

# INTEGRABILITY vs. SUPERSYMMETRY

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## ABSTRACT

We investigate (1,0)-superconformal Toda theories based on simple Lie algebras and find that the classical integrability properties of the underlying bosonic theories do not survive. For several models based on algebras of low rank, we show explicitly that none of the conserved  $\mathcal{W}$ -algebra generators can be generalized to the supersymmetric case. Using these results we deduce that at least one  $\mathcal{W}$ -algebra generator fails to generalize in any model based on a classical Lie algebra. This argument involves a method for relating the bosonic Toda theories and their conserved currents within each classical series. We also scrutinize claims that the (1,0)-superconformal models actually admit (1,1) supersymmetry and find that they do not. Our results are consistent with the belief that all integrable Toda models with fermions arise from Lie superalgebras.

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## 1. Introduction

Bosonic Toda theories [1] have been studied for a number of years as important examples of integrable field theories in two dimensions; see eg. [2,3,4] for introductions to some aspects relevant to this paper. The problem of incorporating fermions in the Toda construction has been considered by many authors [5-17] with much of the attention focused on finding supersymmetric models. One conclusion which has emerged is that the bosonic Toda models based on simple Lie algebras (or their Kac-Moody extensions) cannot be supersymmetrized, except in the simplest case of Liouville (or sinh-Gordon) theory. More precisely, it is believed that there is no  $N = 1$  supersymmetric theory whose bosonic part is a Toda model based on a *simple* Lie algebra of rank bigger than one. To find integrable Toda theories with fermions, a more radical generalization of the bosonic construction is needed in which the underlying Lie algebra is replaced with a Lie superalgebra. Integrability of these models can be understood in a uniform way by casting the field equations as a zero-curvature condition from which one can extract conserved quantities.

The possibility of supersymmetrizing the bosonic Toda models was reconsidered recently in an interesting paper by Papadopoulos [18]. Working within the general framework of sigma-models with a potential term [19], he pointed out that the conformal Toda models based on simple Lie algebras all admit (1,0)-supersymmetric extensions. This means that the theories in question possess a single conserved supercharge of definite two-dimensional chirality, as opposed to (1,1) supersymmetry, which would involve supercharges of both chiralities. It is this second possibility which we referred to above simply as  $N = 1$  supersymmetry and which is believed to be ruled out.

Since the new (1,0) models are supersymmetric extensions of integrable theories it is tempting to think that they too must be integrable. However, it seems that there is no obvious way to write the equations of motion as a zero-curvature condition of the type that guarantees integrability in the bosonic cases. Another puzzle emerges when one examines the known properties of extended chiral algebras appearing in conformal field theory [3]. The conserved quantities in the bosonic Toda models form  $\mathcal{W}$ -algebras—the models based on  $A_n$ , for example, realize the algebras usually referred to as  $\mathcal{W}_{n+1}$ . The (1,0) models, if similarly integrable, would be expected to contain supersymmetric generalizations of these  $\mathcal{W}$ -algebras which should exist at both the classical and quantum levels. Minimal supersymmetric versions of the  $\mathcal{W}_n$  algebras have been constructed [20], but they are “exotic” in the terminology of [3] (“non-deformable” in the terminology of [21]) meaning that they are associative for only a finite set of values of the central charge and therefore have no classical limit.

These observations suggest that the (1,0)-superconformal models are either not integrable, or else they must contain some much larger and more complicated chiral algebra structure in which the bosonic  $\mathcal{W}$ -algebras are embedded. In this paper we aim to settle this issue by showing that the process of supersymmetrizing destroys the  $\mathcal{W}$ -symmetry present in the bosonic models. These are the first examples we know of in which supersymmetry has this destructive effect on integrability. Our arguments are based on knowledge of the bosonic Toda theories together with explicit calculations for (1,0) models corresponding to low rank algebras. We then show how these facts can be put together to draw conclusions about the (1,0) models based on general classical algebras.

