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Intrinsic Regularization Approach to chiral Anomalies

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Abstract

With the help of the intrinsic regularization method, we present a new method which enables us to calculate the chiral anomalies in a much natural and self-consistent way. By checking Ward identities related to various diagrams involving anomaly, we analyze anomalies in the σ model, in Abelian gauge theory, and in non-Abelian gauge theory as well. Our calculations prove to naively preserve all vector Ward identities and accordingly reproduce the famous ABJ-anomaly [1, 2, 3] in no need of introducing any counterterms.

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1 Introduction

Ever since the discovery of chiral anomaly [1, 2, 4], a great progress has been made in this area. Not only has its full structure been found [3], but also the deeper significance of its origin has been investigated from path integral point of view [5] and geometrical point of view [6, 7, 8]. However, the topic still deserves further investigations. One of the main problems in this area is on the perturbative calculation of the chiral anomalies, as can be seen from the following observations:

In the first place, in checking the Ward identities associated with the vector conservation and the ones with axial-vector conservation at, say, one loop level, direct calculations of small loop diagrams (*e.g.*, triangle diagrams), which are linearly or more highly divergent, suffer from the ambiguity of the shift of the integral variable, because a shift of the integral variable will alter linearly or more highly divergent integral by a finite amount. Only after imposing by hand the requirement that the vector Ward identities be held, can we eliminate the above ambiguity and get the anomalous axial-vector Ward identity.

Therefore, to deal with these diagrams properly, one has to adopt an appropriate regularization method, which should be consistent with chiral symmetry, to render them finite. Unfortunately, so far we lack such a method. In the dimensional regularization method, for example, the main difficulty is on the proper definition of γ_5 in d dimensions. In four dimensions, the definition of γ_5 is given by

$$\gamma_5 = \frac{i}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma \tag{1}$$

and the tensor $\epsilon_{\mu\nu\rho\sigma}$ does only make sense in four dimensions. So it seems that the dimensional regularization method do not straightforwardly applicable to the theories involving γ_5. To solve this problem, some artificial rules have to be imposed on the definition of γ_5 [9, 10].

Some other regularization methods were also adopted to analyze chiral anomalies [1, 2, 3, 11]. Most of them, however, proved to be incapable of preserving the vector Ward identities which are essential to the renormalization of gauge theories. To evade this difficulty, counterterms had to be added by hand to diagrams involving anomaly to restore the vector Ward identities.

A few years ago, a new regularization approach named intrinsic regularization was proposed by Wang and Guo [12]. Since then, a series of works have been done along this direction [13, 14, 15, 16]. In ref [15], the issue of dealing with Abelian anomaly was discussed, however, the application of the primitive version of the new regularization approach to non-Abelian gauge theories seems to be difficult.

As an improved version of the intrinsic regularization, recently we presented a new approach, the inserter approach, and succeeded in applying it to the ϕ^4 theory and QED [17], and further to

QCD [18]. The most remarkable feature of this approach is very simple but fundamental, namely, the entire procedure is intrinsic in the QFT. There is nothing changed, the action, the Feynman rules, the spacetime dimensions etc. are all the same as that in the given QFT. This is a very important property, because, in principle, it allows us to preserve all symmetries and topological properties of the theory we are dealing with. It is this property that offers us an opportunity to calculate the chiral anomaly perturbatively in a much natural and self-consistent way. Certainly, this will be helpful to our deeper understanding to the chiral anomaly.

In what follows, with the help of the intrinsic regularization method, we will investigate this issue in detail. By checking Ward identities related to various diagrams involving anomaly, we analyze anomalies in the σ model, in Abelian gauge theory, and in non-Abelian gauge theory as well. Our calculation is up to one loop order since according to theorems given in [19, 20, 21, 22] the structure of the chiral anomaly is not modified in the presence of higher order corrections. The result proves to naively preserve all vector Ward identities and accordingly reproduce the famous ABJ-anomaly [1, 2, 3] in no need of introducing any counterterms.

This paper is organized as follows. In section 2 and section 3, we present in turn the detailed calculations for the triangle diagrams in the σ model and Abelian gauge theories to verify the normal vector Ward identities and reproduce anomalous axial-vector Ward identities as well. Then in section 4, we generalize our method to a much general case, *i.e.*, the non-Abelian gauge anomaly. After calculating various one loop diagrams involving anomaly, we get the anomalous divergence equation which is exactly the same as Bardeen's result. Section 5 contains some concluding remarks.

2 Triangle Anomaly in the σ Model

The research on chiral anomaly was motivated by the study of $\pi^0 \to 2\gamma$ process in the σ model [2]. Let us start with this model. For simplicity we keep only one fermion of charge $+1$ (the proton) and two mesons π^0 and σ. The corresponding Lagrangian is

$$\begin{aligned}\mathcal{L} &= \bar{\psi}[i\partial\!\!\!/ - m + g(\sigma + i\pi\gamma_5)]\psi + \frac{1}{2}(\partial\pi)^2 + \frac{1}{2}(\partial\sigma)^2 - \frac{m_\pi^2}{2}\pi^2 - \frac{m_\sigma^2}{2}\sigma^2 \\ &\quad -\lambda v\sigma(\pi^2+\sigma^2) - \frac{\lambda}{4}(\pi^2+\sigma^2)^2 .\end{aligned} \tag{2}$$

To lowest order,

$$m = -gv , \qquad m_\sigma^2 - m_\pi^2 = 2\lambda v^2 .$$

The axial current

$$J_\mu^5 = \bar{\psi}\gamma_\mu\gamma_5\psi + (\sigma\partial_\mu\pi + \pi\partial_\mu\sigma) + v\partial_\mu\pi \tag{3}$$

is coupled either to the pion with a factor $iq_\rho v$ or directly to the fermion. This feature, together with PCAC, forms the basis of the explanation why the effective coupling constant for $\pi^0 \to 2\gamma$ does not vanish in the soft pion limit. And the key of the explanation is the existence of the triangle anomaly, which arises from the VVA diagrams shown in Fig.1. Let us denote $\hat{T}_{\mu\nu\rho}(p_1, p_2)$ and $\hat{T}_{\nu\mu\rho}(p_2, p_1)$ as the amplitudes of Fig.1(a) and Fig.1(b) respectively, and the total amplitude is

$$\hat{W}_{\mu\nu\rho}(p_1, p_2) = \hat{T}_{\mu\nu\rho}(p_1, p_2) + \hat{T}_{\nu\mu\rho}(p_2, p_1) . \tag{4}$$

In the momentum space, we have

$$\hat{T}_{\mu\nu\rho}(p_1, p_2) = -\int \frac{d^4k}{(2\pi)^4} Tr[\gamma_\mu \frac{1}{k\!\!\!/ - m}\gamma_\nu \frac{1}{k\!\!\!/ - p\!\!\!/_2 - m}\gamma_\rho\gamma_5 \frac{1}{k\!\!\!/ + p\!\!\!/_1 - m}] , \tag{5}$$

and $\hat{T}_{\nu\mu\rho}(p_2, p_1)$ can be obtained from (5) by just interchanging (μ, p_1) and (ν, p_2) with each other. By power counting expressions (5) is linearly divergent. Our goal is show whether the normal vector Ward identities

$$p_1^\mu \hat{W}_{\mu\nu\rho}(p_1, p_2) = p_2^\nu \hat{W}_{\mu\nu\rho}(p_1, p_2) = 0 , \tag{6}$$

and the axial-vector Ward identity (in the limit of $m = 0$)

$$(p_1^\rho + p_2^\rho)\hat{W}_{\mu\nu\rho}(p_1, p_2) = 0 , \tag{7}$$

are held in the presence of radiative corrections. To this end, we use the intrinsic regularization method to render $\hat{T}_{\mu\nu\rho}(p_1, p_2)$ and $\hat{T}_{\nu\mu\rho}(p_2, p_1)$ finite before substituting them into eqs.(6) and (7).

The spirit of intrinsic regularization approach is the same as that stated in [17]. The main step of the approach for the σ model may be stated more concretely as follows. First, we should construct the inserters which are to be inserted into the internal lines in a divergent diagram to render it finite. The inserters for fermion, π and σ can be chosen as follows:

- The fermion-inserter:

$$I^{(f)}(p) = ig. \tag{8}$$

- The π-inserter:

$$I^{(\pi)}(p) = -6i\lambda. \tag{9}$$

- The σ-inserter:

$$I^{[\sigma]}(p) = -6i\lambda. \tag{10}$$

Note that all of these inserters are the vertices originally contained in the theory. Given a divergent 1PI amplitude $\Gamma^{(n_f,n_g)}(p_1,\cdots,p_{n_f};k_1,\cdots,k_{n_g})$ at the one loop order with n_f external fermion lines and n_g external boson lines, we consider a set of 1PI amplitudes $\Gamma^{(n_f,n_g)}(p_1,\cdots,p_{n_f};k_1,\cdots,k_{n_g};q)$ which correspond to the diagrams with, if the loop contained in the diagram purely consists of fermion lines, all possible $2q$ insertions of the fermion inserter in the internal fermion lines, or in other cases, all possible q insertions of the corresponding inserter in the internal boson lines in the original diagram. The divergent degree therefore becomes:

$$\delta = 4 - I_f - 2I_g - 2q.$$

If q is large enough, $\Gamma^{(n_f,n_g)}(p_1,\cdots,p_{n_f};k_1,\cdots,k_{n_g};q)$ are convergent and the original divergent function is the case of $q=0$. Thus we reach a relation between the given divergent 1PI function and a set of convergent 1PI functions at the one loop order. In fact, the function of inserting the inserter(s) into internal lines is simply to raise the power of the propagator of the lines and to decrease the degree of divergence of given diagram. In order to regularize the given divergent function with the help of this relation, we need to deal with those convergent functions on an equal footing and pay attention to their differences due to the insertions. To this end, we introduce a new function:

$$\begin{aligned}&\Gamma^{(n_f,n_g)}(p_1,\cdots,p_{n_f};k_1,\cdots,k_{n_g};\ q;\ \mu)\\ &= (-i\mu)^{2q}(-i\lambda')^{-2q}\tfrac{1}{N_q}\sum\Gamma^{(n_f,n_g)}(p_1,\cdots,p_{n_f};k_1,\cdots,k_{n_g};\ q)\end{aligned} \tag{11}$$

where μ is an arbitrary reference mass parameter, the summation is taken over the entire set of such N_q inserted functions, and the factor $(-i\lambda')^{-2q}$ introduced here, in which λ' stands for g for fermion loop and for $\lambda^{\frac{1}{2}}$ for other cases, is to cancel the ones coming from the inserters. It is clear that this function is the arithmetical average of those convergent functions and has the same dimension in mass, the same order in coupling constant with the original divergent 1PI function. Then we evaluate it and analytically continue q from the integer to the complex number. Finally, the original 1PI function is recovered as its $q\to 0$ limiting case:

$$\Gamma^{(n_f,n_g)}(p_1,\cdots,p_{n_f};k_1,\cdots,k_{n_g}) = \lim_{q\to 0}\Gamma^{(n_f,n_g)}(p_1,\cdots,p_{n_f};k_1,\cdots,k_{n_g};q;\mu), \tag{12}$$

and the original infinity appears as pole in q.

Through the procedure stated above, the regularized amplitude of VVA process can be written as

$$W^{(R)}_{\mu\nu\rho}(p_1,\ p_2;\ q;\ \mu) = \hat{T}^R_{\mu\nu\rho}(p_1,\ p_2;\ q;\ \mu) + \hat{T}^R_{\nu\mu\rho}(p_2,\ p_1;\ q;\ \mu)\ , \tag{13}$$

where the explicit expression of $\hat{T}^R_{\mu\nu\rho}(p_1,\ p_2;\ q;\ \mu)$ is

$$\begin{aligned}\hat{T}^R_{\mu\nu\rho}(p_1,\ p_2;\ q;\ \mu) &= \mu^{2q}g^{-2q}\tfrac{1}{N_q}\sum_{i=0}^{2q}\sum_{j=0}^{2q-i}\hat{T}^R_{\mu\nu\rho}(p_1,\ p_2;\ q)_{ij}\\ &= -\mu^{2q}\tfrac{1}{N_q}\sum_{i=0}^{2q}\sum_{j=0}^{2q-i}\int\tfrac{d^4k}{(2\pi)^4}Tr[\gamma_\mu\Big(\tfrac{1}{\not{k}-m}\Big)^{2q-i-j+1}\gamma_\nu\Big(\tfrac{1}{\not{k}-\not{p}_2-m}\Big)^{i+1}\gamma_\rho\gamma_5\Big(\tfrac{1}{\not{k}+\not{p}_1-m}\Big)^{j+1}],\end{aligned} \tag{14}$$

where $N_q = \frac{1}{2}(q+1)(q+2)$. Of course in the intrinsic regularization scheme γ_5 can be properly defined according to (1), since we have not changed the dimension of the space-time. When q is large enough, the integral expression (14) is convergent and when $q=0$, we have $\lim_{q\to 0}\hat{W}^{(R)}_{\mu\nu\rho}(p_1,\ p_2;\ q;\ \mu) = \hat{W}_{\mu\nu\rho}(p_1,\ p_2)$.

