^2(z) + W'(z)\omega(z) + \frac{1}{4}R(z) = 0 \; . \tag{3.13}$$

This quadratic equation is easily solved, and we obtain the electric field $\omega(z)$ of the Liouville charges at w_p, given by

$$\omega(z) = \frac{1}{2}\left[-W'(z) \pm \sqrt{W'^2(z) - R(z)}\right], \tag{3.14}$$

The sign in front of the square root should be chosen so that the poles in $\omega(z)$, located at the points z_i, are absent when the charges a_i are all real and positive. Indeed, for charges of like sign at z_i and w_p, no Liouville charges at w_p should coincide with any of the z_i.[13] Eq. (3.14) also immediately determines the density of charges by Eq. (3.9). The analysis of the structure of the density will be discussed in the next subsection.

The asymptotic behavior of the various functions yields simple relations between the coefficients R_i, which we now determine. Since $W'(z)$ tends to 0 as $1/z$ for large z, we see from (3.11) that $R(z)$ must tend to 0 as $1/z^2$ for large z. This implies that the sum of all R_i must vanish. Furthermore, using (3.10) and the fact that $\omega(z) \sim -1/z$ as $z \to \infty$, we obtain a second relation between the R_i's. Putting all together, we have

$$\sum_{i=1}^{N-1} R_i = 0 \qquad\qquad \sum_{i=1}^{N-1} z_i R_i = -4 - 4\sum_{i=1}^{N-1} a_i \tag{3.15}$$

The remaining $N-3$ independent parameters R_i appear not to be determined by equation (3.13). Their physical interpretation will be given in Subsection 3.3.

Since R_N defined through (3.12) satisfies $R_N \to 0, z_N R_N \to -4a_N$ when $z_N \to \infty$, the sum in the above formulas may be naturally extended to be from one to N, making these formulas conformally covariant. We also note from (2.10) that the sum of the charges in the high energy limit is simply

$$\sum_{i=1}^{N} a_i = -2 \tag{3.16}$$

3.3. Structure of the Density of Liouville Charges at w_p

From the explicit expression in Eq. (3.14) for the electric field $\omega(z)$ in the limit of large s, we immediately read off the distribution of Liouville charges at w_p for the saddle point. The function $\omega(z)$ is holomorphic throughout the complex plane, except for branch

[13] The cases where a_i are not all real and positive can be obtained by analytic continuation in the charges a_i. It is then in general possible for Liouville charges at w_p to leak into the points z_i, so that also $\omega(z)$ will then have poles at these points.

cuts arising from the square root in (3.14). Thus, the Liouville charges at w_p accumulate to lie on segments of curves that correspond to the branch cuts in the function $\omega(z)$.

The positions of the associated branch points are most easily exhibited by recasting the solution for $\omega(z)$ as follows :

$$W'(z)^2 - R(z) = Q_{2N-4}(z) \times \prod_{i=1}^{N-1} (z - z_i)^{-2} \tag{3.17}$$

Here, $Q_{2N-4}(z)$ is a polynomial in z, which is of degree $2N-4$, in view of the fact that the sum of all R_i vanishes, as shown in (3.15). Using also the second equation in (3.15), we find

$$Q_{2N-4}(z) = (a+2)^2 \prod_{k=1}^{2N-4} (z - x_k) \qquad\qquad a = \sum_{i=1}^{N-1} a_i \tag{3.18}$$

The function $\omega(z)$ thus exhibits $N-2$ branch cuts, $\mathcal{C}_p$, spanned between pairs of branch points x_{2p-1} and x_{2p}, $p = 1, \cdots, N-2$, of $\omega(z)$, which correspond to zeros of the polynomial $Q_{2N-4}(z)$. Next, we shall address the more detailed issue as to exactly where the charges lie.

Since the configuration of the Liouville charges at w_p is one-dimensional, it is convenient to use a notation where this fact is clearly brought out. Since the support of the Liouville charge density is a collection of curve segments, $\mathcal{C} = \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_{N-2}$, the two-dimensional density of charges $\rho^{(2)}(z)$ of (3.9) can be rewritten in terms of a line density of charges $\rho(z)$ as follows

$$\begin{aligned} \rho^{(2)}(z) &= \int_{\mathcal{C}} dw\ \rho(w) \\ &= \int dt\ \dot{w}(t)\ \rho(w(t))\ \delta^{(2)}(z - w(t)) \end{aligned} \tag{3.19}$$

Here, the curve segments of $\mathcal{C}$ are parametrized by a real parameter t, and $\dot{w}$ denotes the t-derivative of w. Given the solution for $\omega(z)$ in (3.14), it is straightforward to calculate the linear density $\rho(w)$, defined when w lies on $\mathcal{C}$.

$$\rho(w) = \frac{1}{2\pi} \sqrt{R(w) - W'(w)^2} \tag{3.20}$$

The requirement that $\mathcal{C}$ lie along branch cuts of ω does not determine the precise position of $\mathcal{C}$. In fact, any analytic curve that joins pairs of branch points would do. To find the saddle point distribution of the charges, an additional ingredient must be clarified.

From the fact that the Liouville charges at the points w_p are all of unit strength times $1/s$, it follows that the charge density must be real and positive along $\mathcal{C}$. This supplementary condition requires that the position of the branch cut $\mathcal{C}$, supporting the Liouville charges at w_p, must be such that $dw\rho(w)$ is real as w is varied along $\mathcal{C}$. Parametrizing $\mathcal{C}$ again by a real parameter t, we have

$$\overline{\dot{w}(t)\rho(w(t))} = \dot{w}(t)\rho(w(t)) \tag{3.21}$$

Equivalently, this condition may be expressed in terms of the (Abelian) integral[14] associated with the (Abelian) differential $dw\rho(w)$:

$$\overline{I(w(t))} = I(w(t)) \qquad\qquad I(w) = \int_{x_1}^{w} dv\rho(v) \tag{3.22}$$

When α and β_i are real and positive , this simply implies that the Liouville charges at w_p are concentrated on the real axis, as was expected. However, (3.21) and (3.22) also provide consistent prescriptions for the case when α, β_i as well as the external momenta, are analytically continued to complex values.

We conclude this subsection by providing a semi-quantitative description of the locations of the branch points and branch cuts of $\omega(z)$. The simplest case is when all the parameters a_i are real and positive, and when all z_i are real as well. On physical grounds, and by symmetry arguments, the Liouville charge density lies on the real axis, and consists of $N-2$ line segments $\mathcal{C}_p$, $p = 1, \cdots, N-2$, located in between the charges at z_i. The charge segments do not touch the points z_i. This configuration is schematically represented in fig. 2.

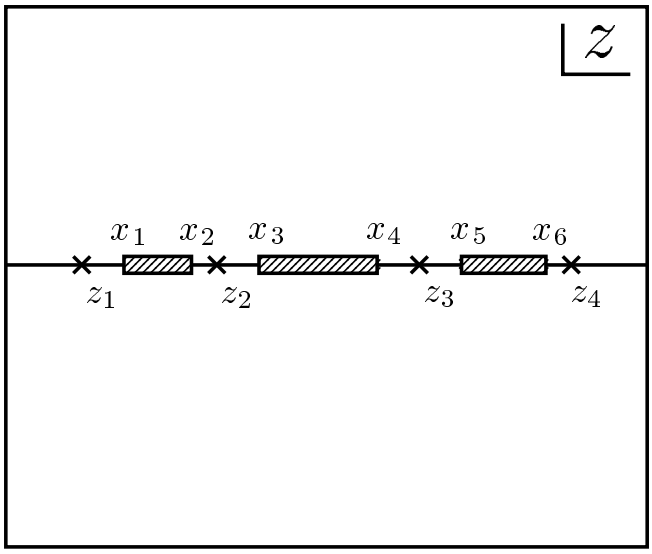

fig. 2 *Schematic representation of the positions of the branch points and the branch cuts for* $N = 5$ *and real charges* $a_i (i = 1, 2, \ldots, 5)$.