## 2. Bosonic conformal Toda models

We recall some details of bosonic Toda theories that will be needed later. Let  $X_n$  be any simple, finite-dimensional Lie algebra of rank  $n$ , and  $\alpha_i$  ( $i = 1, \dots, n$ ) a set of simple roots. The  $X_n$  Toda model can be defined in two-dimensional Minkowski space by a Lagrangian

$$L = \partial\phi \cdot \bar{\partial}\phi - \sum_i \exp(\alpha_i \cdot \phi) \quad (2.1)$$

where  $\phi(z, \bar{z})$  is a field taking values in the Cartan subalgebra of  $X_n$  and a dot denotes the invariant inner-product.<sup>3</sup> We use light-cone coordinates  $z = \frac{1}{2}(t - x)$  and  $\bar{z} = \frac{1}{2}(t + x)$  and we shall refer to quantities depending solely on  $z$  or  $\bar{z}$  as holomorphic or anti-holomorphic respectively.

The Lagrangian above is invariant under the classical conformal transformations

$$\phi(z, \bar{z}) \rightarrow \phi(w, \bar{w}) + \rho \log(\partial w \bar{\partial} \bar{w}) \quad (2.2)$$

where  $w(z)$ ,  $\bar{w}(\bar{z})$  are independent real-analytic functions and the vector  $\rho$  is defined by the property  $\rho \cdot \alpha_i = 1$  for each simple root. The corresponding energy-momentum tensor is traceless, with holomorphic and anti-holomorphic components

$$T = \frac{1}{2} \partial\phi \cdot \partial\phi - \rho \cdot \partial^2\phi, \quad \bar{T} = \frac{1}{2} \bar{\partial}\phi \cdot \bar{\partial}\phi - \rho \cdot \bar{\partial}^2\phi. \quad (2.3)$$

Recall that a quantity  $U(z, \bar{z})$  is said to have scaling-dimensions  $(h, \bar{h})$  if under the transformations  $w = \mu z$  and  $\bar{w} = \bar{\mu} \bar{z}$  it behaves as  $U(z, \bar{z}) \rightarrow \mu^h \bar{\mu}^{\bar{h}} U(\mu z, \bar{\mu} \bar{z})$  where  $\mu, \bar{\mu}$  are positive real numbers. We shall refer to objects with scaling dimension  $(q, 0)$  or  $(0, q)$  as being of spin  $q$ . Notice that the field  $\phi$  does not have definite scaling-dimensions, but that  $\partial\phi$  and  $\bar{\partial}\phi$  have scaling-dimensions  $(1, 0)$  and  $(0, 1)$  respectively. In addition we can construct the potential-like terms  $\exp(\kappa \alpha_i \cdot \phi)$  which have scaling-dimensions  $(\kappa, \kappa)$ .

The  $A_1$  Toda model is just the Liouville theory, for which conformal invariance alone is sufficient to establish integrability. For rank  $n > 1$ , however, integrability of the  $X_n$  model depends on the presence of higher-spin conserved quantities which, together with the energy-momentum tensor, form a  $\mathcal{W}$ -algebra. The simplest example is the  $A_2$  Toda theory in which there is a spin-3 holomorphic current  $W$  in addition to the spin-2 current  $T$ . Explicitly,

$$\begin{aligned} T &= \frac{1}{3} ((\partial\phi_1)^2 + (\partial\phi_2)^2 + \partial\phi_1 \partial\phi_2) - (\partial^2\phi_1 + \partial^2\phi_2) \\ W &= \frac{1}{27} (2\partial\phi_1 + \partial\phi_2)(2\partial\phi_2 + \partial\phi_1)(\partial\phi_1 - \partial\phi_2) \\ &\quad - \frac{1}{3} (\partial\phi_1 \partial^2\phi_1 - \partial\phi_2 \partial^2\phi_2) + \frac{1}{6} (\partial\phi_1 \partial^2\phi_2 - \partial\phi_2 \partial^2\phi_1) + \frac{1}{6} (\partial^3\phi_1 - \partial^3\phi_2) \end{aligned}$$

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<sup>3</sup> It is customary to include an explicit dimensionless coupling  $\beta$  but we have chosen to absorb this in the field  $\phi$ . The normalization of the kinetic terms is also non-standard, to allow a simpler comparison with the supersymmetric models of the next section.

where  $\phi_i = \alpha_i \cdot \phi$ . There are analogous quantities in the anti-holomorphic sector.