Now we are ready to show whether the Ward identities (6) and (7) can be satisfied by direct calculations starting from the regularized expression (14). First, we check the vector Ward identities (6):

$$\begin{aligned}&p_1^\mu\mu^{2q}g^{-2q}\tfrac{1}{N_q}\sum_{i=0}^{2q}\sum_{j=0}^{2q-i}\hat{T}^R_{\mu\nu\rho}(p_1,\ p_2;\ q)_{ij}\\ &= -\mu^{2q}\tfrac{1}{N_q}\sum_{i=0}^{2q}\sum_{j=0}^{2q-i}\int\tfrac{d^4k}{(2\pi)^4}Tr[\not{p}_1\Big(\tfrac{1}{\not{k}-m}\Big)^{2q-i-j+1}\gamma_\nu\Big(\tfrac{1}{\not{k}-\not{p}_2-m}\Big)^{i+1}\gamma_\rho\gamma_5\Big(\tfrac{1}{\not{k}+\not{p}_1-m}\Big)^{j+1}]\ .\end{aligned} \tag{15}$$

Making use of the identity

$$\not{p}_1 = (\not{k}+\not{p}_1-m)-(\not{k}-m)\ , \tag{16}$$

we get

$$\begin{aligned}&p_1^\mu\mu^{2q}g^{-2q}\tfrac{1}{N_q}\sum_{i=0}^{2q}\sum_{j=0}^{2q-i}\hat{T}^R_{\mu\nu\rho}(p_1,\ p_2;\ q)_{ij}\\ &= \mu^{2q}\tfrac{1}{N_q}\sum_{i=0}^{2q}\sum_{j=0}^{2q-i}\int\tfrac{d^4k}{(2\pi)^4}Tr[\Big(\tfrac{1}{\not{k}+\not{p}_1-m}\Big)^{j+1}\Big(\tfrac{1}{\not{k}-m}\Big)^{2q-i-j}\gamma_\nu\Big(\tfrac{1}{\not{k}-\not{p}_2-m}\Big)^{i+1}\gamma_\rho\gamma_5]\\ &\quad-\mu^{2q}\tfrac{1}{N_q}\sum_{i=0}^{2q}\sum_{j=0}^{2q-i}\int\tfrac{d^4k}{(2\pi)^4}Tr[\Big(\tfrac{1}{\not{k}+\not{p}_1-m}\Big)^{j}\Big(\tfrac{1}{\not{k}-m}\Big)^{2q-i-j+1}\gamma_\nu\Big(\tfrac{1}{\not{k}-\not{p}_2-m}\Big)^{i+1}\gamma_\rho\gamma_5]\\ &= \mu^{2q}\tfrac{1}{N_q}\sum_{i=0}^{2q}\int\tfrac{d^4k}{(2\pi)^4}Tr[\Big(\tfrac{1}{\not{k}+\not{p}_1-m}\Big)^{i+1}\gamma_\nu\Big(\tfrac{1}{\not{k}-\not{p}_2-m}\Big)^{2q-i+1}\gamma_\rho\gamma_5]\\ &\quad-\mu^{2q}\tfrac{1}{N_q}\sum_{i=0}^{2q}\int\tfrac{d^4k}{(2\pi)^4}Tr[\Big(\tfrac{1}{\not{k}-m}\Big)^{2q-i+1}\gamma_\nu\Big(\tfrac{1}{\not{k}-\not{p}_2-m}\Big)^{i+1}\gamma_\rho\gamma_5]\ .\end{aligned} \tag{17}$$

Similarly,

$$
\begin{aligned}
&p_1^{\mu}\mu^{2q}g^{-2q}\frac{1}{N_q}\sum_{i=0}^{2q}\sum_{j=0}^{2q-i}\hat{T}^R_{\nu\mu\rho}(p_2,\ p_1;\ q)_{ij} \\
&=\mu^{2q}\frac{1}{N_q}\sum_{i=0}^{2q}\int\frac{d^4k}{(2\pi)^4}Tr[\Big(\frac{1}{\not k+\not p_2-m}\Big)^{i+1}\gamma_\nu\Big(\frac{1}{\not k-m}\Big)^{2q-i+1}\gamma_\rho\gamma_5] \\
&\quad-\mu^{2q}\frac{1}{N_q}\sum_{i=0}^{2q}\int\frac{d^4k}{(2\pi)^4}Tr[\Big(\frac{1}{\not k+\not p_2-m}\Big)^{2q-i+1}\gamma_\nu\Big(\frac{1}{\not k-\not p_1-m}\Big)^{i+1}\gamma_\rho\gamma_5] \ .
\end{aligned}
\tag{18}
$$

When q is large enough, the integrals in (17) and (18) are convergent, so we can make a shift of the integral variable. If we let $k\to k+p_1-p_2$ in the first integral of (17), it equals to the second integral of (18) and they cancel with each other. Similarly, if we make a shift $k\to k-p_2$ in the second integral of (17), it equals to the first integral of (18) and they cancel with each other too. So, we get

$$
p_1^{\mu}\hat{W}^{(R)}_{\mu\nu\rho}(p_1,\ p_2;\ q;\ \mu)=0\ . \tag{19}
$$

Similarly, we can get

$$
p_2^{\nu}\hat{W}^{(R)}_{\mu\nu\rho}(p_1,\ p_2;\ q;\ \mu)=0\ . \tag{20}
$$

Therefore, we conclude that the vector Ward identities (6) are naturally satisfied in the intrinsic regularization method.

In the same way, we can also start from the regularized expression (14) to check the axial vector Ward identities (7):

$$
\begin{aligned}
&(p_1^{\rho}+p_2^{\rho})\hat{W}^{(R)}_{\mu\nu\rho}(p_1,\ p_2;\ q;\ \mu) \\
&=(p_1^{\rho}+p_2^{\rho})\mu^{2q}g^{-2q}\frac{1}{N_q}\sum_{i=0}^{2q}\sum_{j=0}^{2q-i}[\hat{T}^R_{\mu\nu\rho}(p_1,\ p_2;\ q)_{ij}+\hat{T}^R_{\nu\mu\rho}(p_2,\ p_1;\ q)_{ij}] \\
&=-\mu^{2q}\frac{1}{N_q}\sum_{i=0}^{2q}\sum_{j=0}^{2q-i}\int\frac{d^4k}{(2\pi)^4}Tr[\gamma_\mu\Big(\frac{1}{\not k-m}\Big)^{2q-i-j+1}\gamma_\nu\Big(\frac{1}{\not k-\not p_2-m}\Big)^{i+1}(\not p_1+\not p_2)\gamma_5\Big(\frac{1}{\not k+\not p_1-m}\Big)^{j+1}] \\
&\quad-\mu^{2q}\frac{1}{N_q}\sum_{i=0}^{2q}\sum_{j=0}^{2q-i}\int\frac{d^4k}{(2\pi)^4}Tr[\gamma_\nu\Big(\frac{1}{\not k-m}\Big)^{2q-i-j+1}\gamma_\mu\Big(\frac{1}{\not k-\not p_1-m}\Big)^{i+1}(\not p_1+\not p_2)\gamma_5\Big(\frac{1}{\not k+\not p_2-m}\Big)^{j+1}] \ .
\end{aligned}
$$

By using the identity

$$
(\not p_1+\not p_2)\gamma_5=-(\not k-\not p_2-m)\gamma_5-\gamma_5(\not k+\not p_1-m)-2m\gamma_5\ , \tag{21}
$$

we get

$$
\begin{aligned}
&(p_1^{\rho}+p_2^{\rho})\mu^{2q}g^{-2q}\frac{1}{N_q}\sum_{i=0}^{2q}\sum_{j=0}^{2q-i}\hat{T}^R_{\mu\nu\rho}(p_1,\ p_2;\ q)_{ij} \\
&=i[\hat{T}^R_{\nu\mu}(p_2;\ q;\ \mu)-\hat{T}^R_{\mu\nu}(p_1;\ q;\ \mu)]+2m\mu^{2q}\frac{1}{N_q}\sum_{i=0}^{2q}\sum_{j=0}^{2q-i}\int\frac{d^4k}{(2\pi)^4}B(i,j) \\
&\quad+\mu^{2q}\frac{1}{N_q}\sum_{i=1}^{2q}\sum_{j=0}^{2q-i}\int\frac{d^4k}{(2\pi)^4}A(i,j)+\mu^{2q}\frac{1}{N_q}\sum_{i=0}^{2q}\sum_{j=1}^{2q-i}\int\frac{d^4k}{(2\pi)^4}A(i+1,j-1)\ ,
\end{aligned}
\tag{22}
$$

where $\hat{T}^R_{\mu\nu}(p;\ q;\ \mu)$ denotes the two-point VA function, and

$$
\begin{aligned}
A(i,j)&=Tr[\gamma_\mu\Big(\frac{1}{\not k-m}\Big)^{2q-i-j+1}\gamma_\nu\Big(\frac{1}{\not k-\not p_2-m}\Big)^{i}\gamma_5\Big(\frac{1}{\not k+\not p_1-m}\Big)^{j+1}]\ , \\
B(i,j)&=Tr[\gamma_\mu\Big(\frac{1}{\not k-m}\Big)^{2q-i-j+1}\gamma_\nu\Big(\frac{1}{\not k-\not p_2-m}\Big)^{i+1}\gamma_5\Big(\frac{1}{\not k+\not p_1-m}\Big)^{j+1}]\ .
\end{aligned}
\tag{23}
$$

To evaluate the trace of gamma matrices in $A(i,j)$, we divide the summation in $A(i,j)$ into four classes:

$$
\begin{aligned}
\sum_{i=1}^{2q}\sum_{j=0}^{2q-i}A(i,j)\ &=\ \sum_{\alpha=1}^{q}\sum_{\beta=0}^{q-\alpha}A(2\alpha,2\beta)+\sum_{\alpha=1}^{q}\sum_{\beta=1}^{q-\alpha}A(2\alpha,2\beta-1) \\
&\quad+\sum_{\alpha=1}^{q}\sum_{\beta=0}^{q-\alpha}A(2\alpha-1,2\beta)+\sum_{\alpha=1}^{q}\sum_{\beta=0}^{q-\alpha}A(2\alpha-1,2\beta+1)\ ,
\end{aligned}
\tag{24}
$$

where

$$
A(2\alpha,2\beta)=\frac{Tr[\gamma_\mu(\not k+m)^{2q-2\alpha-2\beta+1}\gamma_\nu(\not k-\not p_2+m)^{2\alpha}\gamma_5(\not k+\not p_1+m)^{2\beta+1}]}{(k^2-m^2)^{2q-2\alpha-2\beta+1}[(k-p_2)^2-m^2]^{2\alpha}[(k+p_1)^2-m^2]^{2\beta+1}}\ , \tag{25a}
$$

$$
A(2\alpha,2\beta-1)=\frac{Tr[\gamma_\mu(\not k+m)^{2q-2\alpha-2\beta+2}\gamma_\nu(\not k-\not p_2+m)^{2\alpha}\gamma_5(\not k+\not p_1+m)^{2\beta}]}{(k^2-m^2)^{2q-2\alpha-2\beta+2}[(k-p_2)^2-m^2]^{2\alpha}[(k+p_1)^2-m^2]^{2\beta}}\ , \tag{25b}
$$

$$
A(2\alpha-1,2\beta)=\frac{Tr[\gamma_\mu(\not k+m)^{2q-2\alpha-2\beta+2}\gamma_\nu(\not k-\not p_2-m)(\not k-\not p_2+m)^{2\alpha}\gamma_5(\not k+\not p_1+m)^{2\beta+1}]}{(k^2-m^2)^{2q-2\alpha-2\beta+2}[(k-p_2)^2-m^2]^{2\alpha}[(k+p_1)^2-m^2]^{2\beta+1}}\ , \tag{25c}
$$

$$
A(2\alpha-1,2\beta+1)=\frac{Tr[\gamma_\mu(\not k+m)^{2q-2\alpha-2\beta+1}\gamma_\nu(\not k-\not p_2-m)(\not k-\not p_2+m)^{2\alpha}\gamma_5(\not k+\not p_1+m)^{2\beta+2}]}{(k^2-m^2)^{2q-2\alpha-2\beta+1}[(k-p_2)^2-m^2]^{2\alpha}[(k+p_1)^2-m^2]^{2\beta+2}}\ . \tag{25d}
$$

As our interest is only in anomalous terms which are independent of the mass of fermion, we can work in a more simpler case, *i.e.*, in the massless limit. Therefore, the $B(i,j)$ term in (22) can be neglected, and the mass m in the numerators of eqs.(25a), (25b), (25c), and (25d) can also be set to zero. Nevertheless, the mass m in the denominators of the above equations are remained temporarily to avoid possible infrared divergence for q large enough. As we will see, it has nothing to do with the final result.