On the other hand, when the parameters a_i and the points z_i are complex, the branch points move out into the complex plane, away from the real axis. The positions of the branch points are given by roots of polynomials of degree $2N-4$, and cannot, in general, be exhibited explicitly. Qualitatively however, and using analyticity arguments, we may continuously turn on the complex parts of the parameters a_i and z_i, and follow the branch points and branch cuts as they move off into the complex plane. This effect is represented schematically in fig. 3.

[14] A more complete study of these integrals will be carried out in Subsection 3.5.

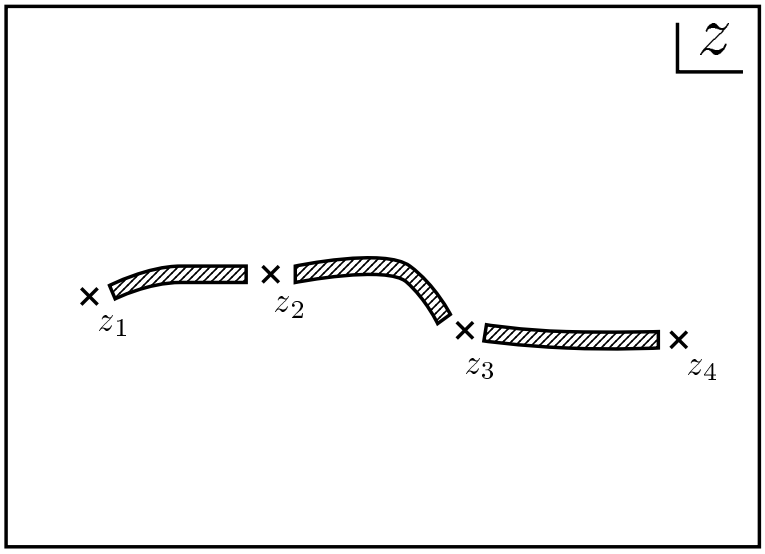

fig. 3 *Schematic representation of the positions of the branch points and the branch cuts for* $N = 5$*, but* a_i *and complex charges* $z_i (i = 1, 2, \ldots, 5)$.

3.4. Closure of Equations

It is convenient to re-express the basic equations we have obtained for the charges at w_p and z_i in terms of the line density $\rho(w)$. With the help of equation (3.6) defining $W(z)$, we have the following expressions for $\rho(w)$

$$\begin{aligned} \rho(w) &= \frac{1}{2\pi}\sqrt{-Q_{2N-4}(w)}\prod_{i=1}^{N-1}\frac{1}{w-z_i} \\ Q_{2N-4}(w) &= \left\{W'(w)^2 - \sum_{i=1}^{N-1}\frac{R_i}{w-z_i}\right\}\prod_{i=1}^{N-1}(w-z_i)^2 \end{aligned} \tag{3.23}$$

The total Liouville charge, obtained by integrating along all the curve segments of $\mathcal{C}$, is normalized to 1 :

$$\int_{\mathcal{C}} dw \rho(w) = 1 \tag{3.24}$$

The electric field $\omega(z)$ is defined by

$$\begin{aligned} \omega(z) &= \int_{\mathcal{C}} dw\ \rho(w)\frac{1}{w-z} \\ &= -\frac{1}{2}W'(z) \pm \frac{1}{2}\sqrt{Q_{2N-4}(z)}\prod_{i=1}^{N-1}\frac{1}{z-z_i} \end{aligned} \tag{3.25}$$

It satisfies the following equations, which are obtained as the $s \to \infty$ limit of equations (3.7) and (3.9).

$$\begin{aligned} W'(w) &= -\{\omega(w^+) + \omega(w^-)\} = 2 \oint_{\mathcal{C}} dw' \rho(w') \frac{1}{w - w'} \\ \rho(w) &= \frac{1}{2\pi i} \{\omega(w^+) - \omega(w^-)\} \end{aligned} \tag{3.26}$$

Here, the integral is performed with principal value prescription, so that $w^{\pm}$ are taken just above and just below the curve $\mathcal{C}$ of Liouville charges; the above equation for $\rho(w)$ holds for w on $\mathcal{C}$.

It remains to clarify the physical significance of the constants R_i, which enter the function $R(z)$ in (3.12) and the polynomial $Q_{2N-4}(z)$ in (3.17). By construction, they are given via the $s \to \infty$ limit of (3.12), which can be expressed in terms of $\omega(z_i)$:

$$R_i = 4a_i \int_{\mathcal{C}} dw \rho(w) \frac{1}{w - z_i} = 4a_i \omega(z_i) \tag{3.27}$$

These relations are *automatically* satisfied by the construction of $\omega(z)$, as can be checked easily by taking the limit of (3.25) when $z \to z_i$. It thus appears that the $N-3$ independent parameters R_i, entering the solution of the second set of equations in (3.4), for the position of the Liouville charges at w_p, are undetermined by these equations.

How can this indeterminacy be understood ? It is easiest to analyze first the case where all z_i, and all a_i are real. By construction, the R_i are then real, as can be seen from (3.12) and (3.23). When all a_i, $i = 1, \cdots, N-1$ are positive, (and only the compensating charge at ∞ is negative) the possible locations for the Liouville charges are on the $N-2$ line segments $\mathcal{C}_p$, $p = 1, \cdots, N-2$, in between pairs of consecutive positive charges at z_i. However, exactly how the total Liouville charge (which is fixed to be 1) is partitioned among the $N-2$ line segments is not à priori determined. Indeed, the positive Liouville charges cannot cross over from one line segment into another, since crossing would involve passing through a charge configuration of infinite electro-static energy when a Liouville charge is on top of a charge z_i. Thus, for any partition of the Liouville charges among the $N-2$ intervals, there must be an equilibrium configuration, and the $N-3$ independent parameters R_i precisely specify the possible partitions of the Liouville charges over the $N-2$ intervals.

When the points z_i move into the complex plane, and the charges a_i are allowed to become complex as well, the allowed line segments on which the Liouville charges can lie move into the complex plane and become more general curve segments. Reality of the Liouville charges continues to impose the constraint that the Abelian integral of (3.22) is real. In particular, the complete Abelian integrals (or A-periods) encircling any given branch cut, must always be real. This condition will be made more explicit in sect. 3.6 : it imposes $N-3$ reality relations between the complex variables R_i. The remaining $N-3$ free parameters in R_i specify the partition of the Liouville charges among the various curve segments, just as was the case for real a_i and z_i. To summarize, $N-3$ real relations exists amongst the $N-3$ complex parameters R_i, and the remaining $N-3$ real parameters of R_i are undetermined and specify the partition of the Liouville charges.

In fact, the values of the parameters R_i are determined by the first set of equations in (3.4), which give the positions of the external vertex charges z_i, $i = 1, \cdots, N-1$.