To construct conserved currents systematically in a general  $X_n$  Toda model, the equations of motion can be written as the zero-curvature condition for an auxiliary, two-dimensional gauge field with values in  $X_n$ . From this gauge field one can then construct a Lax operator which in the simplest cases<sup>4</sup> can be written

$$\mathcal{L}(X_n) = \prod_{\lambda} (\partial + \lambda \cdot \partial\phi) = \sum_r Y_r \partial^r \quad (2.4)$$

where  $\lambda$  are the weights of the fundamental representation of  $X_n$  taken in order of increasing height. The equations of motion imply  $[\bar{\partial}, \mathcal{L}(X_n)] = 0$ , and the coefficients  $Y_r$  written above are exactly the conserved quantities which guarantee integrability of the model. The spins of the conserved currents obtained in this way for the models based on the classical algebras are

$$A_n : 2, 3, \dots, n, n+1 \quad B_n \& C_n : 2, 4, \dots, 2n \quad D_n : 2, 4, \dots, 2n-2, n \quad (2.5)$$

For more details, see [2] and references therein.

The pattern of spins above reflects the fact that it is possible to regard the Toda model based on  $X_n$  as embedded in some sense in the Toda theory based on  $X_{n+1}$ . Conversely, there is a precise way in which we can truncate the  $X_n$  Toda theory to the model one lower in the series based on  $X_{n-1}$  ( $X = A, B, C$  or  $D$ ). This idea will prove important later, so we discuss how it works in more detail.

To truncate the  $X_n$  Toda model to the  $X_{n-1}$  model we must discard one of the Toda fields, but we must do this in a consistent fashion taking due account of the exponential interactions. First, let us agree to label the simple roots of the algebras of each type  $X$  in such a way that we can regard the roots of  $X_{n-1}$  as a subset of those of  $X_n$ . Now let  $\omega$  be the highest weight of the fundamental representation of  $X_n$ . Our choice of labelling means that  $\omega \cdot \alpha_n = \alpha_n^2/2$  and  $\omega \cdot \alpha_i = 0$  for  $i \neq n$ . Next set  $\phi = -k\omega + \tilde{\phi}$  where  $k$  is a constant and  $\tilde{\phi}$  is defined to be orthogonal to  $\omega$ , ie. to be a linear combination of the simple roots of  $X_{n-1}$ . It is easy to check that if  $\phi$  satisfies the  $X_n$  Toda equations, then  $\tilde{\phi}$  satisfies the  $X_{n-1}$  Toda equations in the limit  $k \rightarrow \infty$ . Intuitively this corresponds to taking the component of  $\phi$  along  $\omega$  to be equal to its classical vacuum value.

Now that we have a precise notion of how to truncate the Toda models  $X_n \rightarrow X_{n-1}$ , let us see how this affects the conserved currents. Since the currents depend only on derivatives of  $\phi$ , they are independent of  $k$  and have well-defined limits when we truncate by taking  $k \rightarrow \infty$ . A conserved current of spin  $q$  in the  $X_n$  model clearly descends to a conserved current of spin  $q$  in the  $X_{n-1}$  model. It might become trivial in the truncated theory, of course, in the sense that it might be possible to write it as a combination of lower-spin conserved quantities. Indeed, this must happen when there is no independent conserved quantity of spin  $q$  in the truncated model. For example, the spin-4 current in the  $A_3$  model must descend to a spin-4 current in the  $A_2$  model but, since there is no independent conserved quantity of spin-4, it can only be a combination of  $T^2$ ,  $\partial^2 T$  and  $\partial W$ .