Now we consider one by one the four cases in (25a), (25b), (25c), and (25d). First, it is easy to see that the contribution of the second case, *i.e.*, $A(2\alpha, 2\beta - 1)$, is zero, since $Tr[\gamma_\mu \gamma_\nu \gamma_5] = 0$. For the first case, we have

$$\frac{1}{N_q}\sum_{\alpha=1}^{q}\sum_{\beta=0}^{q-\alpha}\int\frac{d^4k}{(2\pi)^4}A(2\alpha, 2\beta) = \int\frac{d^4k}{(2\pi)^4}Tr[\gamma_\mu \not{k}\gamma_\nu\gamma_5(\not{k}+\not{p}_1)] \times\frac{(k^2)^{q-\alpha-\beta}[(k-p_2)^2]^\alpha[(k+p_1)^2]^\beta}{(k^2-m^2)^{2q-2\alpha-2\beta+1}[(k-p_2)^2-m^2]^{2\alpha}[(k+p_1)^2-m^2]^{2\beta+1}} , \tag{26}$$

By using the formula $Tr[\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_5] = 4i\epsilon_{\mu\nu\rho\sigma}$ to carry out the trace of γ-matrices, and with the help of the Feynman parameterization to perform the integral over loop momentum k, we get

$$\frac{1}{N_q}\sum_{\alpha=1}^{q}\sum_{\beta=0}^{q-\alpha}\int\frac{d^4k}{(2\pi)^4}A(2\alpha, 2\beta) = \frac{1}{24\pi^2}\epsilon_{\mu\nu\rho\sigma}p_1^\rho p_2^\sigma + o(q) . \tag{27}$$

Similarly, for the third and fourth cases, we have

$$\frac{1}{N_q}\sum_{\alpha=1}^{q}\sum_{\beta=0}^{q-\alpha}\int\frac{d^4k}{(2\pi)^4}A(2\alpha + 1, 2\beta) = \frac{1}{24\pi^2}\epsilon_{\mu\nu\rho\sigma}p_1^\rho p_2^\sigma + o(q) , \tag{28}$$

$$\frac{1}{N_q}\sum_{\alpha=1}^{q}\sum_{\beta=0}^{q-\alpha}\int\frac{d^4k}{(2\pi)^4}A(2\alpha - 1, 2\beta + 1) = \frac{1}{24\pi^2}\epsilon_{\mu\nu\rho\sigma}p_1^\rho p_2^\sigma + o(q) . \tag{29}$$

Summing over (27), (28) and (29) yields

$$\frac{1}{N_q}\sum_{i=1}^{2q}\sum_{j=0}^{2q-i}\int\frac{d^4k}{(2\pi)^4}A(i, j) = \frac{1}{8\pi^2}\epsilon_{\mu\nu\rho\sigma}p_1^\rho p_2^\sigma + o(q) . \tag{30}$$

In the same way, we have

$$\frac{1}{N_q}\sum_{i=0}^{2q}\sum_{j=1}^{2q-i}\int\frac{d^4k}{(2\pi)^4}A(i + 1, j - 1) = \frac{1}{8\pi^2}\epsilon_{\mu\nu\rho\sigma}p_1^\rho p_2^\sigma + o(q) . \tag{31}$$

Therefore, from (34), (22), (30) and (31) we conclude that

$$\begin{aligned}&\lim_{q\to 0}(p_1^\rho + p_2^\rho)W^{(R)}_{\mu\nu\rho}(p_1, p_2; q; \mu)\\ &= \lim_{q\to 0}(p_1^\rho + p_2^\rho)\mu^{2q}g^{-2q}\frac{1}{N_q}\sum_{i=0}^{2q}\sum_{j=0}^{2q-i}[\hat{T}^R_{\mu\nu\rho}(p_1, p_2; q)_{ij} + \hat{T}^R_{\nu\mu\rho}(p_2, p_1; q)_{ij}] .\\ &= \frac{1}{2\pi^2}\epsilon_{\mu\nu\rho\sigma}p_1^\rho p_2^\sigma .\end{aligned} \tag{32}$$

The RHS of eq.(32) is the well-known triangle anomaly which violates the axial-vector Ward identity (6).

3 Triangle Anomaly in Abelian Gauge Theory

Now we turn to the anomaly in Abelian gauge theory. The conventional Abelian gauge theory where vector current is coupled to the fermion through the least coupling is QED. Its Lagrangian is given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\not{\partial} - \not{A} - m)\psi . \tag{33}$$

For our purpose, according to the theorems stated in [3, 23, 24], we may only consider the VVA diagrams which consist of an internal fermion loop connected to two vector couplings and one axial-vector coupling.

The total amplitude of VVA process is the addition of the crossed diagrams Fig.2(a), Fig.2(b). The spirit of intrinsic regularization approach is the same as that stated in the previous section, *i.e.*, inserting in all possible ways $2q$ inserters in the internal fermion lines. As was discussed in detail in [17], the inserter is borrowed from the standard model, it is a $ff\phi$-vertex of the Yukawa type with a zero momentum Higgs external line. In the momentum space, the regularized amplitude of VVA process can be written as

$$W^{(R)}_{\mu\nu\rho}(p_1, p_2; q; \mu) = T^R_{\mu\nu\rho}(p_1, p_2; q; \mu) + T^R_{\nu\mu\rho}(p_2, p_1; q; \mu) , \tag{34}$$

where the explicit expression of $T^R_{\mu\nu\rho}(p_1, p_2; q; \mu)$ is given in (68) in the appendix.

Again, what we are going to do now is to check the vector and axial-vector Ward identities related to $W^{(R)}_{\mu\nu\rho}(p_1, p_2; q; \mu)$. It is not hard to see that this task is much easy, because $T^R_{\mu\nu\rho}(p_1, p_2; q; \mu)$ given in (68) is almost the same as $\hat{T}^R_{\mu\nu\rho}(p_1, p_2; q; \mu)$ given in (14) except for an unimportant factor. Thus, following the completely same procedure as that in the σ model, we immediately get

$$p_1^\mu W^{(R)}_{\mu\nu\rho}(p_1, p_2; q; \mu) = 0 , \tag{35}$$

$$p_2^\nu W^{(R)}_{\mu\nu\rho}(p_1, p_2; q; \mu) = 0 , \tag{36}$$

and

$$\begin{aligned}&\lim_{q\to 0}(p_1^\rho + p_2^\rho)W^{(R)}_{\mu\nu\rho}(p_1, p_2; q; \mu)\\ &= \lim_{q\to 0}(p_1^\rho + p_2^\rho)\frac{1}{N_q}\sum_{i=0}^{2q}\sum_{j=0}^{2q-i}[T^R_{\mu\nu\rho}(p_1, p_2; q; \mu)_{ij} + T^R_{\nu\mu\rho}(p_2, p_1; q; \mu)_{ij}]\\ &= \frac{1}{2\pi^2}\epsilon_{\mu\nu\rho\sigma}p_1^\rho p_2^\sigma .\end{aligned} \tag{37}$$

Therefore, we conclude that in QED the normal vector Ward identities related to VVA amplitude are satisfied in the intrinsic regularization method, while the axial-vector Ward identity is violated by an anomaly which is the same as that in the σ model.

For an alternative theory where the gauge field is coupled to the axial-current instead of the vector current [25], *i.e.*, the interaction part of the Lagrangian (33), $-\bar{\psi}\not{A}\psi$, is replaced by $-\bar{\psi}\not{A}\gamma_5\psi$, the only diagrams involving anomaly become the AAA triangle diagrams as shown in Fig.3(a) and Fig.3(b), each of which consists of an internal fermion loop connected to three axial-vector couplings. In the present case, the only work to be done is to check whether the axial-vector Ward identity (in the massless limit)

$$(p_1^\rho + p_2^\rho)W'^{(R)}_{\mu\nu\rho}(p_1,\ p_2;\ q;\ \mu) = 0\ , \tag{38}$$

is held in the presence of radiative corrections, where $W'_{\mu\nu\rho}(p_1,\ p_2)$ denotes the total regularized amplitude of the AAA process which can be written as

$$W'^{(R)}_{\mu\nu\rho}(p_1,\ p_2;\ q;\ \mu) = T'^R_{\mu\nu\rho}(p_1,\ p_2;\ q;\ \mu) + T'^R_{\nu\mu\rho}(p_2,\ p_1;\ q;\ \mu)\ , \tag{39}$$

with the explicit expression of $T'^R_{\mu\nu\rho}(p_1, p_2;\ q; \mu)$ given in (69) in the appendix, and $T'^R_{\nu\mu\rho}(p_2, p_1;\ q; \mu)$ obtained from (69) by just interchanging $(\mu,\ p_1)$ and $(\nu,\ p_2)$ with each other. To check (38), we may following a much similar way as we have done in the σ model, *i.e.*, we may multiply $W'^{(R)}_{\mu\nu\rho}(p_1,\ p_2;\ q;\ \mu)$ by $(p_1^\rho + p_2^\rho)$, then split it into four terms by using the identity (21):

$$\begin{aligned}
&(p_1^\rho + p_2^\rho)W'^{(R)}_{\mu\nu\rho}(p_1,\ p_2,\ p_3;\ q;\ \mu)\\
&= i[T^R_{\mu\nu}(p_1+p_3;\ q;\ \mu) - T^R_{\nu\mu}(p_2+p_3;\ q;\ \mu)] + 2m\mu^{2q}\frac{1}{N_q}\sum_{i=0}^{2q}\sum_{j=0}^{2q-i}\int\frac{d^4k}{(2\pi)^4}B'(i,j)\\
&+\mu^{2q}\frac{1}{N_q}\sum_{i=1}^{2q}\sum_{j=0}^{2q-i}\int\frac{d^4k}{(2\pi)^4}A'(i,j) + \mu^{2q}\frac{1}{N_q}\sum_{i=0}^{2q}\sum_{j=1}^{2q-i}\int\frac{d^4k}{(2\pi)^4}A'(i+1,j-1)\\
&+(\mu, p_1) \leftrightarrow (\nu, p_2)\ .
\end{aligned} \tag{40}$$

where

$$\begin{aligned}
A'(i,j) &= Tr[\gamma_\mu\gamma_5\Big(\frac{1}{\not{k}-m}\Big)^{2q-i-j+1}\gamma_\nu\gamma_5\Big(\frac{1}{\not{k}-\not{p}_2-m}\Big)^{i}\gamma_5\Big(\frac{1}{\not{k}+\not{p}_1-m}\Big)^{j+1}]\ ,\\
B'(i,j) &= Tr[\gamma_\mu\gamma_5\Big(\frac{1}{\not{k}-m}\Big)^{2q-i-j+1}\gamma_\nu\gamma_5\Big(\frac{1}{\not{k}-\not{p}_2-m}\Big)^{i+1}\gamma_5\Big(\frac{1}{\not{k}+\not{p}_1-m}\Big)^{j+1}]\ .
\end{aligned} \tag{41}$$