Expressing them in terms of the high energy limit, where $s \to \infty$, we have

$$\sum_{\substack{j=1 \\ j\neq i}}^{N-1} \frac{2b_{ij}}{z_i - z_j} + \frac{1}{4}R_i = 0 \tag{3.28}$$

Notice that the conditions in (3.15) are automatically satisfied in (3.28), in view of the symmetry of b_{ij}, momentum conservation and (2.10). Therefore, out of the original $N-1$ equations in (3.28), two correspond to the asymptotic conditions (3.15), leaving $N-3$ equations. In view of the analysis of the previous paragraph, only $N-3$ real parameters amongst the $N-3$ complex R_i are determined by the electrostatics equations and the reality conditions, leaving $N-3$ real parameters undetermined. The saddle point equations for the correlation function of non–critical string theory in the high energy limit comprise of (3.26) and (3.28).

3.5. Electro-static Energy

The tree level correlation function, in the saddle point approximation, is given by

$$\langle \prod_{i=1}^{N} \mathcal{V}_i \rangle = \frac{\Gamma(-s)\mu^s}{\alpha(4\pi)^s} e^{-\mathcal{E}_0} \tag{3.29}$$

where all the quantities are to be evaluated at the saddle point. In particular, the electro-static equilibrium energy is given in terms of $\rho(w)$ and $W(w)$ by

$$\begin{aligned} -\frac{\mathcal{E}_0}{2(\alpha s)^2} = \frac{1}{2}\sum_{\substack{i,j=1 \\ i\neq j}}^{N-1} b_{ij} \ln|z_i - z_j|^2 + \int_{\mathcal{C}} dw\rho(w)\{W(w) + \bar{W}(w)\} \\ - \frac{1}{2}\int_{\mathcal{C}} dv\rho(v)\int_{\mathcal{C}} dw\rho(w)\ln|v-w|^2 \end{aligned} \tag{3.30}$$

Now, there is a very important simplification that can be administered to this expression. The key observations were already made previously. First, the saddle point equations for $\bar{z}_i$ and $\bar{w}_p$ are the same as for the quantities z_i and w_p, even when the charges a_i and b_{ij} are complex. Second, the integration measure $dw\rho(w)$ must be real along the line segments of charge density, as pointed out in (3.21).

We see that, as a result, the entire electro-static energy is a sum of a contribution from z_i and w_p on the one hand, and the same functional form, evaluated on $\bar{z}_i$ and $\bar{w}_p$ on the other hand. Thus, given the identity of the equations for barred and unbarred quantities, the electro-static energy is just twice that evaluated on unbarred quantities only :

$$\begin{aligned} -\frac{\mathcal{E}_0}{2(\alpha s)^2} = \sum_{\substack{i,j=1 \\ i\neq j}}^{N-1} b_{ij}\ln(z_i - z_j) + 2\int_{\mathcal{C}} dw\rho(w)W(w) \\ - \int_{\mathcal{C}} dv\rho(v)\int_{\mathcal{C}} dw\rho(w)\ln(v-w) \end{aligned} \tag{3.31}$$

This simplified expression for the electro-static energy at equilibrium has the advantage that it has been recast in terms of complex analytic integrals only, involving z_i, $\rho(v)$ and $\ln(v-w)$.

To make this reformulation more explicit, we introduce the holomorphic potential

$$\Omega(z) = \int_{\mathcal{C}} dw \rho(w) \ln(z-w) \tag{3.32}$$

of the Liouville charges at w_p. This potential is the analogue of the potential W for the charges z_i, and its derivative is $\omega(z) = \Omega(z)'$. In terms of this function, we may evaluate the electro-static potential of the w_p-charges in a simplified way. We begin with[15]

$$\mathrm{Re}\Omega(z) = \frac{1}{2} \int_{\mathcal{C}} dw \rho(w) \ln |z-w|^2 \tag{3.33}$$

By the electro-static equilibrium equation (3.26) for the Liouville charges at w_p, its derivative is related to $W'(v)$, when v is on the curve $\mathcal{C}$:

$$2\frac{\partial}{\partial v} \mathrm{Re}\ \Omega(v) = \fint_{\mathcal{C}} dw \rho(w) \frac{1}{v-w} = \frac{1}{2} W'(v) \tag{3.34}$$

Upon integration along the curve $\mathcal{C}$, we find that

$$\Omega(v) = \frac{1}{2} W(v) + W_0 \tag{3.35}$$

where v is on $\mathcal{C}$, and W_0 is a complex constant. An interesting relation for W_0 is obtained by summing (3.35) over the zeros x_k, $k = 1, \cdots, 2N-4$ of the polynomial $Q_{2N-4}(v)$. We find that

$$\begin{aligned}(2N-4)W_0 = &\sum_{i=1}^{N-1} a_i \ln a_i - (a+2)\ln(a+2) + \sum_{i=1}^{N-1} a_i \sum_{j \neq i} \ln(z_i - z_j) \\ &+ \int_{\mathcal{C}} dw\ \rho(w) \ln Q_{2N-4}(w)\end{aligned} \tag{3.36}$$

As a result, the electro-static energy may be re-written in terms of a single integration over w

$$-\frac{\mathcal{E}_0}{2(\alpha s)^2} = \sum_{\substack{i,j=1 \\ i \neq j}}^{N-1} b_{ij} \ln(z_i - z_j) - W_0 - \sum_{i=1}^{N-1} a_i \Omega(z_i) \tag{3.37}$$

These quantities are now all holomorphic, and as such will not be changed upon continuous changes in the curve $\mathcal{C}$. Thus, any curve $\mathcal{C}$, connecting the branch points can be used in the expression above, which greatly simplifies its calculability.

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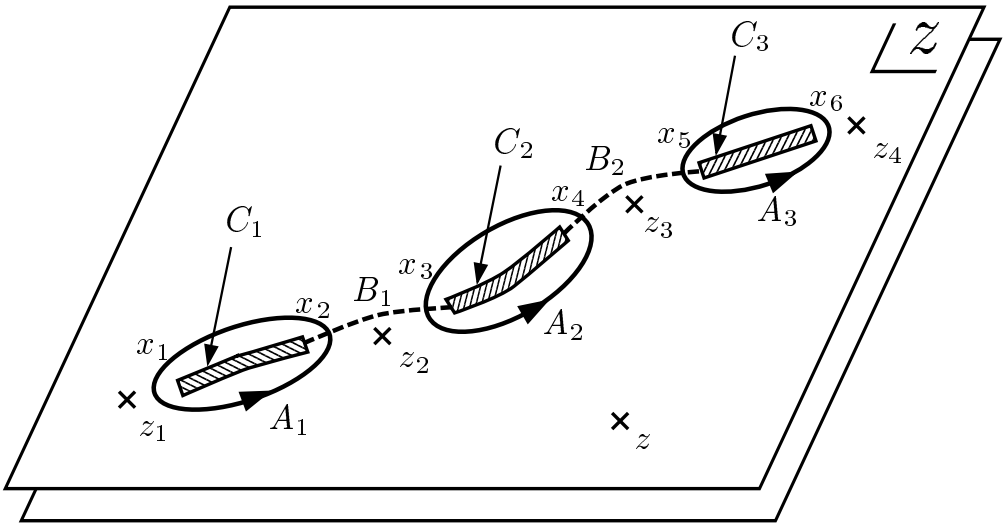
[15] We arrange the logarithm so that $\overline{\ln(z-w)} = \ln(\bar{z} - \bar{w})$.