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<sup>4</sup> This form for  $\mathcal{L}$  holds for the algebras  $A, B, C, G$  which have no weight degeneracies in their fundamental representations. For the remaining algebras there are some additional factors of  $\partial^{-1}$  which make  $\mathcal{L}$  a pseudo-differential operator. Conserved quantities can be found in the same way, however.

In fact it can be shown that all the currents occurring in the sequences given in (2.5) remain non-trivial under truncation whenever possible, ie. any current of spin  $q$  will reduce to something non-trivial, provided there is an independent charge of spin  $q$  in the truncated model. This can be established by considering the effect of truncation on the Lax operator (2.4). Each  $\mathcal{L}(X_n)$  has a well-defined limit when we take  $k \rightarrow \infty$  and, because  $\omega$  is the highest weight of the fundamental representation, it is easy to relate this limit to  $\mathcal{L}(X_{n-1})$ . We find  $\mathcal{L}(A_n) \rightarrow \mathcal{L}(A_{n-1})\partial$  and  $\mathcal{L}(X_n) \rightarrow \partial\mathcal{L}(X_{n-1})\partial$  for  $X = B, C$  or  $D$ . It is therefore a relatively simple matter to keep track of how the conserved quantities are related under truncation and so verify the claim. We will need this result later, specifically for the currents of spins 3 and 4.

### 3. (1,0)-superconformal Toda models

To write down a (1,0) supersymmetric extension of the bosonic  $X_n$  Toda model we use (1,0) superspace. Let  $\theta$  be a real fermionic coordinate transforming as a spinor of definite two-dimensional chirality, which will serve as a superpartner of  $z$ , and define the corresponding superderivative by  $D = \partial_\theta - i\theta\partial$ . Consider the superspace Lagrangian

$$L = iD\Phi \cdot \bar{\partial}\Phi + \sum_a \{ \Psi_a D\Psi_a + 2\Psi_a \exp(\frac{1}{2}\alpha_a \cdot \Phi) \} \quad (3.1)$$

where the real scalar superfield  $\Phi$  takes values in the Cartan subalgebra of  $X_n$  and the fermions  $\Psi_a$  ( $a = 1, \dots, n$ ) are Majorana spinors of the same chirality as  $\theta$  (we write the  $a$  index explicitly because we do not necessarily wish to identify the  $n$ -dimensional space of fermions which it labels with the  $n$ -dimensional Cartan subalgebra). To reduce this manifestly (1,0)-supersymmetric action to components we expand the superfields

$$\Phi = \phi + i\theta\lambda, \quad \Psi_a = \psi_a + \theta\sigma_a, \quad (3.2)$$

and eliminate the scalar auxiliary fields  $\sigma_a$  to arrive at the Lagrangian

$$L = \partial\phi \cdot \bar{\partial}\phi + i\lambda \cdot \bar{\partial}\lambda + \sum_a \{ i\psi_a \partial\psi_a - \exp(\alpha_a \cdot \phi) + i(\alpha_a \cdot \lambda)\psi_a \exp(\frac{1}{2}\alpha_a \cdot \phi) \} \quad (3.3)$$

with equations of motion

$$\begin{aligned} \partial\bar{\partial}\phi &= - \sum_a \frac{1}{2}\alpha_a \exp(\alpha_a \cdot \phi) + \sum_a \frac{1}{4}i(\alpha_a \cdot \lambda)\psi_a \alpha_a \exp(\frac{1}{2}\alpha_a \cdot \phi), \\ \bar{\partial}\lambda &= - \sum_a \frac{1}{2}\psi_a \alpha_a \exp(\frac{1}{2}\alpha_a \cdot \phi), \\ \partial\psi_a &= \frac{1}{2}(\alpha_a \cdot \lambda) \exp(\frac{1}{2}\alpha_a \cdot \phi). \end{aligned} \quad (3.4)$$

When the fermions  $\lambda$  and  $\psi_a$  are set to zero we recover the action and equations of motion for the bosonic  $X_n$  Toda theory. We shall refer to the (1,0)-supersymmetric extension as the  $SX_n$  Toda model.