In evaluating $\sum_{i=1}^{2q}\sum_{j=0}^{2q-i}\int\frac{d^4k}{(2\pi)^4}A'(i,j)$ and $\sum_{i=0}^{2q}\sum_{j=1}^{2q-i}\int\frac{d^4k}{(2\pi)^4}A'(i+1,j-1)$, we may follow similar procedure with (24)–(29), the only difference is that in (41) extra γ_5s appear in the positions next to γ_μ and γ_ν respectively, so we need to move them together to drop them with the help of the

properties of γ_5 before we carry out the trace of γ-matrices. For this sake, the term corresponding to (25c) changes by a minus while the other two terms corresponding to (25a) and (25d) remain unchanged. Therefore, we have

$$\frac{1}{N_q}\sum_{i=1}^{2q}\sum_{j=0}^{2q-i}\int\frac{d^4k}{(2\pi)^4}A'(i,j) = \frac{1}{24\pi^2}\epsilon_{\mu\nu\rho\sigma}p_1^\rho p_2^\sigma + o(q)\ . \tag{42}$$

$$\frac{1}{N_q}\sum_{i=0}^{2q}\sum_{j=1}^{2q-i}\int\frac{d^4k}{(2\pi)^4}A'(i+1,j-1) = \frac{1}{24\pi^2}\epsilon_{\mu\nu\rho\sigma}p_1^\rho p_2^\sigma + o(q)\ . \tag{43}$$

Therefore, from (39), (40), (42) and (43) we conclude that

$$\begin{aligned}
&\lim_{q\to 0}(p_1^\rho + p_2^\rho)W'^{(R)}_{\mu\nu\rho}(p_1,\ p_2;\ q;\ \mu)\\
&= \lim_{q\to 0}(p_1^\rho + p_2^\rho)\frac{1}{N_q}\sum_{i=0}^{2q}\sum_{j=0}^{2q-i}[T'^R_{\mu\nu\rho}(p_1,\ p_2;\ q;\ \mu)_{ij} + T'^R_{\nu\mu\rho}(p_2,\ p_1;\ q;\ \mu)_{ij}]\\
&= \frac{1}{6\pi^2}\epsilon_{\mu\nu\rho\sigma}p_1^\rho p_2^\sigma\ .
\end{aligned} \tag{44}$$

The RHS of eq.(44) is the anomalous term which violates the axial-vector Ward identity (38). As we have known, it differs from (37) by a factor $\frac{1}{3}$.

4 Non-Abelian Gauge Anomaly From Intrinsic Regularization

Now we apply our method to a much general case, namely, the $G_L \times G_R$ non-Abelian chiral gauge theory, which is described by the following Lagrangian:

$$\mathcal{L} = -\frac{1}{2}Tr(\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}) + \bar{\psi}[(i\not{\partial} - \not{\mathcal{A}}) - m]\psi\ . \tag{45}$$

In terms of vector field V_μ and axial-vector field A_μ (VA notation), $\mathcal{A}_\mu$ and $\mathcal{F}_{\mu\nu}$ can be written as

$$\begin{aligned}
\mathcal{A}_\mu &= V_\mu + \gamma_5 A_\mu = V_\mu^a \mathbf{T}_V^a + \gamma_5 A_\mu^a \mathbf{T}_A^a\ ,\\
\mathcal{F}_{\mu\nu} &= F_{V\mu\nu} + \gamma_5 F_{A\mu\nu} = F^a_{V\mu\nu}\mathbf{T}_V^a + \gamma_5 F^a_{A\mu\nu}\mathbf{T}_A^a\ ,
\end{aligned}$$

where

$$\begin{aligned}
F^a_{V\mu\nu} &= \partial_\mu V_\nu^a - \partial_\nu V_\mu^a + i[V_\mu^a,\ V_\nu^a] + i[A_\mu^a,\ A_\nu^a]\ ,\\
F^a_{A\mu\nu} &= \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + i[V_\mu^a,\ A_\nu^a] + i[A_\mu^a,\ V_\nu^a]\ .
\end{aligned}$$

In the present theory, as was pointed out in Ref.[7], the diagrams involving anomaly include not only VVA, AAA triangle diagrams, but also VVVA, VAAA box diagrams, and VVVVA, AAAAA, VVAAA pentagon diagrams. The main step to regularize these diagrams by means of intrinsic regularization approach is the same as that stated in [18], and the regularized integral expressions of the diagrams are presented in the appendix.

The total amplitudes of VVA and AAA processes are the addition of the crossed diagrams Fig.4(a), Fig.4(b) and Fig.5(a), Fig.5(b) respectively. The calculation of the triangle diagrams here is almost the same as in the σ model and QED. The only difference is that here an extra color factor is added to each diagram ($Tr[\mathbf{T}_V\mathbf{T}_V\mathbf{T}_A]$ for VVA diagram and $Tr[\mathbf{T}_A\mathbf{T}_A\mathbf{T}_A]$ for AAA diagram). Therefore, through a much similar evaluation to that in the previous section, we can easily prove that the normal vector Ward identities are naively preserved by the regularized functions in the intrinsic regularization method:

$$p_1^\mu W^{(R)abc}_{\mu\nu\rho}(p_1,\ p_2;\ q;\ \mu) = iTr([\mathbf{T}_V^a,\ \mathbf{T}_V^b]\mathbf{T}_A^c)\{T^R_{\nu\rho}(p_2;\ q;\ \mu) - T^R_{\nu\rho}(p_1+p_2;\ q;\ \mu)\}\ , \tag{46}$$

$$p_2^\nu W^{(R)abc}_{\mu\nu\rho}(p_1,\ p_2;\ q;\ \mu) = iTr([\mathbf{T}_V^a,\ \mathbf{T}_V^b]\mathbf{T}_A^c)\{T^R_{\mu\rho}(p_1+p_2;\ q;\ \mu) - T^R_{\mu\rho}(p_1;\ q;\ \mu)\}\ . \tag{47}$$

And the axial-vector Ward identities are violated by anomaly:

$$\begin{aligned}&\lim_{q\to 0}(p_1^\rho + p_2^\rho)W^{(R)abc}_{\mu\nu\rho}(p_1,\ p_2;\ q;\ \mu)\\ &= iTr([\mathbf{T}_V^a,\ \mathbf{T}_V^b]\mathbf{T}_A^c)\{T^R_{\nu\mu}(p_2;\ q;\ \mu) - T^R_{\mu\nu}(p_1;\ q;\ \mu)\} + \Delta_{VVA}\ ,\end{aligned} \tag{48}$$

$$\begin{aligned}&\lim_{q\to 0}(p_1^\rho + p_2^\rho)W'^{(R)abc}_{\mu\nu\rho}(p_1,\ p_2;\ q;\ \mu)\\ &= iTr([\mathbf{T}_A^a,\ \mathbf{T}_A^b]\mathbf{T}_A^c)\{T^R_{\mu\nu}(p_1+p_3;\ q;\ \mu) - T^R_{\nu\mu}(p_2+p_3;\ q;\ \mu)\} + \Delta_{AAA}\ ,\end{aligned} \tag{49}$$

where

$$\Delta_{VVA} = \frac{1}{4\pi^2}Tr(\{\mathbf{T}_V^a,\ \mathbf{T}_V^b\}\mathbf{T}_A^c)\epsilon_{\mu\nu\rho\sigma}p_1^\rho p_2^\sigma\ , \tag{50}$$

and

$$\Delta_{AAA} = \frac{1}{12\pi^2}Tr(\{\mathbf{T}_A^a,\ \mathbf{T}_A^b\}\mathbf{T}_A^c)\epsilon_{\mu\nu\rho\sigma}p_1^\rho p_2^\sigma\ , \tag{51}$$

are the anomalous terms which violate the axial-vector Ward identities.

The calculations for larger loop diagrams, i.e., box and pentagon diagrams, are also similar to that for triangle ones in principle. In what follows, as a typical example, we will calculate the VVVA amplitude in some detail, then we will directly give the results for other amplitudes.

The total amplitude of the VVVA process is the addition of the crossed diagrams Fig.6(a)-(f). Let us denote the total regularized amplitude of the VVVA process as $W^{(R)abcd}_{\mu\nu\rho\sigma}(p_1,\ p_2,\ p_3;\ q;\ \mu)$. In the momentum space, it can be written as

$$\begin{aligned}&W^{(R)abcd}_{\mu\nu\rho\sigma}(p_1,\ p_2,\ p_3;\ q;\ \mu)\\ &= \sum{}'\Big\{Tr(\mathbf{T}_V^a\mathbf{T}_V^b\mathbf{T}_V^c\mathbf{T}_A^d)\frac{1}{N_q}\sum_{i=0}^{2q}\sum_{j=0}^{2q-i}\sum_{l=0}^{2q-i-j}T^R_{\mu\nu\rho\sigma}(p_1,\ p_2,\ p_3;\ q;\ \mu)_{ijl}\Big\}\ ,\end{aligned} \tag{52}$$

where $\sum'$ denotes summation over all possible permutations of $\{(\mu,a,p_1),\ (\nu,b,p_2),\ (\rho,c,p_3)\}$, $N_q = \frac{1}{6}(q+1)(q+2)(q+3)$ is the total number of possible ways in which the inserters are inserted, and the explicit expression of $T^R_{\mu\nu\rho\sigma}(p_1,\ p_2,\ p_3;\ q;\ \mu)_{ijl}$ is given in (70). When q is large enough, the integral expression (70) is convergent and when $q=0$ the original infinity arises manifestly as pole in q.

Verification of the vector Ward identities related to the amplitude $W^{(R)abcd}_{\mu\nu\rho\sigma}(p_1,\ p_2,\ p_3;\ q;\ \mu)$ is straightforward. As we have done for triangle diagrams, we may consider the product of p_1^μ and the amplitude. With the help of identity (16), we can split it into two terms. After performing a suitable set of shifts for integral variables in these terms, we immediately find that

$$\begin{aligned}&p_1^\mu W^{(R)abcd}_{\mu\nu\rho\sigma}(p_1,\ p_2,\ p_3;\ q;\ \mu)\\ &= \sum{}'\Big\{iTr(\mathbf{T}_V^a\mathbf{T}_V^b\mathbf{T}_V^c\mathbf{T}_A^d)[T^R_{\nu\rho\sigma}(p_2,\ p_3;\ q;\ \mu) - T^R_{\nu\rho\sigma}(p_1+p_2,\ p_3;\ q;\ \mu)]\Big\}\ ,\end{aligned} \tag{53}$$

Eq.(53) is exactly the normal vector Ward identity related to the amplitude of VVVA process.