The above electro-statics problem of charges z_i and w_p on the plane, with $N-2$ quadratic branch cuts, is naturally reformulated in terms of Abelian integrals on an associated hyperelliptic Riemann surface Σ of genus $N-3$. The surface is most easily defined by an algebraic equation in $\mathbf{C}\times\mathbf{C}$ (or more accurately in $\mathbf{CP}^2$), given by

$$y^2 = Q_{2N-4}(w) = (a+2)^2 \prod_{k=1}^{2N-4} (w - x_k) \tag{3.38}$$

The branch cuts, associated with the curve segments $\mathcal{C}_p$, (spanned between pairs of branch points x_{2p-1} and x_{2p}) $p = 1, \cdots, N-2$, may be double covered, and used as a basis for the "A-cycles" of the surface Σ, denoted by A_p. (The cycle A_{N-2} also enters in our calculations since we deal with punctures on the surface.) The remaining curve segments, (spanned between the pairs of branch points x_{2p} and x_{2p+1}) $p = 1, \cdots, N-2$, may also be doubled on the second sheet of Σ and used as a basis for the "B-cycles" of the surface Σ, denoted by B_p, $p = 1, \cdots, N-3$. This construction is schematically represented in fig. 4.



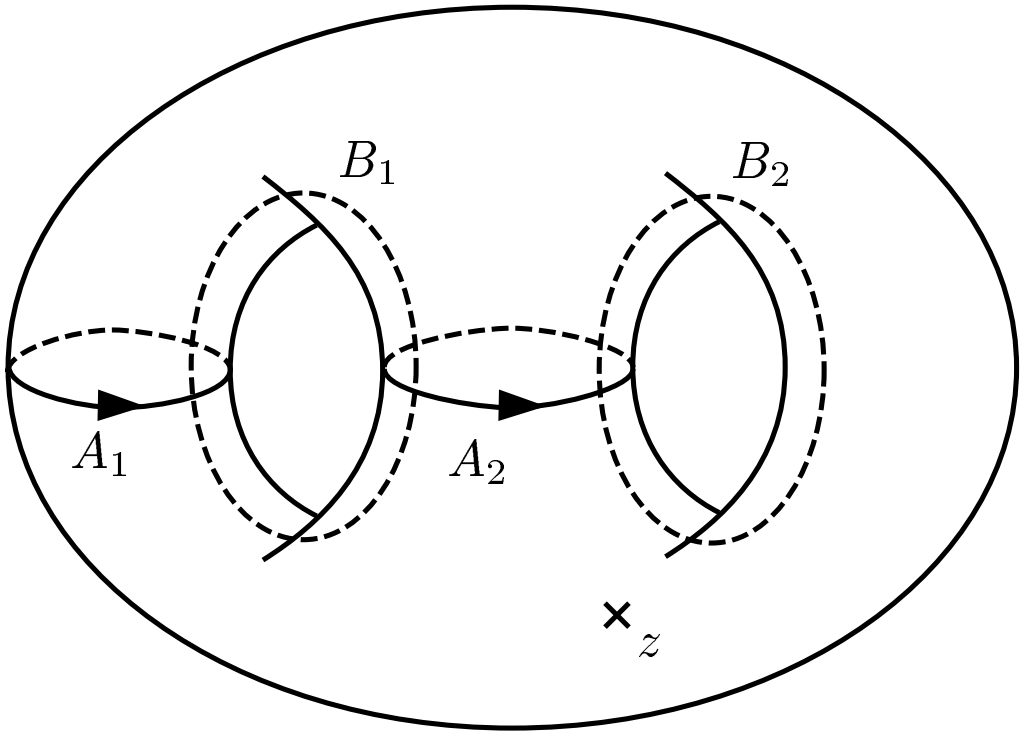


fig. 4 *Double cover of the complex w–plane with branch cuts (a) and its associated hyperelliptic Riemann surface (b).*

On the cut sphere, the one form $\rho(w)dw$ has simple poles at the points z_i, $i = 1, \cdots, N-1$ with residues $a_i/(2\pi i)$ and a simple pole at ∞ with residue $-(a+2)/(2\pi i)$; away from these N poles, $\rho(w)dw$ is holomorphic. The Liouville line charge density $\rho(w)$ naturally defines a meromorphic Abelian differential 1-form, $\rho(w)dw$, on the Riemann surface Σ, with the following properties. $\rho(w)dw$ is holomorphic on Σ apart from the simple poles at the $2(N-1)$ points corresponding to z_i, $i = 1, \cdots, N-1$ with the residues $\pm a_i/(2\pi i)$ and at the two points corresponding to ∞ with the residues $\mp(a+2)/(2\pi i)$. Thus, the differential $\rho(w)dw$ must be a superposition of Abelian differentials on Σ of the first and the third kind. More concretely, denoting the $2N$ points that correspond to z_i $(i = 1, 2, \ldots, N)$ on the Riemann surface itself as $z_i^{\pm}$,

$$dw\rho(w) = \sum_{i=1}^{N} a_i \tau_{z_i^+ z_i^-} + \sum_{p=1}^{N-3} \gamma_p \omega_p \tag{3.39}$$

where ω_p's are the abelian differentials of the first kind normalized in the canonical fashion

$$\int_{A_p} \omega_q = \delta_{pq}, \qquad \int_{B_p} \omega_q = \Omega_{pq}, \qquad p,q = 1,2,\ldots,N-3 \tag{3.40}$$

$(\Omega_{pq})_{p,q=1,2,\ldots N-3}$ is called the period matrix of the Riemann surface. $\tau_{xy}(z)$ is the Abelian differential of the third kind with simple poles at x, y with residues $1, -1$

$$\tau_{xy}(z) \equiv dz \frac{\partial}{\partial z} \ln \frac{\mathcal{E}(z,x)}{\mathcal{E}(z,y)} = dz \frac{\partial}{\partial z} \ln \frac{\vartheta[b](z-x)}{\vartheta[b](z-y)} \tag{3.41}$$

Here, $\mathcal{E}(x,y)$ is the prime form on the Riemann surface and b is any odd half period. τ_{xy} satisfies the normalization condition

$$\int_{A_p} \tau_{xy} = 0, \qquad p = 1,2,\ldots,N-3 \tag{3.42}$$

The reality condition (3.21) corresponds to

$$\gamma_p \in \mathbb{R}, \qquad p = 1,2,\ldots,N-3 \tag{3.43}$$

The γ_p's may be completely determined by computing the charge in each line segment, namely

$$\gamma_p = 2 \int_{x_{2p-1}}^{x_{2p}} dw\, \rho(w) \tag{3.44}$$

Since only $N-3$ of the A_p cycles are homologically independent, the condition on the integral of $\rho(w)$ on A_{N-2} is determined as a consistency condition from the total charge (3.16).

The physical requirement that the density $\rho(w)$ should represent real, positive Liouville charges, demands that $\rho(w)dw$ be real along the equilibrium configuration of the curve segments $\mathcal{C}_p$. As a result, the contour integrals of $\rho(w)dw$ along any $\mathcal{C}_p$, or equivalently, along any A_p-cycle, must be real. In addition, with the proper orientation (clockwise), the integrals should be positive and add up to 1 along the curves $\mathcal{C}_p$, or 2 along the closed cycles A_p.

$$\int_{A_p} \rho(w)dw \geq 0 \qquad\qquad \sum_{p=1}^{N-2} \int_{A_p} \rho(w)dw = 2 \tag{3.45}$$

Of course, this requirement of reality will follow as soon as the reality conditions (3.21) and (3.22) hold. However, the two reality conditions have a somewhat different interpretation. The integral in (3.45) is unchanged under continuous deformations of the cycles A_p, and thus gives a restriction on the differential $\rho(w)dw$ itself, specifically on the parameters R_i, as explained in sect. 3.4. Thus, eq. (3.45) is a necessary condition on the differential $\rho(w)dw$, which must be satisfied if (3.21) and (3.22) are to hold for any curve segments C_p. Then, once (3.45) is satisfied, the proper interpretation of (3.21) and (3.22) is that they determine the curves $\mathcal{C}_p$, within the general homology class $\frac{1}{2}A_p$, such that also the local charge density corresponds to real positive Liouville charges.