The  $SX_n$  Toda theory is conformally-invariant, with the bosons transforming as in (2.2) and fermions transforming as

$$\lambda(z, \bar{z}) \rightarrow (\partial w)^{1/2}\lambda(w, \bar{w}), \quad \psi(z, \bar{z}) \rightarrow (\bar{\partial}\bar{w})^{1/2}\psi(w, \bar{w}). \quad (3.5)$$

Notice that  $\lambda$  and  $\psi_a$  have opposite chirality and scaling dimensions  $(1/2, 0)$  and  $(0, 1/2)$  respectively. In the holomorphic sector the conformal invariance is extended to superconformal invariance by the supersymmetry transformations

$$\delta\phi = i\epsilon\lambda, \quad \delta\lambda = -\epsilon\partial\phi, \quad \delta\psi_a = -\epsilon\exp(\tfrac{1}{2}\alpha_a \cdot \phi), \quad (3.6)$$

where  $\epsilon$  is a real spinor. Corresponding to these symmetries, we have spin-2 holomorphic and anti-holomorphic components of the energy-momentum tensor and a spin-3/2 holomorphic supersymmetry generator:

$$\begin{aligned} T &= \tfrac{1}{2}\partial\phi \cdot \partial\phi + \tfrac{i}{2}\lambda \cdot \partial\lambda - \rho \cdot \partial^2\phi, & \bar{T} &= \tfrac{1}{2}\bar{\partial}\phi \cdot \bar{\partial}\phi + \tfrac{i}{2}\psi_a \bar{\partial}\psi_a - \rho \cdot \bar{\partial}^2\phi, \\ G &= \tfrac{1}{2}\lambda \cdot \partial\phi - \rho \cdot \partial\lambda. \end{aligned} \quad (3.7)$$

It is easy to check using the equations of motion that these are conserved.

We now turn to the question of whether these (1,0) models admit an additional supersymmetry of the opposite chirality, which would mean that they were actually (1,1)-supersymmetric. For this to be the case the fermions  $\psi_a$  must be regarded as living in the tangent bundle to the sigma-model target manifold. The indices  $a$  appearing on  $\psi_a$  must then be taken as labeling a basis for the Cartan sub-algebra of  $X_n$ , and notice that this basis has already been assumed to be *orthonormal*. In our case (a flat target manifold with no torsion) the condition for (1,1) supersymmetry given in [18,19] is that the superpotential terms  $\exp(\tfrac{1}{2}\alpha_a \cdot \phi)$  should be the components of the exterior derivative of some function  $f$  in the orthonormal basis labeled by  $a$ . Now it is natural to write these terms as functions of the coordinates  $\phi_i = \alpha_i \cdot \phi$  on the target manifold, but this coordinate basis is *not* orthonormal. By definition, the two bases are related by the matrix consisting of the components of the vectors  $\alpha_i$  with respect to the orthonormal system:  $M_{ai} = (\alpha_i)_a$ . This matrix satisfies  $\sum_a M_{ai}M_{aj} = \alpha_i \cdot \alpha_j$ , which is the symmetrized Cartan matrix, and we deduce that  $M_{ai}$  is never diagonal unless it has rank one. For (1,1) supersymmetry we now require the existence of a function  $f(\phi_i)$  for which  $\partial f/\partial\phi_i = \sum_j M_{ij} \exp \tfrac{1}{2}\phi_j$ . Clearly no such function exists unless we are dealing with the case of a single scalar field. We conclude that the (1,0)-superconformal models are not (1,1) supersymmetric, except in case based on  $A_1$  which gives the super-Liouville theory. One can also check directly that there is no anti-holomorphic spin-3/2 quantity analogous to  $G$  which could serve as the current corresponding to a second supersymmetry.<sup>5</sup>

#### 4. Higher-spin quantities: low-rank superconformal models

We saw above that in each of the (1,0)-superconformal Toda theories there are holomorphic and anti-holomorphic spin-2 components of the energy-momentum tensor which differ from the bosonic expressions (2.3) by modifications involving fermions. We also saw that in the holomorphic sector the energy-momentum tensor acquires a conserved spin-3/2 superpartner  $G$ . Consider now how this might generalize to the higher-spin conserved quantities present in the bosonic Toda models.