Now we proceed to check the axial-vector Ward identities related to $W^{(R)abcd}_{\mu\nu\rho\sigma}(p_1,\ p_2,\ p_3;\ q;\ \mu)$:

$$\begin{aligned}&(p_1^\sigma + p_2^\sigma + p_3^\sigma)W^{(R)abcd}_{\mu\nu\rho\sigma}(p_1,\ p_2,\ p_3;\ q;\ \mu)\\ &= -i\mu^{2q}Tr(\mathbf{T}_V^a\mathbf{T}_V^b\mathbf{T}_V^c\mathbf{T}_A^d)\frac{1}{N_q}\sum_{i=0}^{2q}\sum_{j=0}^{2q-i}\sum_{l=0}^{2q-i-j}\int\frac{d^4k}{(2\pi)^4}Tr[(\not p_1+\not p_2+\not p_3)\gamma_5\\ &\times\Big(\frac{1}{\not k-m}\Big)^{j+1}\gamma_\mu\Big(\frac{1}{\not k-\not p_1-m}\Big)^{l+1}\gamma_\nu\Big(\frac{1}{\not k-\not p_1-\not p_2-m}\Big)^{2q-i-j-l+1}\gamma_\rho\Big(\frac{1}{\not k-\not p_1-\not p_2-\not p_3-m}\Big)^{i+1}]\\ &+all\ possible\ permutations\ of\ \{(\mu,a,p_1),\ (\nu,b,p_2),\ (\rho,c,p_3)\}.\end{aligned} \tag{54}$$

By using the identity

$$(\not p_1+\not p_2+\not p_3)\gamma_5 = -(\not k-\not p_1-\not p_2-\not p_3-m)\gamma_5 - \gamma_5(\not k-m) - 2m\gamma_5\ ,$$

we get

$$\begin{aligned}&(p_1^\sigma + p_2^\sigma + p_3^\sigma)W^{(R)abcd}_{\mu\nu\rho\sigma}(p_1,\ p_2,\ p_3;\ q;\ \mu)\\ &= i\mu^{2q}Tr(\mathbf{T}_V^a\mathbf{T}_V^b\mathbf{T}_V^c\mathbf{T}_A^d)[T^R_{\nu\rho\mu}(p_2,\ p_3;\ q;\ \mu) - T^R_{\mu\nu\rho}(p_1,\ p_2;\ q;\ \mu)]\\ &+i\mu^{2q}Tr(\mathbf{T}_V^a\mathbf{T}_V^b\mathbf{T}_V^c\mathbf{T}_A^d)\frac{1}{N_q}\sum_{i=1}^{2q}\sum_{j=0}^{2q-i}\sum_{l=0}^{2q-i-j}\int\frac{d^4k}{(2\pi)^4}A(i,j,l)\\ &+i\mu^{2q}Tr(\mathbf{T}_V^a\mathbf{T}_V^b\mathbf{T}_V^c\mathbf{T}_A^d)\frac{1}{N_q}\sum_{i=0}^{2q}\sum_{j=1}^{2q-i}\sum_{l=0}^{2q-i-j}\int\frac{d^4k}{(2\pi)^4}A(i+1,j-1,l)\\ &+2im\mu^{2q}Tr(\mathbf{T}_V^a\mathbf{T}_V^b\mathbf{T}_V^c\mathbf{T}_A^d)\frac{1}{N_q}\sum_{i=0}^{2q}\sum_{j=0}^{2q-i}\sum_{l=0}^{2q-i-j}\int\frac{d^4k}{(2\pi)^4}B(i,j,l)\\ &+all\ possible\ permutations\ of\ \{(\mu,a,p_1),\ (\nu,b,p_2),\ (\rho,c,p_3)\}.\end{aligned} \tag{55}$$

where

$$
\begin{aligned}
A(i,j,l) &= Tr[\gamma_5\Big(\frac{1}{\not{k}-m}\Big)^{j+1}\gamma_\mu\Big(\frac{1}{\not{k}-\not{p}_1-m}\Big)^{l+1}\gamma_\nu\Big(\frac{1}{\not{k}-\not{p}_1-\not{p}_2-m}\Big)^{2q-i-j-l+1}\gamma_\rho\Big(\frac{1}{\not{k}-\not{p}_1-\not{p}_2-\not{p}_3-m}\Big)^{i}] \\
B(i,j,l) &= Tr[\gamma_5\Big(\frac{1}{\not{k}-m}\Big)^{j+1}\gamma_\mu\Big(\frac{1}{\not{k}-\not{p}_1-m}\Big)^{l+1}\gamma_\nu\Big(\frac{1}{\not{k}-\not{p}_1-\not{p}_2-m}\Big)^{2q-i-j-l+1}\gamma_\rho\Big(\frac{1}{\not{k}-\not{p}_1-\not{p}_2-\not{p}_3-m}\Big)^{i+1}].
\end{aligned}
\tag{56}
$$

To evaluate the trace of gamma matrices in $A(i,j,l)$, we divide the summation in $A(i,j,l)$ into eight classes:

$$
\begin{aligned}
\sum_{i=1}^{2q}\sum_{j=0}^{2q-i}\sum_{l=0}^{2q-i-j} A(i,j,l) &= \sum_{\alpha=1}^{q}\sum_{\beta=0}^{q-\alpha}\sum_{\gamma=0}^{q-\alpha-\beta} A(2\alpha,2\beta,2\gamma) + \sum_{\alpha=1}^{q}\sum_{\beta=0}^{q-\alpha}\sum_{\gamma=0}^{q-\alpha-\beta} A(2\alpha-1,2\beta,2\gamma) \\
&+ \sum_{\alpha=1}^{q}\sum_{\beta=1}^{q-\alpha}\sum_{\gamma=0}^{q-\alpha-\beta} A(2\alpha,2\beta-1,2\gamma) + \sum_{\alpha=1}^{q}\sum_{\beta=0}^{q-\alpha}\sum_{\gamma=0}^{q-\alpha-\beta} A(2\alpha-1,2\beta+1,2\gamma) \\
&+ \sum_{\alpha=1}^{q}\sum_{\beta=0}^{q-\alpha}\sum_{\gamma=1}^{q-\alpha-\beta} A(2\alpha,2\beta,2\gamma-1) + \sum_{\alpha=1}^{q}\sum_{\beta=0}^{q-\alpha}\sum_{\gamma=0}^{q-\alpha-\beta} A(2\alpha-1,2\beta,2\gamma+1) \\
&+ \sum_{\alpha=1}^{q}\sum_{\beta=1}^{q-\alpha}\sum_{\gamma=0}^{q-\alpha-\beta} A(2\alpha,2\beta-1,2\gamma+1) + \sum_{\alpha=1}^{q}\sum_{\beta=0}^{q-\alpha}\sum_{\gamma=1}^{q-\alpha-\beta} A(2\alpha-1,2\beta+1,2\gamma-1),
\end{aligned}
\tag{57}
$$

where

$$
\begin{aligned}
A(2\alpha,2\beta,2\gamma) &= Tr[\gamma_5\,\not{k}\gamma_\mu(\not{k}-\not{p}_1)\gamma_\nu(\not{k}-\not{p}_1-\not{p}_2)\gamma_\rho] \\
&\times\frac{(k^2)^\beta[(k-p_1)^2]^\gamma[(k-p_1-p_2)^2]^{q-\alpha-\beta-\gamma}[(k-p_1-p_2-p_3)^2]^\alpha}{(k^2-m^2)^{2\beta+1}[(k-p_1)^2-m^2]^{2\gamma+1}[(k-p_1-p_2)^2-m^2]^{2q-2\alpha-2\beta-2\gamma+1}[(k-p_1-p_2-p_3)^2-m^2]^{2\alpha}}, \qquad (58a)\\
A(2\alpha-1,2\beta,2\gamma) &= Tr[\gamma_5\,\not{k}\gamma_\mu(\not{k}-\not{p}_1)\gamma_\nu\gamma_\rho(\not{k}-\not{p}_1-\not{p}_2-\not{p}_3)] \\
&\times\frac{(k^2)^\beta[(k-p_1)^2]^\gamma[(k-p_1-p_2)^2]^{q-\alpha-\beta-\gamma+1}[(k-p_1-p_2-p_3)^2]^\alpha}{(k^2-m^2)^{2\beta+1}[(k-p_1)^2-m^2]^{2\gamma+1}[(k-p_1-p_2)^2-m^2]^{2q-2\alpha-2\beta-2\gamma+2}[(k-p_1-p_2-p_3)^2-m^2]^{2\alpha}}, \qquad (58b)\\
A(2\alpha,2\beta-1,2\gamma) &= Tr[\gamma_5\gamma_\mu(\not{k}-\not{p}_1)\gamma_\nu\gamma_\rho] \\
&\times\frac{(k^2)^\beta[(k-p_1)^2]^\gamma[(k-p_1-p_2)^2]^{q-\alpha-\beta-\gamma+1}[(k-p_1-p_2-p_3)^2]^\alpha}{(k^2-m^2)^{2\beta}[(k-p_1)^2-m^2]^{2\gamma+1}[(k-p_1-p_2)^2-m^2]^{2q-2\alpha-2\beta-2\gamma+2}[(k-p_1-p_2-p_3)^2-m^2]^{2\alpha}}, \qquad (58c)\\
A(2\alpha-1,2\beta+1,2\gamma) &= Tr[\gamma_5\gamma_\mu(\not{k}-\not{p}_1)\gamma_\nu(\not{k}-\not{p}_1-\not{p}_2)\gamma_\rho(\not{k}-\not{p}_1-\not{p}_2-\not{p}_3)] \\
&\times\frac{(k^2)^{\beta+1}[(k-p_1)^2]^\gamma[(k-p_1-p_2)^2]^{q-\alpha-\beta-\gamma}[(k-p_1-p_2-p_3)^2]^\alpha}{(k^2-m^2)^{2\beta+2}[(k-p_1)^2-m^2]^{2\gamma+1}[(k-p_1-p_2)^2-m^2]^{2q-2\alpha-2\beta-2\gamma+1}[(k-p_1-p_2-p_3)^2-m^2]^{2\alpha}}, \qquad (58d)\\
A(2\alpha,2\beta,2\gamma-1) &= Tr[\gamma_5\,\not{k}\gamma_\mu\gamma_\nu\gamma_\rho] \\
&\times\frac{(k^2)^\beta[(k-p_1)^2]^\gamma[(k-p_1-p_2)^2]^{q-\alpha-\beta-\gamma+1}[(k-p_1-p_2-p_3)^2]^\alpha}{(k^2-m^2)^{2\beta+1}[(k-p_1)^2-m^2]^{2\gamma}[(k-p_1-p_2)^2-m^2]^{2q-2\alpha-2\beta-2\gamma+2}[(k-p_1-p_2-p_3)^2-m^2]^{2\alpha}}, \qquad (58e)\\
A(2\alpha-1,2\beta,2\gamma+1) &= Tr[\gamma_5\,\not{k}\gamma_\mu\gamma_\nu(\not{k}-\not{p}_1-\not{p}_2)\gamma_\rho(\not{k}-\not{p}_1-\not{p}_2-\not{p}_3)] \\
&\times\frac{(k^2)^\beta[(k-p_1)^2]^{\gamma+1}[(k-p_1-p_2)^2]^{q-\alpha-\beta-\gamma}[(k-p_1-p_2-p_3)^2]^\alpha}{(k^2-m^2)^{2\beta+1}[(k-p_1)^2-m^2]^{2\gamma+2}[(k-p_1-p_2)^2-m^2]^{2q-2\alpha-2\beta-2\gamma+1}[(k-p_1-p_2-p_3)^2-m^2]^{2\alpha}}, \qquad (58f)\\
A(2\alpha,2\beta-1,2\gamma+1) &= Tr[\gamma_5\gamma_\mu\gamma_\nu(\not{k}-\not{p}_1-\not{p}_2)\gamma_\rho] \\
&\times\frac{(k^2)^\beta[(k-p_1)^2]^{\gamma+1}[(k-p_1-p_2)^2]^{q-\alpha-\beta-\gamma}[(k-p_1-p_2-p_3)^2]^\alpha}{(k^2-m^2)^{2\beta}[(k-p_1)^2-m^2]^{2\gamma+2}[(k-p_1-p_2)^2-m^2]^{2q-2\alpha-2\beta-2\gamma+1}[(k-p_1-p_2-p_3)^2-m^2]^{2\alpha}}, \qquad (58g)\\
A(2\alpha-1,2\beta+1,2\gamma-1) &= Tr[\gamma_5\gamma_\mu\gamma_\nu\gamma_\rho(\not{k}-\not{p}_1-\not{p}_2-\not{p}_3)] \\
&\times\frac{(k^2)^{\beta+1}[(k-p_1)^2]^\gamma[(k-p_1-p_2)^2]^{q-\alpha-\beta-\gamma+1}[(k-p_1-p_2-p_3)^2]^\alpha}{(k^2-m^2)^{2\beta+2}[(k-p_1)^2-m^2]^{2\gamma}[(k-p_1-p_2)^2-m^2]^{2q-2\alpha-2\beta-2\gamma+2}[(k-p_1-p_2-p_3)^2-m^2]^{2\alpha}}, \qquad (58h)
\end{aligned}
$$

and so does for $A(i+1,j-1,l)$. By neglecting the mass terms in the numerators of the above equations, and considering one by one all cases from (58a) to (58h), we get