Next, we study the integrals in (3.25) and (3.32) defining $\omega(z)$ and $\Omega(z)$, since they are required in the calculation of the electro-static energy. It may, at first, appear surprising that the integral (3.25), for the electric field $\omega(z)$, yields an algebraic result, even though it involves Abelian integrals over the Abelian differential $\rho(w)dw$. The reason that the result is algebraic can be understood from the fact that the integration contour $\mathcal{C}$ surrounds all branch cuts at once. Thus, by contour deformation, $\mathcal{C}$ may be unfolded onto a sum of the pole contributions at points z_i, which may be carried out in an elementary way, yielding algebraic results (see Fig. 4 for the configuration of branch cuts and poles).[16]

By a similar type of contour deformation, we may also simplify the calculation of the integral (3.32), defining $\Omega(z)$, and reexpress it in terms of Abelian integrals only. Since the contributions on the contour from $w \to \infty$ do not vanish in this case, we shall have to proceed with additional care here. To do so, we introduce an auxiliary point z_0, which we may take to ∞ in the end. (see fig. 5.)

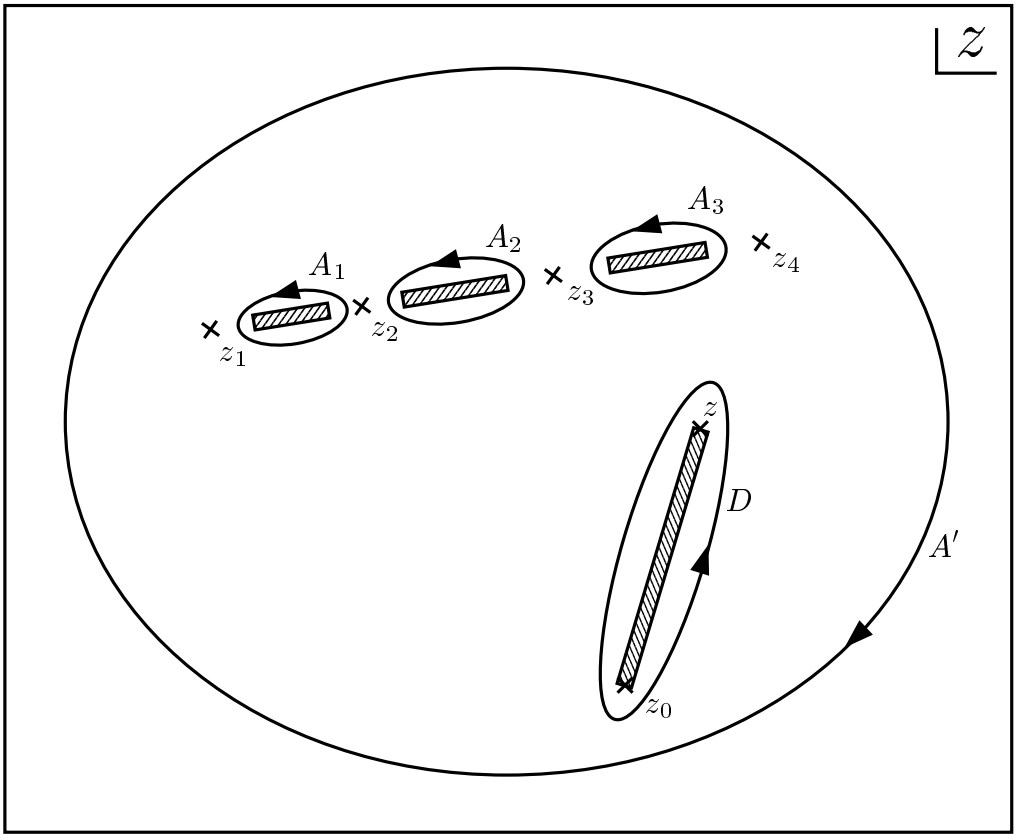


fig. 5 *Complex w–plane with branch cuts due to Liouville charges and due to logarithm for the evaluation of* $\Omega(z)$.

[16] Note that the contributions from the contour at ∞ vanish here.

We evaluate the integral around the sum of all A_p-cycles : $A = A_1 + \cdots + A_{N-2}$, and consider the difference integral

$$\Omega(z) - \Omega(z_0) = \frac{1}{2}\int_A dw\rho(w) \ln \frac{z-w}{z_0-w} \tag{3.46}$$

This integration may be unfolded onto contributions from the poles at z_i and the contour D around the logarithmic cut (see Fig. 5). The contribution from the contour A' vanishes, since it converges to zero in the ∞ radius limit. Thus, we obtain a formula reexpressing the difference of Ω at z and z_0 in terms of an Abelian integral :

$$\Omega(z) - \Omega(z_0) = -\frac{1}{2}\sum_{i=1}^{N-1} a_i \ln \frac{z-z_i}{z_0-z_i} - i\pi \int_{z_0}^{z} dw\rho(w) \tag{3.47}$$

We recover the expression for $\Omega(z)$ alone by considering the limit $z_0 \to \infty$, where $\Omega(z_0) \sim \ln z_0 + \mathcal{O}(1/z_0)$ and we have

$$\Omega(z) = \frac{1}{2}W(z) - i\pi \int_{z_0}^{z} dw\rho(w) + \frac{1}{2}(a+2)\ln z_0 \qquad \text{as } z_0 \to \infty \tag{3.48}$$

This formula reproduces the result, derived in (3.35), that Ω and $\frac{1}{2}W$ are equal up to a constant, as z is restricted to lie on $\mathcal{C}$. To see this, notice that the contribution of the Abelian integral of ρ is purely real as z is varied along $\mathcal{C}$, in view of (3.22). It is easy to see that such integrals are trigonometric when $N = 3$ (the three-point function), elliptic when $N = 4$ (the four-point function), and hyper-elliptic when $N \geq 4$. The tree level correlation function of non-critical string theory in the high energy limit is obtained as (3.29), where $\mathcal{E}_0$ is

$$\begin{aligned} -\frac{\mathcal{E}_0}{2(\alpha s)^2} = &\sum_{\substack{i,j=1 \\ j\neq i}}^{N-1} b_{ij}\ln(z_i - z_j) - \sum_{i=1}^{N-1} a_i\Omega(z_i) \\ &- \frac{1}{2N-4}\left[\sum_{i=1}^{N-1} a_i \ln a_i - a\ln(a+2) + \sum_{i=1}^{N-1} a_i \sum_{\substack{j=1 \\ j\neq i}}^{N-1} \ln(z_i - z_j) + \sum_{l=1}^{2N-4} \Omega(w_l)\right] \end{aligned} \tag{3.49}$$

In view of (3.48), it remains to perform only the integral of the density function. In Sect. 4, we shall produce explicit formulas for the cases $N = 3,\ 4$.

4. Tree Level Three- and Four-Point Functions

To illustrate the general methods developed in the previous sections, we shall here compute the tree level 3- and 4-point amplitudes, in the high energy limit. Actually, the three point function is already known : it was calculated [24] for rational conformal field theories with $c < 1$, and it was shown [23], by analytic continuation, that this formula extends to all values of c. Thus, the calculation of the three point function will provide a non-trivial check that our methods are indeed consistent with known result.