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<sup>5</sup> It was claimed in [18] that all the (1,0) superconformal models are actually (1,1) supersymmetric. The arguments above lead us to disagree with this conclusion. We also find that the additional supercharge proposed in eqn. (21) of [18] is not conserved.

The simplest example is the spin-3 current  $W$  in the  $A_2$  Toda theory. If the (1,0)-extension of this model is to be integrable, it is natural to expect that  $W$  will remain conserved after a suitable modification by terms involving fermions, and it is also natural to expect that it too will acquire a superpartner, this time of spin 5/2. Assume for the moment that any such conserved currents must be polynomials in the fields  $\partial\phi$ ,  $\lambda$  and their holomorphic derivatives. Then it is easy to see that the most general spin-5/2 quantity is a linear combination of the terms

$$\lambda_i \partial\phi_j \partial\phi_k, \quad \lambda_i \partial^2\phi_j, \quad \partial\lambda_i \partial\phi_j, \quad \partial^2\lambda_i, \quad \lambda_i \lambda_j \partial\lambda_k$$

where  $\phi_i = \alpha_i \cdot \phi$  and  $\lambda_i = \alpha_i \cdot \lambda$ . We can look similarly for the most general spin-3 quantity which could arise in modifying  $W$ , and we find the candidate terms

$$\lambda_i \lambda_j \partial\phi_k \partial\phi_l, \quad \lambda_i \lambda_j \partial^2\phi_k, \quad \lambda_i \partial\lambda_j \partial\phi_k, \quad \lambda_i \partial^2\lambda_j, \quad \partial\lambda_i \partial\lambda_j$$

After some lengthy computations, whose details we omit, we find that even when the terms above are taken into account the only holomorphic quantity of spin 5/2 is  $\partial G$  and the only holomorphic quantity of spin 3 is  $\partial T$ . The conclusion is that the  $\mathcal{W}$ -symmetry of the bosonic  $A_2$  theory does not survive in the  $\mathcal{SA}_2$  model.

To clarify our assumptions concerning the nature of the conserved currents, consider the following list of quantities with definite scaling-dimensions which arise in the general  $\mathcal{SX}_n$  Toda model:–

$$\partial\phi : (1, 0), \quad \bar{\partial}\phi : (0, 1), \quad \exp(\kappa\phi_i) : (\kappa, \kappa), \quad \lambda : (\tfrac{1}{2}, 0), \quad \psi_a : (0, \tfrac{1}{2}). \quad (4.1)$$

Our aim is to find the most general holomorphic and anti-holomorphic currents of definite spin which can be constructed as polynomials in the quantities above and in their derivatives. We will not attempt to justify this starting point in any more detail except to say that we consider it to be a rather weak condition on the composition of the currents. We are interested in conserved quantities with scaling-dimensions  $(q, 0)$  or  $(0, q)$ ; but there are many ways of forming non-trivial expressions such as  $\partial\phi_i \bar{\partial}\phi_j \exp(-\phi_k)$  or  $\lambda_i \psi_a \exp(-\frac{1}{2}\phi_k)$  which have  $h = \bar{h} = 0$  and which might therefore be expected to appear with arbitrary powers in the conserved quantities we seek. The crucial point is that such terms can never appear as part of a holomorphic or anti-holomorphic current.