$$
\begin{aligned}
\sum_{\alpha=1}^{q}\sum_{\beta=0}^{q-\alpha}\sum_{\gamma=0}^{q-\alpha-\beta}\int\frac{d^4k}{(2\pi)^4}A(2\alpha,2\beta,2\gamma) &= -\frac{1}{4\pi^2}\epsilon_{\mu\nu\rho\sigma}\Big(\frac{1}{24}p_1^\sigma+\frac{1}{12}p_2^\sigma+\frac{1}{8}p_3^\sigma\Big)+o(q)\,,\\
\sum_{\alpha=1}^{q}\sum_{\beta=0}^{q-\alpha}\sum_{\gamma=0}^{q-\alpha-\beta}\int\frac{d^4k}{(2\pi)^4}A(2\alpha-1,2\beta,2\gamma) &= -\frac{1}{4\pi^2}\epsilon_{\mu\nu\rho\sigma}\Big(\frac{1}{8}p_1^\sigma-\frac{1}{12}p_2^\sigma+\frac{1}{24}p_3^\sigma\Big)+o(q)\,.\\
\sum_{\alpha=1}^{q}\sum_{\beta=1}^{q-\alpha}\sum_{\gamma=0}^{q-\alpha-\beta}\int\frac{d^4k}{(2\pi)^4}A(2\alpha,2\beta-1,2\gamma) &= -\frac{1}{4\pi^2}\epsilon_{\mu\nu\rho\sigma}\Big(\frac{1}{24}p_1^\sigma-\frac{1}{12}p_2^\sigma-\frac{1}{24}p_3^\sigma\Big)+o(q)\,,\\
\sum_{\alpha=1}^{q}\sum_{\beta=0}^{q-\alpha}\sum_{\gamma=0}^{q-\alpha-\beta}\int\frac{d^4k}{(2\pi)^4}A(2\alpha-1,2\beta+1,2\gamma) &= -\frac{1}{4\pi^2}\epsilon_{\mu\nu\rho\sigma}\Big(\frac{1}{8}p_1^\sigma+\frac{1}{12}p_2^\sigma+\frac{1}{24}p_3^\sigma\Big)+o(q)\,,\\
\sum_{\alpha=1}^{q}\sum_{\beta=0}^{q-\alpha}\sum_{\gamma=1}^{q-\alpha-\beta}\int\frac{d^4k}{(2\pi)^4}A(2\alpha,2\beta,2\gamma-1) &= -\frac{1}{4\pi^2}\epsilon_{\mu\nu\rho\sigma}\Big(\frac{1}{8}p_1^\sigma+\frac{1}{12}p_2^\sigma+\frac{1}{24}p_3^\sigma\Big)+o(q)\,,\\
\sum_{\alpha=1}^{q}\sum_{\beta=0}^{q-\alpha}\sum_{\gamma=0}^{q-\alpha-\beta}\int\frac{d^4k}{(2\pi)^4}A(2\alpha-1,2\beta,2\gamma+1) &= -\frac{1}{4\pi^2}\epsilon_{\mu\nu\rho\sigma}\Big(\frac{1}{24}p_1^\sigma-\frac{1}{12}p_2^\sigma+\frac{1}{8}p_3^\sigma\Big)+o(q)\,,\\
\sum_{\alpha=1}^{q}\sum_{\beta=1}^{q-\alpha}\sum_{\gamma=0}^{q-\alpha-\beta}\int\frac{d^4k}{(2\pi)^4}A(2\alpha,2\beta-1,2\gamma+1) &= -\frac{1}{4\pi^2}\epsilon_{\mu\nu\rho\sigma}\Big(-\frac{1}{24}p_1^\sigma-\frac{1}{12}p_2^\sigma+\frac{1}{24}p_3^\sigma\Big)+o(q)\,,\\
\sum_{\alpha=1}^{q}\sum_{\beta=0}^{q-\alpha}\sum_{\gamma=1}^{q-\alpha-\beta}\int\frac{d^4k}{(2\pi)^4}A(2\alpha-1,2\beta+1,2\gamma-1) &= -\frac{1}{4\pi^2}\epsilon_{\mu\nu\rho\sigma}\Big(\frac{1}{24}p_1^\sigma+\frac{1}{12}p_2^\sigma+\frac{1}{8}p_3^\sigma\Big)+o(q)\,,
\end{aligned}
\tag{59}
$$

Summing over the contributions of eqs.(59) yields

$$
\frac{1}{N_q}\sum_{i=1}^{2q}\sum_{j=0}^{2q-i}\sum_{l=0}^{2q-i-j}\int\frac{d^4k}{(2\pi)^4}A(i,j,l) = -\frac{1}{8\pi^2}\epsilon_{\mu\nu\rho\sigma}(p_1^\sigma+p_3^\sigma)+o(q)\,. \tag{60}
$$

In the same way, for $A(i+1,j-1,l)$ we have

$$
\frac{1}{N_q}\sum_{i=0}^{2q}\sum_{j=1}^{2q-i}\sum_{l=0}^{2q-i-j}\int\frac{d^4k}{(2\pi)^4}A(i+1,j-1,l) = -\frac{1}{8\pi^2}\epsilon_{\mu\nu\rho\sigma}(p_1^\sigma+p_3^\sigma)+o(q)\,. \tag{61}
$$

Therefore, from (55), (60), and (61), it is easy to see that the anomaly arising from VVVA diagrams is

$$
\begin{aligned}
\Delta_{VVVA} = -\frac{i}{4\pi^2}\epsilon_{\mu\nu\rho\sigma}\{&Tr(\mathbf{T}_V^a\mathbf{T}_V^b\mathbf{T}_V^c\mathbf{T}_A^d-\mathbf{T}_V^c\mathbf{T}_V^b\mathbf{T}_V^a\mathbf{T}_A^d)(p_1^\sigma+p_3^\sigma)\\
&+Tr(\mathbf{T}_V^b\mathbf{T}_V^c\mathbf{T}_V^a\mathbf{T}_A^d-\mathbf{T}_V^a\mathbf{T}_V^c\mathbf{T}_V^b\mathbf{T}_A^d)(p_1^\sigma+p_2^\sigma)\\
&+Tr(\mathbf{T}_V^c\mathbf{T}_V^a\mathbf{T}_V^b\mathbf{T}_A^d-\mathbf{T}_V^b\mathbf{T}_V^a\mathbf{T}_V^c\mathbf{T}_A^d)(p_2^\sigma+p_3^\sigma)\}\,.
\end{aligned}
\tag{62}
$$

After a straightforward (but tedious) calculation, we find that for all other diagrams, the vector Ward identities are preserved in our regularization method. And the anomalies are as follows:

$$
\Delta_{VAAA} = \frac{i}{12\pi^2}\epsilon_{\mu\nu\rho\sigma}\{Tr(\mathbf{T}_V^a\mathbf{T}_V^b\mathbf{T}_V^c\mathbf{T}_A^d-\mathbf{T}_V^c\mathbf{T}_V^b\mathbf{T}_V^a\mathbf{T}_A^d)(p_1^\sigma-p_3^\sigma)
$$

$$+Tr(\mathbf{T}_V^b\mathbf{T}_V^c\mathbf{T}_V^a\mathbf{T}_A^d - \mathbf{T}_V^a\mathbf{T}_V^c\mathbf{T}_V^b\mathbf{T}_A^d)(p_1^\sigma - p_2^\sigma)$$
$$-Tr(\mathbf{T}_V^c\mathbf{T}_V^a\mathbf{T}_V^b\mathbf{T}_A^d - \mathbf{T}_V^b\mathbf{T}_V^a\mathbf{T}_V^c\mathbf{T}_A^d)(4p_1^\sigma + p_2^\sigma + p_3^\sigma)\} . \qquad (63)$$

$$\Delta_{VVVVA} = -\frac{1}{4\pi^2}\epsilon_{\mu\nu\rho\sigma} \sum_{all\ permu.\ of\ (abce)} \epsilon(abce)Tr(\mathbf{T}_V^a\mathbf{T}_V^b\mathbf{T}_V^c\mathbf{T}_V^e\mathbf{T}_A^d) . \qquad (64)$$

$$\Delta_{AAAAA} = \frac{1}{12\pi^2}\epsilon_{\mu\nu\rho\sigma} \sum_{all\ permu.\ of\ (abce)} \epsilon(abce)Tr(\mathbf{T}_V^a\mathbf{T}_V^b\mathbf{T}_V^c\mathbf{T}_V^e\mathbf{T}_A^d) . \qquad (65)$$

$$\Delta_{VVAAA} = \frac{1}{12\pi^2}\epsilon_{\mu\nu\rho\sigma}\{(Z^{abced} + Z^{acbed} + Z^{baecd} + Z^{beacd} + Z^{aecbd} + 3Z^{cabed})$$
$$-(a \leftrightarrow b)\} . \qquad (66)$$

where $Z^{abced} = Tr(\mathbf{T}_V^a\mathbf{T}_V^b\mathbf{T}_V^c\mathbf{T}_V^e\mathbf{T}_A^d + \mathbf{T}_A^d\mathbf{T}_V^e\mathbf{T}_V^c\mathbf{T}_V^b\mathbf{T}_V^a)$, and $\epsilon(abce) = +1$ for even permutation of (abce), and $\epsilon(abce) = -1$ for odd permutation of (abce).

In the coordinate space, the above equations (62), (63), (64), (65), and (66), together with (50), and (51), amount to the anomalous divergence equation of the axial-vector:

$$\partial^\mu J_{5\mu}^a = \frac{1}{4\pi^2}\epsilon_{\mu\nu\rho\sigma}Tr[\mathbf{T}_A^a(\frac{1}{4}F_V^{\mu\nu}F_V^{\rho\sigma} + \frac{1}{12}F_A^{\mu\nu}F_A^{\rho\sigma}$$
$$-\frac{2i}{3}A^\mu A^\nu F_V^{\rho\sigma} - \frac{2i}{3}A^\mu F_V^{\nu\rho}A^\sigma - \frac{2i}{3}F_V^{\mu\nu}A^\rho A^\sigma - \frac{8}{3}A^\mu A^\nu A^\rho A^\sigma)] , \qquad (67)$$

which is exactly the famous Bardeen anomaly [3]. It should be noted that, unlike other regularization methods [1, 2, 3, 11], in our method, there is no need to introduce any counterterms, since all vector Ward identities are automatically preserved by the regularized amplitudes. As a result, the direct calculations of the diagrams involving anomaly exactly give rise to the Bardeen anomaly.

5 Concluding Remarks

In this paper, we have studied the problem of how to analyze the chiral anomalies from the intrinsic regularization method. In the framework of our method, with the help of the intrinsic relations between the divergent diagrams and the convergent ones which we explained in refs. [17] and [18], a divergent diagram is simply regularized as a limit case of the convergent ones which are originally contained in the theory and share the same loop skeleton with the divergent one, and nothing else changes. Therefore, the ambiguity of shifting the integral variable in a linearly or more highly divergent integral, which has been the only obstacle of verifying the normal Ward identities

related to divergent diagrams, is naively removed, and consequently all vector Ward identities are preserved in a much natural and intuitive way. As a inevitable result, the anomalies occurring to the axial-vector Ward identities are exactly the Bardeen anomalies, in no need of introducing any counterterms.

We are to make a little remarks on this point. As is well known, since the chiral anomalies were analyzed from the path integral point of view [5] and geometrical point of view [6, 7, 8], physicists have realized that the anomalies are actually inescapable non-perturbative effects, but not merely artifacts of the regularization as a result of the violation of some symmetrical and/or topological properties of the original theory. Even so, various regularization methods still affect the results of the calculations of the anomalies in their own ways. It turns out that different regularization methods yield different sets of anomalies, and in general both vector Ward identities and axial-vector Ward identities are violated after the regularization. Only through a procedure of introducing various counterterms to the diagrams involving anomalies to restore the vector Ward identities which are vital to renormalization of the theory, can we extract the "real anomalies" which are independent of the choice of the regularization method. What we would like to comment is that although the violation of symmetrical and/or topological properties of the original theory resulting from the regularization does not account for the existence of the anomalies, it does account for the difference between the anomalies obtained from different regularization methods. With this regard, we should not be surprised about that our method directly yields the "real anomalies" in no need of introducing any counterterms, since the method is "intrinsic" from the viewpoint of the standard model, that is to say, there is nothing changed, the action , the Feynman rules, the spacetime dimensions etc., are all the same as those in the original theory.

Appendix: Integral Expressions of the One Loop Diagrams In Gauge Theories Involving Chiral Anomaly

There are a number of one loop diagrams in gauge theories involving chiral anomaly. In Abelian gauge theory, only two kinds triangle diagrams involve anomaly, *i.e.*, VVA, AAA triangle diagrams. In non-Abelian gauge theory, not only VVA, AAA triangle diagrams, but also VVVA, VAAA box diagrams, and VVVVA, AAAAA, VVAAA pentagon diagrams involve anomaly. All of these diagrams are illustrated in Fig.2–Fig.8. In what follows we present the integral expressions of the regularized diagrams in the momentum space.