The 4-point amplitude, on the other hand, is not known for general values of c, because of the following reasons. For rational conformal field theories with $c < 1$, the 4-point function is known : it satisfies a generalized hypergeometric differential equation. The order of this differential equation varies with c in a discontinuous way : if $c = 1-6(p-q)^2/(p+q)^2$, the order is essentially $p+q+1$, which varies discontinuously with c. In particular, unless c is rational, the order of the equation is in fact infinite. This is why the expressions for the 4-point function for rational conformal field theories cannot, in any direct way, be continued away from c rational. We shall show below, however, that in the high energy limit, our formalism allows us to evaluate these functions, and provides simple analytic continuation rules which allow us to define the amplitudes throughout the complex plane.

4.1. The 3-Point Function

Using conformal invariance, we choose the three external vertex insertion points at $(z_1, z_2, z_3) = (0, 1, \infty)$. Thus, the electro-statics problem of the previous section reduces to finding the equilibrium configuration of a density of charges, in the presence of 3 fixed charges, located at 0, 1 and ∞. The general formalism of the previous section readily provides the answers for the distribution of these charges.

We begin by obtaining the values for the parameters R_i of (3.12) by solving just the asymptotic conditions, given in (3.15). Using the definition $a = a_1 + a_2$ of (3.18), we obtain

$$R_1 = -R_2 = 4(a+1) \tag{4.1}$$

The electric field function $\omega(z)$ is readily given by (3.14), and we find

$$\omega(z) = \frac{1}{2}\left[\frac{a_1}{z} + \frac{a_2}{z-1} - (a+2)\frac{\sqrt{(z-x_1)(z-x_2)}}{z(z-1)}\right] \tag{4.2}$$

The branch points $x_1,\ x_2$, which correspond to the endpoints of the lineal distribution of Liouville charges, are given by an algebraic function of the charges a_i as follows

$$x_{1,2} = \frac{(a_1+2)a + 2 \pm 2\sqrt{(a_1+1)(a_2+1)(a+1)}}{(a+2)^2} \tag{4.3}$$

The distribution of charges is located on a quadratic branch cut curve $\mathcal{C}$ between the points $x_1,\ x_2$, and the lineal density of charges is given by

$$\rho(z) = \frac{1}{2\pi}(a+2)\frac{\sqrt{(z-x_1)(x_2-z)}}{z(1-z)} \tag{4.4}$$

where it is assumed that z is restricted to the branch cut curve $\mathcal{C}$.

When the charges a_i are all real and positive, we have

$$0 < x_1 < x_2 < 1 \tag{4.5}$$

and the branch cut curve $\mathcal{C}$ naturally lies on the real axis, between the points $x_1,\ x_2$, as required by the reality condition (3.21). In fact, the endpoints $x_1,\ x_2$ stay away from the fixed charges at 0 and 1.

When the charges a_i are extended into the complex plane, the end-points $x_1,\ x_2$ also move into the complex plane. In general, the location of the branch cut curve $\mathcal{C}$ is now not naturally given by symmetry arguments alone. Nonetheless, our formalism is still well-defined and the reality requirement of (3.21) determines the position of the branch cut curve $\mathcal{C}$ in a unique way. To find it, we require that the indefinite integral of $\rho(w)$ be real along the curve $\mathcal{C}$ parametrized by some real parameter t, as given in (3.22). Unfortunately, this equation is not easy to solve in any analytic way. However, we do not actually need the precise location of these curves to compute the final electro-static energy of the configuration.

To compute the integrals that yield the electro-static energy $\mathcal{E}_0$, in the correlation function (3.29), we need to evaluate integrals involving $dw\ \rho(w)$, rational functions of w and logarithms. It is standard procedure to *uniformize* the integral by making the following change of variables from w to a variable t (which is not necessarily real)

$$t \equiv \sqrt{\frac{w-x_1}{x_2-w}} \tag{4.6}$$

Clearly, the integration range where w goes from x_1 to x_2 converts into an integration range where t goes from 0 to ∞. Thus, a definite integral in w along the full branch cut may be converted as follows

$$\begin{aligned}\int_{x_1}^{x_2} dw\,\rho(w)f(w) =& \frac{a+2}{\pi}\int_0^\infty dt\left[\frac{1}{t^2+1}-\frac{x_1}{t^2+x_1/x_2}\right.\\ &\left.-\frac{1-x_1}{t^2+(1-x_1)/(1-x_2)}\right] f\left(\frac{x_2t^2+x_1}{t^2+1}\right)\end{aligned} \tag{4.7}$$

We find the following basic integral, valid for $x,\ y$ arbitrary complex variables, to be useful

$$\int_0^\infty dt\,\frac{1}{t^2+x}\ln(t^2+y) = \frac{\pi}{\sqrt{x}}\ln(\sqrt{x}+\sqrt{y}) \tag{4.8}$$

Combining all integrals, we obtain a closed expression for the electric potential

$$\begin{aligned}\Omega(z) =& (a+2)\ln(\sqrt{z-x_1}+\sqrt{z-x_2})/2\\ &-a_1\ln\frac{\sqrt{x_1}\sqrt{z-x_2}+\sqrt{x_2}\sqrt{z-x_1}}{\sqrt{x_1}+\sqrt{x_2}}\\ &-a_2\ln\frac{\sqrt{1-x_1}\sqrt{z-x_2}+\sqrt{1-x_2}\sqrt{z-x_1}}{\sqrt{1-x_1}+\sqrt{1-x_2}}\end{aligned} \tag{4.9}$$

Using the explicit expressions for $\Omega(w)$ in (3.30) we obtain the expression for the three point function as

$$\begin{aligned}\mathcal{E}_0 =& -2(\alpha s)^2\Big[(a_1+1)^2\ln(a_1+1)-a_1^2\ln a_1+(a_2+1)^2\ln(a_2+1)-a_2^2\ln a_2\\ &+(a+1)^2\ln(a+1)-(a+2)^2\ln(a+2)\Big]\\ =& 2(\alpha s)^2\sum_{i=1}^3\left[(a_i+1)^2\ln(a_i+1)-a_i^2\ln a_i\right]\end{aligned} \tag{4.10}$$

which agrees with the result obtained in [23]. In obtaining the last equality, we used the sum of the charges in (3.16). We note that the final formula is symmetric in the three "charges" a_i, as it should be.