We claim that a holomorphic current of type  $(q, 0)$  can only arise as a polynomial in quantities (4.1) and their derivatives which *all* have  $\bar{h} = 0$ ; similarly an anti-holomorphic current of type  $(0, q)$  can only arise as a combination of these quantities which *all* have  $h = 0$ . To see why this is true, consider the holomorphic case. When we apply  $\bar{\partial}$  to any term involving  $\phi$ ,  $\psi_a$ , or their anti-holomorphic derivatives, the result cannot be simplified using the equations of motion (3.4) and the maximum power of  $\bar{\partial}$  appearing in any term is always increased. By contrast, when  $\bar{\partial}$  is applied to expressions involving only  $\partial\phi$ ,  $\lambda$  and their *holomorphic* derivatives, the results *can* be simplified using (3.4) and there is a chance that they will conspire to cancel completely. It is worth noting that these arguments apply just as well to the purely bosonic Toda models and that they provide one way of understanding why only derivatives of  $\phi$  appear in the Lax operators and conserved currents. This establishes our claim and justifies our assumption concerning the composition of the holomorphic quantities which might appear in the the  $\mathcal{SA}_2$  model. The same assumptions now apply to all the  $\mathcal{SX}_n$  models.

We carried out the calculations for the  $\mathcal{SA}_2$  model by hand, but to explore the situation for other low-rank algebras it quickly becomes necessary to use an algebraic manipulation package [22]. Proceeding in a similar way, we searched for all (anti-)holomorphic quantities with spins  $5/2$ ,  $3$ ,  $7/2$  or  $4$  for each of the classical algebras with rank  $n \leq 4$  (there are nine independent cases). For the reasons explained above, we assumed that the currents were polynomial in  $\partial\phi$ ,  $\lambda$  ( $\bar{\partial}\phi$ ,  $\psi_a$ ) and their (anti-)holomorphic derivatives. In each case, we found only those expressions that could be written as combinations of  $G$ ,  $T$ ,  $\bar{T}$  and their derivatives—and as a useful check of the correctness of our calculations we should note that we found *all* such expressions. In the holomorphic sector, for example, we found only

$$\partial G, \quad \partial T, \quad GT, \quad \partial^2 G, \quad \partial^2 T, \quad T^2, \quad G\partial G.$$

We conclude that the bosonic  $\mathcal{W}$ -algebra structure is completely destroyed in these supersymmetric theories.

## 5. Higher-spin quantities: general superconformal models

Having discussed some examples based on algebras of low rank, let us see what can be inferred about the theories based on general classical algebras. For definiteness we focus on the  $A_n$  series. We contend that knowledge of the  $\mathcal{SA}_2$  case is enough to deduce that the  $\mathcal{W}$ -algebra structure is spoiled in each of the  $\mathcal{SA}_n$  Toda models. More precisely, we will show that there is no conserved spin-3 current in the  $\mathcal{SA}_n$  model which reduces to the spin-3 current of the  $A_n$  model when the fermions are set to zero.

We first generalize the truncation procedure of section 2 to the (1,0)-superconformal models. Starting with the  $\mathcal{SA}_n$  theory, we proceed just as before for the bosonic fields but in addition we discard superfluous fermions by demanding that  $\lambda$  lie in the space of roots of  $A_{n-1}$ , setting  $\psi_n = 0$ , and checking that this is compatible with the equations of motion (3.4). This defines a consistent truncation of models  $\mathcal{SA}_n \rightarrow \mathcal{SA}_{n-1}$ .

Suppose there were some spin-3 holomorphic current  $\tilde{W}$  in the  $\mathcal{SA}_n$  model which reduced to  $W$ , the spin-3 current of the  $A_n$  model, when all fermions vanished. If we make the repeated truncation  $\mathcal{SA}_n \rightarrow \mathcal{SA}_2$  then  $\tilde{W}$  must still be a non-trivial holomorphic current, because we know from the arguments of section 2 that its purely-bosonic part  $W$  remains non-trivial under the bosonic truncation  $A_n \rightarrow A_2$ . But we have already shown that there is no independent holomorphic spin-3 current in the  $\mathcal{SA}_2$  model. Hence there is no such current in the  $\mathcal{SA}_n$  model either. We can obviously deduce in a similar fashion that there is no independent holomorphic spin-4 current in the general  $\mathcal{SA}_n$  Toda model because we have shown that there is no such current in the  $\mathcal{SA}_3$  theory.