1. The regularized integral expressions for VVA diagrams in Abelian gauge theory as shown in Fig.2:

For Fig.2a, we have

$$
\begin{aligned}
T^R_{\mu\nu\rho}(p_1,\ p_2;\ q;\ \mu) &= \frac{\mu^{2q}}{N_q}\sum_{i=0}^{2q}\sum_{j=0}^{2q-i} T^R_{\mu\nu\rho}(p_1,\ p_2;\ q)_{ij} \\
&= -\mu^{2q}\frac{1}{N_q}\sum_{i=0}^{2q}\sum_{j=0}^{2q-i}\int\frac{d^4k}{(2\pi)^4}Tr[\gamma_\mu\Big(\frac{1}{\not{k}-m}\Big)^{2q-i-j+1}\gamma_\nu\Big(\frac{1}{\not{k}-\not{p}_2-m}\Big)^{i+1}\gamma_\rho\gamma_5\Big(\frac{1}{\not{k}+\not{p}_1-m}\Big)^{j+1}],
\end{aligned}
\tag{68}
$$

where $N_q = \frac{1}{2}(q+1)(q+2)$ is the total number of possible ways in which the inserters are inserted. The regularized integral expression for Fig.2b can be obtained from (68) by interchanging (μ, p_1) and (ν, p_2) with each other.

2. The regularized integral expressions for AAA diagrams in Abelian gauge theory as shown in Fig.3:

For Fig.3a, we have

$$
\begin{aligned}
T'^R_{\mu\nu\rho}(p_1,\ p_2;\ q;\ \mu) &= \frac{\mu^{2q}}{N_q}\sum_{i=0}^{2q}\sum_{j=0}^{2q-i} T'^R_{\mu\nu\rho}(p_1,\ p_2;\ q)_{ij} \\
&= -\frac{1}{N_q}\sum_{i=0}^{2q}\sum_{j=0}^{2q-i}\int\frac{d^4k}{(2\pi)^4}Tr[\gamma_\mu\gamma_5\Big(\frac{1}{\not{k}-m}\Big)^{2q-i-j+1}\gamma_\nu\gamma_5\Big(\frac{1}{\not{k}-\not{p}_2-m}\Big)^{i+1}\gamma_\rho\gamma_5\Big(\frac{1}{\not{k}+\not{p}_1-m}\Big)^{j+1}],
\end{aligned}
\tag{69}
$$

where N_q is the same as that in (68). The regularized integral expression for Fig.3b can be obtained from (69) by interchanging $(\mu,\ p_1)$ and $(\nu,\ p_2)$ with each other.

3. The regularized integral expressions for VVA diagrams in non-Abelian gauge theory as shown in Fig.4:

The regularized integral expression for Fig.4a is just that for Fig.2a, *i.e.*, eq.(68), multiplied by a color factor $Tr(\mathbf{T}^a_V\mathbf{T}^b_V\mathbf{T}^c_A)$: $Tr(\mathbf{T}^a_V\mathbf{T}^b_V\mathbf{T}^c_A)T^R_{\mu\nu\rho}(p_1,\ p_2;\ q;\ \mu)$. And the regularized integral expression for Fig.4b is $Tr(\mathbf{T}^b_V\mathbf{T}^a_V\mathbf{T}^c_A)T^R_{\nu\mu\rho}(p_2,\ p_1;\ q;\ \mu)$.

4. The regularized integral expressions for AAA diagrams in non-Abelian gauge theory as shown in Fig.5:

The regularized integral expression for Fig.5a is just that for Fig.3a, *i.e.*, eq.(69), multiplied by a color factor $Tr(\mathbf{T}^a_A\mathbf{T}^b_A\mathbf{T}^c_A)$: $Tr(\mathbf{T}^a_A\mathbf{T}^b_A\mathbf{T}^c_A)T'^R_{\mu\nu\rho}(p_1,\ p_2;\ q;\ \mu)$. And the regularized integral expression for Fig.5b is $Tr(\mathbf{T}^b_A\mathbf{T}^a_A\mathbf{T}^c_A)T'^R_{\nu\mu\rho}(p_2,\ p_1;\ q;\ \mu)$.

5. The regularized integral expressions for VVVA diagrams in non-Abelian gauge theory as shown in Fig.6:

For Fig.6a, we have

$$
\begin{aligned}
&Tr(\mathbf{T}^a_V\mathbf{T}^b_V\mathbf{T}^c_V\mathbf{T}^d_A)T^R_{\mu\nu\rho\sigma}(p_1,\ p_2,\ p_3;\ q;\ \mu) \\
&= Tr(\mathbf{T}^a_V\mathbf{T}^b_V\mathbf{T}^c_V\mathbf{T}^d_A)\frac{\mu^{2q}}{N_q}\sum_{i=0}^{2q}\sum_{j=0}^{2q-i}\sum_{l=0}^{2q-i-j} T^R_{\mu\nu\rho\sigma}(p_1,\ p_2,\ p_3;\ q)_{ijl} \\
&= -iTr(\mathbf{T}^a_V\mathbf{T}^b_V\mathbf{T}^c_V\mathbf{T}^d_A)\frac{\mu^{2q}}{N_q}\sum_{i=0}^{2q}\sum_{j=0}^{2q-i}\sum_{l=0}^{2q-i-j}\int\frac{d^4k}{(2\pi)^4}Tr[\gamma_\sigma\gamma_5\Big(\frac{1}{\not{k}-m}\Big)^{j+1}\gamma_\mu\Big(\frac{1}{\not{k}-\not{p}_1-m}\Big)^{l+1} \\
&\quad\times\gamma_\nu\Big(\frac{1}{\not{k}-\not{p}_1-\not{p}_2-m}\Big)^{2q-i-j-l+1}\gamma_\rho\Big(\frac{1}{\not{k}-\not{p}_1-\not{p}_2-\not{p}_3-m}\Big)^{i+1}],
\end{aligned}
\tag{70}
$$

where $N_q = \frac{1}{6}(q+1)(q+2)(q+3)$. The regularized integral expressions for Fig.6b-f can be obtained from (70) by permuting (μ, a, p_1), (ν, b, p_2), and (ρ, c, p_3) in all possible ways.

6. The regularized integral expressions for VAAA diagrams in non-Abelian gauge theory as shown in Fig.7:

For Fig.7a, we have

$$
\begin{aligned}
&Tr(\mathbf{T}^a_V\mathbf{T}^b_A\mathbf{T}^c_A\mathbf{T}^d_A)T'^R_{\mu\nu\rho\sigma}(p_1,\ p_2,\ p_3;\ q;\ \mu) \\
&= Tr(\mathbf{T}^a_V\mathbf{T}^b_A\mathbf{T}^c_A\mathbf{T}^d_A)\frac{\mu^{2q}}{N_q}\sum_{i=0}^{2q}\sum_{j=0}^{2q-i}\sum_{l=0}^{2q-i-j} T'^R_{\mu\nu\rho\sigma}(p_1,\ p_2,\ p_3;\ q)_{ijl} \\
&= -iTr(\mathbf{T}^a_V\mathbf{T}^b_A\mathbf{T}^c_A\mathbf{T}^d_A)\frac{\mu^{2q}}{N_q}\sum_{i=0}^{2q}\sum_{j=0}^{2q-i}\sum_{l=0}^{2q-i-j}\int\frac{d^4k}{(2\pi)^4}Tr[\gamma_\sigma\gamma_5\Big(\frac{1}{\not{k}-m}\Big)^{j+1}\gamma_\mu\Big(\frac{1}{\not{k}-\not{p}_1-m}\Big)^{l+1} \\
&\quad\times\gamma_\nu\gamma_5\Big(\frac{1}{\not{k}-\not{p}_1-\not{p}_2-m}\Big)^{2q-i-j-l+1}\gamma_\rho\gamma_5\Big(\frac{1}{\not{k}-\not{p}_1-\not{p}_2-\not{p}_3-m}\Big)^{i+1}],
\end{aligned}
\tag{71}
$$

where N_q is the same as that in (70). The regularized integral expressions for Fig.6b-f can be obtained from (71) by permuting (μ, a, p_1), (ν, b, p_2), and (ρ, c, p_3) in all possible ways.

7. The regularized integral expression for VVVVA diagram in non-Abelian gauge theory as shown in Fig.8a:

$$
\begin{aligned}
&Tr(\mathbf{T}^a_V\mathbf{T}^b_V\mathbf{T}^c_V\mathbf{T}^e_V\mathbf{T}^d_A)T^R_{\mu\nu\rho\tau\sigma}(p_1,\ p_2,\ p_3,\ p_4;\ q;\ \mu) \\
&= Tr(\mathbf{T}^a_V\mathbf{T}^b_V\mathbf{T}^c_V\mathbf{T}^e_V\mathbf{T}^d_A)\frac{\mu^{2q}}{N_q}\sum_{i=0}^{2q}\sum_{j=0}^{2q-i}\sum_{l=0}^{2q-i-j}\sum_{s=0}^{2q-i-j-l} T^R_{\mu\nu\rho\tau\sigma}(p_1,\ p_2,\ p_3,\ p_4;\ q)_{ijls} \\
&= Tr(\mathbf{T}^a_V\mathbf{T}^b_V\mathbf{T}^c_V\mathbf{T}^e_V\mathbf{T}^d_A)\frac{\mu^{2q}}{N_q}\sum_{i=0}^{2q}\sum_{j=0}^{2q-i}\sum_{l=0}^{2q-i-j}\sum_{s=0}^{2q-i-j-l}\int\frac{d^4k}{(2\pi)^4}Tr[\gamma_\sigma\gamma_5\Big(\frac{1}{\not{k}-m}\Big)^{j+1}\gamma_\mu\Big(\frac{1}{\not{k}-\not{p}_1-m}\Big)^{l+1} \\
&\quad\times\gamma_\nu\Big(\frac{1}{\not{k}-\not{p}_1-\not{p}_2-m}\Big)^{s+1}\gamma_\rho\Big(\frac{1}{\not{k}-\not{p}_1-\not{p}_2-\not{p}_3-m}\Big)^{2q-i-j-l-s+1}\gamma_\tau\Big(\frac{1}{\not{k}-\not{p}_1-\not{p}_2-\not{p}_3-\not{p}_4-m}\Big)^{i+1}],
\end{aligned}
\tag{72}
$$

where $N_q = \frac{1}{24}(q+1)(q+2)(q+3)(q+4)$.

8. The regularized integral expression for AAAAA diagram in non-Abelian gauge theory as shown in Fig.8b:

$$
\begin{aligned}
&Tr(\mathbf{T}^a_A\mathbf{T}^b_A\mathbf{T}^c_A\mathbf{T}^e_A\mathbf{T}^d_A)T''^R_{\mu\nu\rho\tau\sigma}(p_1,\ p_2,\ p_3,\ p_4;\ q;\ \mu)\\
&=Tr(\mathbf{T}^a_A\mathbf{T}^b_A\mathbf{T}^c_A\mathbf{T}^e_A\mathbf{T}^d_A)\frac{\mu^{2q}}{N_q}\sum_{i=0}^{2q}\sum_{j=0}^{2q-i}\sum_{l=0}^{2q-i-j}\sum_{s=0}^{2q-i-j-l}T''^R_{\mu\nu\rho\tau\sigma}(p_1,\ p_2,\ p_3,\ p_4;\ q)_{ijls}\\
&=Tr(\mathbf{T}^a_A\mathbf{T}^b_A\mathbf{T}^c_A\mathbf{T}^e_A\mathbf{T}^d_A)\frac{\mu^{2q}}{N_q}\sum_{i=0}^{2q}\sum_{j=0}^{2q-i}\sum_{l=0}^{2q-i-j}\sum_{s=0}^{2q-i-j-l}\int\frac{d^4k}{(2\pi)^4}Tr[\gamma_\sigma\gamma_5\left(\frac{1}{\not k-m}\right)^{j+1}\gamma_\mu\gamma_5\left(\frac{1}{\not k-\not p_1-m}\right)^{l+1}\\
&\times\gamma_\nu\gamma_5\left(\frac{1}{\not k-\not p_1-\not p_2-m}\right)^{s+1}\gamma_\rho\gamma_5\left(\frac{1}{\not k-\not p_1-\not p_2-\not p_3-m}\right)^{2q-i-j-l-s+1}\gamma_\tau\gamma_5\left(\frac{1}{\not k-\not p_1-\not p_2-\not p_3-\not p_4-m}\right)^{i+1}],
\end{aligned}
\tag{73}
$$

where N_q is the same as that in (72).