4.2. The 4–Point Function

The electrostatics problem corresponding to the 4–point function is to find the electrostatic energy of the equilibrium configuration of the point charges a_i $(i = 1, 2, 3, 4)$ and the Liouville charge density $\rho(z)$. The location of the point charges may be taken to be $(z_1, z_2, z_3, z_4) = (z_1, 0, 1, \infty)$ using conformal invariance. The point z_1 and the Liouville charge density need to be determined from the condition of electrostatic equilibrium. Using the asymptotic conditions on the electric field function $\omega(z)$ given in (3.15), we may eliminate R_1 and R_2 in terms of the a_i and R_3. Making use of the definition $a = a_1+a_2+a_3$ of (3.18), we find

$$R_2 = 4 + 4a + (z_1 - 1)R_1 \qquad\qquad R_3 = -4 - 4a - z_1 R_1 \tag{4.11}$$

Unlike the case of the three point function, we can *not* determine all the R_i's solely from the asymptotics of $\omega(z)$. The density function $\rho(z)$ may be obtained as

$$\rho(z) = \frac{1}{2\pi} \frac{\sqrt{-Q_4(z)}}{z(z-1)(z-z_1)} \tag{4.12}$$

where

$$\begin{aligned} Q_4(z) &= (a+2)^2 \prod_{i=1}^{4} (z - x_k) \\ &= \big[a_1 z(z-1) + a_2(z-z_1)(z-1) + a_3(z-z_1)z\big]^2 \\ &\quad + z(z-1)(z-z_1)\big[(4+4a)(z-z_1) + R_1 z_1(1-z_1)\big] \end{aligned} \tag{4.13}$$

and

$$R_1 = -8\left[\frac{b_{12}}{z_1} + \frac{b_{13}}{z_1 - 1}\right] \tag{4.14}$$

The locations of the zeros of $\rho(z)dz$, $\{x_k | k = 1, 2, 3, 4\}$ may be determined *algebraically* as a function of z_1 by solving the fourth order polynomial equation $Q_4(z) = 0$, albeit cumbersome. These zeros, together with the reality condition (3.21) determine the support for the Liouville charge density $\mathcal{C}$.

The quadratic branch cuts may be chosen along the intervals $[x_1, x_2]$ and $[x_3, x_4]$; A- and B-periods of the elliptic curve may be taken as the contours around $[x_1, x_2]$ and the closed contour (along the first and second sheets) $[x_2, x_3]$.

The modulus τ of the curve is then determined from the periods, which are given in turn by the elliptic integrals

$$\tau = \frac{\int_B dz/\sqrt{Q_4(z)}}{\int_A dz/\sqrt{Q_4(z)}} \tag{4.15}$$

with the reality requirement on the A-period

$$\int_A dz/\sqrt{Q_4(z)} \ \text{ real} \tag{4.16}$$

An alternative determination of the modulus τ that does not require explicit knowledge of the roots of $Q_4(z)$ proceeds from the discriminant Δ of the curve $y^2 = Q_4(z)$. In general, the discriminant is given in terms of the roots x_i as follows

$$\Delta = 16 \prod_{\substack{i<j \\ i,j=1}}^{4} (x_i - x_j)^2 \tag{4.17}$$

The explicit expression for the zeros is however not needed, and Δ may be expressed as a polynomial in the coefficients of $Q_4(z)$. Equivalently, using a Möbius transformation, one of the zeros of $Q_4(z)$ may be moved to ∞ and the polynomial may be put in Jacobi standard form

$$\tilde{Q}_4(w) = 4w^3 - g_2 w - g_3 \tag{4.18}$$

so that $\Delta = g_2^3 - 27 g_3^2$. The coefficients g_2, g_3 may be related to the coefficients of the original polynomial $Q_4(z)$; let

$$Q_4(z) \equiv q_0 z^4 + 4q_1 z^3 + 6q_2 z^2 + 4q_3 z + q_4 \tag{4.19}$$

then

$$g_2 = q_0 q_4 + 3q_2^2 - 4q_1 q_3, \qquad g_3 = \det \begin{pmatrix} q_0 & q_1 & q_2 \\ q_1 & q_2 & q_3 \\ q_2 & q_3 & q_4 \end{pmatrix} \tag{4.20}$$

It is straightforward but cumbersome to reexpress g_2, g_3 in terms of physical quantities using (4.13) and we shall not do so here.

The discriminant then determines the modular invariant function J as follows

$$J = \frac{g_2^3}{g_2^3 - 27 g_3^2} \tag{4.21}$$

and J determines the modulus τ in terms of a hyperelliptic expression

$$\tau = e^{2\pi i/3} \frac{F(J) - \mu e^{i\pi/3} J^{1/3} \tilde{F}(J)}{F(J) - \mu e^{-i\pi/3} J^{1/3} \tilde{F}(J)} \tag{4.22}$$

where

$$F(J) \equiv {}_2F_1(\frac{1}{12}, \frac{1}{12}; \frac{2}{3}; J), \qquad \tilde{F}(J) \equiv {}_2F_1(\frac{5}{12}, \frac{5}{12}; \frac{4}{3}; J) \tag{4.23}$$

and

$$\mu \equiv (2 - \sqrt{3}) \frac{F(1)}{\tilde{F}(1)} = (2 - \sqrt{3}) \frac{\Gamma(11/12)^2 \Gamma(2/3)}{\Gamma(7/12)^2 \Gamma(4/3)} \tag{4.24}$$

While these expressions are not elementary, they are completely explicit.

The density may be obtained in terms of geometric objects as in (3.39)

$$dw\rho(w) = \sum_{i=1}^{N} a_i \tau_{z_i^+ z_i^-} + \gamma_1 \omega_1 \tag{4.25}$$

where

$$\gamma_1 = 2 \int_{x_1}^{x_2} dw\, \rho(w) \tag{4.26}$$

This determines γ_1 in terms of the external charges a_i's and R_1. Given the density $dw\rho$, it remains to integrate this function to obtain the correlation function as in (3.49) — numerically if necessary. We shall not attempt to do this here, but consider a simplified scattering configuration instead.

4.3. Symmetric Scattering : A Solvable Case

While the most general four point function does not seem to be calculable analytically, we will treat an illustrative example of a class of four point functions which is solvable algebraically. It corresponds to the case where the external momentum distributions are symmetrical, and involve only symmetric scattering.

We treat the case when the charge distribution of the electrostatics problem corresponding to the 4–point function has a $\mathbb{Z}_2$ symmetry. For the class of 4–point functions whose saddle points have this property, we may obtain the Liouville charge density $\rho(z)$ and the four point function algebraically. Without loss of generality, we may choose the point charges $\{a_i | i = 1, 2, 3, 4, \ a_3 = a_1\}$ to be at $(z_1, z_2, z_3, z_4) = (-1, 0, 1, \infty)$. The $\mathbb{Z}_2$ symmetry operation interchanges z and $-z$. This class of charge configuration spans a subspace with complex codimension two of the full configuration space. The external momenta need to satisfy

$$k_1 k_2 = k_2 k_3 \tag{4.27}$$

Since $\beta_1 = \beta_3$, particles 1 and 3 should be regarded as both incoming and (4.27) corresponds to the condition of transverse scattering in the center of mass frame.

From the asymptotics of $\omega(z)$, (3.15), and the $\mathbb{Z}_2$ symmetry we obtain R_i's as

$$R_2 = 0, \qquad R_1 = -R_3 = 2\,(a+1) \tag{4.28}$$

Th is allows us to determine the zeros of the $\rho(z)$ and hence $\rho(z)$ itself as

$$\rho(z) = \frac{1}{2\pi}(a+2)\frac{\sqrt{(z^2 - x_1^2)(x_2^2 - z^2)}}{z(1-z^2)} \tag{4.29}$$

where

$$x_1^2, x_2^2 = \frac{a(a_2+2) + 2 \pm \sqrt{2(a_1+1)(a_2+1)(a+1)}}{(a+2)^2} \tag{4.30}$$

The support for the density, $\mathcal{C}$ is determined by the zeros of $\rho(z)$ and the condition (3.22). In particular, for any real values of a_1, a_2 (not necessarily positive) it may be shown that $0 \le x_1, x_2 \le 1$ so that the support for ρ is $\mathcal{C} = [-x_2, -x_1] \cup [x_1, x_2]$. We note that this particular class of the 4–point function is identical to the three point function except that a_1 and z are "doubled" to $2a_1$ and z^2. There are two further conditions that need to be satisfied for consistency; namely R_2 defined by the integral as in (3.27) needs to agree with R_2 obtained above from the asymptotics of $\omega(z)$ and the integral of $\rho(w)$ over $\mathcal{C}$ needs to be one. A straightforward analysis suffices to show that these identities are indeed satisfied.