It is important to emphasize that these arguments forbid the existence of just those higher-spin currents which reduce to the bosonic Toda currents when the fermions vanish, and knowledge of the bosonic case clearly plays a crucial role. For instance, we cannot immediately deduce that there are no spin-5/2 conserved quantities in the general  $\mathcal{SA}_n$  theory, despite having checked explicitly that they are absent in the  $\mathcal{SA}_2$ ,  $\mathcal{SA}_3$  and  $\mathcal{SA}_4$  models—we cannot be sure that there is not a conserved spin-5/2 quantity in some higher-rank model which becomes trivial on truncation to these theories. Having said that, it seems unlikely to us that such quantities exist, since we have no reason to suspect that the higher-rank Toda theories should behave qualitatively differently from the lower-rank ones for which we have carried out explicit calculations.



The arguments used above for the  $A_n$  series can be extended immediately to the other classical algebras. Based on the computations for the cases of rank 4 and below we conclude that there are no generalizations of the bosonic conserved currents of spin 4 in any of the superconformal  $\mathcal{SX}_n$  models with  $X = B, C$  or  $D$ .

## 6. Conclusions

We re-iterate our findings. The  $(1,0)$ -superconformal Toda models do not admit  $(1,1)$  supersymmetry except for the case of the algebra  $A_1$ , corresponding to the super-Liouville theory. Except for this simplest situation, the  $(1,0)$ -superconformal models do not contain generalizations of the conserved currents present in the bosonic Toda theories. We have checked that there are no independent conserved currents with spin  $5/2, 3, 7/2, 4$  in any of the superconformal models based on the classical algebras with ranks 2, 3 or 4, and we have shown that this implies that there are no generalizations of the bosonic conserved currents with spin 3 or 4 in the  $(1,0)$ -superconformal models based on any of the classical algebras.

We think it is fair to interpret these results by saying that the  $(1,0)$ -superconformal models are not integrable in the usual sense of Toda theories. Of course, we cannot rule out the possibility that these models might be integrable in some different sense. We have searched for conserved currents which are either holomorphic or anti-holomorphic and which would therefore appear as part of some extended chiral algebra in the standard fashion [3]. However, it seems that some conformally-invariant field theories possess conserved quantities with both holomorphic and anti-holomorphic components [23] and one could even consider the more general possibility of non-local conserved charges of the type familiar from work on non-linear sigma-models (see eg. [24]). While these may be interesting issues for future study, we have no grounds at present for suspecting that either of these types of conserved quantities are to be found in the models considered here.

Our results confirm the conventional picture of how integrability of Toda theories with fermions seems to be linked inextricably to Lie superalgebras [5-13] and they complement earlier work on the existence, or otherwise, of various types of extended superconformal algebras [3,20]. They also provide a set of cautionary examples which may challenge some preconceptions about the common properties of field theories and their supersymmetric extensions.

We have not considered any exceptional algebras in our analysis. There is no difficulty in principle in checking these models directly, following the approach of section 4. The problem is that the complexity of the calculations increases very rapidly with both the number of fields and the spins of the conserved quantities being sought. Perhaps some useful information about these models could be gleaned from the results we already have, by some suitable truncation to classical algebras. In any case, we see no reason to anticipate any differences in the nature of the results that would be obtained.

Finally, the paper of Papadopoulos [18] also introduces  $(1,0)$ -supersymmetric extensions of the affine Toda models corresponding to  $\hat{A}_n$ . Based on the results obtained here, one might guess that integrability fails for these models too, but it would be interesting to investigate this further. One possibility would be to consider a modification of the truncation procedure discussed above in which the additional term in the potential corresponding to the affine root is scaled away to recover a conformal theory. One might then be able to

deduce the absence of conserved charges from knowledge of the conformal case.

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