9. The regularized integral expression for VVAAA diagram in non-Abelian gauge theory as shown in Fig.8c:

$$
\begin{aligned}
&Tr(\mathbf{T}^a_V\mathbf{T}^b_V\mathbf{T}^c_A\mathbf{T}^e_A\mathbf{T}^d_A)T''^R_{\mu\nu\rho\tau\sigma}(p_1,\ p_2,\ p_3,\ p_4;\ q;\ \mu)\\
&=Tr(\mathbf{T}^a_V\mathbf{T}^b_V\mathbf{T}^c_A\mathbf{T}^e_A\mathbf{T}^d_A)\frac{\mu^{2q}}{N_q}\sum_{i=0}^{2q}\sum_{j=0}^{2q-i}\sum_{l=0}^{2q-i-j}\sum_{s=0}^{2q-i-j-l}T''^R_{\mu\nu\rho\tau\sigma}(p_1,\ p_2,\ p_3,\ p_4;\ q)_{ijls}\\
&=Tr(\mathbf{T}^a_V\mathbf{T}^b_V\mathbf{T}^c_A\mathbf{T}^e_A\mathbf{T}^d_A)\frac{\mu^{2q}}{N_q}\sum_{i=0}^{2q}\sum_{j=0}^{2q-i}\sum_{l=0}^{2q-i-j}\sum_{s=0}^{2q-i-j-l}\int\frac{d^4k}{(2\pi)^4}Tr[\gamma_\sigma\gamma_5\left(\frac{1}{\not k-m}\right)^{j+1}\gamma_\mu\left(\frac{1}{\not k-\not p_1-m}\right)^{l+1}\\
&\times\gamma_\nu\left(\frac{1}{\not k-\not p_1-\not p_2-m}\right)^{s+1}\gamma_\rho\gamma_5\left(\frac{1}{\not k-\not p_1-\not p_2-\not p_3-m}\right)^{2q-i-j-l-s+1}\gamma_\tau\gamma_5\left(\frac{1}{\not k-\not p_1-\not p_2-\not p_3-\not p_4-m}\right)^{i+1}],
\end{aligned}
\tag{74}
$$

where N_q is the same as that in (72).

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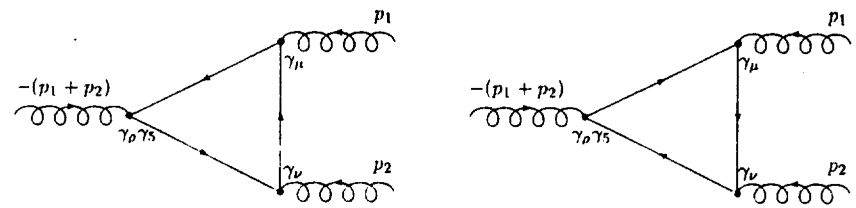


Fig.1a **Fig.1b**

Figure 1: The VVA triangle diagrams in the σ model needed for calculation of the anomaly.

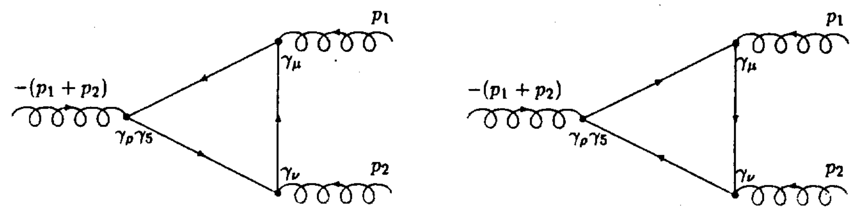


Fig.2a **Fig.2b**

Figure 2: The VVA triangle diagrams in Abelian gauge theory needed for calculation of the anomaly.

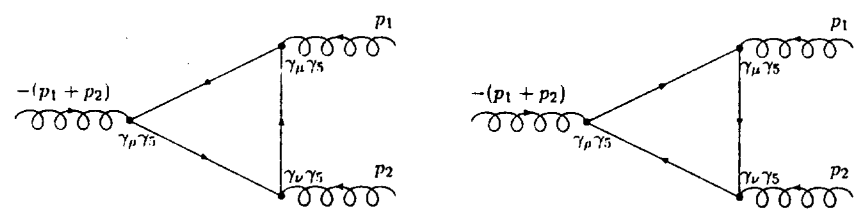


Fig.3a **Fig.3b**

Figure 3: The AAA triangle diagrams in Abelian gauge theory needed for calculation of the anomaly.

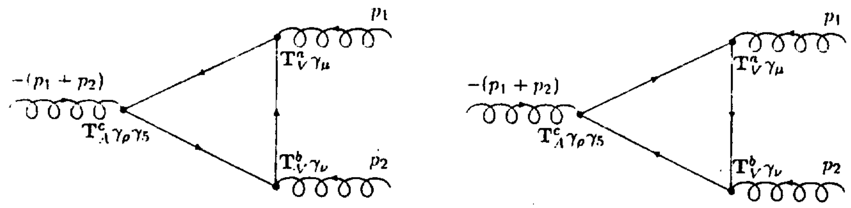


Fig.4a

Fig.4b

Figure 4: The VVA triangle diagrams in non-Abelian gauge theory needed for calculation of the non-Abelian gauge anomaly.

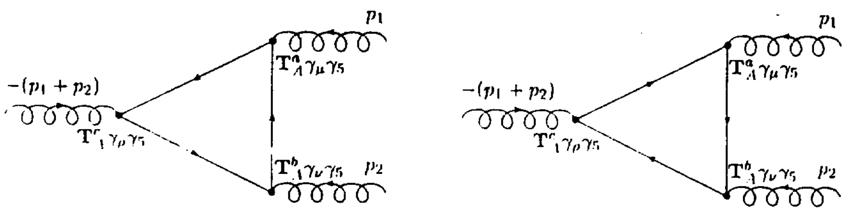


Fig.5a

Fig.5b

Figure 5: The AAA triangle diagrams in non-Abelian gauge theory needed for calculation of the non-Abelian gauge anomaly.

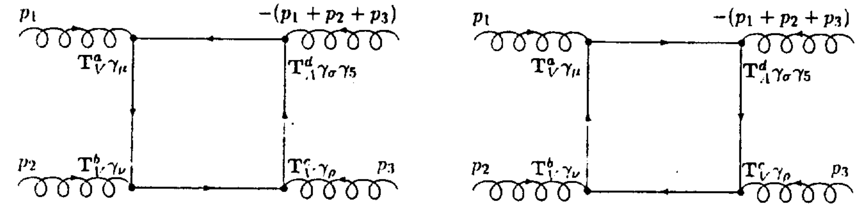


Fig.6a

Fig.6b

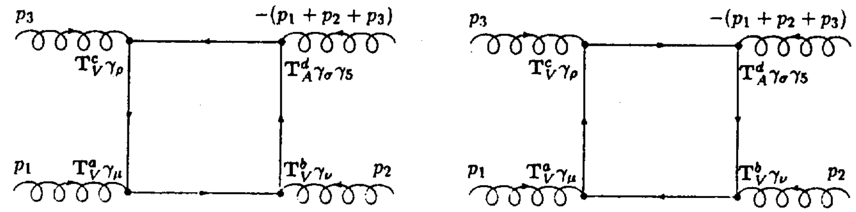


Fig.6c

Fig.6d

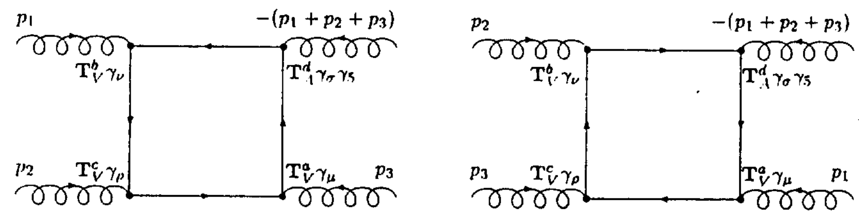


Fig.6e

Fig.6f

Figure 6: The VVVA box diagrams in non-Abelian gauge theory needed for calculation of the non-Abelian gauge anomaly.

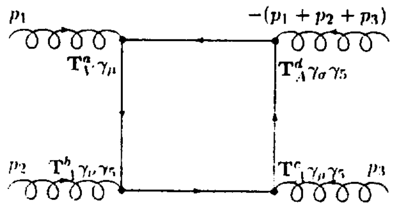


Fig.7a

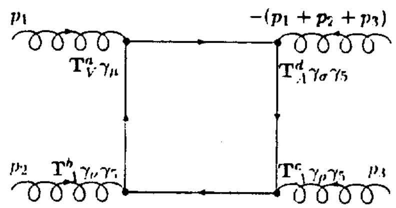


Fig.7b

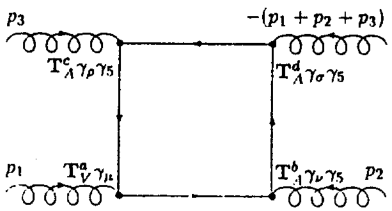


Fig.7c

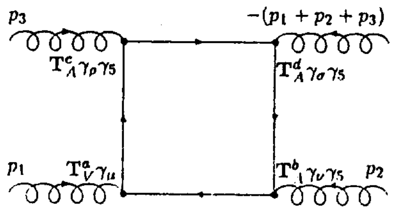


Fig.7d

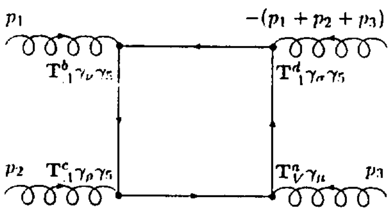


Fig.7e

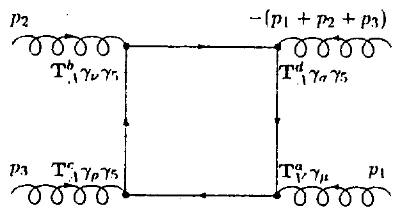
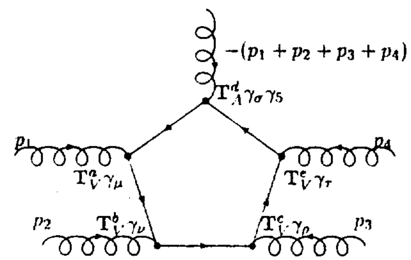


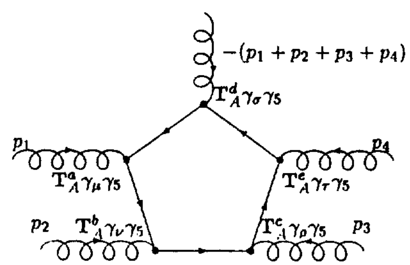
Fig.7f

Figure 7: The VAAA box diagrams in non-Abelian gauge theory needed for calculation of the non-Abelian gauge anomaly.



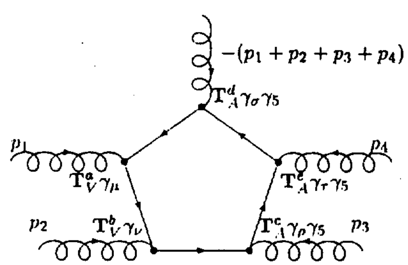
+ all possible permutations of
$\{(a\ \mu\ p_1),\ (b\ \nu\ p_2),\ (c\ \rho\ p_3),\ (e\ \tau\ p_4)\}$

Fig.8a VVVVA diagrams



+ all possible permutations of
$\{(a\ \mu\ p_1),\ (b\ \nu\ p_2),\ (c\ \rho\ p_3),\ (e\ \tau\ p_4)\}$

Fig.8b AAAAA diagrams



+ all possible permutations of
$\{(a\ \mu\ p_1),\ (b\ \nu\ p_2),\ (c\ \rho\ p_3),\ (e\ \tau\ p_4)\}$

Fig.8c VVAAA diagrams

Figure 8: The pentagon diagrams in non-Abelian gauge theory needed for calculation of the non-Abelian gauge anomaly.