Once we have obtained the density $\rho(w)$, we may proceed to compute the 4–point function using (3.30). The computation may be carried out completely analytically as in the 3–point function case:

$$\begin{aligned}
\mathcal{E}_0 = -2(\alpha s)^2 \Big[& (2a_1+1)^2 \ln(2a_1+1) - 2a_1^2 \ln(2a_1) + (a_2+1)^2 \ln(a_2+1) - a_2^2 \ln a_2 \\
& + (a+1)^2 \ln(a+1) - (a+2)^2 \ln(a+2) \Big] \\
= -2(\alpha s)^2 \Big[& (2a_1+1)^2 \ln(2a_1+1) - 2a_1^2 \ln(2a_1) + \sum_{i=2,4} \left((a_i+1)^2 \ln(a_i+1) - a_i^2 \ln a_i \right) \Big]
\end{aligned} \tag{4.31}$$

The above result may be naturally be continued to the complex domain analogously to the three point function case. The electrostatic energy behaves linearly as a function of the Mandelstam variables u_{ij}, in the high energy limit, and thus the 4-point amplitude for symmetrical scattering behaves exponentially as a function of the Mandelstam variables. This behavior is the one obtained for the high energy limit of critical strings, yet the precise behavior of the two amplitudes is clearly different. Also, even though exponential behavior was established through explicit formulas only for the symmetric 4-point function, it is clear that the expression for the general 4-point function also exhibits exponential behavior of the amplitudes. We shall expand further on this behavior in the next section.

5. Discussion

In the present paper, we have derived explicit formulas for the high energy limit of non-critical string theory, on worldsheets with the topology of the sphere. It is found that, for generic values of the matter central charge c, the amplitudes behave exponentially in the Mandelstam variables u, for large energy.

$$\langle \prod_{i=1}^{N} \mathcal{V}_i \rangle \sim \kappa_1 e^{\kappa_0 u} \tag{5.1}$$

This behavior is analogous to that of critical strings at high energies. However, the precise argument κ_0 of the exponential is different than in the critical case. In general, the large energy limit is given in terms of hyper-elliptic integrals, which, as we showed here, can be described concretely. Many very interesting questions remain, which we shall briefly address below.

(1) While the high energy asymptotic behavior of the non-critical amplitudes is generically exponential, it is in principle possible that the argument of the exponential κ_0, or that the prefactor in front of the exponential κ_1 vanishes at certain special values of the central charge c. We have shown, by explicit calculation, that the argument of the exponential κ_0 does not vanish for the 4-point function of a symmetrical scattering process. By analyticity, it is then expected that the argument will in fact be generically non-zero for **any** value of the central charge.

Cancellations of the prefactor κ_1 on the contrary are expected to occur at certain isolated values of the central charge c, such as at $c = 1$, where the behavior of the amplitude is known to be polynomial in the Mandelstam variables. Within the limitations of the leading behavior of the high energy limit of the scattering amplitudes, this prefactor κ_1 is not accessible. Instead, its study would require going to next order and studying the small oscillation problem around the semi-classical dominant configuration. Zeros in κ_1 may then occur through the existence of zero modes of the small oscillation problem (which would be relatively easy to see). They could also arise through the vanishing of the regularized functional determinant of the small oscillation problem, such as could occur in ζ-function regularization (which would be much more difficult to see).

(2) In this paper, we have calculated explicitly only in the case of tree level amplitudes. The higher loop case will be dealt with in a forthcoming publication. We shall just point out here that, in the case of higher loop amplitudes, the high energy limit of the amplitudes again corresponds to an electro-statics problem, this time on a higher

genus Riemann surface. On physical grounds, it is easy to argue that the presence of the Liouville interaction will again produce an infinite number of charges, just as in the tree level case. The Liouville charges accumulate onto curve segments which produce quadratic branch cuts on the Riemann surface. This may be seen, for example, by using the method of images to represent the charges on higher genus surfaces, or may be worked out directly in terms of the prime form on the surface. From these considerations, it is to be expected that the general behavior of the loop amplitudes is again exponential in the Mandelstam variables at high energy.

(3) Finally, the questions of practical importance, such as the implications of our results for the 3-D Ising model and off-shell string theory remain to be worked out in a concrete way. We expect that the high energy limit, derived here, can be used as the starting point for a systematic expansion to physical correlation functions. Clearly, much work remains to be done to achieve this goal, but we regard our results so far as a promising step in the right direction.

Acknowledgments:

We are happy to thank Yoshihisa Kitazawa and Norisuke Sakai for collaboration in the early stages of this research, and we are grateful to Jean-Loup Gervais for discussions of his work on Liouville theory. We have benefited from conversations with S. Higuchi, S. Nussinov, D.H. Phong and H. Sonoda. We thank the Aspen Center for Physics, and the CERN theory division, for their hospitality while part of this work was being carried out. We acknowledge support from a US/Japan exchange grant INT-9315099 from the National Science Foundation and the Japan Society for the Promotion of Science. E. D. was supported in part by the National Science Foundation, under grants PHY-92-18990 and PHY-95–31023, while K. A. was supported in part by the Ministry of Education, Science, Sports and Culture for Grant–in–Aid and Keio University.

Appendix A. 3-Point Function by Jacobi Polynomials

For the case of the 3-point function ($N = 3$), the electro-statics problem may be solved explicitly for any finite number s, and general complex valued charges. The method is a generalization to complex charges of a technique presented in [34]. One introduces the polynomial $P_s(w)$ with roots w_p

$$P_s(w) = \prod_{p=1}^{s} (w - w_p) \tag{A.1}$$

We show that this polynomial satisfies the differential equation for Jacobi polynomials

$$(1 - w^2)y'' + (\beta - \alpha - (\alpha + \beta + 2)w)y' + s(s + \alpha + \beta + 1)y = 0 \tag{A.2}$$

in the following way. Since P_s is of degree s, the polynomial on the left hand side of (A.2) obtained by setting $y(w) = P_s(w)$, is at most of degree s. Thus, it suffices to check that it vanishes on s points, which we choose to be w_p, $p = 1, \cdots, s$. At these points, the third term in y vanishes; working out the derivatives using (A.1), and then using (A.2), we have

$$\frac{P_s''(w_p)}{P_s'(w_p)} = \sum_{q \neq p}^{s} \frac{1}{w_p - w_q} = \frac{-1 - \alpha}{w_p - 1} + \frac{-1 - \beta}{w_p + 1} \tag{A.3}$$

The above equation can be identified with the electro-statics equilibrium equation for the w_p for three external points $(z_1, z_2, z_3) = (-1, 1, \infty)$ and charges related by

$$\alpha = -1 + \frac{a_1 s}{2}, \qquad \beta = -1 + \frac{a_2 s}{2} \tag{A.4}$$

Thus, the Liouville charges at w_p are located at the zeros of the Jacobi polynomials $P_s(w) = P_s^{(\alpha\ \beta)}(w)$. Jacobi polynomials are perfectly well-defined for general complex α and β. For α, β real and > -1, the zeros are real and between -1 and 1; this is in fact the case originally considered in [34] of all repulsive charges. For general complex values of the charges, the zeros move into the complex plane. For large s however, the zeros always accumulate on a curve of charge density, converging towards $\rho(w)dw$ of (4.4).

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